The Big-Dot Product: An Altered Dot Product

John Wrobel

Plymouth State Mathematics Department

Introduction

The dot product is a common vector operation used to find vector projections and perpendicular vectors. Given $\vec{u}$ and $\vec{v}$ with angle θ between them, $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$. We alter the definition to get the big-dot product by replacing the $\cos(\theta)$ with a $\sin(\theta)$.

Definition: The Big-Dot Product

Given $\vec{u}$ and $\vec{v}$ with angle θ between them the big-dot product of $\vec{u}$ and $\vec{v}$ is $\vec{u} \bigcirc \vec{v} = |\vec{u}||\vec{v}|\sin(\theta)$.

Derivation of the Component-Form

Let $\vec{u}$ and $\vec{v}$ be vectors with θ being the angle between them. Also, θ_u is the angle between $\vec{u}$ and $x = 0$, and θ_v is the angle between $\vec{v}$ and $x = 0$. Then,

$$\begin{aligned}\vec{u} \bigcirc \vec{v} &= |\vec{u}||\vec{v}|\sin(\theta_u - \theta_v)\\ &= |\vec{u}||\vec{v}|(\sin(\theta_u)\cos(\theta_v) - \cos(\theta_u)\sin(\theta_v))\\ &= |\vec{u}|\sin(\theta_u)|\vec{v}|\cos(\theta_v) - |\vec{u}|\cos(\theta_u)|\vec{v}|\sin(\theta_v)\\ &= u_y v_x - u_x v_y.\end{aligned}$$

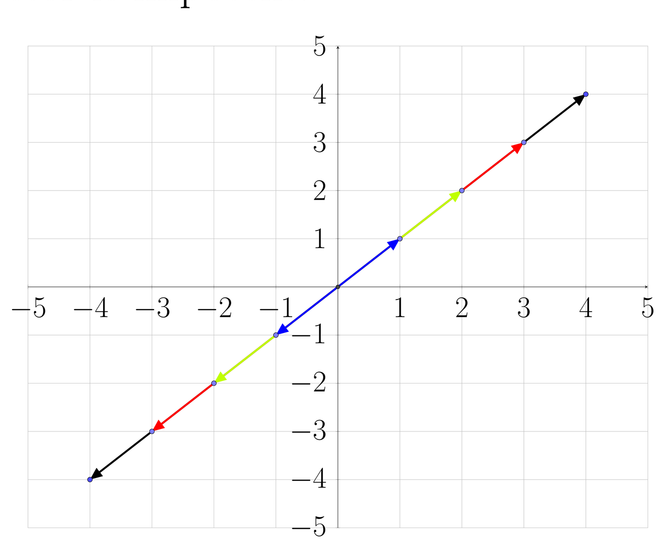
Properties

There are several properties that can be applied to the dot product and big-dot product. Given $\vec{u}$, $\vec{v}$, $\vec{w}$, and constants c and d the following properties are true.

1. $\vec{u} \bigcirc (\vec{v} + \vec{w}) = \vec{u} \bigcirc \vec{v} + \vec{u} \bigcirc \vec{w}$
2. $(\vec{u} + \vec{v}) \bigcirc \vec{w} = \vec{u} \bigcirc \vec{w} + \vec{v} \bigcirc \vec{w}$
3. $\vec{0} \bigcirc \vec{u} = 0$
4. $c\vec{u} \bigcirc d\vec{v} = cd(\vec{u} \bigcirc \vec{v})$

Not every property of the dot product is true for the big-dot product. The following are properties unique to the big-dot product.

5. $\vec{u} \bigcirc \vec{v} = -(\vec{v} \bigcirc \vec{u})$
6. $\vec{u} \bigcirc \vec{u} = 0$.
7. Every vector is nilpotent.



Given $\vec{u}$ and $\vec{v}$ where $\vec{v}$ is a scalar multiple of $\vec{u}$, then $\vec{u} \bigcirc \vec{v} = 0$ by properties 4 and 6.

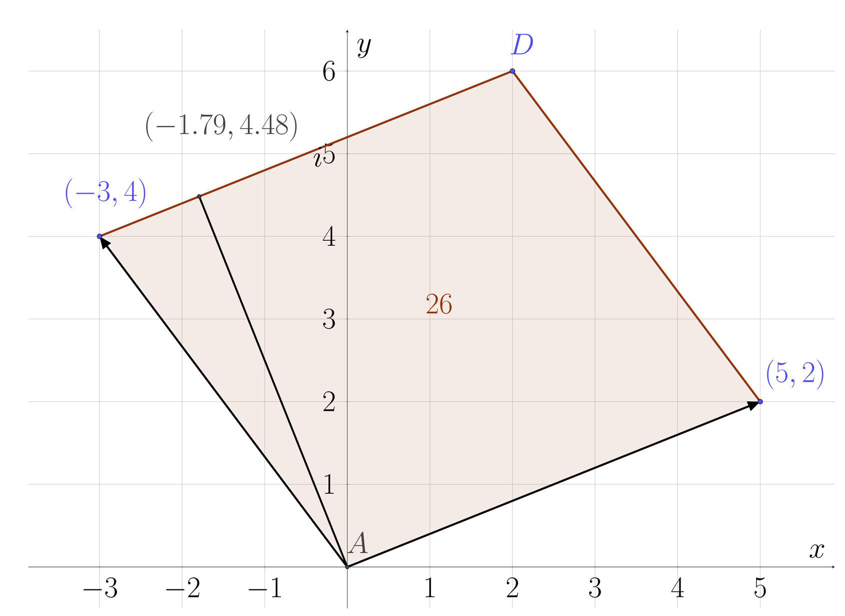
Areas

The big-dot product can result in the areas between the given vectors. Given $\vec{u}$ and $\vec{v}$ with angle θ between them,

- $|\vec{u} \bigcirc \vec{v}|$ is the area of the parallelogram with a pair of sides with lengths equal to $|\vec{u}|$ and the other pair with lengths equal to $|\vec{v}|$.
- $\frac{|\vec{u} \bigcirc \vec{v}|}{2}$ is the area of the triangle with vectors as two of the sides and the third side as the connection of the heads of the two vectors.

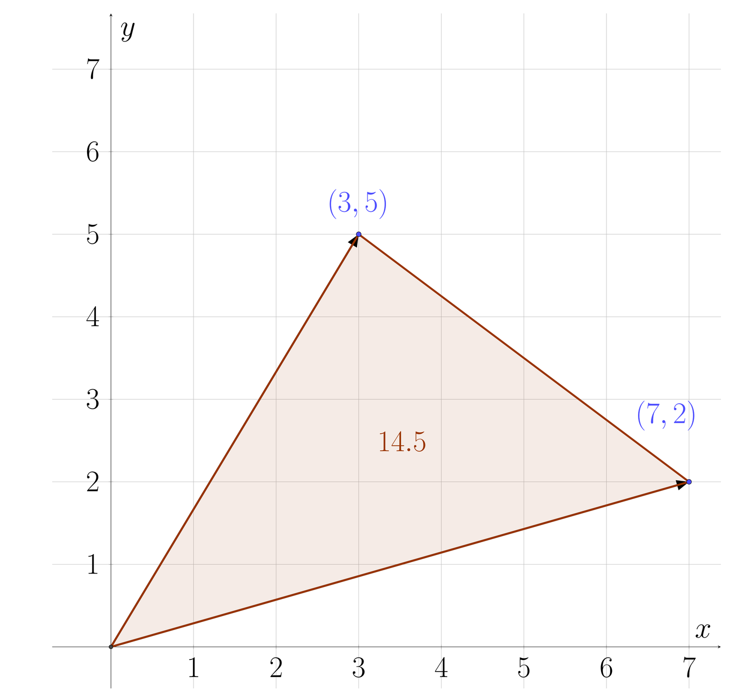
Parallelogram Example

Suppose $\vec{u} = \langle 5, 2\rangle$ and $\vec{v} = \langle -3, 4\rangle$. Then, $\vec{u} \bigcirc \vec{v} = 26$. This is the area of the parallelogram below. A base of the parallelogram, $|\vec{u}|$, is $\sqrt{29}$. Then, the height is $\frac{26}{\sqrt{29}}$. Then, the area is base multiplied by height which gives us 26.



Triangle Example

Suppose $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 7, 2\rangle$. Notice, $\vec{u} \bigcirc \vec{v} = 29$. Taking $\sqrt{53}$ as the base of the triangle, the height is $\frac{29}{\sqrt{53}}$. The area of the triangle is 14.5.



Relationship Between the Dot Product and the Big-Dot Product

Given $\vec{u}$ and $\vec{v}$, $|\vec{u}\cdot\vec{v}| = \sqrt{|\vec{u}|^2|\vec{v}|^2 - (\vec{u} \bigcirc \vec{v})^2}$.

Perpendicular Vectors

- If $\vec{u}$ and $\vec{v}$ are perpendicular vectors, then $\vec{u}\cdot\vec{v} = 0$.
- Using the relationship above, if $\vec{u}$ and $\vec{v}$ are perpendicular vectors, then
$$\sqrt{|\vec{u}|^2|\vec{v}|^2 - (\vec{u} \bigcirc \vec{v})^2} = 0.$$
- Also as a result of the definition when $\theta = \frac{\pi}{2}$, $|\vec{u} \bigcirc \vec{v}| = |\vec{u}||\vec{v}|$.
- This result leads to a couple of cases for more specific vectors.
- Alternatively, if $|\vec{u} \bigcirc \vec{v}| = 0$, then $\vec{u}$ and $\vec{v}$ are scalar multiples of one another.

Vector Projections

Given $\vec{u}$ and $\vec{v}$, the vector projection of $\vec{u}$ onto $\vec{v}$ is
$$\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}.$$
Using the conversion, the projection results in the proper magnitude, but not direction.

Perpendicular Vectors of the Same Magnitude

Given $\vec{u} = \langle u_x, u_y\rangle$ and $\vec{v} = \langle v_x, v_y\rangle$ where $|\vec{u}| = |\vec{v}|$ and $\vec{u}$ is perpendicular to $\vec{v}$,
$$\begin{aligned}|\vec{v} \bigcirc \vec{u}| &= v_x^2 + v_y^2\\ &= u_x^2 + u_y^2.\end{aligned}$$

Perpendicular Unit Vectors

If $\vec{u}$ and $\vec{v}$ are perpendicular, then
$$\begin{aligned}\left|\frac{\vec{u}}{|\vec{u}|} \bigcirc \frac{\vec{v}}{|\vec{v}|}\right| &= \frac{1}{|\vec{u}||\vec{v}|}|\vec{u} \bigcirc \vec{v}|\\ &= \frac{1}{|\vec{u}||\vec{v}|}(|\vec{u}||\vec{v}|)\\ &= 1.\end{aligned}$$

Determinant of a Matrix

Notice, given vectors $\vec{u} = \langle u_x, u_y\rangle$ and $\vec{v} = \langle v_x, v_y\rangle$,
$$\det\left(\begin{vmatrix} v_x & u_x\\ v_y & u_y\end{vmatrix}\right) = v_x u_y - u_x v_y = \vec{u} \bigcirc \vec{v}.$$

Conclusion

Comparing the dot product and big-dot product many of the properties are the same with a few changing. Finding areas involving vectors is a strength of the big-dot product, while currently it falls short with vector projections.

Further Research

- For the results that are not consistent, why do they change? What do they actually represent?
- When the dot product is directly replaced with the big-dot product, what does the vector projection become?
- Considering the conversion, is it possible to derive it without having an absolute value in the end result. If not are there patterns to know the direction of the result?
- Knowing the the big-dot product results in areas, is there a visual representation of the dot-product that can be useful in understanding it?

References

- Kosmala, Witold A. J. A Friendly Introduction to Analysis: Single and Multivariable. 2nd ed., Pearson - Prentice Hall, 2004.
- Weisstein, Eric W."Dot Product." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/DotProduct.html

Contact Information

- Web: http://www.plymouth.edu
- Email: jjw1020@plymouth.edu
- Phone: +1 (603) 233 5473

Plymouth State UNIVERSITY