KENMOTSU MANIFOLDS ADMITTING SCHOUTEN-VAN
KAMPEN CONNECTION

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Abstract. The objective of the present paper is to study the Kenmotsu manifold admitting the Schouten-van Kampen connection. We study the Kenmotsu manifold admitting the Schouten-van Kampen connection satisfying certain curvature conditions. Also, we prove the equivalent conditions for the Ricci soliton in a Kenmotsu manifold to be steady with respect to the Schouten-van Kampen connection.

Keywords: Ricci solitons, Kenmotsu manifolds, Schouten-van Kampen connection, concircular curvature tensor, projective curvature tensor, conharmonic curvature tensor, shrinking.

1. Introduction

The Schouten-van Kampen connection has been introduced for studying non-holomorphic manifolds. It preserves - by parallelism - a pair of complementary distributions on a differentiable manifold endowed with an affine connection [2] [9] [17]. Then, Olszak studied the Schouten-van Kampen connection to adapt it to an almost contact metric structure [14]. He characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection and established certain curvature properties with respect to this connection. Recently, Gopal Ghosh [7] and Yildiz [24] studied the Schouten-van Kampen connection in Sasakian manifolds and $f$-Kenmotsu manifolds, respectively. Kenmotsu manifolds introduced by Kenmotsu in 1971[10] have been extensively studied by many authors [20] [15] [16]. In 1982, Hamilton [8] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Since then the Ricci flow has become a powerful tool for the study of Riemannian manifolds. The Ricci soliton, considered to be a self-similar solution to the Ricci flow is a Riemannian metric $g$ on a manifold $M$, together with a vector field $V$ such that

$$
(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0,
$$

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where $L_V$ denotes the Lie derivative along $V$, and $S$ and $\lambda$ are respectively the Ricci tensor and a constant. A Ricci soliton is said to be shrinking or steady or expanding depending on whether $\lambda$ is negative, zero or positive. A Ricci soliton is said to be a gradient Ricci soliton if the vector field $V$ is the gradient of some smooth function $f$ on $M$. In [18], Sharma started the study of Ricci solitons in the $K$-contact geometry. In 2016, the authors in [21] explained the nature of Ricci solitons in $f$-Kenmotsu manifolds with a semi-symmetric non-metric connection. Ramesh Sharma et al. [18] [19], De et al. [4][1], and Nagaraja et al. [12] [11] [13] extensively studied Ricci solitons in contact metric manifolds in many different ways.

This paper is structured as follows. After a brief review of Kenmotsu manifolds in Section 2, in Section 3 we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature with respect to the Schouten-van Kampen connection, study the curvature properties of the Kenmotsu manifold admitting the Schouten-van Kampen connection, and prove the conditions for the Kenmotsu manifold admitting the Schouten-van Kampen connection to be isomorphic to the hyperbolic space. In the last section we prove the equivalent conditions for the Ricci soliton in a Kenmotsu manifold admitting the Schouten-van Kampen connection to be steady.

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold $M$ is said to be an almost contact metric manifold if it admits an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ compatible with $(\phi, \xi, \eta)$ satisfying

\begin{equation}
\phi^2 X = -X + \eta(X)\xi, \quad \phi \xi = 0, \quad g(X, \xi) = \eta(X), \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0,
\end{equation}

and

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\end{equation}

An almost contact metric manifold is said to be a Kenmotsu manifold [3] if

\begin{equation}
(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,
\end{equation}

where $\nabla$ denotes the Riemannian connection of $g$.

In a Kenmotsu manifold the following relations hold [6].

\begin{align}
\nabla_X \xi &= X - \eta(X)\xi, \\
(\nabla_X \eta)Y &= g(\nabla_X \xi, Y), \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
S(X, \xi) &= -2\eta(X), \\
S(\phi X, \phi Y) &= S(X, Y) + 2\eta(X)\eta(Y),
\end{align}

for any vector fields $X, Y, Z$ on $M$, where $R$ denote the curvature tensor of type $(1, 3)$ on $M$. 

3. Kenmotsu manifolds admitting Schouten-van Kampen connection

Throughout this paper we associate * with the quantities with respect to the Schouten-van Kampen connection. The Schouten-van Kampen connection $\nabla^*$ associated to the Levi-Civita connection $\nabla$ is given by [14]

$$\nabla^*_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi,$$

for any vector fields $X, Y$ on $M$.

Using (2.4) and (2.5), the above equation yields,

$$\nabla^*_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X. \quad (3.2)$$

By taking $Y = \xi$ in (3.2) and using (2.4) we obtain

$$\nabla^*_X \xi = 0. \quad (3.3)$$

We now calculate the Riemann curvature tensor $R^*$ using (3.2) as follows:

$$R^*(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y. \quad (3.4)$$

Using (2.6) and taking $Z = \xi$ in (3.4), we get

$$R^*(X,Y)\xi = 0. \quad (3.5)$$

On contracting (3.4), we obtain the Ricci tensor $S^*$ of a Kenmotsu manifold with respect to the Schouten-van Kampen connection $\nabla^*$ as

$$S^*(Y,Z) = S(Y,Z) + 2ng(Y,Z). \quad (3.6)$$

This gives

$$Q^*Y = QY + 2nY. \quad (3.7)$$

Contracting with respect to $Y$ and $Z$ in (3.6), we get

$$r^* = r + 2n(2n + 1), \quad (3.8)$$

where $r^*$ and $r$ are the scalar curvatures with respect to the Schouten-van Kampen connection $\nabla^*$ and the Levi-Civita connection $\nabla$, respectively.

From the above discussions we state the following:

**Theorem 3.1.** The curvature tensor $R^*$, the Ricci tensor $S^*$ and the scalar curvature $r^*$ of a Kenmotsu manifold $M$ with respect to the Schouten-van Kampen connection $\nabla^*$ are given by (3.4), (3.6) and (3.8), respectively. Further, the curvature tensor $R^*$ of $\nabla^*$ satisfies

i) $R^*(X,Y)Z = -R^*(Y,X)Z$,
ii) $R^*(X,Y,Z,W) + R^*(Y,X,Z,W) = 0$,
iii) $R^*(X,Y,Z,W) + R^*(X,Y,W,Z) = 0$,
iv) $R^*(X,Y)Z + R^*(Y,Z)X + R^*(Z,X)Y = 0$,
v) $S^*$ is symmetric.
From (3.6), it follows that

**Theorem 3.2.** A Kenmotsu manifold $M$ admitting the Schouten-van Kampen connection is Ricci flat with respect to the Schouten-van Kampen connection if and only if $M$ is an Einstein manifold with respect to Levi-Civita connection.

Now, if $R^*(X, Y)Z = 0$, then by virtue of (3.4), we get

$$(3.9) \quad R(X, Y, Z, U) = g(X, Z)g(Y, U) - g(Y, Z)g(X, U).$$

Thus, we state that

**Theorem 3.3.** Let $M$ be a Kenmotsu manifold admitting the Schouten-van Kampen connection. The curvature tensor of $M$ with respect to the Schouten-van Kampen connection vanishes if and only if $M$ with respect to the Levi-Civita connection is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

An interesting invariant of the concircular transformation is concircular curvature tensor. The concircular curvature tensor [22] $C^*$ with respect to the Schouten-van Kampen connection $\nabla^*$ is defined by

$$(3.10) \quad C^*(X, Y)Z = R^*(X, Y)Z - \frac{r^*}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y],$$

for all vector fields $X, Y, Z$ on $M$. If $C^*$ vanishes, the conditions in theorem (3.1) are satisfied.

**Definition 3.1.** A Kenmotsu manifold with respect to the Schouten-van Kampen connection $\nabla^*$ is said to be $\xi$-concircularly flat if $C^*(X, Y)\xi = 0$.

In view of (3.4) and (3.8) in (3.10), we get

$$(3.11) \quad C^*(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y - \frac{r + 2n(2n+1)}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y].$$

By taking $Z = \xi$ in (3.11) and then using (2.1) and (2.6), we find

$$(3.12) \quad C^*(X, Y)\xi = \frac{r + 2n(2n+1)}{2n(2n+1)}R(X, Y)\xi.$$

Thus, from (3.4), (3.8), (3.11) and (3.12), we have the following theorem:

**Theorem 3.4.** Let $M$ be a Kenmotsu manifold admitting the Schouten-van Kampen connection. In $M$, the following three conditions are equivalent:

i) $M$ is $\xi$-concircularly flat,

ii) $r = -2n(2n+1)$,

iii) $r^* = 0$. 

Definition 3.2. A Kenmotsu manifold is said to be $\phi$-concircularly flat with respect to the Schouten-van Kampen connection $\nabla^*$ if

\begin{equation}
g(C^*(\phi X, \phi Y)\phi Z, \phi W) = 0,
\end{equation}

for any vector fields $X, Y, Z$ on $M$.

Using (3.10) in (3.13), we have

\begin{equation}
g(R^*(\phi X, \phi Y)\phi Z, \phi W) = \frac{r^*}{2n(2n+1)}\{g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\}.
\end{equation}

Let $\{e_1, e_2, e_3, \ldots, e_{2n+1}\}$ be a local orthonormal basis of vector fields in $M$. Then $\{\phi e_1, \phi e_2, \phi e_3, \ldots, \phi e_{2n+1}\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (3.14) and summing up with respect to $i$, $1 \leq i \leq 2n+1$, we obtain

\begin{equation}
\sum_{i=1}^{2n} g(R^*(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{r^*}{2n(2n+1)} \sum_{i=1}^{2n} \{g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}.
\end{equation}

From (3.15), it follows that

\begin{equation}
S^*(\phi Y, \phi Z) = \frac{r^*(2n-1)}{2n(2n+1)}g(\phi Y, \phi Z).
\end{equation}

Using (2.1), (3.6) and (3.8) in (3.16), we get

\begin{equation}
S(\phi Y, \phi Z) + 2ng(\phi Y, \phi Z) = \frac{(r + 2n(2n+1))(2n-1)}{2n(2n+1)}g(\phi Y, \phi Z).
\end{equation}

By using (2.2) and (2.8) in (3.17), we obtain

\begin{equation}
S(Y, Z) + 2n\eta(Y)\eta(Z) + \{2n - \frac{(r + 2n(2n+1))(2n-1)}{2n(2n+1)}\}g(\phi Y, \phi Z) = 0.
\end{equation}

Hence by contracting (3.18), we get

\begin{equation}
r = -2n.
\end{equation}

By substituting the equation (3.19) in (3.10), we get

\begin{equation}
C^*(X, Y)Z = R(X, Y)Z + \frac{1}{2n+1}\{g(Y, Z)X - g(X, Z)Y\}.
\end{equation}

This leads to the following:

\textbf{Theorem 3.5.} Let the Kenmotsu manifold $M$ admitting the Schouten-van Kampen connection be $\phi$-concircularly flat. Then $M$ is of constant sectional curvature $-\frac{1}{2n+1}$ if and only if the concircular curvature tensor $C^*$ vanishes.
We consider

By making use of (3.10) and (3.6) in (3.21), we obtain
\[
C^*S^* = S(R(X, Y)Z - \frac{r}{2n(2n + 1)} \{g(Y, Z)X - g(X, Z)Y\}, U) 
+ S(Z, R(X, Y)U - \frac{r}{2n(2n + 1)} \{g(Y, U)X - g(X, U)Y\}).
\]  

Suppose \( C^*S^* = 0 \). Then we have
\[ S^*(C^*(X, Y)Z, U) + S^*(Z, C^*(X, Y)U) = 0. \]  

Taking \( U = \xi \) in (3.23) and using (3.6), it follows that
\[ S^*(Z, C^*(X, Y)\xi) = 0. \]  

Making use of (2.1), (2.6) and (3.11) in (3.24), we get
\[ \frac{r + 2n(2n + 1)}{2n(2n + 1)} S^*(Z, \eta(X)Y - \eta(Y)X) = 0. \]  

Replacing \( X \) by \( \xi \) in (3.25) and using (2.1) and (3.6), we see that
\[ \frac{r + 2n(2n + 1)}{2n(2n + 1)} \{S(Z, Y) + 2ng(Z, Y)\} = 0. \]  

Contracting (3.26) with respect to \( Y \) and \( Z \), we get
\[ r = -2n(2n + 1). \]  

From (3.22) and (3.27), we obtain
\[ S(Y, Z) = -2ng(Y, Z). \]  

Thus \( M \) is an Einstein manifold.

Again, by substituting (3.27) in (3.11), we obtain
\[ C^*(X, Y)Z = R(X, Y)Z + \{g(Y, Z)X - g(X, Z)Y\}. \]  

Thus, from the above discussion and using (3.4), (3.8) and (3.12), we state the following:

**Theorem 3.6.** Let \( M \) be a Kenmotsu manifold admitting the Schouten-van Kampen connection. Then \( C^*S^* = 0 \) if and only if \( S(Y, Z) = -2ng(Y, Z) \). Further if \( C^* = 0 \) then \( M \) is isomorphic to the hyperbolic space \( H^{2n+1}(-1) \).
**Theorem 3.7.** If in a Kenmotsu manifold $M$ admitting the Schouten-van Kampen connection, $C^*S^* = 0$ holds, then the following three conditions are equivalent:

1. $M$ is $\xi$-concircularly flat,
2. $r = -2n(2n + 1)$,
3. $r^* = 0$.

The projective curvature tensor $P^*$ with respect to the Schouten-van Kampen connection $\nabla^*$ is defined by

$$
(3.30) \quad P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n} \{S^*(Y,Z)X - S^*(X,Z)Y\}.
$$

If the projective curvature tensor $P^*$ with respect to the Schouten-van Kampen connection $\nabla^*$ vanishes, then from (3.30), we have

$$
(3.31) \quad R^*(X,Y)Z = \frac{1}{2n} \{S^*(Y,Z)X - S^*(X,Z)Y\}.
$$

Now in view of (3.4) and (3.6), (3.31) takes the form

$$
(3.32) \quad g(R(X,Y)Z,W) + g(Y,Z)g(X,W) - g(X,Z)g(Y,W) = \frac{1}{2n} \{[S(Y,Z) + 2ng(Y,Z)]g(X,W) - [S(X,Z) + 2ng(X,Z)]g(Y,W)\}.
$$

Now taking $W = \xi$ in (3.32), we obtain

$$
(3.33) \quad S(Y,Z)\eta(X) - S(X,Z)\eta(Y) = 2n\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}.
$$

Again, setting $X = \xi$ in (3.33), we get

$$
(3.34) \quad S(Y,Z) = -2ng(Y,Z).
$$

Contracting the above equation (3.34), we get

$$
(3.35) \quad r = -2n(2n + 1).
$$

Using (3.34) in (3.31), we have $R^* = 0$.

Thus we state the following:

**Theorem 3.8.** Let $M$ be a Kenmotsu manifold admitting the Schouten-van Kampen connection. In $M$, the vanishing of the projective curvature tensor with respect to the Schouten-van Kampen connection leads to the vanishing of the curvature tensor with respect to the Schouten-van Kampen connection.

By making use of (3.4) and (3.6) in (3.30), we get

$$
(3.36) \quad P^*(X,Y)Z = R(X,Y)Z - \frac{1}{2n} \{S(Y,Z)X - S(X,Z)Y\}.
$$
Suppose \((P^\ast(X,Y)).S^\ast(Z,U) = 0\) holds in a Kenmotsu manifold \(M\). Then we have
\[
S^\ast(P^\ast(X,Y)Z, U) + S^\ast(Z, P^\ast(X,Y)U) = 0.
\]
Taking \(X = \xi\) in the equation (3.37), we get
\[
S^\ast(P^\ast(\xi,Y)Z, U) + S^\ast(Z, P^\ast(\xi,Y)U) = 0.
\]
By using (3.36), equation (3.38) turns into
\[
S(Y, Z)\eta(U) + S(Y, U)\eta(Z) = 0.
\]
In view of the equation (3.6), (3.39) becomes
\[
S(Y, Z)\eta(U) + S(Y, U)\eta(Z) + 2n\{g(Y, Z)\eta(U) + g(Y, U)\eta(Z)\} = 0.
\]
In (3.40), taking \(U = \xi\) and contracting with respect to \(Y\) and \(Z\), we get
\[
S(Y, Z) = -2ng(Y, Z).
\]
and
\[
r = -2n(2n + 1).
\]
Again, by substituting (3.42) in (3.30), we obtain
\[
P^\ast(X,Y)Z = R(X,Y)Z + \{g(Y,Z)X - g(X,Z)Y\}.
\]
Thus we can state that

**Theorem 3.9.** In a Kenmotsu manifold \(M\) admitting the Schouten-van Kampen connection, \(P^\ast.S^\ast = 0\) if and only if \(S(Y, Z) = -2ng(Y, Z)\).
Further, if \(P^\ast = 0\) then \(M\) is isomorphic to the hyperbolic space \(H^{2n+1}(-1)\).

The conharmonic curvature tensor [5] \(K^\ast\) with respect to the Schouten-van Kampen connection \(\nabla^\ast\) is defined by
\[
K^\ast(X,Y)Z = R^\ast(X,Y)Z - \frac{1}{2n-1}\{S^\ast(Y,Z)X - S^\ast(X,Z)Y\}
\]
If the conharmonic curvature tensor \(K^\ast\) with respect to the Schouten-van Kampen connection \(\nabla^\ast\) vanishes, then from (3.44), we have
\[
R^\ast(X,Y)Z = \frac{1}{2n-1}\{S^\ast(Y,Z)X - S^\ast(X,Z)Y\}
\]
and
\[
g(Y,Z)Q^\ast X - g(X,Z)Q^\ast Y\}.
\]
By using (3.4), (3.6) and (3.7) in (3.45), we get

\[ g(R(X, Y)Z, W) + g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \]
\[ = \frac{1}{2n - 1}[S(Y, Z) + 4ng(Y, Z)]g(X, W) \]
\[ - [S(X, Z) + 4ng(X, Z)]g(Y, W) \]
\[ + S(X, W)g(Y, Z) - S(Y, W)g(X, Z). \] (3.46)

Taking \( W = \xi \) in (3.46), we obtain

\[ S(Y, Z)\eta(X) - S(X, Z)\eta(Y) - 2n\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\} = 0. \] (3.47)

Taking \( X = \xi \) in (3.47), we get

\[ S(Y, Z) = -2ng(Y, Z). \] (3.48)

Contracting the equation (3.48), we get

\[ r = -2n(2n + 1). \] (3.49)

Using (3.48) in (3.45), we have \( R^* = 0 \).

Thus we state the following:

**Theorem 3.10.** Let \( M \) be a Kenmotsu manifold admitting the Schouten-van Kampen connection. In \( M \), the vanishing of the conharmonic curvature tensor with respect to the Schouten-van Kampen connection leads to the vanishing of the curvature tensor with respect to the Schouten-van Kampen connection.

4. **Ricci solitons in Kenmotsu manifold admitting Schouten-van Kampen connection**

Suppose the Kenmotsu manifold \( M \) admits a Ricci soliton with respect to the Schouten-van Kampen connection \( \nabla^* \). Then

\[ (L^* V)g(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0. \] (4.1)

If the potential vector field \( V \) is the structure vector field \( \xi \), then since \( \xi \) is a parallel vector field with respect to the Schouten-van Kampen connection (from (3.3)), the first term in the equation (4.1) becomes zero, hence \( M \) reduces to an Einstein manifold. In this case, the results in Theorem (3.6) and (3.9) hold.

If \( V \) is pointwise collinear with the structure vector field \( \xi \), i.e. \( V = b\xi \), where \( b \) is a function on \( M \), then the equation (1.1) implies that

\[ bg(\nabla^*_X \xi, Y) + (Xb)\eta(Y) + bg(X, \nabla^*_Y \xi) + (Yb)\eta(X) + \]
\[ 2S^*(X, Y) + 2\lambda g(X, Y) = 0. \] (4.2)
Using (3.3) and (3.6) in (4.2), it follows that

\[(Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) + 2\{2n + \lambda\}g(X, Y) = 0.\]

(4.3)

By setting \(Y = \xi\) in (4.3) and using (2.7), we obtain

\[(Xb) = -\{2\lambda + \xi b\}\eta(X).\]

(4.4)

Again replacing \(X\) by \(\xi\) in (4.4), we get

\[(\xi b) = -\lambda.\]

(4.5)

Substituting this in (4.4), we have

\[(Xb) = -\lambda\eta(X).\]

(4.6)

By applying \(d\) on (4.6), we get

\[\lambda d\eta = 0.\]

(4.7)

Since \(d\eta \neq 0\) from (4.7), we have

\[\lambda = 0.\]

(4.8)

Substituting (4.8) in (4.6), we conclude that \(b\) is a constant. Hence it is verified from (4.3) that

\[S(X, Y) = -(2n + \lambda)g(X, Y) + \lambda\eta(X)\eta(Y).\]

(4.9)

This leads to the following:

**Theorem 4.1.** If a Kenmotsu manifold with respect to the Schouten-van Kampen connection admits a Ricci soliton \((g, V, \lambda)\) with \(V\) pointwise collinear with \(\xi\), then the manifold is an \(\eta\)-Einstein manifold and the Ricci soliton is steady.

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**REFERENCES**


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