A note on Łukasiewicz's three-valued logic

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It is well known that Łukasiewicz's three-valued-logic $Ł_3$ admits – unlike classical logic – the definition of two *non trivial, truth-functional* modal operators $\Box$ and $\Diamond$. We address the question of finding a convenient *syntactic* characterization of the "modal content" of $Ł_3$. To this aim, we consider Wajsberg's axiomatization of $Ł_3$ (the calculus $\mathbb{W}$) and prove its equivalence with a modal calculus $\mathbb{W}^{\Box}$ which, essentially, includes: the *BCK+double negation* schemas, the characteristic modal schemas of $\mathbb{S}5$ $(\mathbf{K}, \mathbf{T}, \mathbf{4}, \mathbf{B})$, full contraction for boxed formulas and the "partial collapse" schema $\alpha \to (\alpha \to \Box\alpha)$. As applications, we obtain a simple and natural completeness proof *à la Lindenbaum* for $\mathbb{W}$, as well as a considerable simplification of Wajsberg's original, ingenious completeness proof.

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Introduction

According to Jan Łukasiewicz, both the original motivations and the philosophical significance of the *many-valued* systems of propositional logic he started to develop around 1920 were to be found in two strictly intertwined issues. On the one side, the "spiritual war" against what he considered to be the subtlest form of determinism, namely *logical determinism* as originated from the *bivalence* principle of classical Aristotelian – Crisippean logic (see Łukasiewicz 1918 and 1922). On the other side, the aim to provide an *adequate* logical foundation to modal propositions and, more generally, to the very notions of *possibility* and *necessity.* Where by an 'adequate' foundation he meant not only one which was capable to give a systematic account of the modal principles traditionally recognized as valid, but also one in full accordance with the basic tenets of the *extensional* approach to logic. This second point has to be stressed: the widespread, natural association of – as we would now say – *modal* notions and *intensional* logic was explicitly rejected by Łukasiewicz from the very beginning (see Łukasiewicz 1931), and this rejection was defended and even reinforced in Łukasiewicz 1953: the modal operators $\Box$ and $\Diamond$ have to be *truth-functional* unary connectives, and the *extensionality principle* should be possibly accepted also for modal contexts.

Without discussing this peculiar and – to us – hardly tenable position, we just observe that, *on its basis,* Łukasiewicz's claim concerning the *modal adequacy* of his three-valued system $Ł_3$ (a claim minutely defended in Łukasiewicz 1931) appears to be an almost immediate consequence. In fact, unlike classical bivalent logic, $Ł_3$ is able to define two *non-trivial*, *truth-functional* operators $\Box$ and $\Diamond$ (namely: $\Box\alpha := \neg(\alpha \to \neg\alpha)$, $\Diamond\alpha := \neg\alpha \to \alpha$; this fact was observed by Tarski already in 1921) and to validate, among the "modal principles" emerging from the logic tradition, those which Łukasiewicz considered to be the most significant ones:

(i) the *"modal square" of oppositions;*
(ii) *"Ab oportere ad esse valet consequentia",* and its dual *"Ab esse ad posse valet consequentia";*
(iii) *"Unumquodque, quando est, oportet esse";*
(iv) *"For some p, it is possible that p and it is possible that not-p".*

In Łukasiewicz 1931, he notes that the $Ł_3$-tautologies $\Box\alpha \leftrightarrow \neg\Diamond\neg\alpha$ and $\Diamond\alpha \leftrightarrow \neg\Box\neg\alpha$, resp. $\Box\alpha \to \alpha$ and $\alpha \to \Diamond\alpha$ perfectly correspond to (i) and (ii), as well as the (second-order) $Ł_3$-tautology $\exists p\,(\Diamond p \wedge \Diamond\neg p)$ corresponds to (iv). As for the more questionable (iii), he skilfully takes advantage of the failure of the *contraction law* in $Ł_3$ and proposes, as a (partially) adequate formal version, the $Ł_3$-tautology $\alpha \to (\alpha \to \Box\alpha)$.

By the way, the counterintuitive tautology $\Diamond\alpha \wedge \Diamond\beta \to \Diamond(\alpha \wedge \beta)$ is not considered at all in the 1931 paper. It will be defended in the later paper Łukasiewicz 1953 where, however, the extensional many-valued framework within which modal logic is developed is no more $Ł_3$, but a *four-valued* logic (which is not $Ł_4$!). Actually, the 1953 modal system differs from the one underlying $Ł_3$ under many respects. In particular, the characteristic axiom schema of $\mathbb{S}5$, i.e. von Wright's schema M'' ($\Diamond\Box\alpha \to \Box\alpha$; see von Wright 1951) is rejected as questionable from the intuitive point of view (while it is valid in $Ł_3$, see Proposition 2.1 here). On the contrary, the general extensionality principle $(\alpha \to \beta) \to ((\beta \to \alpha) \to (\gamma[p/\alpha] \to \gamma[p/\beta]))$ is accepted (while it is not valid in Ł$_3$, see the remark after Proposition 1.4, and Proposition 2.8).

The question concerning the *adequacy*, from a broad logical and philosophical perspective, of Łukasiewicz *extensional* reconstruction of modal logic will not be discussed any further in this paper. Instead, our aim is first of all to analyze a bit more deeply – and from a purely formal point of view – the "modal content" of Ł$_3$, and to characterize it *syntactically* within Wajsberg's axiomatic calculus (see Wajsberg 1931) $\mathbb{W}$ for Ł$_3$ (sections 1 and 2). It turns out that $\mathbb{W}$ is equivalent with an axiomatic modal calculus $\mathbb{W}^{\Box}$ (with $\neg, \to$ and $\Box$ as primitives) containing the *transitivity*, the *exchange* and the *weakening* axiom schemas (BCK) and the inference rule of *separation* for $\to$, the *double-negation* schemas and, as far as $\Box$ is concerned, the characteristic schemas of the classical system $\mathbb{S}5$ plus, in addition, a *contraction* schema for *boxed* formulas $(\Box\alpha \to (\Box\alpha \to \beta)) \to (\Box\alpha \to \beta)$ and the *partial collapse* schema $\alpha \to (\alpha \to \Box\alpha)$. Actually, some of the $\mathbb{S}5$ schemas are redundant, but it is not difficult to single out an *independent* axiomatization (see Lemma 2.5).

In the remaining part of the paper we will show how such a syntactic characterization (together with other related facts, namely a *modal version* of the deduction and the replacement theorems) can be fruitfully applied to present *old* completeness results in a new, perhaps more natural, fashion. In section 3, we will try to make fully explicit the modal skeleton which we think is hidden behind Wajsberg's original and extremely ingenious proof of the (special) completeness theorem for Ł$_3$ w.r. to the calculus $\mathbb{W}$. Many years later a different method of proof, which more generally applies to all Łukasiewicz's finite-valued logics Ł$_m$, was proposed by Rosser and Turquette (Rosser and Turquette 1952; see also Ackermann 1967 and Urquhart 1986). The completeness proof presented in section 4, using a modal form of *Lindenbaum's Lemma,* is much in this spirit.

1. *Wajsberg's axiomatization of Ł$_3$*

Preliminaries. $\mathcal{L}$ is the propositional language determined by:

- a countable set $V = \{p_0, p_1, \dots\}$ of *propositional variables* (*atoms*);

- the *connectives* $\neg$ (*negation*) and $\rightarrow$ (*conditional*);
- parentheses as auxiliary symbols.

By $\mathcal{F}$ we denote the set of all $\mathcal{L}$-formulas, which is defined as usual. The letters $p, q, r \ldots$ vary over V, while $\alpha, \beta, \gamma \ldots$ vary over $\mathcal{F}$.
The logical constants $\top$ and $\bot$, as well as the unary connectives (*modalities*) $\Diamond$ and $\Box$, are defined as follows:

- $\top := (p_0 \rightarrow p_0)$;
- $\bot := \neg\top$;
- $\Diamond\alpha := \neg\alpha \rightarrow \alpha$;
- $\Box\alpha := \neg(\alpha \rightarrow \neg\alpha)$.

Moreover, for $n \geq 0$, we use $\alpha^n \rightarrow \beta$ as an abbreviation of:

$$\begin{cases} \beta & \text{if } n = 0, \\ \alpha \rightarrow (\alpha^k \rightarrow \beta) & \text{if } n = k + 1. \end{cases}$$

Further notational conventions include:

- $V(\alpha) := \{p \in V \mid p \textit{ occurs in } \alpha\}$;
- for $p \in V$, $\mathcal{F}_p := \{\alpha \mid V(\alpha) = \{p\}\}$;
- $\beta \preceq \alpha :=$ β *is a subformula of* α;
- $\mathtt{lg}(\alpha) :=$ the *length* of α (number of occurrences of $\neg$ and $\rightarrow$ in α).

Finally, $\equiv$ is used to denote syntactic identity between formulas.

Let us now briefly review the well known semantic characterization of Łukasiewicz's three–valued logic Ł$_3$.
A *trivalent valuation* v for $\mathcal{L}$ ($v \in \mathtt{VAL}$, in symbols) is a map:

$$v : V \longrightarrow \{0, {}^1\!/\!_2, 1\}.$$

Each $v \in \mathtt{VAL}$ is inductively extended to a map (still denoted by v) from the set $\mathcal{F}$ of all formulas into the set $\{0, {}^1\!/\!_2, 1\}$ of truth-values, by means of the following clauses:

- $v(\neg\alpha) = 1 - v(\alpha)$,
- $v(\alpha \rightarrow \beta) = \max\{1, 1 - v(\alpha) + v(\beta)\}$.

Note that, for the defined modal operators $\Box$ and $\Diamond$, it holds:

- $v(\Box\alpha) = \begin{cases} 1 & \text{if } v(\alpha) = 1, \\ 0 & \text{otherwise.} \end{cases}$
- $v(\Diamond\alpha) = \begin{cases} 1 & \text{if } v(\alpha) \geq {}^1\!/\!_2, \\ 0 & \text{otherwise.} \end{cases}$

Finally, given $\alpha \in \mathcal{F}$ and $M \subseteq \mathcal{F}$, we set:

(1) α is a Ł$_3$-*tautology* (in symbols: $\models_3 \alpha$) iff:

$$\forall v \in \mathtt{VAL}\,(\, v(\alpha) = 1\,)\,.$$

(2) α is a Ł$_3$-*logical consequence* of M (in symbols: $M \models_3 \alpha$) iff:

$$\forall v \in \mathtt{VAL}\,[(\forall \beta \in M.\ v(\beta) = 1) \Rightarrow v(\alpha) = 1]\,.$$

Wajsberg's calculus. Wajsberg's *calculus* $\mathbb{W}$ over $\mathcal{L}$ is determined by the four *axiom schemas*:

($\mathbb{W}$.1) $\quad \alpha \to (\beta \to \alpha)$

($\mathbb{W}$.2) $\quad (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

($\mathbb{W}$.3) $\quad (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

($\mathbb{W}$.4) $\quad ((\alpha \to \neg\alpha) \to \alpha) \to \alpha$

and the *inference rule*:

(RS) $\quad \dfrac{\alpha \to \beta \qquad \alpha}{\beta}$ $\quad$ (*separation*).

As usual, given a (possibly empty) set of formulas M, $M \vdash \alpha$ means that α can be *derived* in $\mathbb{W}$ from the assumptions in M (and so $\vdash \alpha$ means that α is *provable* in $\mathbb{W}$). Also, we will write

$$\alpha \dashv\vdash \beta$$

as an abbreviation of: $\vdash \alpha \to \beta$ *and* $\vdash \beta \to \alpha$.

Fact 1.1 (**Validity**). *For every $M \subseteq \mathcal{F}$ and every α, $M \models_3 \alpha \;\Rightarrow\; M \vdash \alpha$.*

As shown by Wajsberg himself (see Wajsberg 1931), the four axiom schemas of $\mathbb{W}$ are *independent*. As a drawback, there is usually a lot of tedious work to do in order to prove in $\mathbb{W}$ a number of useful Ł$_3$-tautologies, like the following ones.

Proposition 1.2. *For every α, β, γ :*

(L1) $\quad \alpha \dashv\vdash \neg\neg\alpha$

(L2) $\quad \vdash \alpha \to \alpha$

(L3) $\quad \bot \dashv\vdash \neg(\alpha \to \alpha)$

(L4) $\quad \neg\alpha \dashv\vdash (\alpha \to \bot)$

(L5) $\vdash (\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

(L6) $\vdash (\alpha \to (\alpha \to \neg\alpha)) \to (\alpha \to \neg\alpha)$

(L7) $\vdash ((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$

(L8) *for* $n \geq 2 : (\alpha^n \to \beta) \dashv\vdash (\alpha^2 \to \beta)$.

Proof. See Wajsberg 1931, or (for a different proof) the Appendix here. □

Ł$_3$ is a *"resource conscious"* logic, in so far as the *contraction law*

$$(p \to (p \to q)) \to (p \to q)$$

is not a Ł$_3$-tautology (assign $1/2$ to p and 0 to q, to get a countermodel). As an immediate consequence, we have that the standard form of the deduction theorem fails for $\mathbb{W}$. However, by making essential use of the provable schema L8 of Proposition 1.2 above, which in fact is a *restricted* form of *contraction*, it is clearly possible to prove a still useful weakened version of the deduction theorem.

Proposition 1.3 ("Weak" deduction theorem). *For every* $M \subseteq \mathcal{F}$ *and every* $\alpha, \beta \in \mathcal{F}$:

(WDT) $M, \alpha \vdash \beta \quad \Leftrightarrow \quad M \vdash \alpha^2 \to \beta$.

Proof. By induction on the length of the derivation $\mathcal{D}$ of β from $M \cup \{\alpha\}$, using $\mathbb{W}$.1, $\mathbb{W}$.2, L2, L5 and L8. □

Proposition 1.4 (Replacement rule). *For every* α, β, γ *and every* p :

(RE) $\alpha \dashv\vdash \beta \quad \Rightarrow \quad \gamma[p/\alpha] \dashv\vdash \gamma[p/\beta]$.

Proof. By straightforward induction on γ, using L1, L2, L5 and $\mathbb{W}$.2. □

Of course, due again to the absence of *full contraction*, the (classically valid) *replacement schema* fails in $\mathbb{W}$. For example

$$\nvdash (\top \to p) \to [(p \to \top) \to (\Box\top \to \Box p)],$$

for otherwise the invalid formula $p \to \Box p$ would be provable.

Remark 1.5. Let $\mathbb{G}$ (see Grigolia 1977) be the calculus obtained from $\mathbb{W}$ by replacing the axiom schema $\mathbb{W}$.4 with the two axiom schemas:

($\mathbb{G}$.4) $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$

($\mathbb{G}$.5) $\quad (\alpha \to (\alpha \to \neg\alpha)) \to (\alpha \to \neg\alpha)$.

It is easily seen that $\mathbb{G}$ and $\mathbb{W}$ are equivalent: $\mathbb{G}$.4 and $\mathbb{G}$.5 are provable in $\mathbb{W}$ (L6, resp. L5 of Proposition 1.2); conversely, $\mathbb{W}$.4 follows by applying $\mathbb{G}$.4 to $\mathbb{G}$.5.
Also, recall that $\{\mathbb{W}.1, \mathbb{W}.2, \mathbb{W}.3, \mathbb{G}.4; \mathrm{RS}\}$ is an independent axiomatization of Łukasiewicz's *infinite valued* logic $Ł_\infty$.

2. *The "modal content" of* $\mathbb{W}$

We will now try to make explicit the behavior of the modalities $\Box$ and $\Diamond$ in $\mathbb{W}$, and then to characterize *axiomatically* the underlying "modal logic" (for an *algebraic* characterization of the $\{\to, \Box\}$–fragment of $Ł_3$, see Figallo 1990).

Proposition 2.1. *For every* α *and* β :

(M1)	$\Box\alpha \dashv\vdash \neg\Diamond\neg\alpha\,,$	$\Diamond\alpha \dashv\vdash \neg\Box\neg\alpha$
(M2)	$\vdash \alpha \to (\alpha \to \Box\alpha)\,,$	$\vdash \neg\alpha \to (\neg\alpha \to \neg\Diamond\alpha)$
(M3)	$\vdash \alpha \;\Rightarrow\; \vdash \Box\alpha\,,$	$\vdash \neg\alpha \;\Rightarrow\; \vdash \neg\Diamond\alpha$
(M4)	$\vdash \alpha \to \beta \;\Rightarrow\; \vdash \Box\alpha \to \Box\beta\,,$	$\vdash \alpha \to \beta \;\Rightarrow\; \vdash \Diamond\alpha \to \Diamond\beta$
(M5)	$\vdash \Box\alpha \to \alpha\,,$	$\vdash \alpha \to \Diamond\alpha$
(M6)	$\vdash \alpha \to \Box\Diamond\alpha\,,$	$\vdash \Diamond\Box\alpha \to \alpha$
(M7)	$\vdash \Box\alpha \to \Box\Box\alpha\,,$	$\vdash \Diamond\Diamond\alpha \to \Diamond\alpha$
(M8)	$\vdash \Diamond\alpha \to \Box\Diamond\alpha\,,$	$\vdash \Diamond\Box\alpha \to \Box\alpha$
(M9)	$\vdash ((\Box\alpha)^2 \to \beta) \to (\Box\alpha \to \beta)$	
(M10)	$\vdash ((\Diamond\alpha)^2 \to \beta) \to (\Diamond\alpha \to \beta)$	
(M11)	$\alpha \to (\alpha \to \beta) \dashv\vdash \Box\alpha \to \beta.$	

Proof. See the Appendix. $\Box$

Proposition 2.2 ("Modal" deduction theorems). *For every* $M \subseteq \mathcal{F}$ *and every formula* α *and* β :

(MDT.1) $\quad M, \alpha \vdash \beta \;\Leftrightarrow\; M \vdash \Box\alpha \to \beta.$

$$\text{(MDT.2)} \quad \begin{cases} (a)\; M, \triangle\alpha \vdash \beta \;\Leftrightarrow\; M \vdash \triangle\alpha \to \beta \\ (b)\; M, \neg\triangle\alpha \vdash \beta \;\Leftrightarrow\; M \vdash \neg\triangle\alpha \to \beta \end{cases} \qquad \triangle \in \{\Box, \Diamond\}.$$

Proof. MDT.1 follows by the weak deduction theorem (WDT) together with M11 above.

MDT.2: (a) is an immediate consequence of MDT.1, M7 and M8; (b) follows from (a) and M1. $\Box$

Proposition 2.3. *For every α and β :*

(M12) $\quad \vdash \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.

Proof. Using M5 and the definition of $\Box$, together with L4, it is easily seen that

$$\Box(\alpha \to \beta), \Box\alpha, \neg\Box\beta \vdash \bot\ .$$

The conclusion follows by applying MDT.2 and L1. $\Box$

By Propositions 2.1 and 2.3 we now see that:

Fact 2.4. *With respect to the defined operators $\Box$ and $\Diamond$, the calculus $\mathbb{W}$:*

(1) *is closed under the* necessitation rule RN (M3)*, and*
(2) *proves the characteristic axiom schemas of the* classical *modal system $\mathbb{S}5$, namely:*

$$\mathbf{K}\ \text{(M12)},\ \ \mathbf{T}\ \text{(M5)},\ \ \mathbf{4}\ \text{(M7)},\ \ \mathbf{B}\ \text{(M6)},\ \ \mathbf{E}\ \text{(M8)},$$

both w.r. to $\Box$ and to $\Diamond$ (in fact, these operators are interdefinable *as in $\mathbb{S}5$, by* M1*).*

Additionally:

(3) *by* M9 *and* M10*,* full contraction *does hold for* "boxed" *formulas (i.e. formulas of the form $\Box\alpha$ or $\Diamond\alpha$);*
(4) *by* M2*, $\mathbb{W}$ proves $\alpha \to (\alpha \to \Box\alpha)$, which trivially is a* collapse *schema in* classical *modal systems containing* **T***.*

Taking advantage of this fact, we obtain an equivalent *"modal"* axiomatization of $Ł_3$ as follows.

Let $\mathcal{L}^\Box$ be the propositional language resulting from $\mathcal{L}$ by addition of $\Box$ as a new primitive unary connective. $\Diamond$ is defined as usual ($\Diamond\alpha := \neg\Box\neg\alpha$), and $\mathcal{F}^\Box$ denotes the set of all formulas of $\mathcal{L}^\Box$.
Next, let $\mathbb{W}^\Box$ be the calculus over $\mathcal{L}^\Box$ which is determined by the *axiom schemas*:

Group I — *BCK + double negation:*

($\mathbb{W}^\Box$.1) $\alpha \to (\beta \to \alpha)$

($\mathbb{W}^\Box$.2) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

($\mathbb{W}^\Box$.3) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

($\mathbb{W}^\Box$.4) $\alpha \to \neg\neg\alpha$ *and* $\neg\neg\alpha \to \alpha$

Group II — *modal schemas:*

($\mathbb{W}^\Box$.5) $\Box\alpha \to \alpha$ (**T**)

($\mathbb{W}^\Box$.6) $\alpha \to \Box\Diamond\alpha$ (**B**)

($\mathbb{W}^\Box$.7) $(\Box\alpha \to (\Box\alpha \to \beta)) \to (\Box\alpha \to \beta)$ (*boxed contraction*)

($\mathbb{W}^\Box$.8) $\alpha \to (\alpha \to \Box\alpha)$ (*partial collapse*).

and the *inference rule* RS.

First of all, note that:

Lemma 2.5. *The four modal schemas* $\mathbb{W}^\Box.5 - \mathbb{W}^\Box.8$ *are independent,* modulo *the Group I schemas.*

Proof. $\mathbb{W}^\Box.8$ is obviously independent, since it is not provable in the classical modal system $\mathbb{S}5$, while $\mathbb{W}^\Box.5 - \mathbb{W}^\Box.7$ are.
ad $\mathbb{W}^\Box.5$: extend each $v \in \mathtt{VAL}$ to $v : \mathcal{F}^\Box \longrightarrow \{0, {}^1\!/_2, 1\}$ by adding to the standard clauses concerning $\neg$ and $\to$ the clause:

$$v(\Box\beta) = 1 \qquad (\textit{and so } \ v(\Diamond\beta) = 0).$$

It is immediately verified that each formula which is provable from $\mathbb{W}^\Box$ *minus* $\mathbb{W}^\Box.5$ is a tautology in the new sense, while $\Box p \to p$ is not.
ad $\mathbb{W}^\Box.6$: as above, this time with the clause:

$$v(\Box\beta) = 0 \qquad (\textit{and so } \ v(\Diamond\beta) = 1).$$

It is easily verified that if α is provable from $\mathbb{W}^\Box$ *minus* $\mathbb{W}^\Box.6$ then *either* $v(\alpha) = 1$ *or* $v(\alpha) = 0$ for every $v \in \mathtt{VAL}$, while $v(p \to \Box\Diamond p) = {}^1\!/_2$ for $v(p) = {}^1\!/_2$.
ad $\mathbb{W}^\Box.7$: trivial, by reading $\Box\alpha$ as α, for every α. $\Box$

To prove the *equivalence* of $\mathbb{W}$ and $\mathbb{W}^\Box$, we consider the *translation*

$$\tau : \mathcal{F}^\Box \longrightarrow \mathcal{F}$$

which is defined inductively as follows:

- $\tau(p) = p$,
- τ commutes with $\neg$ and $\to$,
- $\tau(\Box\alpha) = \neg(\tau(\alpha) \to \neg\tau(\alpha))$.

Theorem 2.6. $\mathbb{W}$ *and* $\mathbb{W}^\Box$ *are equivalent, in the following sense:*

(i) *for every* $\alpha \in \mathcal{F}^\Box$: $\vdash_{\mathbb{W}^\Box} \alpha \to \tau(\alpha)$ *and* $\vdash_{\mathbb{W}^\Box} \tau(\alpha) \to \alpha$;

(ii) *for every* $\alpha \in \mathcal{F}$: $\quad \vdash_{\mathbb{W}} \alpha \;\Leftrightarrow \vdash_{\mathbb{W}^{\Box}} \alpha.$

Proof. First of all, observe that $\mathbb{W}^{\Box}$ is *closed under* RN (by $\mathbb{W}^{\Box}.8$ and separation), and *proves* the modal schema **K**. Indeed, using $\mathbb{W}^{\Box}.5$ we get

(a) $$\vdash \Box(\alpha \to \beta) \to (\Box\alpha \to \beta)$$

and, by applying $\mathbb{W}^{\Box}.8$ to (a),

(b) $$\vdash \Box(\alpha \to \beta) \to (\Box\alpha \to (\beta \to \Box\beta));$$

now, combining (a) and (b), we obtain

(c) $$\vdash (\Box(\alpha \to \beta))^2 \to ((\Box\alpha)^2 \to \Box\beta)$$

whence the conclusion follows by $\mathbb{W}^{\Box}.7$.
Finally, $\mathbb{W}^{\Box}$ *proves* the modal schemas **4** (use $\mathbb{W}^{\Box}.8$, RN, **K** and $\mathbb{W}^{\Box}.7$) and **E** (use, as in classical modal systems, $\mathbf{4}, \mathbf{B}, \mathbf{K}$, RN and transitivity).

(i): it is clearly sufficient to show that, in $\mathbb{W}^{\Box}$, $\Box\alpha \dashv\vdash \neg(\alpha \to \neg\alpha)$. By the axiom schemas of Group I we have

$$\vdash \alpha \to (\alpha \to \neg(\alpha \to \neg\alpha));$$

then, using $\mathbb{W}^{\Box}.5$,

$$\vdash \Box\alpha \to (\Box\alpha \to \neg(\alpha \to \neg\alpha))$$

and finally, by $\mathbb{W}^{\Box}.7$:

$$\vdash \Box\alpha \to \neg(\alpha \to \neg\alpha).$$

The other direction easily follows by $\mathbb{W}^{\Box}.8$, exchange and contraposition.
(ii): in view of (i) and Propositions 2.1 and 2.3, it is sufficient to show that $\mathbb{W}.4$ is provable in $\mathbb{W}^{\Box}$.
By the axioms of Group I, we have:

(1) $\vdash \neg\Box\alpha \to ((\neg\Box\alpha \to \alpha) \to \alpha)$,

(2) $\vdash \alpha \to ((\neg\Box\alpha \to \alpha) \to \alpha)$.

Then, by using RN, **K**, contraposition and the definition of $\Diamond$, we obtain:

(1') $\vdash \neg\Box((\neg\Box\alpha \to \alpha) \to \alpha) \to \Diamond\Box\alpha$,

(2') $\vdash \Box\alpha \to \Box((\neg\Box\alpha \to \alpha) \to \alpha)$.

By (1'), using **E**, we get:

(3) $\vdash \neg\Box((\neg\Box\alpha \to \alpha) \to \alpha) \to \Box\alpha$

whence, by transitivity with (2'):

(4) $\vdash \neg\Box((\neg\Box\alpha \to \alpha) \to \alpha) \to \Box((\neg\Box\alpha \to \alpha) \to \alpha)$.

But, by (i) above, we have $\Diamond\gamma \dashv\vdash \neg\gamma \to \gamma$ in $\mathbb{W}^{\Box}$; so

(5) $\vdash \Diamond\Box((\neg\Box\alpha \to \alpha) \to \alpha)$

follows from (4) and then, using **B** ($= \mathbb{W}^{\Box}.6$):

(6) $\vdash (\neg\Box\alpha \to \alpha) \to \alpha$.

By (i) and (6) we finally get: $\vdash ((\alpha \to \neg\alpha) \to \alpha) \to \alpha$. □

We conclude with two propositions – still concerning modalities – which will be needed in the next sections. Essentially, the first one shows how the truth-table of $\to$ in Ł_3 is reflected in $\mathbb{W}$ (recall that, semantically, $\Box\alpha$ and $\Diamond\alpha$ correspond to $v(\alpha) = 1$, resp. $v(\alpha) \geq 1/2$, while $\Box\neg\alpha$ and $\Diamond\neg\alpha$ correspond to $v(\alpha) = 0$, resp. $v(\alpha) \leq 1/2$). The second proposition deals with two particularly useful *modal replacement schemas* which hold in $\mathbb{W}$.

Proposition 2.7. *For every α and β :*

(M13) $\vdash \Box\neg\alpha \to \Box(\alpha \to \beta)$, $\quad \vdash \Diamond\neg\alpha \to \Diamond(\alpha \to \beta)$.

(M14) $\vdash \Box\beta \to \Box(\alpha \to \beta)$, $\quad \vdash \Diamond\beta \to \Diamond(\alpha \to \beta)$.

(M15) $\vdash \Box\alpha \to (\Diamond\neg\beta \to \Diamond\neg(\alpha \to \beta))$,
$\vdash \Diamond\alpha \to (\Box\neg\beta \to \Diamond\neg(\alpha \to \beta))$.

(M16) $\vdash \Box\alpha \to (\Box\neg\beta \to \Box\neg(\alpha \to \beta))$.

(M17) $\vdash \Diamond\neg\alpha \to (\Diamond\beta \to (\Diamond\neg\beta \to \Box(\alpha \to \beta)))$.

Proof. See the Appendix. □

Proposition 2.8 ("Modal" replacement schemas). *For every $\alpha, \beta, \gamma \in \mathcal{F}$ and every atom p:*

(MRE.1) $\vdash \Box(\alpha \to \beta) \to (\Box(\beta \to \alpha) \to (\gamma[p/\alpha] \to \gamma[p/\beta]))$

(MRE.2) $\vdash \Diamond\neg\alpha \to (\Diamond\alpha \to (\Diamond\neg\beta \to (\Diamond\beta \to (\gamma[p/\alpha] \to \gamma[p/\beta]))))$.

Proof. MRE.1: $\gamma[p/\alpha] \to \gamma[p/\beta]$ is derivable from $\{\alpha \to \beta, \beta \to \alpha\}$ (proof by induction, using L1, L2, L5, $\mathbb{W}$.2), so the conclusion follows by MDT.1.
As for MRE.2: by MRE.1

$$\vdash \Box(\alpha \to \beta) \to (\Box(\beta \to \alpha) \to (\gamma[p/\alpha] \to \gamma[p/\beta])),$$

so by M17 we get:

$$\vdash (\Diamond\neg\alpha)^2 \to [(\Diamond\neg\beta)^2 \to (\Diamond\alpha \to (\Diamond\beta \to (\gamma[p/\alpha] \to \gamma[p/\beta])))].$$

The conclusion now follows by boxed contraction M10. □

3. *Wajsberg's completeness proof revisited*

Wajsberg's original proof of the *special* completeness theorem for $\mathbb{W}$ (i.e: $\models_3 \alpha \Rightarrow \vdash \alpha$) runs by induction on the number of distinct atoms occurring in α. Although the Author makes no mention at all of the definable modal operators $\Box$ and $\Diamond$ and their underlying logic, yet a careful inspection of his proof of both the *basis* and the *induction step* shows that the key lemmas he employs (Theorems T9, T10, T12, whose laborious and often intricate verification takes up a considerable part of the paper; see pp. 20–21 of the English translation in Borkowski 1970), actually concern clear-cut properties of the modal operators. Indeed, T9 and T12 correspond exactly to the two forms of the *modal replacement schema* (MRE.1, resp. MRE.2) we proved in sect. 2. On the other side, T10 corresponds to:

Lemma 3.1. *The following inference rule* ("Wajsberg's rule") *is eliminable in* $\mathbb{W}$*:*

$$\text{(WR)} \qquad \frac{\Box\beta \to \alpha \qquad \Box\neg\beta \to \alpha \qquad \Diamond\neg\beta \to (\Diamond\beta \to \alpha)}{\alpha}$$

Proof. Assume:

(1) $\vdash \Box\beta \to \alpha$

(2) $\vdash \Box\neg\beta \to \alpha$

(3) $\vdash \Diamond\neg\beta \to (\Diamond\beta \to \alpha)$.

Then by RN and $\mathbf{K}, \mathbf{4}, \mathbf{E}$ (i.e. M12, M7, M8) we get:

(4) $\vdash \Box\beta \to \Box\alpha$

(5) $\vdash \Box\neg\beta \to \Box\alpha$

(6) $\quad \vdash \Diamond\neg\beta \to (\Diamond\beta \to \Box\alpha)$.

By (4) and (5), using contraposition and M1:

(7) $\quad \vdash \Diamond\neg\alpha \to \Diamond\neg\beta$

(8) $\quad \vdash \Diamond\neg\alpha \to \Diamond\beta$,

and by (6), (7) and (8):

(9) $\quad \vdash \Diamond\neg\alpha \to (\Diamond\neg\alpha \to \Box\alpha)$.

By boxed contraction M10, (9) gives

(10) $\quad \vdash \Diamond\neg\alpha \to \Box\alpha$,

i.e., by M1 and the definition of $\Diamond$,

(11) $\quad \vdash \Diamond\Box\alpha$,

whence $\vdash \alpha$ finally follows by **B**, i.e. M6. $\Box$

Note that WR is a sort of *argument by "excluded fourth"*.

We shall now reconstruct Wajsberg's proof from a modal point of view — quite faithfully, save for a simplification in the proof of the *basis* of the induction (Lemmas 3.2 and 3.3).

Let us denote by $\mathcal{F}_p^*$ the set of all formulas $\alpha \in \mathcal{F}_p$ containing at least one subformula β s.t. $\beta \dashv\vdash \neg p$.

Lemma 3.2. *If* $\alpha \in \mathcal{F}_p \smallsetminus \mathcal{F}_p^*$, *then* either $\alpha \dashv\vdash p$, or $\alpha \dashv\vdash p \to p$, or $\alpha \dashv\vdash \neg(p \to p)$.

Proof. Straightforward induction on α. $\Box$

Lemma 3.3. *Let* $\alpha \in \mathcal{F}_p, u \in \mathtt{VAL}$, *and* $u(p) = 1/2$. *If* $u(\alpha) = 1$ *then*

$$\vdash \Diamond p \to (\Diamond\neg p \to \alpha).$$

Proof. By induction on $\mathtt{lg}(\alpha)$. Under the assumptions, there are two cases two distinguish.

<u>Case 1</u>: $\alpha \in \mathcal{F}_p^*$. Let then $\beta \preceq \alpha$ be such that

(1) $\quad \beta \dashv\vdash \neg p, \qquad$ *so also* $\models_3 \beta \to \neg p, \models_3 \neg p \to \beta$.

Then $\mathtt{lg}(\beta) > 0$ and, for some γ and $q, \alpha \equiv \gamma[q/\beta]$. Consider:

$$\alpha' := \gamma[q/\neg p], \qquad \alpha'' := \gamma[q/p].$$

Then $\mathtt{lg}(\alpha'') < \mathtt{lg}(\alpha)$ and $u(\alpha'') = 1$, since by (1) and the assumptions $1/2 = u(p) = u(\neg p) = u(\beta)$. Therefore, by the induction hypothesis:

$$\vdash \Diamond p \to (\Diamond\neg p \to \alpha''). \tag{2}$$

On the other side, by modal replacement MRE.2:

$$\vdash (\Diamond p)^2 \to ((\Diamond\neg p)^2 \to (\alpha'' \to \alpha')). \tag{3}$$

Combining (2) and (3) we get

$$\vdash (\Diamond p)^3 \to ((\Diamond\neg p)^3 \to \alpha'), \tag{4}$$

and by modal contraction M10:

$$\vdash \Diamond p \to (\Diamond\neg p \to \alpha'). \tag{5}$$

Since $\alpha \dashv\vdash \alpha'$ (by (1) and weak replacement RE), by (5) and RE again we finally get the conclusion.

Case 2: $\alpha \in \mathcal{F}_p \smallsetminus \mathcal{F}_p^*$. Then, by Lemma 3.2 and the assumption $u(\alpha) = 1$, necessarily $\alpha \dashv\vdash p \to p$, whence the conclusion immediately follows by $\mathbb{W}$.1 and replacement. □

Lemma 3.4. *If $\alpha \in \mathcal{F}_p$ and $\beta \in \{\alpha[p/p \to p], \alpha[p/\neg(p \to p)]\}$, then either $\beta \dashv\vdash p \to p$ or $\beta \dashv\vdash \neg(p \to p)$.*

Proof. By straightforward induction on α, using L1–L4. □

Lemma 3.5. *For every $\alpha \in \mathcal{F}$ and $p, q \in V$: if*

(i) $\vdash \alpha[p/q \to q]$, (ii) $\vdash \alpha[p/\neg(q \to q)]$,

(iii) $\vdash \alpha[q/p]$, (iv) $\vdash \alpha[q/p \to p]$, (v) $\vdash \alpha[q/\neg(p \to p)]$,

then $\vdash \alpha$.

Proof. By the modal replacement theorem MRE.1 we have:

(1) $\vdash \Box(p \to (q \to q)) \to (\Box((q \to q) \to p) \to (\alpha[p/q \to q] \to \alpha))$,

(2) $\vdash \Box(p \to \neg(q \to q)) \to (\Box(\neg(q \to q) \to p) \to$
$\to (\alpha[p/\neg(q \to q)] \to \alpha))$.

Now, by L2, L3, L4, M3, M4, we have:

$\vdash \Box(p \to (q \to q))$, $\vdash \Box(\neg(q \to q) \to p)$,

$\Box((q \to q) \to p) \dashv\vdash \Box p$, $\Box(p \to \neg(q \to q)) \dashv\vdash \Box\neg p$;

so (1), (2) and the assumptions (i) and (ii) give:

(3) $\vdash \Box p \to \alpha,$

(4) $\vdash \Box\neg p \to \alpha.$

Exactly in the same way, from the assumptions (iv) and (v) we get:

(5) $\vdash \Box q \to \alpha,$ so also $\vdash \Box q \to (\Diamond p \to (\Diamond\neg p \to \alpha)),$

(6) $\vdash \Box\neg q \to \alpha,$ so also $\vdash \Box\neg q \to (\Diamond p \to (\Diamond\neg p \to \alpha)).$

Furthermore, by the modal replacement theorem MRE.2, we have:

(7) $\vdash \Diamond q \to (\Diamond\neg q \to (\Diamond p \to (\Diamond\neg p \to (\alpha[q/p] \to \alpha))))$

and so by the assumption (iii)

(8) $\vdash \Diamond q \to (\Diamond\neg q \to (\Diamond p \to (\Diamond\neg p \to \alpha))).$

By a first application of Wajsberg's rule WR to (5), (6), (8) we obtain

(9) $\vdash \Diamond p \to (\Diamond\neg p \to \alpha).$

A second application of WR to (3), (4), (9) gives the conclusion $\vdash \alpha$. □

Theorem 3.6 (**Special completeness**). *For every* α :

$$\models_3 \alpha \;\Rightarrow \vdash \alpha.$$

Proof. Assuming $\models_3 \alpha$, we argue by induction on the number n of distinct atoms occurring in α.
$\underline{n = 1}$: then $\alpha \in \mathcal{F}_p$ for some atom p, and by Lemma 3.3 we have:

(1) $\vdash \Diamond p \to (\Diamond\neg p \to \alpha),$

while MRE.1 and Lemma 3.4, exactly as in the proof of Lemma 3.5, give:

(2) $\vdash \Box p \to \alpha\,, \qquad \vdash \Box\neg p \to \alpha.$

The conclusion follows by (1), (2) and Wajsberg's rule WR.
$\underline{n > 1}$: let p and q be *distinct* atoms occurring in α. Since α is valid, also its substitution instances

$$\alpha[p/q \to q], \quad \alpha[p/\neg(q \to q)],$$
$$\alpha[q/p], \quad \alpha[q/p \to p], \quad \alpha[q/\neg(p \to p)]$$

are valid, and the conclusion immediately follows by the induction hypothesis and Lemma 3.5. □

4. *A general completeness theorem for* $\mathbb{W}$

As everyone knows, standard proofs of the *general completeness theorem* for classical propositional logic essentially rely upon *Lindenbaum's extension lemma*, saying that every consistent set of formulas has a *maximal* (i.e.: consistent and *syntactically complete*) extension. Maximal sets of formulas are in fact into one–one correspondence with *bivalent* valuations.

As for Ł$_3$, it turns out quite naturally, on the basis of the results of sections 1 and 2, that: (i) the sets of formulas being into one–one correspondence with *trivalent* valuations are exactly those consistent sets (*3-maximal* sets, as we will say) which, for every formula α, either prove $\Box\alpha$ (α has the value 1), or prove both $\Diamond\alpha$ and $\Diamond\neg\alpha$ (α has the value $1/2$), or prove $\Box\neg\alpha$ (α has the value 0); (ii) for 3-maximality, an analogous of Lindenbaum's lemma does hold. As a consequence, we can prove a general completeness theorem for $\mathbb{W}$ ($M \models_3 \alpha \;\Rightarrow\; M \vdash \alpha$) much in the same way as for classical logic.

Definition 4.1. Let M be a set of formulas:

(1) M is *consistent* iff $M \nvdash \bot$;

(2) M is *3-maximal* iff M is consistent and, for every formula α, the following condition (MAX$_3$) is satisfied:

$$\textit{either } M \vdash \Box\alpha, \textit{ or } (M \vdash \Diamond\alpha \textit{ and } M \vdash \Diamond\neg\alpha), \textit{ or } M \vdash \Box\neg\alpha.$$

Lemma 4.2. *Let M be a consistent set of formulas. Then:*

(i) *For every α, either $M + \Box\alpha$ is consistent, or $M + \Diamond\alpha + \Diamond\neg\alpha$ is consistent, or $M + \Box\neg\alpha$ is consistent.*

(ii) *The three cases of condition* MAX$_3$ *are pairwise incompatible.*

(iii) *If M is 3-maximal then, for every α:*

$$M \vdash \alpha \quad \Rightarrow \quad M \vdash \Box\alpha.$$

Proof. (i): assume M is consistent and suppose, by absurd, that:

(a) $M, \Box\alpha \vdash \bot$; (b) $M, \Diamond\alpha, \Diamond\neg\alpha \vdash \bot$; (c) $M, \Box\neg\alpha \vdash \bot$.

Using MDT.2 and M1, we get:

$M \vdash \Diamond\alpha \to \Box\alpha$, by (b);

$M \vdash \Diamond\alpha,$ by (c);

therefore $M \vdash \Box\alpha$ and, by (a), $M \vdash \bot$. This contradicts the consistency of M.
(ii): assume M is consistent and suppose, by absurd, that two out of the three cases of MAX$_3$ hold simultaneously:

- if $M \vdash \Box\alpha$ and $M \vdash \Diamond\neg\alpha$ then, by M1, $M \vdash \neg\Box\alpha$, so $M \vdash \bot$;
- if $M \vdash \Box\alpha$ and $M \vdash \Box\neg\alpha$ then, by M5, $M \vdash \alpha$ and $M \vdash \neg\alpha$, so $M \vdash \bot$;
- if $M \vdash \Diamond\alpha$ and $M \vdash \Box\neg\alpha$ then, by M1, $M \vdash \neg\Diamond\alpha$, so also $M \vdash \bot$.

In all cases M is inconsistent, in contrast with the assumption.
(iii): suppose M is 3-maximal, $M \vdash \alpha$, and $M \nvdash \Box\alpha$. Then, by MAX$_3$,

$$\text{either } M \vdash \Diamond\alpha \ \textit{ and } \ M \vdash \Diamond\neg\alpha \,, \qquad \text{or } M \vdash \Box\neg\alpha.$$

In the first case, by definition of $\Diamond$ and $M \vdash \Diamond\neg\alpha$, we have $M \vdash \alpha \to \neg\alpha$ and so, by $M \vdash \alpha$, $M \vdash \neg\alpha$ and $M \vdash \bot$: contradiction.
In the second case, by M5 we have $M \vdash \neg\alpha$ and so, by $M \vdash \alpha$, $M \vdash \bot$: contradiction again. □

Lemma 4.3. *Let M be 3-maximal. For every α and β :*

(i) *if $M \vdash \Box(\alpha \to \beta)$, one of the following cases does hold:*
  (a) $M \vdash \Box\beta$,
  (b) $M \vdash \Box\neg\alpha$,
  (c) $M \vdash \Diamond\alpha$ *and* $M \vdash \Diamond\neg\alpha$ *and* $M \vdash \Diamond\beta$ *and* $M \vdash \Diamond\neg\beta$;

(ii) *if $M \vdash \Box\neg(\alpha \to \beta)$, then* $M \vdash \Box\alpha$ *and* $M \vdash \Box\neg\beta$;

(iii) *if $M \vdash \Diamond(\alpha \to \beta)$ and $M \vdash \Diamond\neg(\alpha \to \beta)$, one of the following cases does hold:*
  (a) $M \vdash \Box\alpha$ *and* $M \vdash \Diamond\beta$ *and* $M \vdash \Diamond\neg\beta$,
  (b) $M \vdash \Diamond\alpha$ *and* $M \vdash \Diamond\neg\alpha$ *and* $M \vdash \Box\neg\beta$.

Proof. Straightforward verification, using MAX$_3$ and M13–M17 of Proposition 2.7. □

Lemma 4.4 (**Extension**). *Every consistent set of formulas M can be extended to a 3-maximal set M^*.*

Proof. The standard proof of *Lindenbaum's lemma* is adapted in the obvious way, using Lemma 4.2. Namely, having fixed an enumeration $\{\alpha_n\}_{n\geq 0}$ of $\mathcal{F}$, we define inductively the infinite chain $\{M_n\}_{n\geq 0}$

of sets of formulas:

$$\begin{cases} M_0 := M \\ M_{k+1} := \begin{cases} M_k + \Box\alpha_k & \text{if this set is consistent,} \\ M_k + \Diamond\alpha_k + \Diamond\neg\alpha_k & \text{if this set is consistent,} \\ M_k + \Box\neg\alpha_k & \text{otherwise.} \end{cases} \end{cases}$$

The definition is correct by (i) and (ii) of Lemma 4.2, and $M^* := \bigcup_{n\geq 0} M_n$ is clearly a 3-maximal extension of M. □

Theorem 4.5 (**General completeness**). *For every $M \subseteq \mathcal{F}$ and every formula α:*

$$M \models_3 \alpha \;\Rightarrow\; M \vdash \alpha.$$

Proof. Suppose $M \nvdash \alpha$. Then $M + \neg\Box\alpha$ is consistent, for otherwise $M \vdash \Box\alpha$ by MDT.2 and L1, and then $M \vdash \alpha$ by M5.
By Lemma 4.4, let M^* be a 3-maximal extension of $M + \neg\Box\alpha$.
Next, let $v^* \in \mathtt{VAL}$ be the valuation induced by M^* as follows:

$$(1) \qquad v^*(p) := \begin{cases} 1 & , \text{ if } M^* \vdash \Box p \\ 1/2 & , \text{ if } M^* \vdash \Diamond p \;\; \textit{and} \;\; M^* \vdash \Diamond\neg p \\ 0 & , \text{ if } M^* \vdash \Box\neg p. \end{cases}$$

v^* is well defined and total by (ii) of Lemma 4.2 and 3-maximality of M^*.
By (1) and Lemma 4.3 it is easily verified, by induction on β, that:

(2)

$$\textit{for all } \beta \in \mathcal{F}: \quad v^*(\beta) := \begin{cases} 1 & , \text{ if } M^* \vdash \Box\beta \\ 1/2 & , \text{ if } M^* \vdash \Diamond\beta \;\; \textit{and} \;\; M^* \vdash \Diamond\neg\beta \\ 0 & , \text{ if } M^* \vdash \Box\neg\beta. \end{cases}$$

Since $M + \neg\Box\alpha \subseteq M^*$ and M^* is 3-maximal, we have:

(3) *for every* $\gamma \in M$, $v^*(\gamma) = 1$, by (iii) of Lemma 4.2 and (2);

(4) $v^*(\alpha) \neq 1$, by (2) and $M^* \nvdash \Box\alpha$.

We conclude, by (3) and (4), that $M \not\models_3 \alpha$. □

5. *Appendix*

Proof of Proposition 1.2. Here are the formal proofs of L1 – L8 in $\mathbb{W}$ (together with the proofs of the auxiliary schemas and eliminable inference rules A1 – A6).

[A1] $\vdash \neg\alpha \to (\alpha \to \beta)$

1	$\neg\alpha \to (\neg\beta \to \neg\alpha)$	$\mathbb{W}.1$
2	$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$	$\mathbb{W}.3$
3	A1	$\mathbb{W}.2 : 1, 2$

[L1←] $\vdash \neg\neg\alpha \to \alpha$

1	$\neg\neg\alpha \to (\neg\alpha \to \neg(\alpha \to \neg\alpha))$	A1
2	$(\neg\alpha \to \neg(\alpha \to \neg\alpha)) \to ((\alpha \to \neg\alpha) \to \alpha)$	$\mathbb{W}.3$
3	$\neg\neg\alpha \to ((\alpha \to \neg\alpha) \to \alpha)$	$\mathbb{W}.2 : 1, 2$
4	$((\alpha \to \neg\alpha) \to \alpha) \to \alpha$	$\mathbb{W}.4$
5	L1 $\leftarrow$	$\mathbb{W}.2 : 3, 4$

[L1→] $\vdash \alpha \to \neg\neg\alpha$

1	$\vdash \neg\neg\neg\alpha \to \neg\alpha$	L1 $\leftarrow$
2	L1 $\to$	$\mathbb{W}.3 : 1$

[L2] $\vdash \alpha \to \alpha$

1	$\alpha \to \neg\neg\alpha$	L1
2	$\neg\neg\alpha \to \alpha$	L1
3	L2	$\mathbb{W}.2 : 1, 2$

[A2] $\vdash (\top \to \alpha) \to \alpha$

1	$\top \to ((\alpha \to \neg\alpha) \to \top)$	$\mathbb{W}.1$
2	$\top$	L2
3	$(\alpha \to \neg\alpha) \to \top$	RS $: 1, 2$
4	$(\top \to \alpha) \to ((\alpha \to \neg\alpha) \to \alpha)$	$\mathbb{W}.2 : 3$
5	$((\alpha \to \neg\alpha) \to \alpha) \to \alpha$	$\mathbb{W}.4$
6	A2	$\mathbb{W}.2 : 4, 5$

[A3] $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$

1	$(\top \to \alpha) \to ((\alpha \to \beta) \to (\top \to \beta))$	$\mathbb{W}.2$
2	$\alpha \to (\top \to \alpha)$	$\mathbb{W}.1$
3	$\alpha \to ((\alpha \to \beta) \to (\top \to \beta))$	$\mathbb{W}.2 : 1, 2$
4	$((\alpha \to \beta) \to (\top \to \beta)) \to [((\top \to \beta) \to \beta) \to$ $\to ((\alpha \to \beta) \to \beta)]$	$\mathbb{W}.2$
5	$\alpha \to [((\top \to \beta) \to \beta) \to ((\alpha \to \beta) \to \beta)]$	$\mathbb{W}.2 : 3, 4$
6	$\top \to ((\top \to \beta) \to \beta)$	$\mathrm{A2}, \mathbb{W}.1$
7	$[((\top \to \beta) \to \beta) \to ((\alpha \to \beta) \to \beta)] \to$ $\to (\top \to ((\alpha \to \beta) \to \beta))$	$\mathbb{W}.2 : 6$
8	$\alpha \to (\top \to ((\alpha \to \beta) \to \beta))$	$\mathbb{W}.2 : 5, 7$
9	$(\top \to ((\alpha \to \beta) \to \beta)) \to ((\alpha \to \beta) \to \beta)$	A2
10	A3	$\mathbb{W}.2 : 8, 9$

[L5] $\vdash (\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

1	$(\alpha \to (\beta \to \gamma)) \to [((\beta \to \gamma) \to \gamma) \to (\alpha \to \gamma)]$	$\mathbb{W}.2$
2	$\beta \to ((\beta \to \gamma) \to \gamma)$	A3
3	$[((\beta \to \gamma) \to \gamma) \to (\alpha \to \gamma)] \to (\beta \to (\alpha \to \gamma))$	$\mathbb{W}.2 : 2$
4	L5	$\mathbb{W}.2 : 1, 3$

Note: in the following, the abbreviation 'LL : ...' means: provable from lines ... by using (a combination of) RS, $\mathbb{W}.1$ – $\mathbb{W}.3$, L1, L2, L5, A1–A3.

[L3] $\bot \dashv\vdash \neg(\alpha \to \alpha)$

1	$(\alpha \to \alpha) \to \top$	$\mathbb{W}.1, \mathrm{L2}$
2	$\top \to (\alpha \to \alpha)$	$\mathbb{W}.1, \mathrm{L2}$
3	$\neg(\alpha \to \alpha) \to \bot$	LL : 2
4	$\bot \to \neg(\alpha \to \alpha)$	LL : 1
5	L3	3, 4

[L4] $\neg\alpha \dashv\vdash (\alpha \to \bot)$

1	$\neg\alpha \to (\alpha \to \bot)$	A1

2	$(\top \to \neg\alpha) \to \neg\alpha$	A2
3	$(\alpha \to \bot) \to (\neg\bot \to \neg\alpha)$	LL
4	$\top \to \neg\bot$	L1
5	$(\alpha \to \bot) \to (\top \to \neg\alpha)$	LL : 3, 4
6	$(\alpha \to \bot) \to \neg\alpha$	$\mathbb{W}$.2 : 2, 5
7	L4	1, 6

[L6] $\vdash (\alpha \to (\alpha \to \neg\alpha)) \to (\alpha \to \neg\alpha)$

1	$(((\alpha \to \neg\alpha) \to \alpha) \to \alpha) \to [(\alpha \to (\alpha \to \neg\alpha)) \to$ $\to (((\alpha \to \neg\alpha) \to \alpha) \to (\alpha \to \neg\alpha))]$	$\mathbb{W}$2
2	$(\alpha \to (\alpha \to \neg\alpha)) \to$ $\to (((\alpha \to \neg\alpha) \to \alpha) \to (\alpha \to \neg\alpha))$	$\mathbb{W}$.4, 1
3	$\neg\alpha \to \neg\neg(\alpha \to \neg\alpha)$	LL
4	$\neg(\alpha \to \neg\alpha) \to \alpha$	$\mathbb{W}$.3 : 3
5	$((\alpha \to \neg\alpha) \to \neg(\alpha \to \neg\alpha)) \to ((\alpha \to \neg\alpha) \to \alpha)$	LL : 4
6	$(\alpha \to (\alpha \to \neg\alpha)) \to$ $\to (((\alpha \to \neg\alpha) \to \neg(\alpha \to \neg\alpha)) \to (\alpha \to \neg\alpha))$	LL : 2, 5
7	$[((\alpha \to \neg\alpha) \to \neg(\alpha \to \neg\alpha)) \to$ $\to (\alpha \to \neg\alpha)] \to (\alpha \to \neg\alpha)$	$\mathbb{W}$.4
8	L6	$\mathbb{W}$.2 : 6, 7

[A4] $\dfrac{\vdash \alpha \to \beta \qquad \vdash (\alpha \to \neg\alpha) \to \beta}{\vdash \beta}$

1	$\alpha \to \beta$	$\vdash$
2	$(\alpha \to \neg\alpha) \to \beta$	$\vdash$
3	$(\beta \to \neg\alpha) \to (\alpha \to \neg\alpha)$	$\mathbb{W}$.2 : 1
4	$(\beta \to \neg\alpha) \to \beta$	$\mathbb{W}$.2 : 2, 3
5	$(\beta \to \neg\beta) \to ((\neg\beta \to \neg\alpha) \to (\beta \to \neg\alpha))$	$\mathbb{W}$.2
6	$\neg\beta \to \neg\alpha$	LL : 1
7	$(\beta \to \neg\beta) \to (\beta \to \neg\alpha)$	LL : 5, 6
8	$(\beta \to \neg\beta) \to \beta$	$\mathbb{W}$.2 : 4, 7
9	$((\beta \to \neg\beta) \to \beta) \to \beta$	$\mathbb{W}$.4

10	β	$\mathrm{RS}:8,9$

[A5] $$\frac{\vdash \alpha \to \gamma \qquad \vdash \beta \to \gamma}{\vdash ((\alpha \to \beta) \to \beta) \to \gamma}$$

1	$((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))$	LL
2	$\neg(\alpha \to \beta) \to \alpha$	LL
3	$\alpha \to \gamma$	$\vdash$
4	$\neg(\alpha \to \beta) \to \gamma$	$\mathbb{W}.2:2,3$
5	$((\alpha \to \beta) \to \beta) \to (\neg\beta \to \alpha)$	$\mathrm{LL}:1,2$
6	$(\neg\beta \to \alpha) \to ((\alpha \to \neg\alpha) \to (\alpha \to \beta))$	LL
7	$(\alpha \to \beta) \to [((\alpha \to \beta) \to \neg(\alpha \to \beta)) \to \neg(\alpha \to \beta)]$	A3
8	$(\neg\beta \to \alpha) \to [(\alpha \to \neg\alpha) \to$ $\to (((\alpha \to \beta) \to \neg(\alpha \to \beta)) \to \neg(\alpha \to \beta))]$	$\mathrm{LL}:6,7$
9	$(\neg\beta \to \alpha) \to [(\alpha \to \neg\alpha) \to$ $\to (((\alpha \to \beta) \to \neg(\alpha \to \beta)) \to \gamma)]$	$\mathrm{LL}:4,8$
10	$((\alpha \to \beta) \to \beta) \to [(\alpha \to \neg\alpha) \to$ $\to (((\alpha \to \beta) \to \neg(\alpha \to \beta)) \to \gamma)]$	$\mathbb{W}.2:5,9$
11	$((\alpha \to \beta) \to \neg(\alpha \to \beta)) \to$ $\to [(\alpha \to \neg\alpha) \to (((\alpha \to \beta) \to \beta) \to \gamma))]$	$\mathrm{LL}:10$
12	$\beta \to \gamma$	$\vdash$
13	$(\alpha \to \beta) \to [((\alpha \to \beta) \to \beta) \to \beta]$	A3
14	$(\alpha \to \beta) \to [((\alpha \to \beta) \to \beta) \to \gamma]$	$\mathrm{LL}:12,13$
15	$(\alpha \to \beta) \to [(\alpha \to \neg\alpha) \to$ $\to (((\alpha \to \beta) \to \beta) \to \gamma)]$	$\mathrm{LL}:14$
16	$(\alpha \to \neg\alpha) \to (((\alpha \to \beta) \to \beta) \to \gamma)$	$\mathrm{A4}:11,15$
17	$\gamma \to (((\alpha \to \beta) \to \beta) \to \gamma)$	$\mathbb{W}.1$
18	$\alpha \to (((\alpha \to \beta) \to \beta) \to \gamma)$	$\mathbb{W}.2:3,17$
19	$((\alpha \to \beta) \to \beta) \to \gamma$	$\mathrm{A4}:16,18$

[L7] $\vdash ((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$

1	$\alpha \to ((\beta \to \alpha) \to \alpha)$	$\mathbb{W}.1$
2	$\beta \to ((\beta \to \alpha) \to \alpha)$	A3

3	L7	$A5 : 1, 2$

[A6] $\vdash (\alpha \to (\alpha \to (\alpha \to \beta))) \to (\alpha \to (\alpha \to \beta))$

1	$\neg\alpha \to (\alpha \to \beta)$	L1
2	$(\alpha \to \neg\alpha) \to (\alpha \to (\alpha \to \beta))$	$LL : 1$
3	$((\alpha \to (\alpha \to \beta)) \to \alpha) \to ((\alpha \to \neg\alpha) \to \alpha)$	$\mathbb{W}.2 : 2$
4	$((\alpha \to \neg\alpha) \to \alpha) \to \alpha$	$\mathbb{W}.4$
5	$((\alpha \to (\alpha \to \beta)) \to \alpha) \to \alpha$	$\mathbb{W}.2 : 3, 4$
6	A6	$L7 : 5$

[L8]: for $n \geq 2$: $(\alpha^n \to \beta) \dashv\vdash (\alpha^2 \to \beta)$

1	$(\alpha^n \to \beta) \to (\alpha^2 \to \beta)$	$A6, LL$
2	$(\alpha^2 \to \beta) \to (\alpha^n \to \beta)$	LL
3	L8	$1, 2$

$\square$

Proof of Proposition 2.1. The proofs of M1 – M5 are straightforward, by LL and the definitions of $\square$ and $\Diamond$.

[M6] (a) $\vdash \alpha \to \square\Diamond\alpha$ (b) $\vdash \Diamond\square\alpha \to \alpha$

1	$\alpha \to \Diamond\alpha$	M5
2	$\neg\Diamond\alpha \to \neg\alpha$	$LL : 1$
3	$(\Diamond\alpha \to \neg\Diamond\alpha) \to (\Diamond\alpha \to \neg\alpha)$	$LL : 2$
4	$((\neg\alpha \to \alpha) \to \neg\alpha) \to \neg\alpha$	$\mathbb{W}4, LL$
5	$(\Diamond\alpha \to \neg\alpha) \to \neg\alpha$	Def. : 4
6	$(\Diamond\alpha \to \neg\Diamond\alpha) \to \neg\alpha$	$\mathbb{W}.2 : 3, 5$
7	$\alpha \to \neg(\Diamond\alpha \to \neg\Diamond\alpha)$	$LL : 6$
8	M6 (a)	Def. : 7

[M7] (a) $\vdash \square\alpha \to \square\square\alpha$ (b) $\vdash \Diamond\Diamond\alpha \to \Diamond\alpha$

1	$(\alpha \to (\alpha \to \neg\alpha)) \to (\alpha \to \neg\alpha)$	L6
2	$(\alpha \to (\alpha \to (\alpha \to \neg\alpha))) \to (\alpha \to (\alpha \to \neg\alpha))$	L8
3	$(\alpha \to (\alpha \to (\alpha \to \neg\alpha))) \to (\alpha \to \neg\alpha)$	$\mathbb{W}.2 : 1, 2$
4	$(\alpha \to (\neg(\alpha \to \neg\alpha) \to \neg\alpha)) \to \neg\neg(\alpha \to \neg\alpha)$	$LL : 3$

5	$(\alpha \to (\Box\alpha \to \neg\alpha)) \to \neg\Box\alpha$	Def. : 4
6	$(\Box\alpha \to \neg\neg(\alpha \to \neg\alpha)) \to \neg\Box\alpha$	LL : 5
7	$(\Box\alpha \to \neg\Box\alpha) \to \neg\Box\alpha$	Def. : 6
8	$\Box\alpha \to \neg(\Box\alpha \to \neg\Box\alpha)$	LL : 7
9	M7 (a)	Def. : 8

[M8] (a) $\vdash \Diamond\alpha \to \Box\Diamond\alpha$ (b) $\vdash \Diamond\Box\alpha \to \Box\alpha$

1	$\Diamond\Diamond\alpha \to \Diamond\alpha$	M7
2	$\Box\Diamond\Diamond\alpha \to \Box\Diamond\alpha$	M4 : 1
3	$\Diamond\alpha \to \Box\Diamond\Diamond\alpha$	M6
4	M8 (a)	$\mathbb{W}.2 : 2, 3$

[M9] $(\Box\alpha \to (\Box\alpha \to \beta)) \to (\Box\alpha \to \beta)$

1	$(\Box\alpha \to (\Box\alpha \to \beta)) \to (\Box\alpha \to (\neg\beta \to \neg\Box\alpha))$	LL
2	$(\Box\alpha \to (\Box\alpha \to \beta)) \to (\neg\beta \to (\Box\alpha \to \neg\Box\alpha))$	LL : 1
3	$(\Box\alpha \to (\Box\alpha \to \beta)) \to (\neg\beta \to \neg\Box\Box\alpha)$	Def. : 2
4	$(\Box\alpha \to (\Box\alpha \to \beta)) \to (\Box\Box\alpha \to \beta)$	LL : 3
5	$\Box\alpha \to \Box\Box\alpha$	M7
6	M9	LL : 4, 5

[M10] $(\Diamond\alpha \to (\Diamond\alpha \to \beta)) \to (\Diamond\alpha \to \beta)$

1	$(\Diamond\alpha \to (\Diamond\alpha \to \beta)) \to (\Diamond\alpha \to (\neg\beta \to \neg\Diamond\alpha))$	LL
2	$(\Diamond\alpha \to (\Diamond\alpha \to \beta)) \to (\neg\beta \to (\Diamond\alpha \to \neg\Diamond\alpha))$	LL : 1
3	$(\Diamond\alpha \to (\Diamond\alpha \to \beta)) \to (\neg\beta \to \neg\Box\Diamond\alpha)$	Def. : 2
4	$(\Diamond\alpha \to (\Diamond\alpha \to \beta)) \to (\Box\Diamond\alpha \to \beta)$	LL : 3
5	$\Diamond\alpha \to \Box\Diamond\alpha$	M8
6	M10	LL : 4, 5

[M11] $(\alpha \to (\alpha \to \beta)) \dashv\vdash (\Box\alpha \to \beta)$

1	$(\alpha \to \beta) \to (\Box\alpha \to \beta)$	LL, M5
2	$(\Box\alpha \to (\alpha \to \beta)) \to (\Box\alpha \to (\Box\alpha \to \beta))$	LL : 1
3	$(\alpha \to (\alpha \to \beta)) \to (\Box\alpha \to (\alpha \to \beta))$	LL, M5
4	$(\alpha \to (\alpha \to \beta)) \to (\Box\alpha \to (\Box\alpha \to \beta))$	$\mathbb{W}.2 : 2, 3$

5	$(\Box\alpha \to (\Box\alpha \to \beta)) \to (\Box\alpha \to \beta)$	M9
6	$(\alpha \to (\alpha \to \beta)) \to (\Box\alpha \to \beta)$	$\mathbb{W}.2 : 4, 5$
7	$(\alpha \to \Box\alpha) \to ((\Box\alpha \to \beta) \to (\alpha \to \beta))$	$\mathbb{W}.2$
8	$(\alpha \to (\alpha \to \Box\alpha)) \to (\alpha \to ((\Box\alpha \to \beta) \to (\alpha \to \beta)))$	LL : 7
9	$\alpha \to (\alpha \to \Box\alpha)$	M2
10	$\alpha \to ((\Box\alpha \to \beta) \to (\alpha \to \beta))$	RS : 8, 9
11	$(\Box\alpha \to \beta) \to (\alpha \to (\alpha \to \beta))$	LL : 10
12	M11	5, 11

$\Box$

Proof of Proposition 2.7. The proofs of M13, M14 and M16 are immediate, by using M4, M12 and LL. Note that we may now use MDT.1 and MDT.2 (Proposition 2.2).

[M15] (a) $\vdash \Box\alpha \to (\Diamond\neg\beta \to \Diamond\neg(\alpha \to \beta))$

1	$\Box(\alpha \to \beta)$	ass.
2	$\Box\alpha$	ass.
3	$\Diamond\neg\beta$	ass.
4	$\Box\beta$	M12 : 1, 2
5	$\neg\Diamond\neg\beta$	M1 : 4
6	$\perp$	L1 : 3, 5
7	M15 (a)	MDT.2, M1

[M15] (b) $\vdash \Diamond\alpha \to (\Box\neg\beta \to \Diamond\neg(\alpha \to \beta))$: *analogously.*

[M17] $\vdash \Diamond\neg\alpha \to (\Diamond\beta \to (\Diamond\neg\beta \to \Box(\alpha \to \beta)))$

1	$\Diamond\neg(\alpha \to \beta)$	ass.
2	$(\alpha \to \beta) \to \neg(\alpha \to \beta)$	Def., LL : 1
3	$\neg(\alpha \to \beta) \to \neg\beta$	LL
4	$(\alpha \to \beta) \to \neg\beta$	$\mathbb{W}.2 : 2, 3$
5	$\Diamond\beta$	ass.
6	$(\alpha \to \beta) \to \beta$	Def., $\mathbb{W}.2 : 4, 5$
7	$(\beta \to \alpha) \to \alpha$	L7 : 6
8	$\neg\beta \to (\beta \to \alpha)$	L1

9	$\neg\beta \to \alpha$	$\mathbb{W}.2 : 7, 8$
10	$\Diamond\neg\beta$	ass.
11	$\beta \to \alpha$	Def., $\mathbb{W}.2 : 9, 10$
12	α	RS $: 7, 11$
13	$\Diamond\neg\alpha$	ass.
14	$\neg\alpha$	Def., RS $: 12, 13$
15	$\perp$	LL $: 12, 14$
16	$\Diamond\neg\alpha \to (\Diamond\beta \to (\Diamond\neg\beta \to (\Diamond\neg(\alpha \to \beta) \to \perp)))$	MDT.2
17	M17	LL, M1 $: 16$

$\square$

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