Multivariate Normality of Cartesian Frame Covariances: Evaluation and Operational Significance

2019 AAS/AIAA Astrodynamics Specialist Conference

Travis Lechtenberg (Omitron)

Agenda and Overview

- **Introduction**
  - Motivation
  - Objectives
- **Methodology**
  - Data Sets
  - Test Statistics
- **Analysis**
  - Test Passage CDF Plot – Historic Conjunctions
  - Test Statistic Correlation
  - Mean Anomaly Uncertainty Cutoffs
  - Test Passage CDF Plot – High Pc Conjunctions
  - Pc Miscarriage Detection Rate
- **Conclusions and Recommendations**
  - Conclusions and Recommendations
  - Future Work

INTRODUCTION

Motivation [1 of 2]

- **Collision risk assessment approaches are largely based on Probability of Collision (*Pc*) assessments**
- **Typically this assessment is performed using a 2D method**
  - The conjunction is transformed to a 2D plane with the relative velocity vector between two objects coming out of the plane
- **2D Pc calculations require 3 assumptions to be satisfied over an encounter:**
  - The relative object trajectories can be approximated as linear
  - The position covariance matrices of the two objects can be approximated as constant
  - The covariance matrices of the two objects are multivariate normal, and can be combined in a common reference frame
- **Covariance matrices are typically assumed as multivariate normal, or Gaussian, in the inertial reference frame in which they are combined**

Motivation [2 of 2]

- **This assumption is known to be flawed due to the curvilinear nature of satellite orbits**
  - The assumption is likely valid for relatively small covariance matrices
  - Covariance matrices are reported for future events and typically incorporate a method of state uncertainty growth
  - This growth typically is most pronounced in the intrack uncertainty, and can cause position uncertainties to become non-Gaussian relatively quickly
  - Primary object state uncertainties are typically more confident than secondary objects, and hence the secondary object is where this problem crops up most frequently
  - Curvilinear reference frames remain Gaussian representations of the state uncertainty longer, but these frames do not lend themselves to forming combined covariance matrices
- **Identification of when this covariance normality assessment is flawed can provide one measure of when 2D Pc calculations may be suspect**

Objectives [1 of 2]

- **Examine proposed methods of assessing when the multivariate normality assumption is flawed**
  - Kolmogorov-Smirnov (K-S) 2-Sample Test
  - Henze-Zirkler (H-Z) Multivariate normality test
- **Cross examine test results for consistency and robustness**
- **The Henze-Zirkler Test was selected based on prior research which showed that the test was the most robust of a large number of multivariate normality tests**
- **Both tests require a sampling of object states using a Monte Carlo methodology**

Objectives [2 of 2]

- **Determine multivariate normality failure rates for operational data sets**
  - Examine a large number of unrestricted, reported operational events to determine failure rates of multivariate normality
  - Examine failure rates with regards to normalized intrack uncertainty
  - Recommend when intrack uncertainty measures near definitively indicate multivariate normality failures
- **Determine test efficacy in identification of 2D Pc failures**
  - Examine a large number of limited, reported operational events to determine failure rates of multivariate normality
    - 2D Pc estimates > 1E-07
    - Relatively recent orbit determination epochs with respect to TCA

METHODOLOGY

Data Sets [1 of 2]

- **A first set of ~50,000 recent conjunctions over the past year were examined for multivariate normality to assess failure rates**
  - These conjunctions spanned many orbit regimes from LEO to GEO orbits
  - The time between the secondary object orbit determination epoch and the time of close approach (TCA) had no restrictions imposed
  - No restrictions on 2D Pc were imposed
- **A second set of ~44,000 conjunctions coincident with Hall[1] were examined for operational significance**
  - Data with older secondary object orbit determination epochs was omitted
  - All conjunctions had 2D Pc estimates greater than 1E-07

[1] Hall, D.T., "Implementation Recommendations and Usage Boundaries for the Two-Dimensional Probability of Collision Calculation," 2019 AAS/AIAA Astrodynamics Specialist Conference, Portland ME, August 11-15, 2019.

Data Sets [2 of 2]

- **For each conjunction, two sets of 10,000 state samples were generated for the secondary object using the reported state and uncertainty**
  - One set of state samples was generated by transforming the state and covariance matrix to the equinoctial frame and sampling in the equinoctial frame
  - A second set of state samples was generated using the secondary object's inertial, Cartesian state and covariance matrix
  - The sample size was dictated by a desire to achieve 99.7% confidence in multivariate normality test results
- **Multivariate normality was examined with regards to mean anomaly uncertainty to normalize for various operational altitudes**

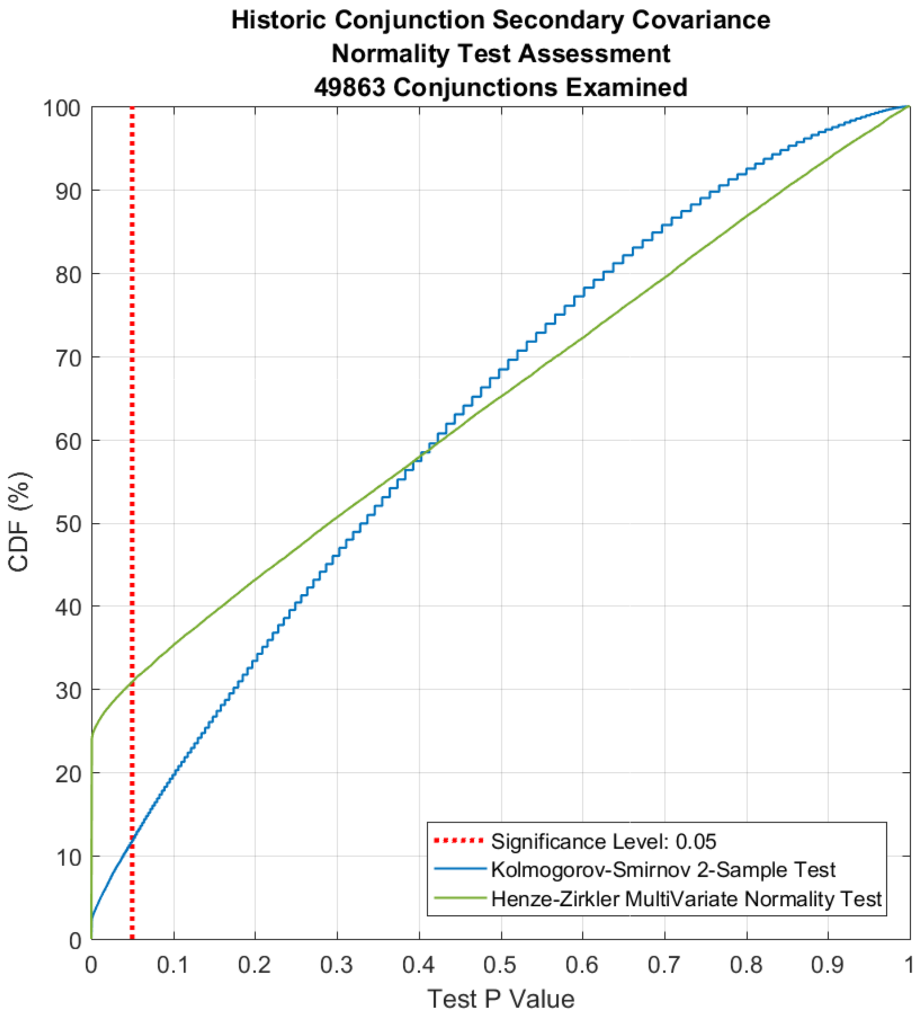
Test Statistics

- **The Henze-Zirkler test requires only one sample distribution of states to assess multivariate normality**
  - The equinoctially sampled states were transformed back to inertial Cartesian states and then used within the Henze-Zirkler test to assess multivariate normality
- **The Kolmogorov-Smirnov 2-Sample test requires two sample distributions of states to assess multivariate normality**
  - The test examines the hypothesis that both distributions come from the same parent distribution
  - The equinoctially sampled states and Cartesian sampled states were compared in the Cartesian frame for consistency
  - Each of the X, Y and Z element distributions were examined independently, then the most detrimental test result was used as an assessment of the entire test
- **Tests were only executed with regards to position state elements, as the order of magnitude differences in position and velocity distributions caused higher test failure rates**
  - Position distributions are the primary driver of 2D Pc calculations
- **Tests were modified to output P-Value test statistics for which to compare to a significance threshold**
  - Typical significance threshold is 0.05

ANALYSIS

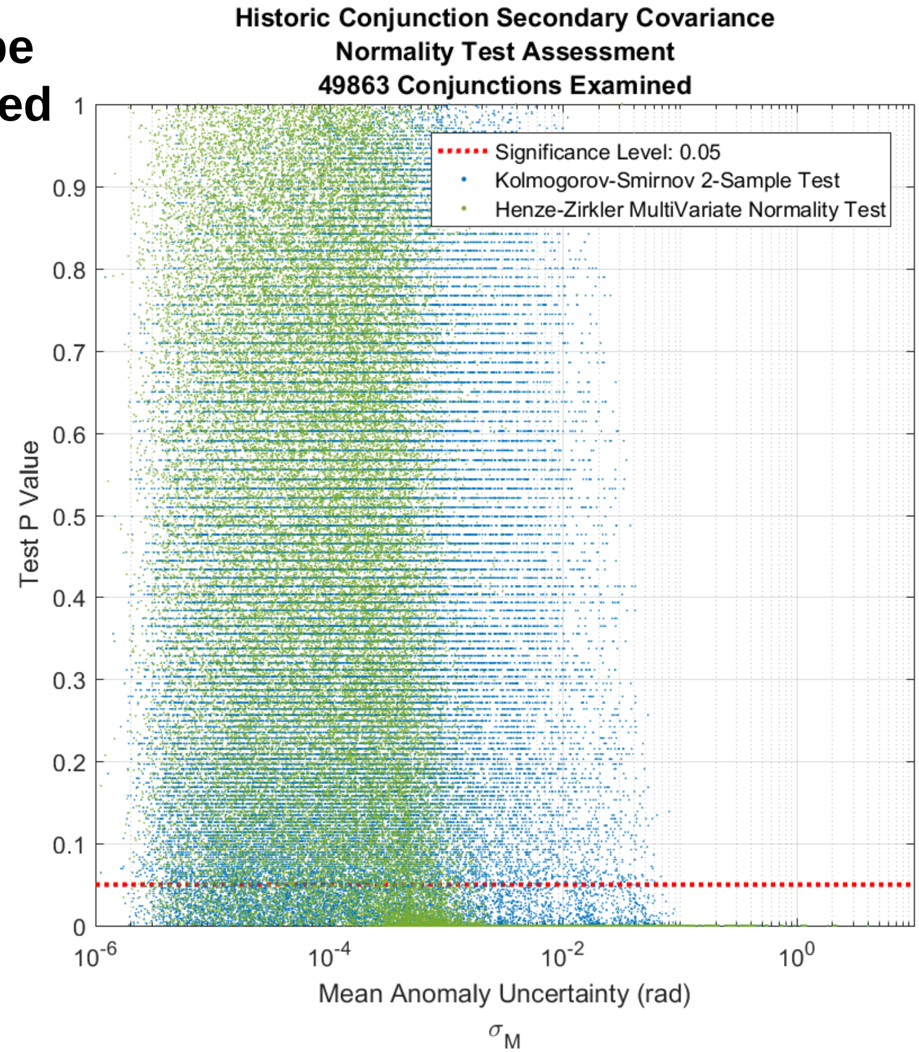
Test Passage CDF Plot – Historic Conjunctions

- The two normality tests show little commonality in results
- H-Z test shows a roughly 70% pass rate
- K-S test shows a 88% pass rate



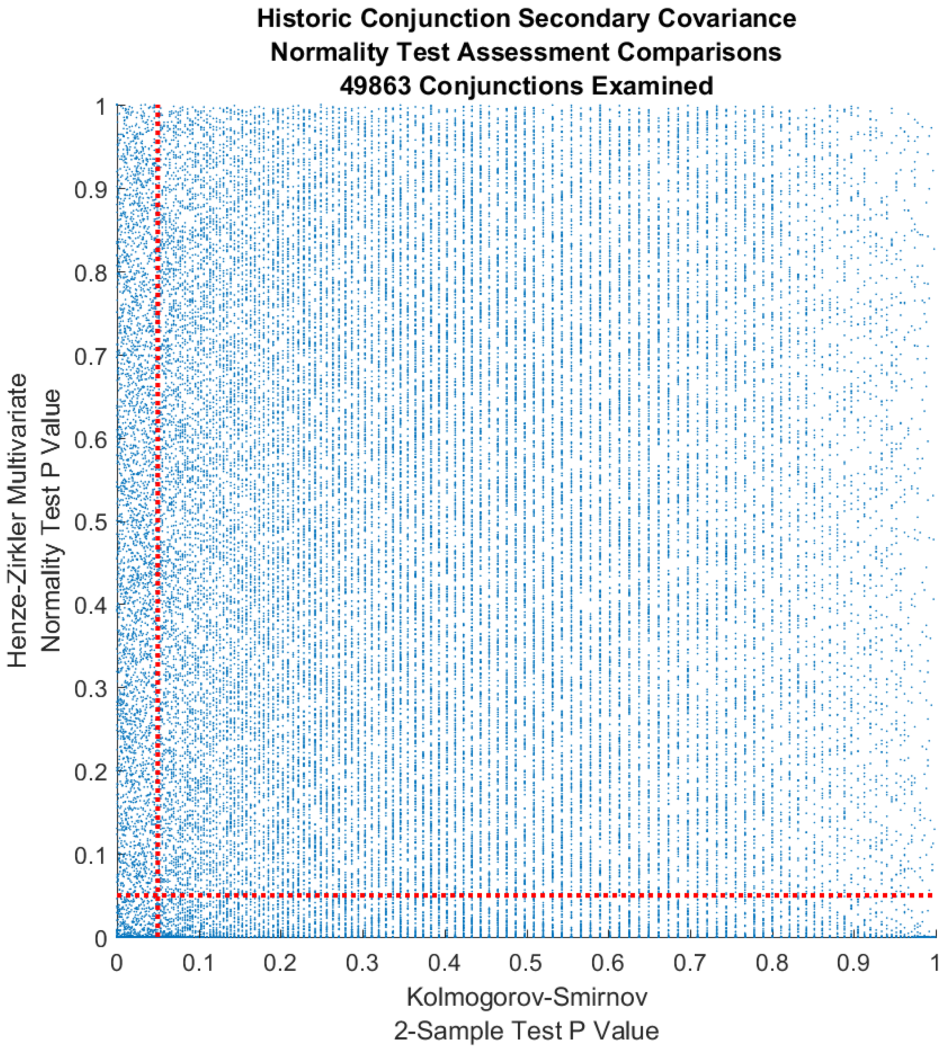
Mean Anomaly Uncertainty Dependence Both Tests

- **The Henze-Zirkler test appears to be the stringent test with fewer sampled states with high mean anomaly uncertainties passing the multivariate normality test**
- **The Kolmogorov-Smirnov test has many instances indicating test passage for secondary objects which have mean anomaly uncertainties of greater than 0.05 radians**
  - Roughly an order of magnitude more tolerant than the H-Z test
- **For this reason, use of the Henze-Zirkler test is recommended for assessing multivariate normality**



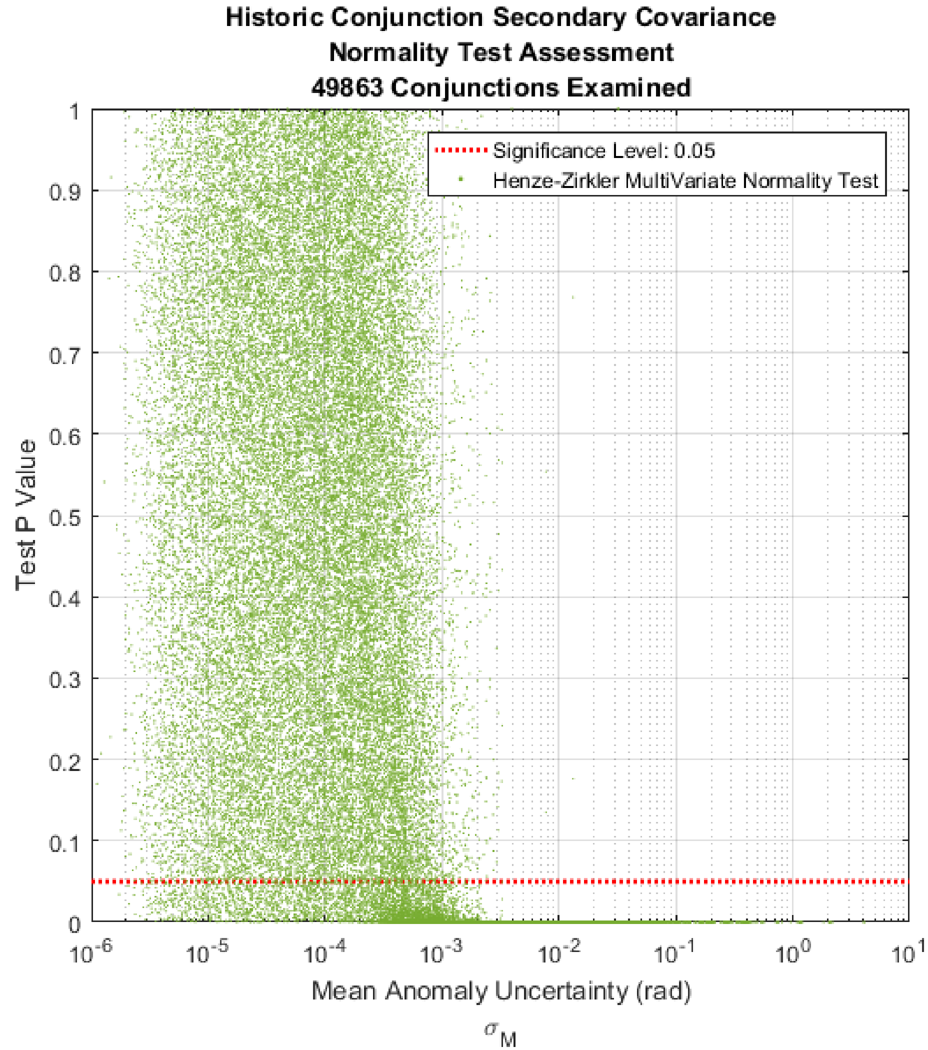
Test Statistic Correlation

- **Examination of test statistic correlation between the two tests revealed little observable correlation**
  - Correlation Coefficient: 0.008
- **Expected behavior should be relatively linear**
- **Or at minimum, without large numbers of conjunctions passing one test and failing the other**
- **As the H-Z test is the more stringent test, it is used as the primary decision criteria for future examinations**



Mean Anomaly Uncertainty Dependence

- **The K-S test statistic results are removed from the prior figure**
- **A cutoff value for mean anomaly uncertainty presents itself as a method to readily identify when covariance matrices can be assumed non-Gaussian**
  - This alleviates the need to run the full test for all conjunctions
- **The likelihood that the Cartesian covariance is realistically Gaussian drops sharply at around:**
  - 1.30E-03 radian mean anomaly uncertainty

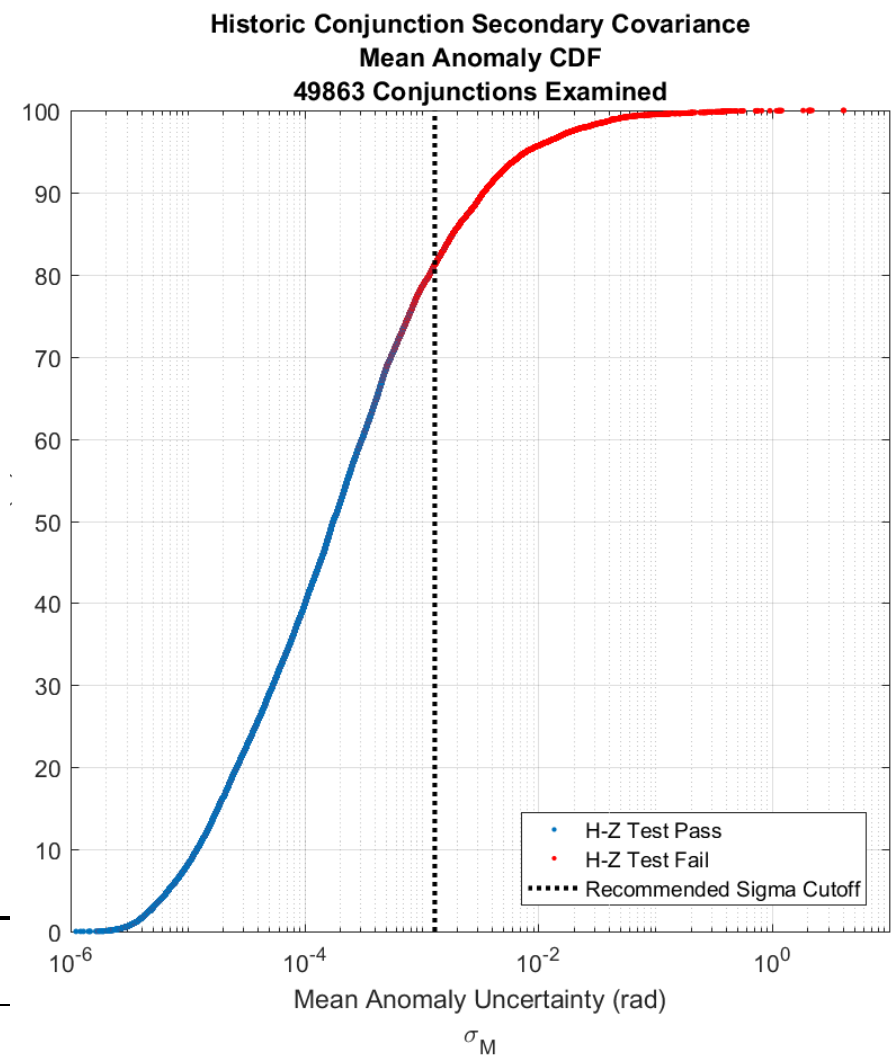


Mean Anomaly Uncertainty Cutoffs

- **Additional significance thresholds were examined for mean anomaly uncertainty cutoff values**
- **Cutoffs are reported based on a quantile basis referring to the H-Z pass rate**
  - *E.g.* What is the mean anomaly uncertainty below which 99% of conjunctions which passed the multivariate normality test exist?
- **Significance levels have little effect on cutoff values**
- **99% quantile recommended for operational usage**

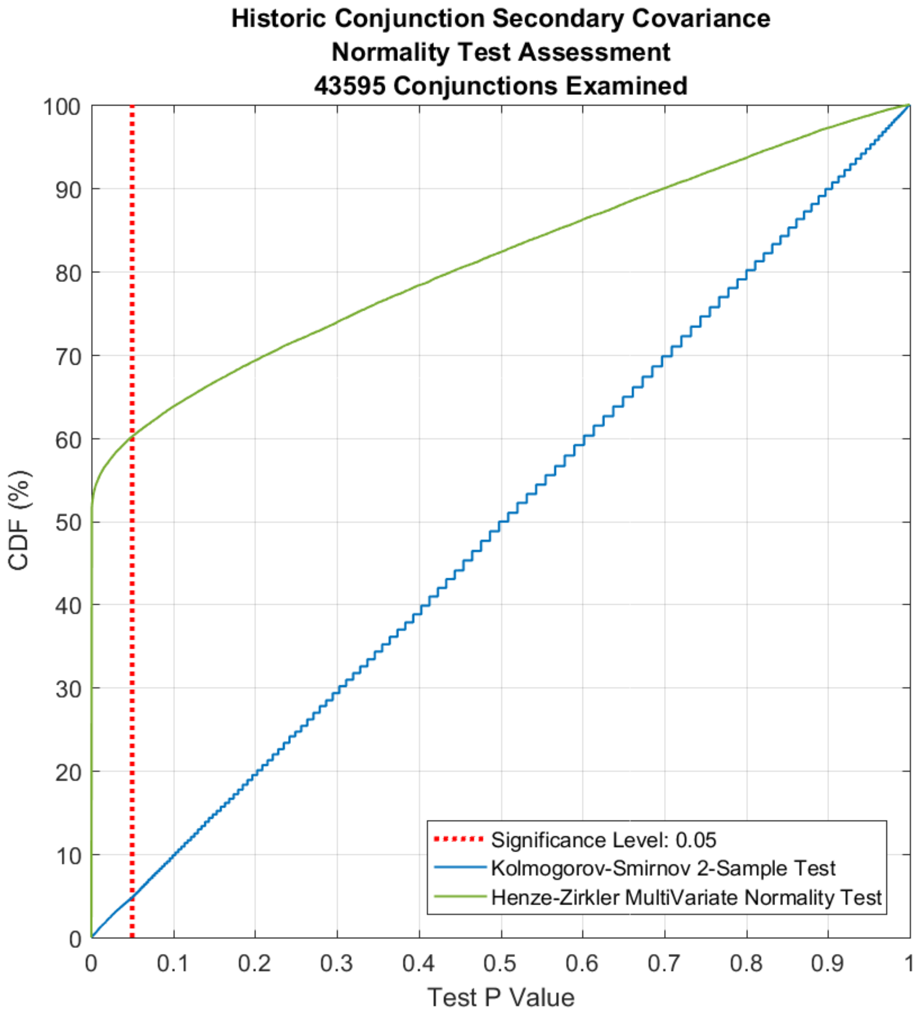
Table 1: Recommended Mean Anomaly Uncertainty Cutoffs

Significance Level	*95% Quantile*	*99% Quantile*	*99.9% Quantile*	*100% Quantile*
0.1	6.90E-04	1.30E-03	2.70E-03	3.20E-02
0.05	6.50E-04	1.30E-03	2.60E-03	3.20E-02
0.01	6.20E-04	1.30E-03	2.60E-03	3.20E-02



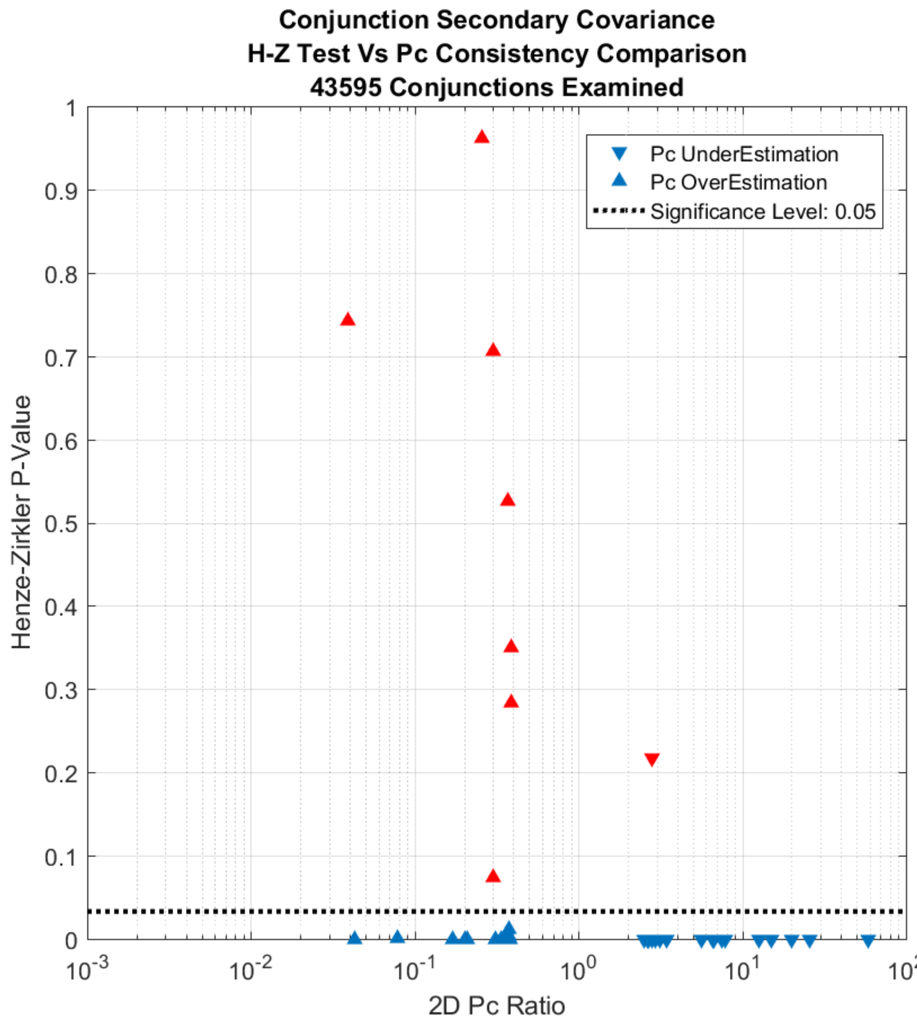
Test Passage CDF Plot – High Pc Conjunctions

- **Multivariate normality test passage investigated for second data set of high Pc conjunctions**
- **Only ~40% of conjunctions passed the H-Z test for multivariate normality**
- **This data shows a lower test pass rate for high Pc events**
  - Indicating that high Pc values are often driven by secondary objects which fail multivariate normality tests



Pc Miscarriage Detection Rate

- **Concurrent work identified Pc miscarriages by examining the difference between 2D Pc values and those determined using a Monte Carlo analysis**
  - 22 Cases of underestimation
  - 35 Cases of overestimation
- **Differences of an order of magnitude of 2.5 were considered miscarriages**
- **Pc underestimation results are the more worrisome**
- **The H-Z multivariate normality test caught all but 1 of the Pc underestimation miscarriages**



CONCLUSIONS AND RECOMMENDATIONS

Conclusions and Recommendations

- **The Henze-Zirkler multivariate normality test can be used to identify objects where the reported Cartesian covariance matrices at TCA are not accurate representations**
- **Objects with mean anomaly uncertainties exceeding a cutoff threshold should be assumed not multivariate normal**
  - Recommended Cutoff: 1.30E-03 Radians
- **A multivariate normality test of secondary object covariance matrices may be used in tandem with other tests to identify suspect 2D Pc calculations**
  - The 2D Pc boundary condition test identified by Doyle[1] is less computationally intensive and shows a large degree of overlap with this test
- **This test should be limited to High Interest Events (HIEs) due to the computational load imposed by it**

Table 3: Recommended Intrack Uncertainty Cutoff Values for Predefined Orbit Regimes

Orbit Regime	*Regime Definition*	*Intrack Uncertainty Cutoff*
LEO1	Perigee ≤ 500 km Eccentricity < 0.25	8.94 km
LEO2	500 km < Perigee ≤ 750 km Eccentricity < 0.25	9.27 km
LEO3	750 km < Perigee ≤ 1200 km Eccentricity < 0.25	9.85 km
LEO4	1200 km < Perigee ≤ 2000 km Eccentricity < 0.25	10.89 km
MEO	600 min < Period < 800 min Eccentricity < 0.25	37.11 km
GEO	1300 min < Period < 1800 min Eccentricity < 0.25 Inclination < 35°	54.81 km

Future Work

- **Examine Pc sensitivity more thoroughly to multivariate normality test failures**
- **Incorporation of this test into a more thorough test suite**
- **Examine more computationally efficient methods of assessing multivariate normality**
  - Sample size dependency
  - Sigma Point Analysis

QUESTIONS