# A Little Charity Guarantees Almost Envy-Freeness

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**Abstract**

Fair division of indivisible goods is a very well-studied problem. The goal of this problem is to distribute $m$ goods to $n$ agents in a "fair" manner, where every agent has a valuation for each subset of goods. We assume general valuations.

Envy-freeness is the most extensively studied notion of fairness. However, envy-free allocations do not always exist when goods are indivisible. The notion of fairness we consider here is "envy-freeness up to any good" (EFX) where no agent envies another agent after the removal of any single good from the other agent's bundle. It is not known if such an allocation always exists even when $n = 3$.

We show there is always a partition of the set of goods into $n+1$ subsets $(X_1, \ldots, X_n, P)$ where for $i \in [n]$, $X_i$ is the bundle allocated to agent $i$ and the set $P$ is unallocated (or donated to charity) such that we have:

- envy-freeness up to any good,
- no agent values $P$ higher than her own bundle, and
- fewer than $n$ goods go to charity, i.e., $|P| < n$ (typically $m \gg n$).

Our proof is constructive. When agents have additive valuations and $|P|$ is large (i.e., when $|P|$ is close to $n$), our allocation also has a good maximin share (MMS) guarantee. Moreover, a minor variant of our algorithm also shows the existence of an allocation which is 4/7 *groupwise* maximin share (GMMS): this is a notion of fairness stronger than MMS. This improves upon the current best bound of 1/2 known for an approximate GMMS allocation.

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# 1 Introduction

Fair division of goods among competing agents is a fundamental problem in Economics and Computer Science. There is a set $M$ of $m$ goods and the goal is to *allocate* goods among $n$ agents in a *fair* way. An allocation is a partition of $M$ into disjoint subsets $X_1, \ldots, X_n$ where $X_i$ is the set of goods given to agent $i$. When can an allocation be considered "fair"? One of the most well-studied notions of fairness is *Envy-freeness*. Every agent has a *value* associated with each subset of $M$ and agent $i$ *envies* agent $j$ if $i$ values $X_j$ more than $X_i$. An allocation is *envy-free* if no agent envies another. An envy-free allocation can be regarded as a fair and desirable partition of $M$ among the $n$ agents since no agent envies another; as mentioned in [26], such a mechanism of partitioning land dates back to the Bible.

Unlike land which is divisible, goods in our setting are *indivisible* and an envy-free allocation of the given set of goods need not exist. Consider the following simple example with two agents and a single good: one of the agents has to receive this good and the other agent envies her. Since envy-free allocations need not exist, several relaxations have been considered.

**Relaxations.** Budish [11] introduced the notion of *EF1*: this is an allocation of goods that is "envy-free up to *one* good". In an EF1 allocation, agent $i$ may envy agent $j$, however this envy would vanish as soon as *some* good is removed from $X_j$. Note that no good is really removed from $X_j$: this is simply a way of assessing how much $i$ values $X_j$ more than $X_i$. That is, if $i$ values $X_j$ more than $X_i$, then there exists some $g \in X_j$ such that $i$ values $X_i$ at least as much as $X_j \setminus \{g\}$. Going back to the example of two agents and a single good, the allocation where one agent receives this good is EF1. It is known that EF1 allocations always exist; as shown by Lipton et al. [25], such an allocation can be efficiently computed.

Caragiannis et al. [13] introduced a notion of envy-freeness called *EFX* that is stronger than EF1. An EFX allocation is one that is "envy-free up to *any* good". In an EFX allocation, agent $i$ may envy agent $j$, however this envy would vanish as soon as *any* good is removed from $X_j$. Thus every EFX allocation is also EF1 but not every EF1 allocation is EFX. Consider this simple example: there are three goods $a, b, c$ and two agents with *additive valuations* (defined in Section 1.1) as described below.

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| *Agent* 1 | 1 | 1 | 2 |
| *Agent* 2 | 1 | 1 | 2 |

Both agents value $c$ twice as much as $a$ or $b$. The allocation where agent 1 gets $\{a\}$ and agent 2 gets $\{b, c\}$ is EF1 but not EFX. On the other hand, the allocation where agent 1 gets $\{a, b\}$ and agent 2 gets $\{c\}$ is EFX. Indeed, the latter allocation seems fairer than the former allocation. As said in [12]: "*Arguably, EFX is the best fairness analog of envy-freeness of indivisible items*". While it is known that EF1 allocations always exist, the question of whether EFX allocations always exist is still an open problem (despite significant effort, as per [13]).

Plaut and Roughgarden [26] showed that EFX allocations always exist (i) when there are only two agents or (ii) when all $n$ agents have the same valuations. Moreover, it was shown in [26] that exponentially many value queries may be needed to determine EFX allocations even in the restricted setting where there are only two agents with identical *submodular* valuation functions[1]. It is not known if an EFX allocation always exists even when there are only three agents with additive valuations. It was remarked in [26]: "*We suspect that at least for general valuations, there exist instances where no EFX allocation exists*".

[1]These are valuation functions with decreasing marginal values.

**A relaxation of EFX.** Very recently, Caragiannis et al. [12] introduced a more relaxed notion of EFX called *EFX-with-charity*. This is a partial allocation that is EFX, i.e., the entire set of goods need not be distributed among the agents. So some goods may be left *unallocated* and it is assumed that these unallocated goods are donated to charity. There is a very simple allocation that is EFX-with-charity where *no* good is assigned to any agent—thus all goods are donated to charity. Obviously, this is not an interesting allocation and one seeks allocations with better guarantees and one such allocation was shown in [12].

Let $X^* = \langle X_1^*, \ldots, X_n^* \rangle$ be an optimal *Nash social welfare* allocation[2] on the entire set of goods. It was shown in [12] that there always exists an EFX-with-charity allocation $X = \langle X_1, \ldots, X_n \rangle$ where every agent receives at least half the value of her allocation in $X^*$. Interestingly, as shown in [12], $X_i \subseteq X_i^*$ for all $i$. Unfortunately, there are no upper bounds on how *complete* this allocation is (wrt bounding the number of unallocated goods) or on the value any agent has for the set of goods donated to charity.

We believe these are important questions to ask. The ideal allocation is one that is EFX and *complete*; so we would like a guarantee that a large number of goods have been allocated to agents. Moreover, since EFX allocations guarantee envy-freeness once any good is removed from another agent's set, it is in the same spirit that we seek an EFX (partial) allocation where nobody envies the set of unallocated goods. The allocation in [12] gives no guarantee either on the number of unallocated goods or on whether any agent values the set of unallocated goods more than her own bundle. Here we consider the notion of *EFX-with-bounded-charity*. That is, we seek EFX-with-charity allocations with bounds on the set given to charity, i.e., a bound on the *size* and a bound on the *value* of the set of goods donated to charity.

## 1.1 Our Results

Let $N = [n]$ be the set of agents. Every agent $i \in [n]$ has a valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$, where $M$ is the set of $m$ goods. We show our main existence result for *general* valuation functions, i.e., the only assumptions we make on any valuation function $v_i$ is that (i) it is *normalized*, i.e., $v_i(\emptyset) = 0$, and (ii) it is *monotone*, i.e., $S \subseteq T$ implies $v_i(S) \leq v_i(T)$. In contrast, the EFX-with-charity allocation in [12] works only for *additive valuations*, i.e., $v_i(S) = \sum_{g \in S} v_i(\{g\})$.

We show there always exists an allocation[3] $X = \langle X_1, \ldots, X_n \rangle$ that satisfies the following properties:

1. $X$ is EFX, i.e., for any two agents $i, j$: $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for any $g \in X_j$;
2. $v_i(X_i) \geq v_i(P)$ for all agents $i$, where $P = M \setminus \cup_{i=1}^n X_i$ is the set of unallocated goods;
3. $|P| < n$ (recall that $n$ is the number of agents).

Our proof is constructive. We start with no goods being allocated to the agents and find the claimed allocation by at most $nmV/\Delta$ applications of three simple update rules. Here, $n$ is the number of agents, $m$ is the number of goods, $V = \max_i v_i(M)$ is the maximum valuation of any agent, and $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ is the minimum difference between distinct valuations. The update rules use a *minimum-size-valuable-set*-oracle: Given $S \subseteq M$, agent $i$, and $\alpha \in \mathbb{R}$ such that $v_i(S) > \alpha$, find a minimum cardinality subset $Z \subseteq S$ such that $v_i(Z) > \alpha$. If this oracle can be realized by an algorithm, our proof is algorithmic.

It also follows from our proof that when all agents have the same valuation function, our allocation is *complete*. That is, $|P| = 0$. This is an alternate proof to the existence of complete EFX allocations for identical (general) valuations, originally shown in [26].

[2]This is an allocation that maximizes $\Pi_{i=1}^n v_i(X_i^*)$, where $v_i$ is agent $i$'s valuation function.

[3]Henceforth, allocations are *partial* and we will use "complete allocation" to refer to one where all goods are allocated.

Our next result is a pseudo-polynomial time algorithm to find an allocation that obeys properties 1-3 given above. For this, we assume that all agents have *gross substitute valuations* (defined in Section 3). For gross substitute valuations, the minimum-size-valuable-set-oracle can be realized by a simple greedy algorithm. Every additive valuation is a gross substitute valuation and gross substitute valuations form a subclass of the set of submodular valuations [22].

- Suppose all agents have gross substitute valuations. Then an EFX allocation with properties 1-3 can be computed with $\mathsf{poly}(n, m, V, 1/\Delta)$ queries.

### 1.1.1 Additive valuations

The most well-understood class of valuation functions is the set of *additive* valuations. We consider the case when all agents have additive valuations and show that our allocation or very minor variants of our allocation can guarantee several other notions of fairness.

**Ensuring high Nash social welfare.** We show that modifying the starting step of our algorithm ensures that our allocation $X$, which satisfies properties 1-3 stated above, also has a high Nash social welfare. That is, $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ as promised in [12], where $X^* = \langle X_1^*, \dots, X_n^* \rangle$ is an optimal Nash social welfare allocation.

**Number of Unallocated Goods and MMS Guarantee.** Another interesting and well-studied notion of fairness is *maximin share*. Suppose agent $i$ has to partition $M$ into $n$ *bundles* (or sets) knowing that she would receive the worst bundle with respect to her valuation. Then $i$ will choose a partition of $M$ that maximizes the valuation of the worst bundle (wrt her valuation). The value of this worst bundle is the maximin share of agent $i$. An important question here is: does there always exist an allocation of $M$ where every agent gets a bundle worth at least her maximin share?

Formally, let $N$ and $M$ be the sets of $n$ agents and $m$ goods, respectively. We define the maximin share of an agent $i$ as follows: (here $\mathcal{X}$ is the set of all complete allocations)

$$MMS_i(n, M) = \max_{\langle X_1, \dots, X_n \rangle \in \mathcal{X}} \min_{j \in [n]} v_i(X_j).$$

The goal is to determine an allocation $\langle X_1, X_2, \dots, X_n \rangle$ of $M$ such that for every $i$ we have $v_i(X_i) \geq MMS_i(n, M)$. This question was first posed by Budish [11]. Procaccia and Wang [27] showed that such an allocation need not exist, even in the restricted setting of only three agents! Thereafter, *approximate-MMS* allocations were studied [27, 18, 20, 19] and we know polynomial time algorithms to find allocations where for all $i$, agent $i$ gets a bundle of value at least $\alpha \cdot MMS_i(n, M)$; the current best guarantee for $\alpha$ is $3/4 - \epsilon$ by Ghodsi et al. [20] (for any $\epsilon > 0$) and this was very recently improved to $3/4$ by Garg and Taki [19].

Amanatadis et al. [2] showed that any complete EFX allocation is also a $\frac{4}{7}$-MMS allocation. We show that our allocation promises better MMS guarantees when the number of unallocated goods is large. Let $X = \langle X_1, \dots, X_n \rangle$ be our allocation as described by properties 1-3 above and let $P$ be the set of unallocated goods. For any agent $i \in [n]$, we have:

$$v_i(X_i) \geq \frac{1}{2 - |P|/n} MMS_i(n, M).$$

Hence, the larger the number of unallocated goods, the better guarantees we get on MMS. The extreme values are $|P| = 0$ and $|P| = n - 1$. When $|P| = 0$, we have a complete EFX allocation and when $|P| = n-1$, we have an EFX allocation that is an *almost-MMS* allocation: $v_i(X_i) \geq (1 - 1/n) \cdot MMS_i(n, M)$ for all $i$.

**Improved Guarantees for Groupwise MMS.** Barman et al. [6] recently introduced a notion of fairness called *groupwise maximin share* (GMMS) which is stronger than MMS. An allocation is said to be GMMS if the MMS condition is satisfied for *every subgroup* of agents and the union of the sets of goods allocated to them. Formally, a complete allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is $\alpha$-GMMS if for any $N' \subseteq N$, we have $v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{i \in N'} X_i)$ where $n' = |N'|$. Every GMMS allocation, i.e. $\alpha = 1$, is also a complete EFX allocation [6].

It is known [6] that there are instances where GMMS is arbitrarily better than MMS. Naturally, it is a harder problem to approximate GMMS than MMS. While $\frac{3}{4}$-MMS allocations always exist, the largest $\alpha$ for which $\alpha$-GMMS allocations are known to exist is $\frac{1}{2}$ [6]. We extend the result of Amanatadis et al. [2] for MMS to show the following:

- A $\frac{4}{7}$-GMMS allocation always exists and can be computed in pseudo-polynomial time.

In particular, we show that modifying the last step of our algorithm results in a complete allocation that is $\frac{4}{7}$-GMMS.

## 1.2 Our Techniques

**Envy-Graph.** We now give an overview of the main ideas used to find our EFX allocation. We first recall the algorithm of Lipton et al. [25] for finding an EF1 allocation[4]. They use the notion of an *envy-graph*: here each vertex corresponds to an agent and there is an edge $(i, j)$ iff $i$ envies $j$. The invariant maintained is that the envy-graph is a DAG: a cycle corresponds to a cycle of envy and by swapping bundles along a cycle, every agent becomes better-off and the number of envy edges decreases. More precisely, if $i_0 \to i_1 \to i_2 \to \ldots \to i_{\ell-1} \to i_0$ is a cycle in the envy graph, then reassigning $X_{i_{j+1}}$ to agent $i_j$ for $0 \leq j < \ell$ (indices are to be read modulo $\ell$) will increase the valuation of every agent in the cycle. Also if there was an edge $s$ to some $i_k$ where $s$ is not a part of the cycle, this edge just gets directed now from $s$ to $i_{k+1}$ after we exchange bundles along the cycle. Thus the number of envy edges in the graph does not increase and the valuations of the agents in the cycle goes up. Thus cycles can be eliminated.

The algorithm in [25] runs in rounds and always maintains an allocation that is also EF1. At the beginning of every round, an unenvied agent $s$ (this is a source vertex in this DAG) is identified and an unallocated good $g$ is allocated to $s$. The new allocation is also EF1, as nobody will envy the bundle of $s$ after removing the good $g$.

**The Cut-and-Merge Operation.** We now highlight a key difference between an EF1 allocation and an EFX allocation. From the algorithm of Lipton et al. [25], it is clear that given an EF1 allocation on a set $M_0$ of goods, one can determine an EF1 allocation on $M_0 \cup M_1$, for any $M_1 \subseteq M \setminus M_0$, by simply adding goods from $M_1$ one-by-one to the existing bundles and changing the owners (if necessary) in a clever way. Intuitively, we never need to cut or merge the bundles formed in any EF1 allocation. We can just append the unallocated goods appropriately to the current bundles.

The above strategy is very far from true for EFX. Consider the example illustrated below with three agents with additive valuations and four goods $a, b$, $c$, and $d$.

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| *Agent* 1 | 0 | 1 | 1 | 2 |
| *Agent* 2 | 1 | 0 | 1 | 2 |
| *Agent* 3 | 1 | 1 | 0 | 2 |

[4]The algorithm in [25] was published in 2004 with a different property and EF1 was proposed in 2011.

An EFX allocation for the first three goods has to give exactly one of $a, b, c$ to each of the three agents. However an EFX allocation for all the four goods has to allocate the singleton set $\{d\}$ to some agent (say, agent 1) and say, $\{a\}$ to agent 2 and $\{b, c\}$ to agent 3. Thus the allocation needs to *cut and merged.* When there are many agents - each with her own valuation, figuring out the cut-and-merge operations is the difficult step. Here we implement a "merge-and-cut" operation as follows.

**Improving Social Welfare.** Suppose we have an EFX allocation $X = \langle X_1, \ldots, X_n \rangle$ on some subset $M_0 \subset M$. We would now like to add a good $g \in M \setminus M_0$. However we will not be able to guarantee an EFX allocation on $M_0 \cup \{g\}$. What we will ensure is that either case (i) or case (ii) occurs:

(i) We have an EFX allocation $X' = \langle X'_1, \ldots, X'_n \rangle$ on a subset of $M_0 \cup \{g\}$ such that $v_i(X'_i) \geq v_i(X_i)$ for all $i$ and for at least one agent $j$ we have $v_j(X'_j) > v_j(X_j)$. Thus $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$; in other words, *social welfare* strictly improves.

(ii) We have an EFX allocation on $M_0 \cup \{g\}$ and the social welfare remains unchanged.

Hence in each step of our algorithm, we either increase the number of allocated goods or we increase social welfare—thus we always make progress. This is similar to the approach used by Plaut and Roughgarden [26] to guarantee the existence of $\frac{1}{2}$-EFX[5] when agents have subadditive valuations. We now outline how we ensure one of case (i), case (ii) has to happen:

For simplicity of exposition, we assume the envy-graph corresponding to our starting EFX allocation $X$ has a single source $s$. Add $g$ to $s$'s bundle: if nobody envies $s$ up to any good then we are in an easy case as we have an EFX allocation on $M_0 \cup \{g\}$. In this case, we "decycle" the envy-graph (if cycles are created) and continue. Observe that swapping bundles along a cycle in the envy-graph increases social welfare.

**Most Envious Agent.** So assume there are one or more agents who envy $s$ up to any good after $g$ is allocated to $s$. To resolve this, we introduce the concept of a *most envious agent.* Let $i$ be an agent who envies $s$ up to any good, so $v_i(X_i) < v_i(S')$ for some $S' \subset X_s \cup \{g\}$. Let $S_i$ be the subset of $X_s \cup \{g\}$ with the *minimum cardinality* such that $v_i(X_i) < v_i(S_i)$. So for any $T_i \subset S_i$, we have $v_i(X_i) \geq v_i(T_i)$. The agent $i$ with the smallest value of $|S_i|$ (break ties arbitrarily) is the most envious agent of $s$. Call this agent $t$.

The crucial observation is that no agent envies $S_t$ *up to any good*—otherwise it would contradict $t$ being the agent with the smallest value of $|S_t|$. Recall the assumption that $s$ is the only source, so there is a path $s \to i_1 \to \cdots \to i_{k-1} \to t$ in the envy-graph. We do a leftwise shift of bundles along this path: so $s$ gets $i_1$'s bundle, and for $1 \leq r \leq k-1$: $i_r$ gets $i_{r+1}$'s bundle (where $i_k = t$), and finally $t$ gets $S_t$. The goods in $X_s \cup \{g\} \setminus S_t$ are thrown back into the pool of unallocated goods.

Observe that every agent in this path is strictly better-off now than in the allocation $X$ and nobody is worse-off. Moreover, by the definition of $S_t$, there are no agents envying any agent up to any good. Thus we have a desired EFX allocation $X'$. When there are multiple sources, we can adapt this technique *provided* there are enough unallocated goods; in particular, the number of unallocated goods must be at least the number of sources in the envy-graph. We describe this in detail in Section 2.

We would like to contrast the above approach with other EFX algorithms [26, 12]. The $\frac{1}{2}$-EFX algorithm by Plaut and Roughgarden [26] either merges $g$ (the new good) with an existing bundle or allocates the singleton set $\{g\}$ to an agent while the EFX-with-charity algorithm by Caragiannis et al. [12] takes an allocation of maximum Nash social welfare as input and then permanently removes some goods from the instance. We regard the notion of "most envious

[5] An allocation $X = (X_1, \ldots, X_n)$ is $\frac{1}{2}$-EFX if for any two agents $i, j$: $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_j \setminus \{g\})$ for all $g \in X_j$.

agent" that shows a natural way of breaking up a bundle to preserve envy-freeness up to any good as one of the innovative contributions of our work.

**Our Other Results.** Regarding our result with approximate MMS guarantee, the larger the number of unallocated goods in our EFX allocation, the larger are the number of sources: these are unenvied agents. Moreover, no agent envies the set of unallocated goods. Suppose for now that $|P| = n - 1$. This means every agent is a source. So no agent envies the bundle of any other agent and also the set of unallocated goods. Thus for each agent $i$, we have:

$$v_i(X_i) \;\geq\; \frac{v_i(M)}{n+1} \;\geq\; (1 + 1/n)^{-1} \cdot \frac{v_i(M)}{n} \;\geq\; (1 - 1/n) \cdot MMS_i(n, M),$$

where the constraint that $v_i(M)/n \geq MMS_i(n, M)$ holds for additive valuations. We show our result for approximate-MMS allocation and our improved bound for approximate-GMMS allocation in Section 4.

### 1.3 Related Work

Fair division of divisible resources is a classical and well-studied subject starting from 1940's [28]. Fair division of indivisible goods among competing agents is a young and exciting topic with recent work on EF1 and EFX allocations [13, 8, 26, 9, 12], approximate maximin share allocations [11, 10, 3, 7, 23, 20, 18], and approximation algorithms for maximizing Nash social welfare and generalizations [16, 15, 14, 4, 17, 5]. As mentioned earlier, Caragiannis et al. [13] introduced the notion of EFX: whether such allocations always exist is an enigmatic open problem. It was shown in [13] that there always exists an EF1 allocation that is also Pareto-optimal[6] and Barman et al. in [8] showed a weakly-polynomial time algorithm to compute such an allocation.

**Applications.** Fair division of goods or resources occurs in many real-world scenarios and this is demonstrated by the popularity of the website Spliddit (`http://www.spliddit.org`) that implements mechanisms for fair division where users can log in, define what needs to be divided, and enter their valuations. This website guarantees an EF1 allocation that is also Pareto-optimal and since its launch in 2014, it has been used tens of thousands of times [13]. We refer to [21, 26] for details on the diverse applications for which Spliddit has been used: these range from rent division and taxi fare division to credit assignment for an academic paper or group project. Another such website is Fair Outcomes, Inc. (`http://www.fairoutcomes.com`). Another interesting application is *Course Allocate* used at Wharton School that guarantees certain fairness properties to allocate courses among students [26].

## 2 Existence of an EFX-Allocation with Bounded Charity

We prove our main result on EFX-with-bounded-charity allocations in this section. We will define three *update rules*. Each update rule takes a pair $(X, P)$ consisting of an allocation $X$ and a set $P$ of unallocated goods (we will call $P$ the *pool*) and returns a modified pair $(X', P')$.

Each application of an update rule will ensure that either (i) the *social welfare* $\phi(X) = \sum_{i\in[n]} v_i(X_i)$ of the current allocation increases or (ii) the size of the pool decreases and the social welfare is left unchanged, so $|P'| < |P|$ in this case. Hence the update process will terminate. The overall structure of the algorithm is given in Algorithm 1.

In order to define our update rules, we need the concepts of *envy-graph* and the *most envious agent* for a bundle of goods. These were discussed in Section 1.2 and we formally define them below.

[6] An allocation $X = \langle X_1, \ldots, X_n\rangle$ is Pareto-optimal if there is no allocation $Y = \langle Y_1, \ldots, Y_n\rangle$ where $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$ and $v_i(Y_j) > v_i(X_j)$ for some $j$.

**Algorithm 1** Algorithm for Computing an EFX-Allocation

**Postcondition**: $X$ is EFX, $|P| < n$ and $v_i(P) \leq v_i(X_i)$ for all $i \in [n]$.

```
X_i ← ∅ for i ∈ [n]; P ← M;
while one of the update rules shown in Algorithm 2 is applicable do
Invariant: X is EFX and the envy-graph G_X is acyclic
    Let U_ℓ be an applicable update rule;
    (X, P) ← U_ℓ(X, P);
    Decycle the envy-graph;
end while
```

**Definition 1.** *The* envy-graph $G_X$ *for an allocation* $X = \langle X_1, X_2, \ldots, X_n \rangle$ *has the set of agents as vertices and there is a directed edge from agent* $i$ *to agent* $j$ *if and only if* $v_i(X_i) < v_i(X_j)$.

The notion of envy-graph was introduced in [25] and it is well-known that cycles can be removed from the envy-graph without destroying desirable properties (see Lemma 2). Thus we can maintain $G_X$ as a DAG. For an agent $s$, the *reachability component* $C(s)$ consists all agents reachable from $s$ in the envy-graph. The *sources* in the envy-graph are the vertices with indegree zero.

For ease of notation, we will use $B \setminus g$ and $B \cup g$ to denote $B \setminus \{g\}$ and $B \cup \{g\}$, respectively.

**Lemma 2.** *Let* $i_0 \to i_1 \to \cdots \to i_{k-1} \to i_0$ *be a cycle in the envy-graph. Consider the allocation* $X'$ *where* $X'_{i_\ell} = X_{i_{\ell+1}}$ *(indices are modulo* $k$*) for* $\ell \in \{0, \ldots, k-1\}$ *and* $X'_j = X_j$ *for* $j \notin \{i_0, \ldots, i_{k-1}\}$*. If* $X$ *is EFX, then* $X'$ *is also EFX. Moreover,* $\phi(X') > \phi(X)$.

*Proof.* Consider any agent $i$. We have $v_i(X'_i) \geq v_i(X_i)$ with strict inequality if $i$ lies on the cycle. So $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$. Thus $\phi(X') > \phi(X)$.

Since $X'$ is just a permutation of $X$, for any agent $j$ there exists some agent $j'$ such that $X'_j = X_{j'}$. Therefore, since $X$ is EFX, for any good $g \in X_{j'}$ (or equivalently $X'_j$) we have $v_i(X'_j \setminus g) = v_i(X_{j'} \setminus g) \leq v_i(X_i) \leq v_i(X'_i)$. Thus $X'$ is also EFX. □

**Definition 3.** *Let* $X$ *be an allocation and* $S \subseteq M$*. For an agent* $i$ *with* $v_i(S) > v_i(X_i)$*, let* $\kappa_X(i, S)$ *be the minimum* $k$ *such that there is a set* $Z \subseteq S$ *of size* $k$ *with* $v_i(Z) > v_i(X_i)$*. We define* $\kappa_X(S) = \min_{i \in [n]} \kappa_X(i, S)$.

The following definition formalizes the notion of "most envious agents". Let $S \subseteq M$, then we define

$$A_X(S) = \{i \in [n] : \kappa_X(i, S) = \kappa_X(S)\}.$$

If there is no $i$ with $v_i(S) > v_i(X_i)$, then $A_X(S)$ is empty. We make a simple observation.

**Lemma 4.** *Consider an agent* $i \in A_X(S)$ *and let* $Z \subseteq S$ *be such that* $|Z| = \kappa_X(S)$ *and* $v_i(Z) > v_i(X_i)$*. Then for any agent* $j$ *(incl.* $i$*) we have that* $v_j(Z \setminus g) \leq v_j(X_j)$ *for all* $g \in Z$.

*Proof.* Let $j$ be any agent. There are two cases: either $v_j(X_j) \geq v_j(S)$ or $v_j(X_j) < v_j(S)$. In the former case, we have $v_j(X_j) \geq v_j(Z \setminus g)$ by monotonicity. In the latter case, $v_j(X_j) \geq v_j(Z')$ for all sets $Z' \subseteq S$ of size at most $\kappa_X(j, S) - 1$ (by definition of $\kappa_X(j, S)$). Note that the set $Z \setminus g$ has size $\kappa_X(S) - 1 \leq \kappa_X(j, S) - 1$ since $\kappa_X(S) \leq \kappa_X(j, S)$ (by definition of $\kappa_X(S)$). Thus $v_j(Z \setminus g) \leq v_j(X_j)$. □

We are now ready to present our three update rules $U_0$, $U_1$, and $U_2$, see Algorithm 2.

**Rule** $U_0$ : Rule $U_0$ is the easiest of the update rules. It is applicable whenever adding a good from the pool to some source of $G_X$ does not destroy the EFX-property (see Algorithm 2).

**Lemma 5** (Rule $U_0$)**.**

**Algorithm 2** The Update Rules

1: **function** $U_0$(allocation $X$, pool $P$)
    **Precondition:** There is a good $g \in P$ and an agent $i$ such that allocating $g$ to $i$ results in an EFX allocation.
2: Allocate $g$ to $i$, i.e., $X'_i \leftarrow X_i \cup g$, $P' \leftarrow P \setminus g$, and $X'_j = X_j$ for $j \neq i$.
3: **return** $(X', P')$.
4: **end function**

5: **function** $U_1$(allocation $X$, pool $P$)
    **Precondition**: There is an agent $i$ such that $v_i(P) > v_i(X_i)$.
6: Let $i$ be an agent in $A_X(P)$ and let $Z \subseteq P$ be a set of size $\kappa_X(P)$ such that $v_i(Z) > v_i(X_i)$.
7: Set $X'_i = Z$ and $X'_j = X_j$ for $j \neq i$.
8: Set $P' = X_i \cup (P \setminus Z)$.
9: **return** $(X', P')$.
10: **end function**

11: **function** $U_2$(allocation $X$, pool $P$)
    **Precondition**: There is an $\ell \geq 1$, distinct goods $g_0, g_1, \ldots, g_{\ell-1}$ in $P$, distinct sources $s_0, s_1, \ldots, s_{\ell-1}$ of $G_X$ and distinct agents $t_1, t_2, \ldots, t_\ell$ such that $t_i \in C(s_i) \cap A_X(X_{s_{i-1}} \cup g_{i-1})$ for $0 \leq i \leq \ell - 1$ (indices are to be interpreted modulo $\ell$).
12: Let $Z_i \subseteq X_{s_i} \cup g_i$ of size $\kappa_X(X_{s_i} \cup g_i)$ be such that $v_{t_{i+1}}(Z_i) > v_{t_{i+1}}(X_{t_{i+1}})$ for $0 \leq i \leq \ell - 1$.
13: Set $P' = (P \setminus \{g_0, \ldots, g_{\ell-1}\}) \bigcup_{i=0}^{\ell-1}((X_{s_i} \cup \{g_i\}) \setminus Z_i)$.
14: Let $u^i_0 \to \cdots \to u^i_{m_i}$ be the path of length $m_i$ from $s_i = u^i_0$ to $t_i = u^i_{m_i}$ in $C(s_i)$ for $0 \leq i \leq \ell - 1$.
15: Set $X'_{u^i_k} = X_{u^i_{k+1}}$ for all $k \in \{0, \ldots, m_i - 1\}$ and all $i \in \{0, \ldots, \ell - 1\}$.
16: Set $X'_{t_i} = Z_{i-1}$ for all $i \in \{1, \ldots, \ell\}$.
17: Set $X'_j = X_j$ for all other $j$.
18: **return** $(X', P')$.
19: **end function**

*a) Rule $U_0$ returns an EFX allocation. An application of the rule does not decrease social welfare and decreases the size of the pool.*

*b) If rule $U_0$ is not applicable then for any source $i$ of $G_X$ and good $g \in P$, there will be an agent $j \neq i$ such that $v_j(X_i \cup g) > v_j(X_j)$. In particular, $A_X(X_i \cup g)$ will not be empty, and $\kappa_X(j, X_i \cup g) \leq |X_i|$ for all $j \in A_X(X_i \cup g)$.*

*Proof.* The first part of a) follows directly from the precondition of the rule. The second part holds since the valuations are monotone and because $|P'| = |P| - 1$.

The first two sentences in part b) are obvious. We come to the third sentence. Since adding $g$ to $X_i$ destroys the EFX-property, there must be some $g' \in X_i \cup g$ and some $j \in [n]$ such that $v_j(X_i \cup g \setminus g') > v_j(X_j)$. Thus $\kappa_X(j, X_i \cup g) \leq |X_i|$. □

**Rule** $U_1$ : Rule $U_1$ is applicable whenever there is an agent that values the pool higher than her current bundle (see Algorithm 2).

**Lemma 6** (Rule $U_1$)**.** *Rule $U_1$ increases the social welfare and returns an EFX allocation.*

*Proof.* Since there is an agent that values the pool higher than her own bundle, $A_X(P)$ is non-empty. Choose $i$ from $A_X(P)$ arbitrarily. Let $X'$ be the allocation defined in Algorithm 2, line 7. Then $v_i(X'_i) > v_i(X_i)$ and $v_j(X'_j) = v_j(X_j)$ for $j \neq i$. Thus $\phi(X') > \phi(X)$.

It remains to show that the allocation $X'$ is EFX, i.e., for every pair of agents $j$ and $k$ and any good $g \in X'_k$, we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$. Since $X$ is EFX, this is obvious if neither $j$ nor $k$ is equal to $i$. If $j = i$, then $v_i(X'_i) > v_i(X_i) \geq v_i(X_k \setminus g) = v_i(X'_k \setminus g)$ for all $g \in X'_k$ (or equivalently $g \in X_k$). Finally, we consider $k = i$. Since $k = i \in A_X(P)$, we have $v_j(X'_j) = v_j(X_j) \geq v_j(Z \setminus g) = v_j(X'_i \setminus g)$ for any $g \in Z$ (where $Z$ is defined in Algorithm 2, line 6) by Lemma 4. ☐

**Rule $U_2$** : Rule $U_2$ is our most complex rule. It is applicable if for some $\ell \geq 1$, there are distinct goods $g_0$, $g_1$, ..., $g_{\ell-1}$ in $P$, distinct sources $s_0$, $s_1$, ..., $s_{\ell-1}$ of $G_X$ and distinct agents $t_1$, $t_2$,..., $t_\ell$ (indices are to be interpreted modulo $\ell$) such that for each $i$: (1) $t_i$ is a most envious agent when $g_{i-1}$ is added to $s_{i-1}$ and (2) $t_i$ is reachable from $s_i$. We first show that rule $U_2$ is applicable if rule $U_0$ is not applicable and the pool contains at least $n$ goods.

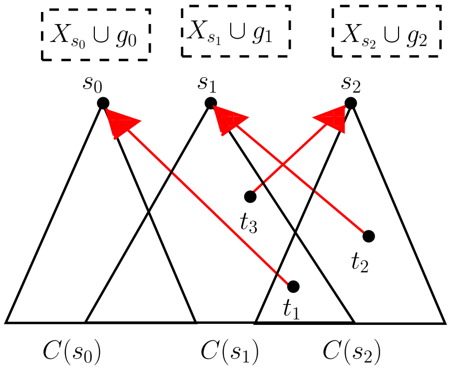


Figure 1: We have $t_i \in A_X(X_{s_{i-1}} \cup g_{i-1}))$. Moreover, $t_i \notin C(s_0) \cup \ldots \cup C(s_{i-1})$ for $i = 1, 2$ and $t_3 \in C(s_0) \cup \ldots \cup C(s_2)$. $j = 1$ is largest such that $t_3 \in C(s_j)$. The cycle is defined by $s_1$, $s_2$, $g_1$, $g_2$, $t_2$ and $t_3$.

**Lemma 7.** *If $|P| \geq n$ and rule $U_0$ is not applicable then there is an $\ell \geq 1$, distinct goods $g_0, g_1, \ldots, g_{\ell-1}$ in $P$, distinct sources $s_0, s_1, \ldots, s_{\ell-1}$ of $G_X$, and distinct agents $t_1, t_1, \ldots, t_\ell$ such that $t_i \in C(s_i) \cap A_X(X_{s_{i-1}} \cup g_{i-1})$ for $i \in \{0, \ldots, \ell-1\}$ (indices are modulo $\ell$).*

*Proof.* Since rule $U_0$ is not applicable, $A_X(X_s \cup g)$ is non-empty for every source $s$ of $G_X$ and every good $g \in P$. Construct a sequence of triples $(s_i, g_i, t_{i+1})$, $i \geq 0$ defined as follows. Let $s_0$ be an arbitrary source of $G_X$ and let $g_0$ be an arbitrary good in $P$. Assume we have defined $s_{i-1}$ and $g_{i-1}$. Let $t_i \in A_X(X_{s_{i-1}} \cup g_{i-1})$ be arbitrary. If $t_i \in C(s_0) \cup \cdots \cup C(s_{i-1})$ then stop the construction of the sequence and let $j$ be maximum such that $t_i \in C(s_j)$. Set $\ell = i - j$ and return $s_j, \ldots, s_{i-1}$, $g_j, \ldots, g_{i-1}$ and $t_{j+1}, \ldots, t_i$; see Figure 1 for an illustration. If $t_i \notin C(s_0) \cup \cdots \cup C(s_{i-1})$, let $s_i$ be such that $t_i \in C(s_i)$. Also, let $g_i$ be a good in $P$ distinct from $g_0$ to $g_{i-1}$.

The construction is well-defined since $|P| \geq n$ and hence we cannot run out of goods. The sources and goods are pairwise distinct by construction. The agents $t_1$ to $t_{i-1}$ are distinct by construction. The agent $t_i$ is distinct from $t_{j+1}$ to $t_{i-1}$ since $t_k \in C(s_k) \setminus (C(s_0) \cup \ldots \cup C_{s_{k-1}})$ for $k < i$ and $j$ is maximum such that $t_i \in C(s_j)$. ☐

For each $i$, let $u^i_0 \to u^i_1 \to \cdots \to u^i_{m_i}$ be the path of length $m_i$ from $s_i = u^i_0$ to $t_i = u^i_{m_i}$ in $C(s_i)$. Rule $U_2$ assigns (i) $X'_{u^i_k} = X_{u^i_{k+1}}$ for all $k \in \{0, \ldots, m_i - 1\}$ and all $i \in \{0, \ldots, \ell-1\}$ and (ii) $X'_{t_i} = Z_{i-1}$ for all $i \in \{1, \ldots, \ell\}$, where $Z_i$ is defined in Algorithm 2 (see line 12). For all other $j$, we have $X'_j = X_j$.

**Lemma 8** (Rule $U_2$)**.** *Rule $U_2$ increases social welfare and returns an EFX allocation.*

*Proof.* We first observe that the valuations of the agents for their bundles have either increased or remained the same (since either the agents are left with their old bundles or assigned bundles that they envied). In particular, the valuations of all the agents in $\bigcup_{i=0}^{\ell-1} \bigcup_{k=0}^{m_i}\{u_k^i\}$ are strictly larger, where the vertices $u_k^i$ are defined above. Thus $\phi(X') > \phi(X)$.

It remains to show that the allocation $X'$ is EFX, i.e., for every pair of agents $j$ and $k$ and any good $g \in X'_k$ we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$. Let $T = \{t_1, t_2, \ldots, t_\ell\}$. For every agent $r \notin T$ we have $X'_r = X_{r'}$ for some $r'$. Now consider two cases depending $k$:

– $k \notin T$: Note that valuations of the agents for their current bundles (in $X'$) is at least as good as their old bundles (in $X$). We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(X_{k'} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$ (or equivalently $g \in X_{k'}$).

– $k \in T$: Let $k = t_i$. We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(Z_{i-1} \setminus g)$ for any $g \in Z_{i-1}$ (by Lemma 4) and $v_j(Z_{i-1} \setminus g) = v_j(X'_{t_i} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$. ☐

We can now summarize. Let $V = \max_i v_i(M)$ be the maximum valuation of any agent and let $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ be the minimum difference between distinct valuations. Each application of rule $U_1$ or rule $U_2$ increases the social welfare by at least $\Delta$ and hence there can be no more than $nV/\Delta$ applications of these rules. Each application of rule $U_0$ decreases the size of the pool by one and hence there cannot be more than $m$ successive applications of this rule. We conclude that the number of iterations is at most $nmV/\Delta$. Thus we have shown the following theorem.

**Theorem 9.** *For normalized and monotone valuations, there is always an allocation $X$ and a pool $P$ of unallocated goods such that*

- *$X$ is EFX,*
- *$v_i(X_i) \geq v_i(P)$ for all agents $i$, and*
- *$|P|$ is less than the number of sources in the envy-graph; in particular, $|P| < n$.*

*Algorithm 1 determines such an allocation in at most $nmV/\Delta$ iterations.*

We also claimed in Section 1 that our algorithm gives another proof that when all the agents have identical (general) valuations then an EFX allocation with $P = \emptyset$ always exists. This proof is given in the appendix

# 3 Finding the Desired Allocation in Pseudo-Polynomial Time

In this section we describe how to find in time polynomial in $n$, $m$, $V$, and $1/\Delta$, the EFX-with-bounded-charity allocation described in Theorem 9 for all gross substitute valuations. A *minimum-size-valuable-set*-oracle will be used here; it is defined as: given $S \subseteq M$, agent $i$, and $\alpha \in \mathbb{R}$ such that $v_i(S) > \alpha$, find a minimum cardinality subset $Z \subseteq S$ such that $v_i(Z) > \alpha$.[7]

For additive valuations, the oracle is easy to realize. We initialize $Z$ to the empty set and as long as $v_i(Z) \leq \alpha$, we select $g \in S \setminus Z$ with maximum $v_i(g)$ and add $g$ to $Z$.

We now show that the oracle can also be realized for gross substitutes valuations. For the definition of the gross substitutes property, we use the notion of demand correspondence.

[7]Alternatively, a *size-constrained-optimal-valuation* oracle would suffice too, where given a set $S$, agent $i$ and an integer $k$, find $Z \subseteq S$ such that $|Z| \leq k$ and $v_i(Z)$ is maximum. We can simulate the minimum-size-valuation-oracle with the size-constrained-optimal-valuation oracle: All we need to do is to determine the smallest $k \in [n_0]$ (where $n_0 = |S|$) such that the valuation of agent $i$ for the optimal set returned by the size-constrained-optimal-valuation oracle is larger than $\alpha$. This can be realized with $n_0$ queries of this oracle (enumerating over all $k \in [n_0]$).

**Definition 10.** (Demand Correspondence $D(v,p)$) *Given a valuation function $v : 2^M \to \mathbb{R}_{\geq 0}$ and a price vector $p \in \mathbb{R}^m_{\geq 0}$, define the demand correspondence as*

$$D(v,p) = \left\{ S \subseteq M : v(S) - \sum_{g \in S} p_g \;\geq\; v(S') - \sum_{g \in S'} p_g \quad \text{for all } S' \subseteq M \right\}.$$

*That is, the demand correspondence is the family of sets that maximize the utility under prices $p$.*

**Definition 11.** (Gross Substitutes (GS) [1]) *A valuation function $v : 2^M \to \mathbb{R}_{\geq 0}$ satisfies the gross substitutes (GS) property if for any price vectors $p, p' \in \mathbb{R}^m_{\geq 0}$ with $p \leq p'$ (i.e. $p_g \leq p'_g$ for all $g \in [m]$) and any set $S \in D(v,p)$, there is a set $T \in D(v,p')$ such that $S \cap \{g : p_g = p'_g\} \subseteq T$.*

A useful consequence of the GS property [24] is that the greedy approach shown in Algorithm 3 computes a set $S \in D(v,p)$. It considers goods in order of non-increasing incremental value of $v(g|S) - p_g$ where $v(g|S) = v(S \cup g) - v(S)$ and $S$ is the current set. The algorithm is non-deterministic in the choice of $g^*$ in line 4 and whether to terminate in line 6.

**Algorithm 3** Greedy Demand Oracle

1: **Input**: $v : 2^M \to \mathbb{R}_{\geq 0}$ ($v$ satisfies GS), $p \in \mathbb{R}^m_{\geq 0}$
2: Initialize $S = \emptyset$
3: **Repeat**
4:     Let $g^* \in M \setminus S$ maximize $\Delta_g = v(g|S) - p_g$
5:     If $\Delta_{g^*} > 0$ then set $S = S \cup g^*$
6:     If $\Delta_{g^*} = 0$ then either set $S = S \cup g^*$ or **return** $S$
7:     If $S = M$ or $\Delta_{g^*} < 0$ then **return** $S$

If all prices are the same (equivalently, zero), the greedy approach (Algorithm 4) computes for each cardinality $k$, a set $S_k$ of maximum value (this is $Z$ after $k$ rounds of Algorithm 4).

**Algorithm 4** Most Valuable Sets

**Input**: $\alpha \geq 0$, $v : 2^M \to \mathbb{R}_{\geq 0}$ ($v$ satisfies GS)
Initialize $Z = \emptyset$
**While** $Z \neq M$ **do**
    Let $g^* \in \arg\max_{g \in M \setminus Z}\{v(g|Z)\}$
    $Z = Z \cup g^*$

**Lemma 12.** *Let $g_k$ be the good added in the $k$-th round of Algorithm 4 and let $S_k = \{g_1, g_2, \ldots, g_k\}$. Then $S_k$ is a set of cardinality $k$ of maximum value, i.e., $v(S_k) \geq v(T)$ for every set $T$ with $|T| \leq k$.*

*Proof.* Consider any $k$. If $v(g_{k+1}|S_k) = 0$, then $v(g \cup S_k) = v(S_k)$ for every $g \notin S_k$ and hence $v(S_k) = v(M)$ by the *submodularity*[8] of $v$ [22]. Thus $S_k$ is a most valuable set of size $k$. This is also true for $k = m$. So assume $k < m$ and $v(g_{k+1}|S_k) > 0$. Then $v(g_k|S_{k-1}) \geq v(g_{k+1}|S_{k-1}) \geq v(g_{k+1}|S_k) > 0$, where the first inequality holds since $g_k$ is chosen in round $k$ and the second inequality follows from the submodularity of $v$. Let $p$ be a price vector with $p_g = v(g_k|S_{k-1})$ for all $g \in M$; we abuse the notation and use also $p$ for the common price. We will show that $S_k \in D(v,p)$.

**Claim 13.** $S_k \in D(v,p)$.

[8] So $v(S \cup g) - v(S) \geq v(T \cup g) - v(T)$ whenever $S \subseteq T$.

*Proof.* We will use Algorithm 3 to derive that $S_k \in D(v,p)$. Observe that the price is the same (this is $p$) for all goods. So in each round, Algorithm 3 chooses a good $g$ that maximizes $v(g|S)$. This is also what Algorithm 4 does. We may assume that ties are broken in the same way and hence both algorithms add goods in the same order. We need to guarantee that Algorithm 3 can return the set $S_k$. This holds because $v(g_k|S_{k-1}) - p = 0$, so we can add this item and terminate the algorithm in the next round where $\Delta_{g_{k+1}} \leq 0$. □

We are now ready to show that $v(T) \leq v(S)$ for any set $T$ of size at most $k$. We have

$$v(T) - \sum_{g \in T} p_g \quad \leq \quad v(S_k) - \sum_{g \in S_k} p_g = \ v(S_k) - kp,$$

for every set $T \subseteq M$ since $S_k \in D(v,p)$. Hence we have:

$$v(T) \ \leq \ v(S_k) + (|T| - k)p \ \leq \ v(S_k)$$

for every set $T$ with $|T| \leq k$. □

The minimum-size-valuable-set oracle is now readily realized. We simply run Algorithm 4 on the set $S$ until a set of value greater $\alpha$ is obtained. We can conclude the following theorem.

**Theorem 14.** *For gross substitute valuations, the allocation defined in Theorem 9 can be determined with* $\mathsf{poly}(n,m,V,1/\Delta)$ *value queries.*

**An FPTAS to Determine an "Almost" Desired Allocation.** Our algorithm is pseudo-polynomial, since the increase in individual valuations of the agents when we perform the update rules could be very small. Suppose we just wanted an "almost" EFX property, i.e., for every pair of agents $i$ and $j$, we are happy to ensure that $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(X_j)$ and also $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(P)$ for every $i$. Then we have an algorithm that runs in $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ time and finds a desired allocation.

**Theorem 15.** *For normalized and gross substitute integral valuations, given any* $\varepsilon > 0$*, in time* $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$*, we can determine an allocation* $X = \langle X_1, X_2, \ldots, X_n \rangle$ *and a pool of unallocated goods* $P$ *such that*

- *for any pair of agents* $i$ *and* $j$ *we have* $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ *for all* $g \in X_j$*,*
- *for any agent* $i$*, we have* $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(P)$*, and*
- $|P| < n$*.*

The proof follows in a straightforward manner from the proof of Theorem 9 in Section 2. The key idea is that the "almost" EFX property is violated if and only if $(1+\varepsilon) \cdot v_i(X_i) < v_i(X_j \setminus g)$ for some $i, j \in [n]$ or $(1+\varepsilon) \cdot v_i(X_i) < v_i(P)$ for some $i \in [n]$. So every time we apply the update rules $U_1$ or $U_2$ there is a multiplicative improvement (by a factor of $1+\varepsilon$) in the valuation of some agents. Since these valuations are upper-bounded by $V$ we get a bound of $\text{poly}(n, m, \log_{(1+\varepsilon)} V)$ on the number of iterations.

## 4 Guarantees on Other Notions of Fairness

In this section we assume that all agents have additive valuations. We show that a minor variant of our algorithm finds an allocation with a good guarantee on Nash social welfare *and* groupwise maximin share (GMMS).

**Guarantee in Terms of Nash Social Welfare.** We claimed in Section 1 that for additive valuations, it can also be ensured that for each $i$, we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ where $X^* = \langle X_1^*, \ldots, X_n^* \rangle$ is an optimal *Nash social welfare* allocation and $X$ is the allocation in Theorem 9. This is easy to see from Algorithm 1: rather than initialize $X_i = \emptyset$, we will initialize $X_i$ to the bundle corresponding to the allocation determined by the algorithm in [12]. So we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$, to begin with. As the algorithm progresses, our invariant is that $v_i(X_i)$ never decreases for any $i$. So if $X' = \langle X'_1, \ldots, X'_n \rangle$ is the final allocation computed by our algorithm, then we have $v_i(X'_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ for $i \in [n]$.

**Lemma 16.** *Given a set $N$ of agents with additive valuations and a set $M$ of goods, there exists an allocation $X = \langle X_1, \ldots, X_n \rangle$ and a pool $P$ of unallocated goods that satisfy all the conditions of Theorem 9 and $v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where $X^* = \langle X_1^*, \ldots, X_n^* \rangle$ is an optimal Nash social welfare allocation.*

## 4.1 An Approximate MMS Allocation for Large $|P|$

We now show that if $|P|$ (the number of unallocated goods in our allocation) is sufficiently large, then our EFX allocation $X$ has a very good MMS guarantee. Recall that our algorithm continues till $|P|$ is smaller than the number of sources in the envy-graph $G_X$ and recall that sources are *unenvied* agents. In particular, if $|P| = n - 1$, then the number of sources in $G_X$ is $n$; so no agent envies another. That is, for each $i$, we have $v_i(X_i) \geq v_i(X_j)$ for all $j \in [n]$. Moreover, $v_i(X_i) \geq v_i(P)$. So we have

$$v_i(X_i) \;\geq\; \frac{v_i(M)}{n+1} \;\geq\; \left(1 + \frac{1}{n}\right)^{-1} \cdot \frac{v_i(M)}{n} \;\geq\; \left(1 + \frac{1}{n}\right)^{-1} \cdot MMS_i(n, M),$$

where for every agent $i$, the inequality $MMS_i(n, M) \leq v_i(M)/n$ holds for additive valuations. We formalize the above intuition in Theorem 18. The following proposition will be useful.

**Proposition 17** ([18])**.** *Let $N$ be a set of $n$ agents with additive valuations and let $M$ be a set of $m$ goods. If $N' \subseteq N$ and $M' \subseteq M$ are such that $|N \setminus N'| \geq |M \setminus M'|$ then for any agent $i \in N'$, we have $MMS_i(n', M') \geq MMS_i(n, M)$ where $n' = |N'|$.*

**Theorem 18.** *Given a set $N$ of $n$ agents with additive valuations and a set $M$ of $m$ goods, there exists an allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ and set $P$ of unallocated goods that satisfies:*

- *the 3 conditions stated in Theorem 9;*
- *$v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where $X^*$ is an optimal Nash social welfare allocation;*
- *$v_i(X_i) \geq MMS_i(n, M) / \left(2 - \frac{k}{n}\right)$ for every $i \in N$, where $k = |P|$.*

*Proof.* Let $(X, P)$ be the allocation guaranteed by Lemma 16. Hence the first two conditions given in the theorem statement are satisfied by $(X, P)$. So what we need to show now is that for any agent $i$, we have $v_i(X_i) \geq MMS_i(n, M) / \left(2 - \frac{k}{n}\right)$.

Let $N' \subseteq N$ be the set of agents $j$ for which either $v_i(X_j) \leq v_i(X_i)$ or $|X_j| \geq 2$. Then $i \in N'$ and all sources of $G_X$ belong to $N'$. Also, $|X_j| = 1$ and $v_i(X_j) > v_i(X_i)$ for $j \in N \setminus N'$. Let $M'$ be the set of goods allocated to the agents in $N'$. The agents in $N \setminus N'$ are allocated the goods in $M \setminus (M' \cup P)$. Observe that every agent in $N \setminus N'$ is allocated at most one good. So we have $|N \setminus N'| \geq |M \setminus (M' \cup P)|$. We know from Proposition 17 that $MMS_i(n', M' \cup P) \geq MMS_i(n, M)$ where $n' = |N'|$. Thus it suffices to show that $v_i(X_i) \geq MMS_i(n', M' \cup P) / \left(2 - \frac{k}{n}\right)$.

Consider any $j \in N'$ with $v_i(X_j) > v_i(X_i)$. So $|X_j| \geq 2$. Since valuations are additive and $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$, we have

$$v_i(X_i) \;\geq\; \left(1 - \frac{1}{|X_j|}\right) \cdot v_i(X_j) \;\geq\; \frac{1}{2} \cdot v_i(X_j).$$

We know the following inequalities hold:

$$\begin{aligned} v_i(X_i) &\geq v_i(P), && (1)\\ v_i(X_i) &\geq v_i(X_j) \quad \text{for all } j \text{ that were sources in } G_X, && (2)\\ 2v_i(X_i) &\geq v_i(X_j) \quad \text{for all other } j \in N'. && (3) \end{aligned}$$

Recall that the number of sources is at least $|P|+1 = k+1$. Summing up all inequalities in (1)-(3) and using the fact that $v_i$ is additive, we have $(2(n'-(k+1))+k+2)\cdot v_i(X_i) \geq v_i(M' \cup P)$. Hence we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M' \cup P)}{2n'-k} \\ &\geq \frac{v_i(M' \cup P)}{n'} \cdot \frac{n'}{2n'-k} \\ &\geq MMS_i(n', M' \cup P) \cdot \frac{n'}{2n'-k} && \text{since } v_i \text{ is additive} \\ &= MMS_i(n', M' \cup P) / \left(2 - \tfrac{k}{n'}\right) \\ &\geq MMS_i(n', M' \cup P) / \left(2 - \tfrac{k}{n}\right) && \text{since } n' \leq n. \end{aligned}$$

□

### 4.2 An Improved Bound for Approximate-GMMS

As mentioned in Section 1, a new notion of fairness called *groupwise maximin share* (GMMS) was recently introduced by Barman et al. [6]. We formally define a GMMS allocation below.

**Definition 19.** *Given a set $N$ of $n$ agents and a set $M$ of $m$ goods, an allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is $\alpha$-GMMS if for every $N' \subseteq N$, we have $v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{i \in N'} X_i)$ where $n' = |N'|$.*

Observe that a GMMS allocation is also an MMS allocation. Since MMS allocations do not always exist in a given instance [27], GMMS allocations also need not always exist. Interestingly, $\frac{1}{2}$-GMMS allocations always exist [6]. We now describe how to modify our allocation so that the resulting allocation is $\frac{4}{7}$-GMMS.

Let $X = \langle X_1, \ldots, X_n \rangle$ be the allocation and let $P$ be the pool of unallocated goods that satisfy the conditions of Lemma 16. Without loss of generality, assume that agent 1 is a source in the envy-graph $G_X$. Define the complete allocation $Y = \langle Y_1, \ldots, Y_n \rangle$ as follows:

* $Y_1 = X_1 \cup P$ and $Y_i = X_i$ for all $i \neq 1$.

Theorem 20 shows that $Y$ is our desired allocation. The proof of Theorem 20 is similar to [2, Proposition 3.4].

**Theorem 20.** *Given a set $N$ of $n$ agents with additive valuations and a set $M$ of $m$ goods, there exists a complete allocation $Y = \langle Y_1, Y_2, \ldots, Y_n \rangle$ of $M$ such that*

- *$Y$ is $\frac{4}{7}$-GMMS.*
- *$v_i(Y_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$ where $X^*$ is the optimal Nash social welfare allocation.*[9]

*Proof.* Observe that the bound on Nash social welfare holds for allocation $X$ and thus for allocation $Y$ (since $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$). So what we need to show now is the

[9] In private communication we are aware that Jugal Garg and Setareh Taki have obtained independently related results. For **additive** valuations, they can show that there is an EFX-allocation after donating at most $n-1$ goods to charity. However, there is no bound on the value of the goods donated to charity. Thus they obtain a 4/7-GMMS-allocation after removing $n-1$ goods from the original set of goods.

guarantee on GMMS. That is, we need to show that for every $\tilde{N} \subseteq N$ and all $i \in \tilde{N}$, we have $v_i(Y_i) \geq \frac{4}{7} MMS_i(\tilde{n}, \tilde{M})$ where $\tilde{n} = |\tilde{N}|$ and $\tilde{M} = \bigcup_{j \in \tilde{N}} Y_j$.

Fix some $i \in \tilde{N}$. Define $N'$ as the subset of $\tilde{N}$ that contains $i$ and all other agents that have been allocated at least two goods in $Y$, i.e., $j \in N' \iff j = i$ or $|Y_j| \geq 2$. Let $M' = \bigcup_{j \in N'} Y_j$.

Note that $Y$ allocates all goods in $\tilde{M} \setminus M'$ to agents in $\tilde{N} \setminus N'$. Since every agent in $\tilde{N} \setminus N'$ has been allocated at most one good, we have $|\tilde{N} \setminus N'| \geq |\tilde{M} \setminus M'|$. Proposition 17 tells us that $MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$ where $n' = |N'|$. Thus it suffices to show $v_i(Y_i) \geq 4/7 \cdot MMS_i(n', M')$.

Let $j \in N' \setminus \{1, i\}$. Call $Y_j$ a *bad* bundle if $|Y_j| = 2$ and and the goods in $Y_j$ will be called *bad* goods. Call all the remaining bundles *good* bundles and analogously, call the goods in these bundles *good* goods. We make some helpful observations below.

**Observation 21.** *For any bad good $g$, we have $v_i(g) \leq v_i(Y_i)$.*

*Proof.* Let $g \in Y_j$, where $Y_j$ is a bad bundle. So $|Y_j| = 2$, let $Y_j = \{g, g'\}$. Since $j \neq 1$ (by definition of a bad bundle), we have $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus g') = v_i(Y_j \setminus g') = v_i(g)$. □

**Observation 22.** *For any $i \in N'$, we have $v_i(Y_i) \geq \frac{1}{2} v_i(Y_1)$.*

*Proof.* Since agent 1 was a source, we have $v_i(X_i) \geq v_i(X_1)$. By Theorem 9, we have $v_i(X_i) \geq v_i(P)$. Therefore, we have $v_i(Y_1) = v_i(X_1 \cup P) = v_i(X_1) + v_i(P) \leq v_i(X_i) + v_i(X_i) = 2v_i(X_i) = 2v_i(Y_i)$. □

**Observation 23.** *Let $j \neq 1$. If $Y_j$ is not a* bad *bundle then $v_i(Y_j) \leq \frac{3}{2} v_i(Y_i)$ for any $i \in N$.*

*Proof.* Let $j \neq 1$. If $Y_j$ is not a bad bundle then $|Y_j| \geq 3$. Since $j \neq 1$ we have $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus \{g\}) = v_i(Y_j \setminus g)$ for any $g \in Y_j$. In particular, let $g \in Y_j$ be such that $v_i(g)$ is the least. Then we have:

$$v_i(Y_i) \;\geq\; (1 - \frac{1}{|Y_j|}) \cdot v_i(Y_j) \;\geq\; (1 - \frac{1}{3}) \cdot v_i(Y_j) \;=\; \frac{2}{3} \cdot v_i(Y_j).$$ □

Now we are ready to show the bound on GMMS. Let $x$ be the number of bad goods in $M'$. Then we have $x/2$ bad bundles and $n' - x/2$ good bundles. We first assume $x \leq n'$. For any good bundle $Y_j$ we have:

$$\begin{aligned} v_i(Y_i) &= v_i(Y_j) && \text{when } j = i, \\ v_i(Y_i) &\geq \frac{1}{2} \cdot v_i(Y_j) && \text{when } j = 1 \text{ (by Obs. 22)}, \\ v_i(Y_i) &\geq \frac{2}{3} \cdot v_i(Y_j) && \text{otherwise (by Obs. 23)}. \end{aligned}$$

Thus the total valuation agent $i$ has for the *good* goods is at most $\frac{3}{2}(n' - \frac{x}{2} - 2) \cdot v_i(Y_i) + v_i(Y_i) + 2v_i(Y_i) = \frac{3}{2}(n' - \frac{x}{2}) \cdot v_i(Y_i)$. Also, the total valuation agent $i$ has for the bad goods is at most $x \cdot v_i(Y_i)$ (since there are $x$ many bad goods and each bad good is worth at most $v_i(Y_i)$ by Obs. 21). Therefore we have

$$\begin{aligned} v_i(M') &= v_i(\text{bad goods}) \;+\; v_i(\text{good goods}) \\ &\leq x \cdot v_i(Y_i) \;+\; \frac{3}{2}(n' - \frac{x}{2}) \cdot v_i(Y_i) \\ &= (x + \frac{3}{2}n' \;-\; \frac{3x}{4}) \cdot v_i(Y_i) \\ &= \frac{6n' + x}{4} \cdot v_i(Y_i) \\ &\leq \frac{7n'}{4} \cdot v_i(Y_i) \quad (\text{since } x \leq n') \end{aligned}$$

So $v_i(M') \le \frac{7n'}{4} \cdot v_i(Y_i)$, which gives us the desired bound: $v_i(Y_i) \ge \frac{4}{7} MMS_i(n', M')$ since $MMS_i(n', M') \le v_i(M)/n'$.

We come to the case $x > n'$. In that case we will prune $M'$ into $M''$ and $N'$ into $N''$ so that $M''$ has at most $x' \le n''$ bad goods, $n'' - \frac{x'}{2}$ many good bundles of $Y$ where $n'' = |N''|$, and $MMS_i(n', M') \le MMS_i(n'', M'')$.

Let $Z = \langle Z_1, Z_2, \dots Z_{n'} \rangle$ be an optimal MMS partition for agent $i$ on the set $M'$ of goods. Since there are more than $n'$ bad goods in $M'$, there is a set $Z_k$ with at least *two* bad goods: let $g_1, g_2 \in Z_k$ be a pair of bad goods. The following observation will be useful.

**Observation 24.** *$MMS_i(n', M') \le MMS_i(n' - 1, M' \setminus \{g_1, g_2\})$.*

*Proof.* Distribute the goods in $Z_k \setminus \{g_1, g_2\}$ arbitrarily among the other sets in $Z$. So we have a partition of the set $M' \setminus \{g_1, g_2\}$ of goods into $n' - 1$ many sets corresponding to agents in $N' \setminus \{j\}$ for some $j \in N'$ and $j \neq i$ (note that we can choose any $j \in N'$ such that $Y_j$ is a bad bundle). The value of any set for agent $i$ is at least $MMS_i(n', M')$. □

So let us update $N'$ to $N' \setminus \{j\}$ and $M'$ to $M' \setminus \{g_1, g_2\}$. By Obs. 24, the MMS value for $i$ does not decrease. We keep repeating this reduction until $M'$ has at most $|N'|$ bad goods. Since each step decreases $|N'|$ by 1 and the number of bad goods in $|M'|$ by 2, there will be a step when $M'$ has at most $|N'|$ bad goods. □

# 5 Conclusions and Open Problems

We studied the existence of EFX allocations when agents have general valuations. We showed that we can ensure such an allocation always exists when we donate a small number of goods that nobody envies to charity. The major open problem here is whether EFX allocations always exist. Our main result implies that among the $n$ agents, if there is just *one* agent who is "beyond the feeling of envy" (say, for some $i$, we have $v_i(S) = v_i(T)$ for all non-empty $S, T \subseteq M$) then an EFX allocation always exists for general valuations. Plaut and Roughgarden [26] remarked that an instance with *no* EFX allocation may be easier to find in the setting of general valuations. Our result on "almost-EFX" allocations for general valuations allows one to hope that EFX allocations always exist, at least for more structured valuations such as additive.

Plaut and Roughgarden [26] showed that an exponential number of "value queries" are required to determine an EFX allocation even for two agents with identical submodular valuations. From our proof it is evident that we can determine a $(1 - \varepsilon)$ EFX allocation (as in Theorem 15) with polynomially many *size-constrained-optimal-valuation* queries (where for a given $k$, $S$ and agent $i$ we need to find the subset of $S$ of size at most $k$ that maximizes agent $i$'s valuation). Studying the complexity of determining approximate EFX allocations under other queries is a line of direction for future work.

We also showed that we get guarantees in terms of other notions of fairness when agents have additive valuations. To the best of our knowledge, allocations with good guarantees (i.e., constant factor approximation) on Nash social welfare and MMS (as well as GMMS) were not known prior to our work. It would also be interesting to investigate whether these guarantees can be improved or if instances can be constructed where our guarantees are tight. We believe that our work is just the beginning towards determining an allocation that gives good guarantees with respect to several notions of fairness: an allocation that is *universally* fair.

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## Appendix: New Proof of a Result from [26]

For agents with identical (general) valuations, it was shown by Plaut and Roughgarden [26] that an allocation that maximizes the the minimum valuation, then maximizes the size of this bundle, then maximizes the second minimum valuation, then maximizes the size of this bundle, and so on is EFX. We now show that Algorithm 1 gives another proof that when all the agents have identical valuations, a complete allocation that is EFX always exists.

Recall that Algorithm 1 consists of applying 3 update rules: $U_0, U_1, U_2$ – whichever of these is applicable. Moreover, if a certain precondition is satisfied (see Algorithm 2), then rule $U_2$ is applicable.

We will now show that when all the agents have identical valuations and rule $U_0$ is not applicable, then the precondition of rule $U_2$ is satisfied as long as there is some unallocated good. Let $X = \langle X_1, \ldots, X_n \rangle$ be the current allocation and let $P = M \setminus \cup_{i=1}^{n} X_i$ be the set of unallocated goods in $X$.

**Lemma 25.** *Let $s$ be any source vertex in the envy-graph $G_X$. If $|P| \geq 1$ and rule $U_0$ is not applicable then $s \in A_X(X_s \cup g)$ for any $g \in P$.*

*Proof.* Let $g \in P$ and $s$ be any source in the envy-graph $G_X$. Since rule $U_0$ is not applicable, $A_X(S) \neq \emptyset$, where $S = X_s \cup g$. Let $t \in A_X(S)$. So $v(X_t) < v(S)$, where $v$ is the common valuation function of all agents. Let $Z \subseteq S$ be the subset of size $\kappa_X(S)$ such that $v(X_t) < v(Z)$. Since $s$ is a source in $G_X$, we have $v(X_s) \leq v(X_t)$. So $v(X_s) < v(Z)$; thus $\kappa_X(s, S) \leq \kappa_X(S)$. Hence $s \in A_X(X_s \cup g)$. □

Lemma 25 implies that while $P \neq \emptyset$, either rule $U_0$ or rule $U_2$ is applicable. Whenever we apply rule $U_2$, we add any good $g$ in $P$ to the bundle of a source $s$ in $G_X$ and determine $Z \subseteq X_s \cup g$ of size $\kappa_X(X_s \cup g)$ such that $v(Z) > v(X_s)$. We then throw the goods in $(X_s \cup g) \setminus Z$ back into the pool $P$ and set $X_s = Z$.

This makes agent $s$ strictly better-off and no agent is worse-off: thus we have made progress. So when Algorithm 1 terminates, we have an EFX allocation with $P = \emptyset$. Thus we have a complete allocation $X = \langle X_1, \ldots, X_n \rangle$ that is EFX.