Exponential and Cayley maps for Dual Quaternions

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Abstract. In this work various maps between the space of twists and the space of finite screws are studied.

Dual quaternions can be used to represent rigid-body motions, both finite screw motions and infinitesimal motions, called twists. The finite screws are elements of the group of rigid-body motions while the twists are elements of the Lie algebra of this group. The group of rigid-body displacements are represented by dual quaternions satisfying a simple relation in the algebra. The space of group elements can be though of as a six-dimensional quadric in seven-dimensional projective space, this quadric is known as the Study quadric. The twists are represented by pure dual quaternions which satisfy a degree 4 polynomial relation. This means that analytic maps between the Lie algebra and its Lie group can be written as a cubic polynomials. In order to find these polynomials a system of mutually annihilating idempotents and nilpotents is introduced. This system also helps find relations for the inverse maps.

The geometry of these maps is also briefly studied. In particular, the image of a line of twists through the origin (a screw) is found. These turn out to be various rational curves in the Study quadric, a conic, twisted cubic and rational quartic for the maps under consideration.

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1. Introduction

Dual quaternions were introduced by Clifford in [3] to transform what he called rotors. These were vectors bound to points in space, essentially directed lines with an associated magnitude. These rotors were intended to model angular velocities and wrenches. The sum of two rotors is in general a motor, what would now be called a twist. In this work the term dual quaternion will be used rather than Clifford's name 'biquaternion' since this seems to refer to several possible cases.

In modern notation dual quaternions can be thought of as elements of the degenerate Clifford algebra $Cl(0, 2, 1)$ or perhaps more conveniently as the even subalgebra of $Cl(0, 3, 1)$. The Spin group for this algebra is the double cover of the group of proper Euclidean displacements.

The group of proper Euclidean displacements itself can be realised as a quadric in the projectivisation of the Clifford algebra, this quadric is usually known as the Study quadric.

The Lie algebra of both groups also lies in the Clifford algebra. Lie algebra elements, sometimes called twists, are represented by dual pure quaternions. That is dual quaternions of the form $s = a + \varepsilon b$, where ε is the dual unit satisfying $\varepsilon^2 = 0$ and a, b are quaternions with no real part.

The aim of this paper is to investigate some mappings between the Lie algebra of the rigid-body displacement group and the group itself. These maps are of fundamental importance in many areas. Apart from its theoretical importance, the exponential map connects mechanical joints, specified by a Lie algebra element with the possible displacements allowed by the joint. However, for general helical joints the map is not algebraic. On the other hand the Cayley map is a rational map. This has several practical advantages, numerical methods based on the Cayley map do not need so many trigonometric function calls and are hence more efficient. Many problems in the group can be linearised by mapping them to the space of twists. Actually there are several different Cayley maps defined by different matrix representations of the group. In this work we look at two of these based on the 4×4 homogeneous representation of the group and the 6×6 adjoint representation of the group. A Cayley map based on the dual quaternion representation of the group is also introduced and studied here. In particular, since the pure dual quaternions satisfy a degree 4 relation, it is possible to find cubic polynomial relation for all of these maps. Moreover, the inverse maps, from the group back to the Lie algebra can also written as cubic polynomials in the group elements. Many of the computations in this work are somewhat lengthy and therefore the computer algebra package *Mathematica* has been used in several places.

The study of the maps follows a brief section explaining the dual quaternions and their relation to rigid-body displacements in a little more detail, and a short section looking at the exponential and Cayley maps for ordinary quaternions.

2. Dual Quaternions

Let us denote the algebra of quaternions by $\mathbb{H}$, elements of this algebra have the form, $h = h_0 + h_x\mathrm{i} + h_y\mathrm{j} + h_z\mathrm{k}$. The ring of dual numbers $\mathbb{D}$, consists of pairs of real numbers λ, μ in the form $\lambda + \varepsilon\mu$. Here ε is the dual unit which satisfies $\varepsilon^2 = 0$ and commutes with the real coefficients. Now the dual quaternions are the elements of the algebra $\mathbb{H} \otimes \mathbb{D}$. It is simple to show that this algebra is isomorphic to the

Clifford algebra $Cl(0,2,1)$. Elements of $\mathbb{H}\otimes\mathbb{D}$ have the form, $h_a+\varepsilon h_b$ where h_a and h_b are both quaternions and the dual unit ε commutes with the quaternion generators i, j and k.

The Clifford conjugate of a dual quaternion is given by $(h_a+\varepsilon h_b)^- = h_a^- + \varepsilon h_b^-$ where the conjugates on the right of this equation are the standard quaternion conjugates.

Elements of the Spin group satisfy the relation, $gg^- = 1$. These can be written in the form,

$$g = r + \frac{1}{2}\varepsilon t r$$

where r is a quaternion representing a rotation, and t is a pure quaternion, that is one with no real part, which represents the translation. The group elements g and $-g$ both represent the same rigid displacement, that is the spin group double covers the group of rigid-body displacements.

Recall that quaternions representing rotations have the form,

$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\omega} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\hat{\omega}_x\mathrm{i} + \hat{\omega}_y\mathrm{j} + \hat{\omega}_z\mathrm{k})$$

where θ is the angle of rotation and $\hat{\omega}$ is a unit vector (a pure quaternion with unit modulus) directed along the rotation axis.

By Chasles's theorem, a general rigid-body displacement is a finite screw motion. That is, a rotation about a line followed by a translation in the same direction as the rotation axis. Such a finite screw motion is specified by a line in space, the screw axis; a rotation angle; and a pitch. The pitch gives the distance translated along the screw axis for a complete turn. The dual quaternion representing such a displacement can be found as follows. First we will assume that the rotation is about an axis through the origin and so is given by the quaternion r. The translation in the direction of the screw axis will be given by the dual quaternion,

$$1 + \frac{1}{2}\varepsilon\frac{\theta p}{2\pi}\hat{\omega}$$

where p is the pitch of the displacement, θ the angle of rotation and $\hat{\omega}$ the direction of the screw axis. The screw motion about the line through the origin is represented by the dual quaternion product,

$$\big(1 + \frac{1}{2}\varepsilon\frac{\theta p}{2\pi}\hat{\omega}\big)r = r + \frac{1}{2}\varepsilon\frac{\theta p}{2\pi}\hat{\omega}r.$$

Now to find the group element corresponding to a screw motion about an arbitrary line in space we can use a group conjugation to move the line through the origin to an arbitrary point in space. Suppose that q is a (pure quaternion representing a) point on the arbitrary line, then the screw motion about this line can be found by translating this point to the origin, performing the screw motion about the line through the origin and finally translating the line back to its original position.

This is represented by the product,

$$\begin{aligned} g &= (1+\frac{1}{2}\varepsilon q)\big(r+\frac{1}{2}\varepsilon\frac{\theta p}{2\pi}\hat{\omega}r\big)(1-\frac{1}{2}\varepsilon q) \\ &= r+\frac{1}{2}\varepsilon\big(\frac{\theta p}{2\pi}\hat{\omega}r+qr-rq\big) \end{aligned} \quad (2.1)$$

Next suppose we write the group element as,

$$g = (a_0 + a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k}) + \varepsilon(c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k})$$

then the relation $gg^- = 1$, for the dual quaternion to be a group element, simplifies to two relations,

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 1, \\ a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 &= 0. \end{aligned}$$

Thinking of these variables as homogeneous coordinates in seven-dimensional projective space $\mathbb{P}^7$, means that group elements g and $-g$ are identified. The first equation above is redundant and only the second equation is meaningful. In this way elements of the group of rigid-body displacements are identified with point in a six-dimensional quadric, called the Study quadric after its discoverer E. Study [7]. Not every point in the Study quadric corresponds to a rigid-body displacement, there is a 3-plane of elements satisfying $a_0 = a_1 = a_2 = a_3 = 0$ which do not represent any rigid displacement, all other point in the Study quadric correspond to distinct rigid displacements.

3. Map for Pure Quaternions

Before studying the exponential and Cayley maps for dual quaternions we take a brief look at the corresponding maps for quaternions.

Suppose that ω is a pure quaternion, that is a quaternion with no real part. The exponential of such an element is given by the familiar Maclaurin series,

$$e^{\omega} = 1 + \omega + \frac{1}{2!}\omega^2 + \frac{1}{3!}\omega^3 + \cdots$$

Since $\omega = \omega_x\mathrm{i} + \omega_y\mathrm{j} + \omega_z\mathrm{k}$ is a pure quaternion it satisfies the quadratic relation,

$$\omega^2 = -(\omega_x^2 + \omega_y^2 + \omega_z^2) = -|\omega|^2$$

Substituting this in the Maclaurin series above it is straightforward to recognise the series for the sine and cosine functions and simplify the exponential to,

$$e^{\omega} = \cos|\omega| + \sin|\omega|\frac{\omega}{|\omega|}$$

Again this is easily recognised as a quaternion representing a rotation, the axis of the rotation is $\hat{\omega} = \omega/|\omega|$ and the angle of rotation satisfies, $|\omega| = \theta/2$.

Now we turn to the Cayley map, let a be another pure dual quaternion. The Cayley map is defined as,

$$\mathrm{Cay}(a) = (1+a)(1-a)^{-1},$$

this is a straightforward generalisation of the well known Cayley map for $SO(3)$, see [2].

This can be developed as an infinite power series,

$$\mathrm{Cay}(a) = (1+a)(1+a+a^2+a^3+\cdots)$$

Then substituting the quadratic relation for pure quaternions we get,

$$\mathrm{Cay}(a) = (1+a)\Big(\frac{1}{1+|a|^2} + \frac{a}{1+|a|^2}\Big)$$

So that finally we can write,

$$\mathrm{Cay}(a) = \frac{1-|a|^2}{1+|a|^2} + \frac{2a}{1+|a|^2}$$

Again this can be recognised as a rotation, using the 'tan-half-angle' formulas. The axis of rotation is $\hat{a}$ and the angle of rotation will satisfy $|a| = \tan(\theta/4)$.

4. Idempotents and Nilpotents

For dual quaternions the power series are not so easily simplified. However the pure dual quaternions or twists, do satisfy a degree 4 relation. Let $s = \omega + \varepsilon v$ be a pure dual quaternion, that is ω and v are pure quaternions, then it is straightforward to verify that these elements satisfy the relation,

$$s^4 + 2|\omega|^2 s^2 + |\omega|^4 = 0.$$

Notice that this relation factorises easily,

$$s^4 + 2|\omega|^2 s^2 + |\omega|^4 = (s+i|\omega|)^2(s-i|\omega|)^2,$$

where here i is the imaginary unit. (This should not be confused with the quaternion i, note the different typefaces used).

Using the above it is possible to find a system of idempotents and nilpotents p_+, p_-, n_+ and n_-, satisfying,

$$p_+^2 = p_+, \quad p_-^2 = p_-, \quad p_+n_+ = n_+, \quad p_-n_- = n_-, \quad n_+^2 = n_-^2 = 0$$

and all other products are zero, [6]. This system can be used to write infinite power series in s as cubic polynomials in s.

The system of idempotents and nilpotents is given by,

$$p_+ = \frac{1}{4i|\omega|^3}(s-2i|\omega|)(s+i|\omega|)^2, \tag{4.1}$$

$$p_- = \frac{-1}{4i|\omega|^3}(s+2i|\omega|)(s-i|\omega|)^2, \tag{4.2}$$

$$n_+ = \frac{-1}{4|\omega|^2}(s+i|\omega|)^2(s-i|\omega|), \tag{4.3}$$

$$n_- = \frac{-1}{4|\omega|^2}(s+i|\omega|)(s-i|\omega|)^2. \tag{4.4}$$

The idempotents satisfy the simple relation,

$$1 = p_+ + p_-$$

The original twist is recovered from the relation

$$s = i|\omega|p_+ + n_+ - i|\omega|p_- + n_-$$

The point of using such a representation is that the powers of the twist are easily found. Using the properties of the idempotents and nilpotent (4.1)–(4.4), we have that the powers of s are given by,

$$s^k = (i|\omega|)^k p_+ + k(i|\omega|)^{k-1} n_+ + (-i|\omega|)^k p_- + k(-i|\omega|)^{k-1} n_-$$

for integer $k \geq 1$.

5. The Exponential and Cayley Maps

5.1. The Exponential Map

The last relation above can be used in an infinite series to substitute for powers of s. For example, using this in the Maclaurin series for the exponential of s gives,

$$e^s = e^{i|\omega|}p_+ + e^{i|\omega|}n_+ + e^{-i|\omega|}p_- + e^{-i|\omega|}n_-$$

This can be rewritten as a polynomial in s by substituting for the idempotents and nilpotents from (4.1)–(4.4). The resulting expressions, containing complex exponential, can be simplified using Euler's formula from complex analysis the following result is obtained,

$$\begin{aligned} e^s \quad = \quad & \frac{1}{2}\big(2\cos|\omega| + |\omega|\sin|\omega|\big) - \frac{1}{2|\omega|}\big(|\omega|\cos|\omega| - 3\sin|\omega|\big)s + \\ & \frac{1}{2|\omega|}\big(\sin|\omega|\big)s^2 - \frac{1}{2|\omega|^3}\big(|\omega|\cos|\omega| - \sin|\omega|\big)s^3. \end{aligned} \tag{5.1}$$

This is similar to the well known Rodrigues formula for rotations.

There are perhaps simpler ways of producing these results using eigenvalues methods, see [8], however here the idemponents and nilpotents will be needed later.

5.2. The Cayley Map

Another map from the Lie algebra to the group of proper Euclidean transformations is given by the Cayley map. As above, this is defined as $\mathrm{Cay}_q(s) = (1+s)(1-s)^{-1}$. In a similar manner this map can also be expressed as a cubic polynomial in the twist $s = a + \varepsilon b$. In terms of the idempotents and nilpotents we have,

$$\mathrm{Cay}_q(s) = \frac{(1+i|a|)}{(1-i|a|)}p_+ + \frac{2}{(1-i|a|)^2}n_+ + \frac{(1-i|a|)}{(1+i|a|)}p_- + \frac{2}{(1+i|a|)^2}n_-$$

Expanding the idempotents and nilpotents as polynomials in s using (4.1)–(4.4), and simplifying produces the cubic polynomial,

$$\mathrm{Cay}_q(s) = \frac{1+2|a|^2-|a|^4}{(1+|a|^2)^2} + \frac{2+4|a|^2}{(1+|a|^2)^2}s + \frac{2}{(1+|a|^2)^2}s^2 + \frac{2}{(1+|a|^2)^2}s^3 \quad (5.2)$$

In order to understand this map in a little more detail a different approach is useful. Notice that the powers of the twist can be written,

$$\begin{aligned} s &= a+\varepsilon b \\ s^2 &= a^2+\varepsilon(ab+ba) \\ s^3 &= a^3+\varepsilon(a^2b+aba+ba^2) \\ &\vdots \end{aligned}$$

From this it can be seen that,

$$(1-s)^{-1} = (1-a)^{-1} + \varepsilon(1-a)^{-1}b(1-a)^{-1}$$

and hence the Cayley map can be written,

$$\mathrm{Cay}_q(s) = (1+s)(1-s)^{-1} = (1+a)(1-a)^{-1} + \varepsilon 2(1-a)^{-1}b(1-a)^{-1}.$$

Now we compare this result to the dual quaternion representing a general screw motion found in (2.1) above.

$$r + \frac{1}{2}\varepsilon\big(\frac{\theta p}{2\pi}\hat{\omega}r + qr - rq\big) = (1+a)(1-a)^{-1} + \varepsilon 2(1-a)^{-1}b(1-a)^{-1}$$

From the quaternion part we have $r = (1+a)(1-a)^{-1}$ so that the direction of the screw axis is given by, $\hat{\omega} = \hat{a}$ and the angle of rotation satisfies, $|a| = \tan(\theta/4)$. From the dual part of the equation we have,

$$\frac{1}{2}\big(\frac{\theta p}{2\pi}\hat{a}r + qr - rq\big) = 2(1-a)^{-1}b(1-a)^{-1}.$$

Rearranging this to isolate b gives,

$$b = \frac{1}{4}(1-a)\big(\frac{\theta p}{2\pi}\hat{a}r + qr - rq\big)(1-a)$$

and using the fact that $r = (1+a)(1-a)^{-1}$ gives,

$$b = \frac{1}{4}\big(\frac{\theta p}{2\pi}\hat{a}(1+|a|^2) + 2(qa - aq)\big)$$

Finally we can turn this into a vector equation using the Gibbs relation, so that,

$$b = q\times a + \frac{\theta p}{8\pi}(1+\tan^2\frac{\theta}{4})\hat{a} = q\times a + \frac{\theta/2}{1-\cos\theta/2}\left(\frac{p}{2\pi}\right)a$$

So the twist s and the finite screw motion $\mathrm{Cay}(s)$ have the same screw axis, with direction $\hat{a}$ passing through the point q. But while the finite screw has pitch p the pitch of the twist is given by,

$$h_q = \frac{a\cdot b}{a\cdot a} = \frac{\theta/2}{1-\cos\theta/2}\left(\frac{p}{2\pi}\right)$$

Unlike the exponential map, the Cayley map depends on the representation of the group. In [5] Cayley maps for the 4×4 homgeneous representation of $SE(3)$ and the 6×6 adjoint representation were computed. In all of these maps and also for the exponential map, the twist and its image in the group share the same screw axis. The difference between the maps is their pitches, for the exponential map the pitch of the twist is the same as the pitch of the finite screw motion. For the 4×4 homogeneous representation a pitch p finite screw is produced by a twist with pitch,

$$h_4 = \frac{\theta/2}{\tan\theta/2}\left(\frac{p}{2\pi}\right)$$

This quatity is the quasi-pitch or 'quatch' discussed in [4]. The pitch of a twist associated to a pitch p screw motion by the Cayley map using the adjoint representation of the group is,

$$h_6 = \frac{\theta}{\sin\theta}\left(\frac{p}{2\pi}\right).$$

The dual quaternion representation give yet another map from the Lie algebra of the rigid-body displacement group to the group itself.

6. Geometry of the Cayley Maps

It is possible to say a little more about the geometry of these different Cayley maps. We begin with the 4×4 Cayley map. Consider a general group element, a pitch p screw motion about an axis with direction $\hat{\omega}$ and moment $v = q\hat{\omega} - \hat{\omega}q = q\times\hat{\omega}$,

$$g = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\omega}) + \varepsilon(\sin\frac{\theta}{2}v - \frac{1}{2}\left(\frac{\theta p}{2\pi}\right)\sin\frac{\theta}{2} + \frac{1}{2}\left(\frac{\theta p}{2\pi}\right)\cos\frac{\theta}{2}\hat{\omega},)$$

see (2.1) above. Now take the 2-plane determined by this point and the two other points 1 and ε. This 2-plane meets the 5-plane $a_0 = c_0 = 0$ of screws in a point,

$$s = \hat{\omega} + \varepsilon\left(v + \frac{\theta/2}{\tan\theta/2}\left(\frac{p}{2\pi}\right)\hat{\omega}\right)$$

This projection determines a map from the group to the space of screws, that is to the projective 5-space formed by taking the lines through the origin in the Lie algebra. This map is almost the inverse of the 4×4 Cayley map, 'almost' because the screws have no amplitude. Notice that the intersection of the group with the 2-plane determines a conic of points in the group which all map to the same screw. This can be interpreted as follows, consider the line of twists in the Lie algebra which determine the same screw, the 4×4 Cayley map of this line will be a conic in the space of group elements.

This can be viewed in another way, consider the finite screw motions with pitch p about the z-axis. Restricting attention to rigid displacements about this particular axis will not affect the results here since we can always use a group conjugation to put the screw axis in general position. Moreover it is clear that the action of the Cayley maps commute with the conjugation by a group element. The dual quaternion representing such motion is,

$$g = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathrm{k}) + \frac{1}{2}\varepsilon\big(- \frac{\theta p}{2\pi}\sin\frac{\theta}{2} + \frac{\theta p}{2\pi}\cos\frac{\theta}{2}\mathrm{k}\big)$$

Now we seek the group elements which have the same quatch, that is the group element that project to the same screw. The quatch is given by,

$$h_4 = \frac{\theta/2}{\tan\theta/2}\left(\frac{p}{2\pi}\right)$$

so if we substitute

$$\frac{\theta p}{2\pi} = 2h_4\tan\theta/2$$

we get

$$g = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathrm{k}) + \varepsilon(-h_4\tan\frac{\theta}{2}\sin\frac{\theta}{2} + h_4\sin\frac{\theta}{2}\mathrm{k}) \tag{6.1}$$

These group elements lie on the Study quadric $a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0$ of course, but also on the 5 hyperplanes, $a_1 = a_2 = c_1 = c_2 = 0$ and $h_4a_3 - c_3 = 0$. Hence these points do form a conic and we can give a rational parameterisation of the curve,

$$g = (c^2 + sc\mathrm{k}) + \varepsilon(-h_4s^2 + h_4sc\mathrm{k})$$

where s and c are homogeneous parameters.

We can perform a similar analysis for the other Cayley maps. For the 6×6 Cayley map the pitch of the screw is given by,

$$h_6 = \frac{\theta}{\sin\theta}\left(\frac{p}{2\pi}\right).$$

and hence the finite screws about the z-axis which map to this screw have the form,

$$g = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathrm{k}) + \varepsilon(-h_6\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} + h_6\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\mathrm{k})$$

These points can be parameterised by a homogeneous cubic,

$$g = ((c^2 + s^2)c + (c^2 + s^2)s\mathrm{k}) + \varepsilon(-h_6cs^2 + h_6c^2s\mathrm{k})$$

Hence, under this map the group elements which result from a line of twists about a given screw form a twisted cubic curve. Such curves always lie in the intersection of three linearly independent quadrics, here one of the quadrics is given by the Study quadric and the other two are $a_0c_3 - a_3c_0 - ha_0a_3 = 0$ and $ha_0c_0 + c_0^2 + c_3^2 = 0$. The curve lies in the 3-plane defined by the linear equations, $a_1 = a_2 = c_1 = c_2 = 0$. Notice that the motion defined by these curves are vertical Darboux motions, see [1, p. 321].

Lastly the dual quaternion Cayley map introduced above determines the pitch,

$$h_q = \frac{\theta/2}{1-\cos\theta/2}\left(\frac{p}{2\pi}\right)$$

so the finite screw motion about the z-axis can be written,

$$g = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathrm{k}) + \varepsilon(-h_q(1-\cos\frac{\theta}{2})\sin\frac{\theta}{2} + h_q(1-\cos\frac{\theta}{2})\cos\frac{\theta}{2}\mathrm{k})$$

This curve can be parameterised by homogeneous quartic forms,

$$g = ((c^2+s^2)(c^2-s^2) + 2(c^2+s^2)cs\mathrm{k}) + \varepsilon(-4h_q cs^3 + 2h_q(c^2-s^2)s^2\mathrm{k})$$

This shows that the dual quaternion Cayley map sends a line of twists to a rational quartic curve in the group.

7. Polynomial Expressions for the Cayley Maps

In this section polynomials in the dual quaternions will be found for the 4×4 and 6×6 Cayley maps.

In order to carry out the computations it is easiest to look at the case of motion about the z-axis, as usual the action of the group can be used to generalise the results found to motions about arbitrary axes. So begin with a pitch p motion about the z-axis, substituting for the pitch using the relation for the pitch of the corresponding twist under the 4×4 Cayley map yields (6.1).

Now the corresponding twist will have the form $s = |a|(\mathrm{k} + h_4\varepsilon\mathrm{k})$. For such a twist the idempotents and nilpotents will be,

$$\begin{aligned}
p_+ &= \frac{1}{2}(1 - i\mathrm{k}),\\
p_- &= \frac{1}{2}(1 + i\mathrm{k}),\\
n_+ &= \frac{1}{2}h_4|a|\varepsilon(i + \mathrm{k}),\\
n_- &= \frac{1}{2}h_4|a|\varepsilon(-i + \mathrm{k}).
\end{aligned}$$

In these cases the rotation angle is related to the Lie algebra element by the relation, $|a| = \tan(\theta/2)$. Using this and the relations above equation (6.1) can be written in terms of the idempotents and nilpotents as,

$$g = \frac{1+i|a|}{\sqrt{1+|a|^2}}p_+ + \frac{1+i|a|}{\sqrt{1+|a|^2}}n_+ + \frac{1-i|a|}{\sqrt{1+|a|^2}}p_- + \frac{1-i|a|}{\sqrt{1+|a|^2}}n_-.$$

Now, as argued above, this is a general relation for motion about an arbitrary axis, so we may assume that the idempotents and nilpotents are given by (4.1)–(4.4) again. Expanding the idempotents and nilpotents then gives the polynomial,

$$\mathrm{Cay}_4(s) = \frac{1}{2\sqrt{1+|a|^2}}\big((2+|a|^2) + 2s + s^2\big). \tag{7.1}$$

Notice that map is only quadratic in s, the cubic term has disappeared.

The 6×6 Cayley map can be treated in the same manner. This yields an expression in terms of idempotents and nilpotents,

$$g = \frac{1+i|a|}{(1+|a|^2)^{1/2}}p_+ + \frac{1+i|a|}{(1+|a|^2)^{3/2}}n_+ + \frac{1-i|a|}{(1+|a|^2)^{1/2}}p_- + \frac{1-i|a|}{(1+|a|^2)^{3/2}}n_-.$$

This leads to the following dual quaternion expression for the 6×6 Cayley map,

$$\mathrm{Cay}_6(s) = \frac{1}{2(1+|a|^2)^{3/2}}\big((2+3|a|^2) + (2+3|a|^2)s + s^2 + s^3\big). \tag{7.2}$$

8. Inverse Maps

Here we compute polynomials in the group elements for the inverse maps. The group elements also satisfy a polynomial equation. The polynomial satisfied by unit quaternions is simple to spot. Consider a general unit quaternion,

$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\omega}$$

where $\hat{\omega}$ is unit vector, so as a pure quaternion satisfies $\hat{\omega}^2 = -1$. Such a unit quaternion represents a rotation by and angle θ about an axis $\hat{\omega}$. Is is simple to verify that these unit quaternions satisfy the quadratic relation,

$$r^2 - 2\cos\frac{\theta}{2}r + 1 = 0. \tag{8.1}$$

To find the polynomial satisfied by the unit dual quaternions is not so straightforward. The system of idempotents and nilpotents can be used to find it however. Suppose that a unit dual quaternion is given in terms of the system of idempotents and nilpotents as, $g = \alpha p_+ + \beta n_+ + \gamma p_- + \delta n_-$. The unit dual quaternions must satisfy the relation, $g^{-1} = g^-$, from the relations for the idempotents and nilpotents (4.1)–(4.4) above it is easy to see that the Clifford conjugates of these elements are $p_+^- = p_-$ and $n_+^- = -n_-$. Hence a unit dual quaternion will be given in terms of the idempotents and nilpotents by,

$$g = (\lambda p_+ + \lambda\mu n_+ + \frac{1}{\lambda}p_- + \frac{\mu}{\lambda}n_-) \tag{8.2}$$

where λ and μ are arbitrary. The first four powers of such a group element are given by,

$$\begin{aligned}
1 &= p_+ + p_- \\
g &= \lambda p_+ + \lambda\mu n_+ + \frac{1}{\lambda}p_- + \frac{\mu}{\lambda}n_- \\
g^2 &= \lambda^2 p_+ + 2\lambda^2\mu n_+ + \frac{1}{\lambda^2}p_- + 2\frac{\mu}{\lambda^2}n_- \\
g^3 &= \lambda^3 p_+ + 3\lambda^3\mu n_+ + \frac{1}{\lambda^3}p_- + 3\frac{\mu}{\lambda^3}n_- \\
g^4 &= \lambda^4 p_+ + 4\lambda^4\mu n_+ + \frac{1}{\lambda^4}p_- + 4\frac{\mu}{\lambda^4}n_-
\end{aligned}$$

Now suppose we assume that the group elements satisfy a quartic relation: $g^4 = \alpha g^3 + \beta g^2 + \gamma g + \delta g$. Comparing the coefficients of the idempotents and nilpotents gives a set of linear equations in the unknown coefficients of the quartic.

$$\begin{array}{ccccccccc}
\alpha\lambda^3 & + & \beta\lambda^2 & + & \gamma\lambda & + & \delta & = & \lambda^4 \\
3\alpha\lambda^3\mu & + & 2\beta\lambda^2\mu & + & \gamma\lambda\mu & & & = & 4\lambda^4\mu \\
\alpha\lambda^{-3} & + & \beta\lambda^{-2} & + & \gamma\lambda^{-1} & + & \delta & = & \lambda^{-4} \\
3\alpha\lambda^{-3}\mu & + & 2\beta\lambda^{-2}\mu & + & \gamma\lambda^{-1}\mu & & & = & 4\lambda^{-4}\mu
\end{array}$$

These equations could be solved by standard techniques from linear algebra however it not too difficult to see that the equation are in the same form as the equations for a Hermite interpolation problem. The coefficients sought in the problem above are the same as the coefficients in the polynomial through the point (λ, λ^4) with gradient $4\lambda^3$ and through the point $(\lambda^{-1}, \lambda^{-4})$ with gradient $4\lambda^{-3}$. Note that the second and forth equations above can be divided by $\lambda\mu$ and $\lambda^{-1}\mu$ respectively.

Standard techniques for solving Hermite interpolation problems can be applied and this yields the following result for the quartic polynomial satisfied by the unit dual quaternions,

$$g^4 = 2(\lambda^1 + \lambda^{-1})g^3 - (\lambda^2 + 4 + \lambda^{-2})g^2 + 2(\lambda^1 + \lambda^{-1})g - 1. \tag{8.3}$$

Comparing this with the results for the exponential map given above, we can identify the parameter λ with $e^{i\theta/2}$ where θ is the rotation angle of the transformation g.

$$g^4 - 4\cos\frac{\theta}{2}g^3 + (4\cos^2\frac{\theta}{2} + 4)g^2 - 4\cos\frac{\theta}{2}g + 1 = 0 \tag{8.4}$$

Notice that this relation will also give a polynomial expression for the Clifford conjugate of a general group element, since for group elements $gg^- = 1$. That is,

$$g^- = -g^3 + 4\cos\frac{\theta}{2}g^2 - (4\cos^2\frac{\theta}{2} + 4)g + 4\cos\frac{\theta}{2} \tag{8.5}$$

8.1. The Logarithm

The same technique can be used to compute polynomials for the inverse functions to other maps from the Lie algebra to the group. The inverse to the exponential function is of course the Logarithm, this can be found by Hermite interpolation through the points $(e^{i\theta/2},\, i\theta/2)$ and $(e^{-i\theta/2},\, -i\theta/2)$ with gradients $e^{-i\theta/2}$ and $e^{i\theta/2}$ respectively.

This produces the polynomial,

$$\begin{aligned}\log(g) \quad &= \quad \frac{1}{8\sin^3(\theta/2)}\big(2(\theta-\sin\theta)g^3+4(\sin\frac{3\theta}{2}-\frac{3\theta}{2}\cos\frac{\theta}{2})g^2+\\ &2(3\theta-8\sin\theta-\sin 2\theta)g+\big(2\sin\frac{3\theta}{2}-2\sin\frac{\theta}{2}+\theta\cos\frac{3\theta}{2}-3\theta\cos\frac{\theta}{2}\big)\big).\end{aligned} \tag{8.6}$$

8.2. The Inverse Cayley Maps

Similar results can be produced for the inverse Cayley maps. For the dual quaternion Cayley map the points for the Hermite interpolation are $\big((1+i|a|)(1-i|a|)^{-1},\, i|a|\big)$ and $\big((1-i|a|)(1+i|a|)^{-1},\, -i|a|\big)$ with $(1/2)(1-i|a|)^2$ and $(1/2)(1+i|a|)^2$ as the gradients at the respective points.

$$\begin{aligned}\mathrm{Cay}_q^{-1}(g) \quad &= \quad \frac{1}{8}\big((|a|^2+1)^2g^3+(3|a|^4-2|a|^2-5)g^2+\\ &(3|a|^4-2|a|^2+11)g+(|a|^4+2|a|^2-7)\big).\end{aligned} \tag{8.7}$$

Recall that the rotation angle of the group element satisfies $|a| = \tan(\theta/4)$. However, the sine and cosine of the half-angles are more immediately accessible, note that $\cos(\theta/2) = a_0$ and $\sin^2(\theta/2) = a_1^2 + a_2^2 + a_3^2$, where a_i are the first four homogenous coordinates in $\mathbb{P}^7$, see section 2 above. Using the relation $\tan(\theta/4) = \big(1-\cos(\theta/2)\big)/\sin(\theta/2)$, the inverse map can be written,

$$\begin{aligned}\mathrm{Cay}_q^{-1}(g) \quad &= \quad \frac{1}{2(\cos(\theta/2)+1)^2}\big(g^3-(4\cos\frac{\theta}{2}+1)g^2+\\ &(4\cos^2\frac{\theta}{2}+4\cos\frac{\theta}{2}+3)g+(2\cos^2\frac{\theta}{2}+4\cos\frac{\theta}{2}+1)\big).\end{aligned} \tag{8.8}$$

The other Cayley maps can be treated in the same way, except for the fact that now we have that $|a| = \tan(\theta/2)$. For the 4×4 Cayley map the result is,

$$\mathrm{Cay}_4^{-1}(g) = \frac{1}{2\cos(\theta/2)}\big(g^3-4\cos\frac{\theta}{2}g^2+(4\cos^2\frac{\theta}{2}+3)g-4\cos\frac{\theta}{2}\big). \tag{8.9}$$

And the 6×6 Cayley map gives,

$$\mathrm{Cay}_6^{-1}(g) = \frac{-1}{2\cos^2(\theta/2)}\big(g^2-4\cos\frac{\theta}{2}g+(2\cos^2\frac{\theta}{2}+1)\big). \tag{8.10}$$

Again notice that in this case the map is only quadratic in the group element.

9. Conclusions

In this work cubic polynomials have been given for the exponential and Cayley maps from the space of twists to the space of finite screw motions. Polynomials for two other Cayley maps have also been found. It has also been possible to find polynomials for the inverses of these maps.

A little about the geometry of these maps has also been found. The exponential of a zero or infinite pitch screw is well known to map to a line in the Study quadric. Screws of general pitch do not map to algebraic curve. Here it has been shown that the 4×4 Cayley maps sends screws to conics in the Study quadric. The 6×6 Cayley map sends screws to twisted cubic curves in the Study quadric. The image of a screw under the dual quaternion Cayley map is a twisted quartic curve. Moreover, the twisted cubic curves in the group correspond to motions already studied in the kinematics literature, they are known as vertical Darboux motions. A Darboux motion is a rigid-body motion under which the points of space trace conic trajectories, not all lying in the same plane. Rigid body motions corresponding to conics in the Study quadric are also know. The motions of the coupler bar in a Bennett mechanism is known to follow a conic curve in the Study quadric. It has not been determined if the the image of a screw under the 4×4 Cayley map is one of these Bennett motions. Motions corresponding to rational quartic curves in the Study quadric are less well understood (the well known Bricard-Borel motion corresponds to an elliptic quartic).

In the introduction it was stated that the Cayley maps could be used in efficient numerical methods. In order to carry this out it would usually be necessary to have polynomial relations for the derivatives of these maps and their inverses. Using the methods of this paper it would be a straightforward matter to complete these computations.

It is possible that there are other Cayley maps resulting from other representations. Notice that the dual quaternions could be viewed as an 8×8 matrix representation of the group by writing a dual quaternion as an 8-dimensional vector and representing the right and left products as matrix operations. It may be possible to find all possibilities by studying the possible representations of the maps in terms of the system of idempotents and nilpotents described in the paper.

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