Group consensus control for networked multi-agent systems with communication delays[☆]

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Abstract

This paper investigates group consensus problems in networked multi-agent systems (NMAS) with communication delays. Based on the proposed state prediction scheme, the group consensus control protocol is designed to compensate the communication delay actively. In light of algebraic graph theories and matrix theories, necessary and(or) sufficient conditions of group consensus with respect to a given admissible control set are obtained for the NMAS with communication delays under mild assumptions. Finally, simulations are worked out to demonstrate the efficacy of the theoretical results.

Keywords: networked multi-agent systems, group consensus, communication delays, time delay compensation scheme

1. Introduction

In recent years, consensus analysis, which is a very important fundamental research focus in the field of cooperative control of multi-agent systems (MASs), has been widely used in many scientific projects, such as distributed sensor networks[1, 2, 3, 4], intelligent traffic management systems[5, 6, 7],

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power converters[8], PEM fuel cell air-feed system[9], unmanned aerial vehicles (UAVs)[10], etc. Generally speaking, the main goal of consensus problems is to design an appropriate control protocol based on the information exchange between an agent and its neighbors so that all the agents in a MAS converge to a common state.

Since the first systemic framework for consensus analysis of a continuous multi-agent system was established in 2004 by Olfati-Saber and Murray[11], fruitful research explorations in consensus control of a MAS have begun in various research communities. Ren et al.[12] considered the consensus problem under directed interaction typologies and gave discrete and continuous update schemes and convergent conditions. Xiao et al.[13] designed a valid distributed consensus algorithm for first-order discrete-time multi-agent systems with switching topologies and time-varying delays, and analysis some applicable conditions. Subsequently, good progress has also been made on consensus problems of second-order and higher-order multi-agent systems with delays and noises[14, 15, 16]. Moreover, the existing theoretical results involve some extended consensus problems. Xiao et al.[17] studied asynchronous consensus problems of multi-agent systems in discontinuous information transmission. Wang et al.[18] discussed the finite-time state consensus problems using the theory of finite-time Lyapunov stability. Tahbaz-Salehi et al.[19] presented a sufficient and necessary condition for the stochastic consensus of discrete-time linear multi-agent systems. Zhou et al.[20] designed constrained consensus controllers of the asynchronous multi-agent systems,where each agent is required to lie in a closed convex constraint set. For more details, see[21, 22, 23].

It is worth noting that the above research results only discuss how to design appropriate protocols and algorithms for guaranteeing consensus requirements of all the agents in a network. However, in complex practical applications of multi-agent control systems, changes of external environment, cooperative task allocation or even time may lead to the fact that agents in a network converge to more than one consistent states. Especially, when the analysis and design of large-scale complex network system is carried out, the complex large-scale network is decomposed into several smaller sub-network according to the specific cooperative requirements. In nature, there are foraging and migration among multiple species, for instance, birds, fish and primates often coordinate their behavior in interaction with peers and other species[24]. In the study of social networks, the analytical results from some dynamic evolution models of human opinion show that all the agents evolve into groups in some cases, and agents in the same group asymptotically reach a consistent state[25]. Due to some potential applications of group consensus into multi-group formation, flocking and swarming control[26], some scholars have studied group consensus. Group consensus is more general than complete consensus, whose main goal is to design appropriate control protocols and algorithms so that the agents in the same subgroup reach a state agreement, but the consistent states of different subgroups may not be the same[27]. Yu et al.[28, 29] investigate group consensus problems of first-order continuous-time multi-agent systems with connected undirected/directed topology and several criterions are established by using graph theories and matrix theories. On this basis, double-tree-form transformations are introduced to analyze the case of switching topology and communication delay, and consequently the group consensus problem is converted into a stability problem of switched linear systems. Wang et al.[30] studied the maximum (weight) stable set and vertex coloring problems with application to the group consensus of MASs, and presented a new protocol design procedure by the matrix semi-tensor product method. Wang et al.[31] designed competition-based control protocol for a class of first-order continuous-time multi-agent systems with time-delay and connected bipartite graph topologies. Ji et al.[32] further analyzed the coupling relationship between group consensus and time delays based on the results from Wang et al. Futhermore, Guan et al.[33] pointed out that the multiagent network may have several consistent states for a special grouping or no grouping and group consensus can be recognized as one special case of multiconsensus. For more details on second-order continuous-time multi-agent systems, please refer to Ref.[34, 35].

Among all existing works, there are few literatures which concern group consensus of a MAS with high-order discrete-time dynamics. In addition, a strict constraint is that the sum of adjacent weights from each agent in one group to all agents in other groups equals zero at any moment, which has great conservation in practical application. Moreover, the issue of communication delays that degrade the consensus performance is only focused on a first-order or second-order integral-type MAS. To overcome the negative effects of communication delays on group consensus, most approaches have been presented directly using theoretical results on time delay systems. These approaches commonly aim at obtaining the upper bound of the delay tolerance that guarantee group consensus, which are so passive to compensate for communication delays.

The paper investigates the group consensus problem in a discrete-time MAS with general linear dynamics and communication delays. Based on state predictive scheme, a novel group consensus control protocol is designed to compensate for communication delays actively. By means of graph theory and matrix approach, necessary and (or) sufficient conditions are established for the realization of group consensus in the time-varying delay case.

This paper is organized as follows. Some background of problem formulation and group consensus control protocols are presented in Section 2. Section 3 describes necessary and(or) sufficient conditions of group consensus with respect to a given admissible control set for the NMAS with communication delays. Simulation results are given in Section 4 to demonstrate the feasibility and efficiency of the proposed group consensus control protocol. The conclusion drawn from the present study is presented in Section 5.

Notation: Throughout the paper, $\mathcal{R}$ denotes the set of real numbers, $\mathcal{Z}^+$ denotes the set of non-negative integers, $\mathcal{C}$ denotes the set of complex numbers and $\mathcal{R}^{m\times n}$ stands for the set of $m \times n$ real matrices. $\boldsymbol{I}_n$ is an n-dimensional identity matrix. $\boldsymbol{1}_n = [\ 1\ \cdots\ 1\ ]^T \in \mathcal{R}^{n\times 1}$ means an $n \times 1$ column vector with all elements equal to unity. The symbol $\otimes$ represents the Kronecker product. If $\boldsymbol{W} = (w_{ij})$ is an $m \times n$ matrix and $\boldsymbol{V} = (v_{ij})$ is a $p \times q$ matrix, then the Kronecker product $\boldsymbol{W} \otimes \boldsymbol{V}$ represents the $mp \times nq$ block matrix:

$$\boldsymbol{W} \otimes \boldsymbol{V} = \begin{bmatrix} w_{11}\boldsymbol{V} & \cdots & w_{1n}\boldsymbol{V} \\ \vdots & \ddots & \vdots \\ w_{m1}\boldsymbol{V} & \cdots & w_{mn}\boldsymbol{V} \end{bmatrix}.$$

2. Problem Formulation and Group Consensus Control Protocol

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order $N(N \geq 2)$ with a set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a nonnegative weighted adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}] \in \mathcal{R}^{N\times N}$. The node indexes belong to a finite index set $\ell = \{1, 2, \cdots, N\}$. A directed edge that starts from the node v_i and ends on the node v_j is denoted by $e_{ij} = (v_i, v_j)$. The adjacency element a_{ji} associated with the edge e_{ij} is nonzero, i.e., $e_{ij} \in \mathcal{E}$ if and only if $a_{ji} \neq 0$. Moreover, it is assumed that $a_{ii} \equiv 0$, $\forall i \in \ell$. The neighborhood set of the node v_i is represented by $\mathcal{N}_i = \{\mathcal{V}_j \in \mathcal{V} | (v_j, v_i) \in \mathcal{E}\}$. The indegree of the node v_i is defined as $d_{in}(i) = \sum_{j\in\mathcal{N}_i} a_{ij}$. The degree matrix of $\mathcal{G}$ is a diagonal matrix $\mathcal{D}(\mathcal{G}) = diag\big(d_{in}(1), d_{in}(2), \cdots, d_{in}(N)\big)$. Then the Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$.

Given a network composed of N agents, the state of the ith agent is denoted by $\boldsymbol{x}_i \in \mathcal{R}^n$, thus $(\mathcal{G}, \boldsymbol{x})$ stands for the network with a state vector $\boldsymbol{x} = [\boldsymbol{x}_1^T \ \boldsymbol{x}_2^T \ \cdots \ \boldsymbol{x}_N^T]^T \in \mathcal{R}^{nN}$ and a topological graph $\mathcal{G}$. The state of each node can describe physical quantities in practical scene, such as position, velocity, acceleration, temperature, pressure, flow, liquid level, etc.. Considering each agent as a node in the network $(\mathcal{G}, \boldsymbol{x})$, an available communication link from agent v_j to agent v_i is corresponding to a directed edge $e_{ij} \in \mathcal{E}$. Each agent updates its current state based on its own information and the received from its neighboring agents.

Definition 1: A network $(\mathcal{G}_1, \boldsymbol{x}_1)$ with a topological graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ is said to be a sub-network of a network $(\mathcal{G}, \boldsymbol{x})$ with a topological graph with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ if **(i)**$\mathcal{V}_1 \subseteq \mathcal{V}$, **(ii)**$\mathcal{E}_1 \subseteq \mathcal{E}$ and **(iii)**$\mathcal{A}_1$ inherits $\mathcal{A}$. In a similar way, $\mathcal{G}_1$ is called a sub-graph of $\mathcal{G}$. Moreover, if the conditions **(i)** and **(ii)** are strictly satisfied, and $\mathcal{E}_1 = \{(v_i, v_j)|(v_i, v_j) \in \mathcal{E}, i, j \in \mathcal{V}_1\}$, the network $(\mathcal{G}_1, \boldsymbol{x}_1)$ is called a proper sub-network of the network $(\mathcal{G}, \boldsymbol{x})$ and $\mathcal{G}_1$ is said to be a proper sub-graph of $\mathcal{G}$.

Considering a complex network $(\mathcal{G}, \boldsymbol{x})$ composed of $N+M(N, M > 1)$ agents, denote its state vector $\boldsymbol{x} = [\ \boldsymbol{x}_1^T\ \boldsymbol{x}_2^T\ \cdots\ \boldsymbol{x}_{N+M}^T\ ]^T \in \mathcal{R}^{n(N+M)}$, and its topological graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a weighted directed graph composed of the sub-graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ and the sub-graph $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2)$, where $\mathcal{V}_1 = \{v_1, v_2, \cdots, v_N\}$, $\mathcal{V}_2 = \{v_{N+1}, v_{N+2}, \cdots, v_{N+M}\}$, $\ell_1 = \{1, 2, \cdots, N\}$, $\ell_2 = \{N+1, N+2, \cdots, N+M\}$. For node v_i, the set of its neighbor nodes in two sub-graphs are defined as $\mathcal{N}_{1i} = \{\mathcal{V}_j \in \mathcal{V}_1|(v_j, v_i) \in \mathcal{E}\}$ and $\mathcal{N}_{2i} = \{\mathcal{V}_j \in \mathcal{V}_2|(v_j, v_i) \in \mathcal{E}\}$, respectively. It is clear that $\mathcal{V} = \mathcal{V}_1 \bigcup \mathcal{V}_2$, $\mathcal{N}_i = \mathcal{N}_{1i} \bigcup \mathcal{N}_{2i}$, $\ell = \ell_1 \bigcup \ell_2$. Therefore, $(\mathcal{G}, \boldsymbol{x})$ is considered to be composed of two sub-network $(\mathcal{G}_1, \chi_1)$ and $(\mathcal{G}_2, \chi_2)$, where $\chi_1 = [\ \boldsymbol{x}_1^T\ \boldsymbol{x}_2^T\ \cdots\ \boldsymbol{x}_N^T\ ]^T \in \mathcal{R}^{nN}$, $\chi_2 = [\ \boldsymbol{x}_N^T\ \boldsymbol{x}_{N+1}^T\ \cdots\ \boldsymbol{x}_{N+M}^T\ ]^T \in \mathcal{R}^{nM}$, and $\boldsymbol{x} = [\ \chi_1^T\ \chi_2^T\ ]^T$. It implies information exchanges among agents exist not only in the same sub-network but also in different sub-networks.

Suppose the discrete-time dynamics of the ith agent in the network $(\mathcal{G}, \boldsymbol{x})$ are described by a general linear system:

$$\begin{aligned} \boldsymbol{x}_i(t+1) &= \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{B}\boldsymbol{u}_i(t) \\ \boldsymbol{y}_i(t) &= \boldsymbol{C}\boldsymbol{x}_i(t), \qquad \forall\ i \in \ell. \end{aligned} \tag{1}$$

where $\boldsymbol{x}_i(t) \in \mathcal{R}^n$ is the state vector, $\boldsymbol{y}_i(t) \in \mathcal{R}^m$ is the measured output vector, $\boldsymbol{u}_i(t) \in \mathcal{R}^r$ is the control input vector, $\boldsymbol{A} \in \mathcal{R}^{n\times n}$, $\boldsymbol{B} \in \mathcal{R}^{n\times r}$ and $\boldsymbol{C} \in \mathcal{R}^{m\times n}$.

In this paper, the objective is to design a suitable group consensus control protocol for a MAS subject to communication constraints in two sub-networks so that the first N agents converge to one consistent state asymptotically while the last M agents converge to another consistent state asymptotically. In order to describe systemic structure and network characteristics of the NMAS (1), the following assumptions are made:

Assumption 1 The matrix pair $(\boldsymbol{A},\ \boldsymbol{C})$ is detectable.

Assumption 2 The ith agent $(\forall i \in \ell)$ can receive the information of its own and the information of the jth agent ($v_j \in \mathcal{N}_i$) through a communication network with time-varying delays $\tau_{ij}(t)$, and

$$0 \le \tau_0 \le \check{\tau}_{ij}(t) \le \tau_{ij}(t) \le \hat{\tau}_{ij}(t) \le \tau, \ \forall \ i, j \in \ell, \tag{2}$$

where $\check{\tau}_{ij}(t)$ and $\hat{\tau}_{ij}(t)$ is known bounded functions, τ_0 and τ are the lower and upper bounds of communication delays, respectively.

Assumption 3 The data packets transmitted in the network are with time stamps.

Assumption 4 The adjacent weights a_{ij} of the sub-network $(\mathcal{G}_1, \chi_1)$ and $(\mathcal{G}_2, \chi_2)$ satisfy

$$\begin{aligned} \sum_{v_j \in \mathcal{N}_{2i}} a_{ij} = \alpha, \quad \forall \ i \in \ell_1, \\ \sum_{v_j \in \mathcal{N}_{1i}} a_{ij} = \beta, \quad \forall \ i \in \ell_2, \end{aligned} \tag{3}$$

where $\alpha, \ \beta \in \mathcal{R}$ are constant numbers.

Due to the presence of time delay $\tau_{ij}(t)$ in the transmission channel from the jth agent to the ith agent, the ith agent at time t might only receive the jth agent's information at time $t - \tau_{ij}(t)$. At present, many different types of control protocols directly using delayed information have been developed for solving the group consensus problem for instance

$$\boldsymbol{u}_i(t) = \begin{cases} \boldsymbol{K}\Bigg(\sum\limits_{v_j \in \mathcal{N}_{1i}} a_{ij}\big(\boldsymbol{x}_j(t - \tau_{ij}(t)) - \boldsymbol{x}_i(t - \tau_{ij}(t))\big) \\ \qquad\qquad + \sum\limits_{v_j \in \mathcal{N}_{2i}} a_{ij}\boldsymbol{x}_j(t - \tau_{ij}(t))\Bigg), \forall \ i \ \in \ell_1, \\ \boldsymbol{K}\Bigg(\sum\limits_{v_j \in \mathcal{N}_{2i}} a_{ij}\big(\boldsymbol{x}_j(t - \tau_{ij}(t)) - \boldsymbol{x}_i(t - \tau_{ij}(t))\big) \\ \qquad\qquad + \sum\limits_{v_j \in \mathcal{N}_{1i}} a_{ij}\boldsymbol{x}_j(t - \tau_{ij}(t))\Bigg), \forall \ i \ \in \ell_2. \end{cases} \tag{4}$$

As is known, the past information cannot express the current of a control system in time. Consequently, these design methods ignore network characteristics and attempt to exert control passively within upper bounds to the maximal tolerable delay that can guarantee group consensus. In order to overcome adverse effects caused by network delay, a network delay compensation scheme is proposed to predict the state and control input of each agent.

On the basis of ***Assumptions 1–3***, suppose the state of the ith agent ($\forall\ i \in \ell$) is not measurable. For the state estimation of its neighboring agent j, a state observer on the ith agent is constructed as

$$\begin{aligned} \hat{\boldsymbol{x}}_j(t-\tau+1|t-\tau) &= \boldsymbol{A}\hat{\boldsymbol{x}}_j(t-\tau|t-\tau-1)+\boldsymbol{B}\boldsymbol{u}_j(t-\tau) \\ &\quad + \boldsymbol{L}\big(\boldsymbol{y}_j(t-\tau)-\hat{\boldsymbol{y}}_j(t-\tau)\big) \\ \hat{\boldsymbol{y}}_j(t-\tau) &= \boldsymbol{C}\hat{\boldsymbol{x}}_j(t-\tau|t-\tau-1), \qquad \forall\ i \in \ell. \end{aligned} \tag{5}$$

where $\hat{\boldsymbol{x}}_j(t-p|t-q) \in \mathcal{R}^n$ $(p<q)$ means the state estimation of the agent j for time $t-p$ on the basis of the information up to time $t-q$, $\hat{\boldsymbol{y}}_j(t) \in \mathcal{R}^m$ is the observer output at time t. Although Equation (5) provides a one-step ahead state estimation of the agent j using the output at time $t-\tau$, the state estimations from time $t-\tau+2$ to time $t+\tau$ can be calculated by

$$\begin{aligned} \hat{\boldsymbol{x}}_j(t-\tau+2|t-\tau) &= \boldsymbol{A}\hat{\boldsymbol{x}}_j(t-\tau+1|t-\tau)+\boldsymbol{B}\boldsymbol{u}_j(t-\tau+1) \\ &\ \vdots \\ \hat{\boldsymbol{x}}_j(t|t-\tau) &= \boldsymbol{A}\hat{\boldsymbol{x}}_j(t-1|t-\tau)+\boldsymbol{B}\boldsymbol{u}_j(t-1). \end{aligned} \tag{6}$$

On the ith agent, a consensus control protocol using the above state estimation scheme is constructed as

$$\boldsymbol{u}_i(t) = \begin{cases} \boldsymbol{K}\Bigg(\displaystyle\sum_{v_j \in \mathcal{N}_{1i}} a_{ij}\big(\hat{\boldsymbol{x}}_j(t|t-\tau)-\hat{\boldsymbol{x}}_i(t|t-\tau)\big) \\ \qquad\qquad + \displaystyle\sum_{v_j \in \mathcal{N}_{2i}} a_{ij}\hat{\boldsymbol{x}}_j(t|t-\tau)\Bigg), \forall\ i\ \in \ell_1, \\ \boldsymbol{K}\Bigg(\displaystyle\sum_{v_j \in \mathcal{N}_{2i}} a_{ij}\big(\hat{\boldsymbol{x}}_j(t|t-\tau)-\hat{\boldsymbol{x}}_i(t|t-\tau)\big) \\ \qquad\qquad + \displaystyle\sum_{v_j \in \mathcal{N}_{1i}} a_{ij}\hat{\boldsymbol{x}}_j(t|t-\tau)\Bigg), \forall\ i\ \in \ell_2, \end{cases} \tag{7}$$

where $\boldsymbol{K} \in \mathcal{R}^{r\times n}$ denotes the control gain matrix, and the adjacency weight a_{ij}

satisfies the following conditions:

$$\begin{aligned}&\textbf{(i) } a_{ij} \geq 0, \forall\, i,j \in \ell_1;\\&\textbf{(ii) } a_{ij} \geq 0, \forall\, i,j \in \ell_2;\\&\textbf{(iii) } a_{ij} \in \mathcal{R}, \forall\, (i,j) \in \phi = \{(i,j) : i \in \ell_1, j \in \ell_2\} \bigcup \{(i,j) : i \in \ell_2, j \in \ell_1\}.\end{aligned} \tag{8}$$

Remark 1: By the above state estimation scheme, the state of the agent j at time t can be predicted using the information up to time $t-\tau$. Active compensation of communication delays can be made by employing the sate predictions into the design of group consensus control protocols. On account of $\tau_{ij}(t) < \tau$, measured output and control input of the agent j from time $t-\tau$ to $t-\tau_{ij}(t)$ are obtained on the ith agent at time t. Meanwhile, on the basis of the state estimation equations (5)–(6) and the control protocol (7), the information of agent j from time $t-\tau+1$ to $t-\tau_{ij}(t)$ is not involved in the calculation of control input $\boldsymbol{u}_i(t)$. Although redundant state estimations lead to the increased amount of computation on the ith agent to a certain extent, the above network delay compensation scheme provides a unified prediction process and overcomes time-varying delay effects on consensus performance. Moreover, in practical application, embedded microprocessors with powerful computing ability can ensure smooth execution of the whole estimation algorithm, and the controller of each agent can be combined into an integrated controller with a quite powerful function at low cost.

Let

$$\boldsymbol{u}(t) = \left[\ \boldsymbol{u}_1^T(t)\ \boldsymbol{u}_2^T(t)\ \cdots\ \boldsymbol{u}_{N+M}^T(t)\ \right]^T. \tag{9}$$

If ***Assumptions 1–4*** stand, for system (1), considering the following admissible control set:

$\mathcal{U} = \Big\{\boldsymbol{u}(t) : \mathcal{Z}^+ \longrightarrow \mathcal{R}^{r(N+M)} | \boldsymbol{u}_i(t)$ satisfies Equations (7)–(8), $\forall\, i \in \ell, \forall\, t = 0, 1, \cdots\Big\}$. An important question is that under what conditions, the NMAS (1) achieves group consensus with respect to (w.r.t.) the admissible control set $\mathcal{U}$? Aiming at this question, this paper defines the group consensusability of an MAS w.r.t. the admissible control set $\mathcal{U}$.

Definition 2 For the NMAS (1), if there exists $\boldsymbol{u}(t) \in \mathcal{U}$ such that for any initial state $\boldsymbol{x}_i(0)$, $i \in \ell$, the NMAS (1) is said to be group consensusable w.r.t. $\mathcal{U}$, if the following conditions holds:

$$\begin{aligned}
&\textbf{(i)} \lim_{t\to+\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| = 0,\ \forall\ i, j \in \ell_1; \\
&\textbf{(ii)} \lim_{t\to+\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| = 0,\ \forall\ i, j \in \ell_2; \\
&\textbf{(iii)} \lim_{t\to+\infty} \|\boldsymbol{x}_i(t) - \hat{\boldsymbol{x}}_i(t|t-1)\| = 0,\ \forall\ i \in \ell.
\end{aligned} \tag{10}$$

For the convenience of theoretical analysis, define the following new variables:

$$\begin{aligned}
\zeta_i(t) &= \boldsymbol{x}_1(t) - \boldsymbol{x}_i(t),\ \forall\ i \in \ell_1, \\
\eta_i(t) &= \boldsymbol{x}_{N+1}(t) - \boldsymbol{x}_i(t),\ \forall\ i \in \ell_2, \\
\varepsilon_i(t) &= \boldsymbol{x}_i(t) - \hat{\boldsymbol{x}}_i(t|t-1),\ \forall\ i \in \ell, \\
\zeta(t) &= \begin{bmatrix} \zeta_2^T(t) & \zeta_3^T(t) & \cdots & \zeta_N^T(t) \end{bmatrix}^T, \\
\eta(t) &= \begin{bmatrix} \eta_{N+2}^T(t) & \eta_{N+3}^T(t) & \cdots & \eta_{N+M}^T(t) \end{bmatrix}^T, \\
\delta(t) &= \begin{bmatrix} \zeta^T(t) & \eta^T(t) \end{bmatrix}^T.
\end{aligned}$$

From *Definition 2*, it is obvious that the equation (10) stands if and only if $t \to \infty$, $\|\delta(t)\| \to 0$, $\|\varepsilon_i(t)\| \to 0$.

Before the presentation of main results, some relevant matrices are labeled as follows.

$$\mathcal{L}(\mathcal{G}_1) = \left(\begin{array}{c:c} \mathcal{L}_{11}(\mathcal{G}_1) & \mathcal{L}_{12}(\mathcal{G}_1) \\ \hdashline \mathcal{L}_{21}(\mathcal{G}_1) & \mathcal{L}_{22}(\mathcal{G}_1) \end{array}\right) = \left(\begin{array}{c} \mathcal{L}_1(\mathcal{G}_1) \\ \hdashline \mathcal{L}_2(\mathcal{G}_1) \end{array}\right),$$

$$\mathcal{L}(\mathcal{G}_2) = \left(\begin{array}{c:c} \mathcal{L}_{11}(\mathcal{G}_2) & \mathcal{L}_{12}(\mathcal{G}_2) \\ \hdashline \mathcal{L}_{21}(\mathcal{G}_2) & \mathcal{L}_{22}(\mathcal{G}_2) \end{array}\right) = \left(\begin{array}{c} \mathcal{L}_1(\mathcal{G}_2) \\ \hdashline \mathcal{L}_2(\mathcal{G}_2) \end{array}\right),$$

$$\Omega(\mathcal{G}_1) = \left(\begin{array}{c:ccc} -a_{1(N+1)} & -a_{1(N+2)} & \cdots & -a_{1(N+M)} \\ \hdashline -a_{2(N+1)} & -a_{2(N+2)} & \cdots & -a_{2(N+M)} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{N(N+1)} & -a_{N(N+2)} & \cdots & -a_{N(N+M)} \end{array}\right) = \left(\begin{array}{c:c} \Omega_{11}(\mathcal{G}_1) & \Omega_{12}(\mathcal{G}_1) \\ \hdashline \Omega_{21}(\mathcal{G}_1) & \Omega_{22}(\mathcal{G}_1) \end{array}\right) = \left(\begin{array}{c} \Omega_{1}(\mathcal{G}_1) \\ \hdashline \Omega_{2}(\mathcal{G}_1) \end{array}\right),$$

$$\Omega(\mathcal{G}_2) = \left(\begin{array}{c:ccc} -a_{(N+1)1} & -a_{(N+1)2} & \cdots & -a_{(N+1)N} \\ \hdashline -a_{(N+2)1} & -a_{(N+2)2} & \cdots & -a_{(N+2)N} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{(N+M)1} & -a_{(N+M)2} & \cdots & -a_{(N+M)N} \end{array}\right) = \left(\begin{array}{c:c} \Omega_{11}(\mathcal{G}_2) & \Omega_{12}(\mathcal{G}_2) \\ \hdashline \Omega_{21}(\mathcal{G}_2) & \Omega_{22}(\mathcal{G}_2) \end{array}\right) = \left(\begin{array}{c} \Omega_{1}(\mathcal{G}_2) \\ \hdashline \Omega_{2}(\mathcal{G}_2) \end{array}\right),$$

$$\mathcal{L}(\mathcal{G}) = \left(\begin{array}{c:c} \mathcal{L}(\mathcal{G}_1) & \Omega(\mathcal{G}_1) \\ \hdashline \Omega(\mathcal{G}_2) & \mathcal{L}(\mathcal{G}_2) \end{array}\right),$$

where $\mathcal{L}_{11}(\mathcal{G}_1) \in \mathcal{R}$, $\mathcal{L}_{12}(\mathcal{G}_1) \in \mathcal{R}^{1\times(N-1)}$, $\mathcal{L}_{21}(\mathcal{G}_1) \in \mathcal{R}^{(N-1)\times 1}$, $\mathcal{L}_{22}(\mathcal{G}_1) \in \mathcal{R}^{(N-1)\times(N-1)}$, $\mathcal{L}_{11}(\mathcal{G}_2) \in \mathcal{R}$, $\mathcal{L}_{12}(\mathcal{G}_2) \in \mathcal{R}^{1\times(M-1)}$, $\mathcal{L}_{21}(\mathcal{G}_2) \in \mathcal{R}^{(M-1)\times 1}$, $\mathcal{L}_{22}(\mathcal{G}_2) \in \mathcal{R}^{(M-1)\times(M-1)}$, $\Omega_1(\mathcal{G}_1) \in \mathcal{R}^{1\times M}$, $\Omega_2(\mathcal{G}_1) \in \mathcal{R}^{(N-1)\times M}$, $\Omega_1(\mathcal{G}_2) \in \mathcal{R}^{1\times N}$, $\Omega_2(\mathcal{G}_2) \in \mathcal{R}^{(M-1)\times N}$.

3. Main Results

In this section, group consensus control protocols (7) in NMAS (1) with network topologies satisfying Equations (3) and (8) has been investigated. Necessary and/or sufficient conditions for reaching group consensus in the case of time-varying communication delays are given.

Theorem 1 When ***Assumptions*** (1)–(4) stand, if there exists the time-varying delay (2) in the networked multi-agent system (1) with two sub-networks

$(\mathcal{G}_1, \chi_1)$ and $(\mathcal{G}_2, \chi_2)$, the control protocol (7) solves the admissible group consensus problem if and only if the matrices $\varUpsilon$ and $\boldsymbol{A} - \boldsymbol{LC}$ are Schur stable, i.e. their eigenvalues lie in the unit circle, where

$$\begin{aligned}
\varUpsilon &= \left(\begin{array}{c:c} \Lambda_1 & \Pi_1 \\ \hdashline \Pi_2 & \Lambda_2 \end{array}\right), \\
\Lambda_1 &= I_{N-1} \otimes A - \big(\mathcal{L}_{22}(\mathcal{G}_1) - \boldsymbol{1}_{N-1}\mathcal{L}_{12}(\mathcal{G}_1)\big) \otimes \big(BK\big), \\
\Pi_1 &= -\big(\Omega_{22}(\mathcal{G}_1) - \boldsymbol{1}_{N-1}\Omega_{12}(\mathcal{G}_1)\big) \otimes \big(\boldsymbol{B}K\big), \\
\Pi_2 &= -\big(\Omega_{22}(\mathcal{G}_2) - \boldsymbol{1}_{N-1}\Omega_{12}(\mathcal{G}_2)\big) \otimes \big(BK\big), \\
\Lambda_2 &= I_{M-1} \otimes A - \big(\mathcal{L}_{22}(\mathcal{G}_2) - \boldsymbol{1}_{N-1}\mathcal{L}_{12}(\mathcal{G}_2)\big) \otimes \big(BK\big).
\end{aligned}$$

Proof: By the state observer equation (5), it is easy to get the following state error equation:

$$\varepsilon_i(t+1) = (\boldsymbol{A} - \boldsymbol{LC})\varepsilon_i(t). \tag{11}$$

At the same time, using the state estimation equation (6) recursively leads to

$$\begin{aligned}
\hat{\boldsymbol{x}}_i(t|t-\tau) &= \hat{\boldsymbol{x}}_i(t|t-1) - \sum_{s=0}^{\tau-2} \boldsymbol{A}^s \boldsymbol{LC}\varepsilon_{t-s-1} \\
&= \boldsymbol{x}_i(t) - \varepsilon_i(t) - \sum_{s=0}^{\tau-2} \boldsymbol{A}^s \boldsymbol{LC}\varepsilon_{t-s-1} \\
&= \boldsymbol{x}_i(t) - \Theta(\tau)\boldsymbol{E}_i(t),
\end{aligned} \tag{12}$$

where $\Theta(\tau) = \big[\boldsymbol{I}\ \boldsymbol{LC}\ \boldsymbol{ALC}\ \cdots\ \boldsymbol{A}^{\tau-2}\boldsymbol{LC}\big] \in \mathcal{R}^{n\times n\tau}$, $\boldsymbol{E}_i(t) = \big[{\varepsilon_i}^T(t)\ {\varepsilon_i}^T(t-1)\ {\varepsilon_i}^T(t-2)\ \cdots\ {\varepsilon_i}^T(t-\tau+1)\big]^T \in \mathcal{R}^{n\tau\times 1}$.

Let $\hat{\zeta}_i(t|t-\tau) = \hat{\boldsymbol{x}}_i(t|t-\tau) - \hat{\boldsymbol{x}}_1(t|t-\tau)$, $\forall\ i \in \ell_1$, $\hat{\eta}_i(t|t-\tau) = \hat{\boldsymbol{x}}_i(t|t-\tau) - \hat{\boldsymbol{x}}_{N+1}(t|t-\tau)$, $\forall\ i \in \ell_2$. Based on the above equation, it can be obtained that

$$\begin{aligned}
\hat{\zeta}_i(t|t-\tau) &= \zeta_i(t) - \Theta(\tau)\boldsymbol{E}_i(t) + \Theta(\tau)\boldsymbol{E}_1(t), \quad \forall\ i \in \ell_1, \\
\hat{\eta}_i(t|t-\tau) &= \eta_i(t) - \Theta(\tau)\boldsymbol{E}_i(t) + \Theta(\tau)\boldsymbol{E}_{N+1}(t), \quad \forall\ i \in \ell_2
\end{aligned} \tag{13}$$

Combining with Equations (7) and (13) results in

$$
\begin{aligned}
& \zeta_i(t+1) \\
= \; & \boldsymbol{A}\zeta_i(t) + \boldsymbol{B}\big(\boldsymbol{u}_i(t) - \boldsymbol{u}_1(t)\big) \\
= \; & \boldsymbol{A}\zeta_i(t) + \boldsymbol{B}\boldsymbol{K}\Bigg(\sum_{v_j \in \mathcal{N}_{1i}} a_{ij}\big(\hat{\boldsymbol{x}}_j(t|t-\tau) - \hat{\boldsymbol{x}}_i(t|t-\tau)\big) \\
& + \sum_{v_j \in \mathcal{N}_{2i}} a_{ij}\hat{\boldsymbol{x}}_j(t|t-\tau) \\
& - \sum_{v_j \in \mathcal{N}_{11}} a_{1j}\big(\hat{\boldsymbol{x}}_j(t|t-\tau) - \hat{\boldsymbol{x}}_1(t|t-\tau)\big) \\
& - \sum_{v_j \in \mathcal{N}_{21}} a_{1j}\hat{\boldsymbol{x}}_j(t|t-\tau)\Bigg) \\
= \; & \boldsymbol{A}\zeta_i(t) + \boldsymbol{B}\boldsymbol{K}\Bigg(\sum_{v_j \in \mathcal{N}_{1i}} a_{ij}\big(\hat{\zeta}_j(t|t-\tau) - \hat{\zeta}_i(t|t-\tau)\big) \\
& + \sum_{v_j \in \mathcal{N}_{2i}} a_{ij}\big(\hat{\eta}_j(t|t-\tau) + \hat{\boldsymbol{x}}_{N+1}(t|t-\tau)\big) - \sum_{v_j \in \mathcal{N}_{11}} a_{1j}\hat{\zeta}_j(t|t-\tau) \\
& - \sum_{v_j \in \mathcal{N}_{21}} a_{1j}\big(\hat{\eta}_j(t|t-\tau) + \hat{\boldsymbol{x}}_{N+1}(t|t-\tau)\big)\Bigg), \forall \; i \in \ell_1
\end{aligned}
\tag{14}
$$

According to ***Assumption 4***, it is easily to be obtained

$$
\begin{aligned}
\sum_{v_j \in \mathcal{N}_{2i}} a_{ij}\hat{\boldsymbol{x}}_{N+1}(t|t-\tau) &= \alpha\hat{\boldsymbol{x}}_{N+1}(t|t-\tau), \\
\sum_{v_j \in \mathcal{N}_{21}} a_{1j}\hat{\boldsymbol{x}}_{N+1}(t|t-\tau) &= \alpha\hat{\boldsymbol{x}}_{N+1}(t|t-\tau).
\end{aligned}
$$

Substituting Equation (13) in Equation (14) drives

$$\begin{aligned}
\zeta_i(t+1) &= \boldsymbol{A}\zeta_i(t) + \boldsymbol{BK}\bigg(\sum_{v_j \in \mathcal{N}_{1i}} a_{ij}\big(\zeta_j(t) - \mathit{\Theta}(\tau)\boldsymbol{E}_j(t) - \zeta_i(t) + \mathit{\Theta}(\tau)\boldsymbol{E}_i(t)\big) \\
&\quad + \sum_{v_j \in \mathcal{N}_{2i}} a_{ij}\big(\eta_j(t) - \mathit{\Theta}(\tau)\boldsymbol{E}_j(t) + \mathit{\Theta}(\tau)\boldsymbol{E}_{N+1}(t)\big) \\
&\quad - \sum_{v_j \in \mathcal{N}_{11}} a_{1j}\big(\zeta_j(t) - \mathit{\Theta}(\tau)\boldsymbol{E}_j(t) + \mathit{\Theta}(\tau)\boldsymbol{E}_1(t)\big) \\
&\quad - \sum_{v_j \in \mathcal{N}_{21}} a_{1j}\big(\eta_j(t) - \mathit{\Theta}(\tau)\boldsymbol{E}_j(t) + \mathit{\Theta}(\tau)\boldsymbol{E}_{N+1}(t)\big)\bigg) \\
&= \boldsymbol{A}\zeta_i(t) + \boldsymbol{BK}\bigg(\sum_{j=1}^{N} a_{ij}\big(\zeta_j(t) - \zeta_i(t)\big) \\
&\quad - \sum_{j=1}^{N} a_{1j}\zeta_j(t) + \sum_{j=N+1}^{N+M} (a_{ij} - a_{1j})\eta_j(t)\bigg) \\
&\quad + \boldsymbol{BK}\mathit{\Theta}(\tau)\bigg(\sum_{j=1}^{N} a_{ij}\big(\boldsymbol{E}_i(t) - \boldsymbol{E}_j(t)\big) + \sum_{j=1}^{N} a_{1j}\big(\boldsymbol{E}_j(t) - \boldsymbol{E}_1(t)\big) \\
&\quad - \sum_{j=N+1}^{N+M} \big(a_{ij} - a_{1j}\big)\boldsymbol{E}_j(t)\bigg), \forall\ i \in \ell_1. \qquad (15)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\eta_i(t+1) &= \boldsymbol{A}\eta_i(t) + \boldsymbol{B}\boldsymbol{K}\Big(\sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\big(\eta_j(t) - \Theta(\tau)\boldsymbol{E}_j(t) - \eta_i(t) \\
&\quad + \Theta(\tau)\boldsymbol{E}_i(t)\big) + \sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\big(\zeta_j(t) - \Theta(\tau)\boldsymbol{E}_j(t) + \Theta(\tau)\boldsymbol{E}_1(t)\big) \\
&\quad - \sum_{v_j\in\mathcal{N}_{2(N+1)}} a_{(N+1)j}\big(\eta_j(t) - \Theta(\tau)\boldsymbol{E}_j(t) + \Theta(\tau)\boldsymbol{E}_{N+1}(t)\big) \\
&\quad - \sum_{v_j\in\mathcal{N}_{1(N+1)}} a_{(N+1)j}\big(\zeta_j(t) - \Theta(\tau)\boldsymbol{E}_j(t) + \Theta(\tau)\boldsymbol{E}_1(t)\big)\Big) \\
&= \boldsymbol{A}\eta_i(t) + \boldsymbol{B}\boldsymbol{K}\Big(\sum_{j=N+1}^{N+M} a_{ij}\big(\eta_j(t) - \eta_i(t)\big) - \sum_{j=N+1}^{N+M} a_{(N+1)j}\eta_j(t) \\
&\quad + \sum_{j=1}^{N}(a_{ij} - a_{(N+1)j})\zeta_j(t)\Big) \\
&\quad + \boldsymbol{B}\boldsymbol{K}\Theta(\tau)\Big(\sum_{j=N+1}^{N+M} a_{ij}\big(\boldsymbol{E}_i(t) - \boldsymbol{E}_j(t)\big) \\
&\quad + \sum_{j=N+1}^{N+M} a_{(N+1)j}\big(\boldsymbol{E}_j(t) - \boldsymbol{E}_{N+1}(t)\big) \\
&\quad - \sum_{j=1}^{N}(a_{ij} - a_{(N+1)j})\boldsymbol{E}_j(t)\Big),\ \forall\ i \in \ell_2. \qquad (16)
\end{aligned}$$

Let

$$\boldsymbol{E}(t) = \Big[\ \boldsymbol{E}_1^T(t)\ \cdots\ \boldsymbol{E}_N^T(t)\ \boldsymbol{E}_{N+1}^T(t)\ \cdots\ \boldsymbol{E}_{N+M}^T(t)\ \Big]^T,$$

In the light of Equations (11), (15) and (16), one obtains a compact representation with intragroup tracking error vectors and estimation error vectors as

$$\begin{pmatrix} \zeta(t+1) \\ \hdashline \eta(t+1) \\ \hdashline \boldsymbol{E}(t+1) \end{pmatrix} = \left(\begin{array}{c:c:c} \Lambda_1 & \Pi_1 & \Gamma_1 \\ \hdashline \Pi_2 & \Lambda_2 & \Gamma_2 \\ \hdashline 0 & 0 & \boldsymbol{I}_{n(N+M)\tau} \otimes (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}) \end{array}\right) \begin{pmatrix} \zeta(t) \\ \hdashline \eta(t) \\ \hdashline \boldsymbol{E}(t) \end{pmatrix}, \quad (17)$$

where

$$\begin{aligned}
\Gamma_1 &= \Big[\mathcal{L}_2(\mathcal{G}_1) - \boldsymbol{1}_{N-1} \otimes \mathcal{L}_1(\mathcal{G}_1) \quad \Omega_2(\mathcal{G}_1) - \boldsymbol{1}_{N-1} \otimes \Omega_1(\mathcal{G}_1)\Big] \otimes \big(\boldsymbol{B}\boldsymbol{K}\Theta(\tau)\big), \\
\Gamma_2 &= \Big[\Omega_2(\mathcal{G}_2) - \boldsymbol{1}_{M-1} \otimes \Omega_1(\mathcal{G}_2) \quad \mathcal{L}_2(\mathcal{G}_2) - \boldsymbol{1}_{M-1} \otimes \mathcal{L}_1(\mathcal{G}_2)\Big] \otimes \big(\boldsymbol{B}\boldsymbol{K}\Theta(\tau)\big).
\end{aligned}$$

Clearly Equation (17) describes an upper block triangular matrix. It is well-known that a block triangular matrix is Schur stable if and only if its submatrices on the diagonal line are Schur stable. Based on ***Definition 2***, the control protocol given by (7) can solve the admissible group consensus problem for the networked multi-agent system (1) if and only if an upper system (17) is Schur stable which implies that the matrices $\varUpsilon$ and $\boldsymbol{A}-\boldsymbol{LC}$ are required to have their eigenvalues with magnitude less than one. The proof is completed.

Remark 2 It is easy to see from ***Theorem 1*** that the state-estimation-based group consensus protocol (7) provides an active time-delay compensation of NMAS (1) in a complex network with multiple subnetworks. In the design of the group consensus control protocol (7), the stability of the augmented error system (17) is only determined by the matrices $\varUpsilon$ and $\boldsymbol{A}-\boldsymbol{LC}$, which simplifies the controller design process of each agent without taking into account time delay. This brings great flexibility and efficiency for theoretical design and engineering implementation.

Let

$$\boldsymbol{u}(t)=\left[\ \boldsymbol{u}_1^T(t)\ \boldsymbol{u}_2^T(t)\ \cdots\ \boldsymbol{u}_{N+M}^T(t)\ \right]^T. \tag{18}$$

Corollary 1 When ***Assumptions*** (**1**)–(**4**) stand, for NMAS (1) with two sub-networks $(\mathcal{G}_1,\chi_1)$ and $(\mathcal{G}_2,\chi_2)$ and the time-varying network delay (2), the following conclusions are equivalent:

(i) The control protocol (7) solves the group consensus problem;

(ii) Suppose an arbitrarily chosen pair of matrices $\boldsymbol{U}_{12}\in\mathcal{R}^{N\times(N-1)}$ and $\boldsymbol{U}_{22}\in\mathcal{R}^{M\times(M-1)}$, satisfying that $\boldsymbol{U}_1=\left[\frac{1}{\sqrt{N}}\boldsymbol{1}_N\ \ \boldsymbol{U}_{12}\right]$ and $\boldsymbol{U}_2=\left[\frac{1}{\sqrt{M}}\boldsymbol{1}_M\ \ \boldsymbol{U}_{22}\right]$ are orthogonal matrices, and the two matrices $\varPsi$ and $\boldsymbol{A}-\boldsymbol{LC}$ are Schur stable, where

$$\widehat{\boldsymbol{U}}=\left(\begin{array}{c:c}\boldsymbol{U}_{12} & 0\\ \hdashline 0 & \boldsymbol{U}_{22}\end{array}\right),\ \varPhi=\widehat{\boldsymbol{U}}^T\boldsymbol{L}(\mathcal{G})\widehat{\boldsymbol{U}}\ \mathit{and}\ \varPsi=\boldsymbol{I}_{N+M-2}\otimes\boldsymbol{A}-\varPhi\otimes(\boldsymbol{BK}).$$

Proof: Let

$$S = \left(\begin{array}{c:c} S_1 & 0 \\ \hdashline 0 & S_2 \end{array}\right), \; S_1 = \begin{bmatrix} 1_{N-1} & -I_{N-1} \end{bmatrix} \; and \; S_2 = \begin{bmatrix} 1_{M-1} & -I_{M-1} \end{bmatrix}.$$

It is easy to be obtained that

$$\Upsilon = I_{N+M-2} \otimes A - \big(S\mathcal{L}(\mathcal{G})S^T(SS^T)^{-1}\big) \otimes BK. \tag{19}$$

Define

$$U = \left(\begin{array}{c:c} U_1 & 0 \\ \hdashline 0 & U_2 \end{array}\right), \quad T = \left(\begin{array}{c:c} S_1U_{12} & 0 \\ \hdashline 0 & S_2U_{22} \end{array}\right),$$

Because S_1 and S_2 have full row rank, it is easy to be deduced that S_1U_{12} and S_2U_{22} are invertible. Also,

$$\begin{aligned} & S\mathcal{L}(\mathcal{G})S^T(SS^T)^{-1} \\ = \;& (SU)\Big(U^T\mathcal{L}(\mathcal{G})U\Big)(SU)^T\Big((SU)(SU)^T\Big)^{-1} \\ = \;& (SU)\Big(U^T\mathcal{L}(\mathcal{G})U\Big)(SU)^T\Big((SU)(SU)^T\Big)^{-1} \\ = \;& (SU)\Big(U^T\mathcal{L}(\mathcal{G})U\Big)(SU)^T\Big((SU)(SU)^T\Big)^{-1} \\ = \;& \left(\begin{array}{c:c} (S_1U_{12})U_{12}^T\mathcal{L}(\mathcal{G}_1)U_{12}(S_1U_{12})^{-1} & (S_1U_{12})U_{12}^T\Omega(\mathcal{G}_1)U_{22}(S_2U_{22})^{-1} \\ \hdashline (S_2U_{22})U_{22}^T\Omega(\mathcal{G}_1)U_{12}(S_1U_{12})^{-1} & (S_2U_{22})U_{22}^T\mathcal{L}(\mathcal{G}_2)U_{22}(S_2U_{22})^{-1} \end{array}\right) \\ = \;& T\Phi T^{-1}. \end{aligned} \tag{20}$$

From the above equation, it follows that

$$\begin{aligned} & (T \otimes I_n)^{-1}\Upsilon(T \otimes I_n) \\ = \;& (T \otimes I_n)^{-1}\Big(I_{N+M-2} \otimes A - \big(S\mathcal{L}(\mathcal{G})S^T(SS^T)^{-1}\big) \otimes BK\Big)(T \otimes I_n) \\ = \;& I_{N+M-2} \otimes A - \Big(T^{-1}\big(S\mathcal{L}(\mathcal{G})S^T(SS^T)^{-1}\big)T\Big) \otimes \big(BK\big) \\ = \;& I_{N+M-2} \otimes A - \Phi \otimes (BK) \\ = \;& \Psi. \end{aligned} \tag{21}$$

It means that Υ is similar to Ψ. Since similar matrices have the same eigenvalues, Υ is Schur stable if and only if Ψ is Schur stable. Based on ***Theorem 1***, it is obvious that NMAS (1) subjected to control protocol (7) can achieve group consensus asymptotically if and only if Ψ and $A - LC$ are Schur stable. This completes the proof.

Corollary 2 When ***Assumptions*** (**1**)–(**4**) stand, a NMAS (1) with two sub-networks $(\mathcal{G}_1, \chi_1)$ and $(\mathcal{G}_2, \chi_2)$ is considered. If $\boldsymbol{A}$ is unstable, the necessary condition of group consensusability w.r.t. the admissible control set $\mathcal{U}$ is that $\mathcal{L}(\mathcal{G})$ has the zero eigenvalue with algebraic multiplicity 2, furthermore,

(i) If $\Omega_1 = 0$ or $\Omega_2 = 0$, then both of $(\mathcal{G}_1, \chi_1)$ and $(\mathcal{G}_2, \chi_2)$ contain a spanning tree;

(ii) If $\boldsymbol{A}$ is nonsingular, then $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable.

Proof: Denote

$$
\begin{aligned}
P &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1_{N-1} & 0 & \boldsymbol{I}_{N-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \boldsymbol{1}_{M-1} & 0 & \boldsymbol{I}_{M-1} \end{pmatrix}, \\
\boldsymbol{Q} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_{N-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_{M-1} \end{pmatrix}, \\
\Gamma &= \begin{pmatrix} 0 & \mathcal{L}_{12}(\mathcal{G}_1) & 0 & \Omega_{12}(\mathcal{G}_1) \\ 0 & \mathcal{L}_{22}(\mathcal{G}_1) - \boldsymbol{1}_{N-1}\mathcal{L}_{12}(\mathcal{G}_1) & 0 & \Omega_{22}(\mathcal{G}_1) - \boldsymbol{1}_{N-1}\Omega_{12}(\mathcal{G}_1) \\ 0 & \Omega_{12}(\mathcal{G}_2) & 0 & \mathcal{L}_{12}(\mathcal{G}_2) \\ 0 & \Omega_{22}(\mathcal{G}_2) - \boldsymbol{1}_{M-1}\Omega_{12}(\mathcal{G}_2) & 0 & \mathcal{L}_{22}(\mathcal{G}_2) - \boldsymbol{1}_{N-1}\mathcal{L}_{12}(\mathcal{G}_2) \end{pmatrix}.
\end{aligned}
$$

By computation and comparison, an equivalence relation is constructed as

$$
\begin{aligned}
\boldsymbol{P}^{-1}\mathcal{L}(\mathcal{G})\boldsymbol{P} &= \left(\begin{array}{c:c} 0_{2\times 2} & \boldsymbol{F} \\ \hdashline 0 & \boldsymbol{S}\mathcal{L}(\mathcal{G})\boldsymbol{S}^T(\boldsymbol{S}\boldsymbol{S}^T)^{-1} \end{array}\right) \\
&= \boldsymbol{Q}^{-1}\Gamma\boldsymbol{Q}, \qquad (22)
\end{aligned}
$$

where

$$\boldsymbol{F} = \begin{pmatrix} \mathcal{L}_{12}(\mathcal{G}_1) & \Omega_{12}(\mathcal{G}_1) \\ \Omega_{12}(\mathcal{G}_2) & \mathcal{L}_{12}(\mathcal{G}_2) \end{pmatrix}.$$

Let the eigenvalues of $\mathcal{L}(\mathcal{G})$ be $\lambda_1 = \lambda_2 = 0,\ \lambda_3, \cdots,\ \lambda_{N+M}$. From the Equation (22), it is easy to verify that the eigenvalues of the matrix $\boldsymbol{S}\mathcal{L}(\mathcal{G})\boldsymbol{S}^T(\boldsymbol{S}\boldsymbol{S}^T)^{-1}$ are $\lambda_3, \cdots, \lambda_{N+M}$. Thus, there exists a transformation matrix $\boldsymbol{T}_1 \in \mathcal{R}^{(N+M-2)\times(N+M-2)}$ such that

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{T}_1^{-1}\big(\boldsymbol{S}\mathcal{L}(\mathcal{G})\boldsymbol{S}^T(\boldsymbol{S}\boldsymbol{S}^T)^{-1}\big)\boldsymbol{T}_1 \\ &= diag\big(\boldsymbol{J}_3(\lambda_3), \boldsymbol{J}_4(\lambda_4), \cdots, \boldsymbol{J}_s(\lambda_s)\big), \end{aligned}$$

where $\boldsymbol{J}_i(\lambda_i)$ is the upper triangular Jordan block corresponding to λ_i, $i = 3, 4, \cdots, N+M$.

Therefore

$$(\boldsymbol{T}_1 \otimes \boldsymbol{I}_n)^{-1}\Upsilon(\boldsymbol{T}_1 \otimes \boldsymbol{I}_n) = \boldsymbol{I}_{N+M-2} \otimes \boldsymbol{A} - \boldsymbol{J} \otimes \boldsymbol{B}\boldsymbol{K}, \tag{23}$$

which implies the eigenvalues of Υ are given by the eigenvalues of $\boldsymbol{A} - \lambda_i\boldsymbol{B}\boldsymbol{K}$, $i = 3, 4, \cdots, N+M$. Hence, all the eigenvalues of $\boldsymbol{A} - \lambda_i\boldsymbol{B}\boldsymbol{K}$ lie in the unit circle. Following the condition that $\boldsymbol{A}$ is unstable, it is easy to get $\lambda_i \neq 0$, $i = 3, 4, \cdots, N+M$, the zero eigenvalue of $\mathcal{L}(\mathcal{G})$ has algebraic multiplicity of two.

If $\Omega_1 = 0$ or $\Omega_2 = 0$, then $\sigma\big(\mathcal{L}(\mathcal{G})\big) = \sigma\big(\mathcal{L}(\mathcal{G}_1)\big) \bigcup \sigma\big(\mathcal{L}(\mathcal{G}_2)\big)$. Since $\mathcal{L}(\mathcal{G}_1)\boldsymbol{1}_N = 0$ and $\mathcal{L}(\mathcal{G}_2)\boldsymbol{1}_M = 0$, the eigenvalue of $\mathcal{L}(\mathcal{G})$ has algebraic multiplicity equal to 2. To be sure for $\mathcal{L}(\mathcal{G}_1)$ and $\mathcal{L}(\mathcal{G}_2)$, each of them has exactly one zero eigenvalue. From the Equation (8), $\boldsymbol{A}(\mathcal{G}_1)$ and $\boldsymbol{A}(\mathcal{G}_2)$ are nonnegative matrices. Based on Lemma 3.3 in [36], $\mathcal{G}_1$ and $\mathcal{G}_2$ contain a spanning tree, respectively.

In the following sections, the above conclusion **(ii)** will be proved.

The proving process is similar to Theorem 4 in [37]. For all $i = 3, 4, \cdots, N+M$, if there exists a real number denoted by λ_3, all the eigenvalues of $\boldsymbol{A} - \lambda_3\boldsymbol{B}\boldsymbol{K}$ lie in the unit circle. It is obvious that $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable. If λ_i ($i \in \{3, 4, \cdots, N+M\}$) is an imaginary number, i.e. the imaginary part is nonzero, all the eigenvalues of the real matrix $\mathcal{L}(\mathcal{G})$ appear in the form of conjugate pairs. Without loss of generality, let $\lambda_3 = e + jd$ and $\lambda_4 = e - jd$ be a pair of conjugate eigenvalues. Noticing, $\forall \lambda \in \mathcal{C}$,

$$\begin{vmatrix} \lambda I_n - (A - eBK) & -dBK \\ dBK & \lambda I_n - (A - eBK) \end{vmatrix}$$
$$= |\lambda I_n - (A - \lambda_3 BK)| \cdot |\lambda I_n - (A - \lambda_4 BK)|,$$

Since the eigenvalues of $A - \lambda_3 BK$ and $A - \lambda_4 BK$ are all in the unit circle, the eigenvalues of $\begin{pmatrix} A - eBK & dBK \\ -dBK & A - eBK \end{pmatrix}$ lie in the unit circle, further organization is

$$\begin{pmatrix} A - eBK & dBK \\ -dBK & A - eBK \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} -eK & dK \\ -dK & -eK \end{pmatrix},$$

which implies that $\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \right)$ is stabilizable. Meanwhile,

$$Rank \begin{pmatrix} sI_n - A & 0 & B & 0 \\ 0 & sI_n - A & 0 & B \end{pmatrix} = 2n, \ \forall s \in \mathcal{C}, \ |s| \geq 1,$$

is equivalent to $Rank(sI_n - A \ B) = n, \ \forall s \in \mathcal{C}, \ |s| \geq 1$. Therefore, $(A, \ B)$ is stabilizable. This completes the proof.

4. Simulated Example

In this section, some numerical simulated examples are presented to describe the effectiveness of the above theoretical results.

Consider a NMAS (1) composed of four agents, where $N = 2$, $M = 2$, $\ell_1 = \{1, 2\}$, $\ell_2 = \{3, 4\}$. The communication delay is shown in Fig. 1, which implies that $\alpha = -1$, $\beta = -1$. The adjacency matrix $\mathcal{A}(\mathcal{G})$ of the network $(\mathcal{G}, \boldsymbol{x})$

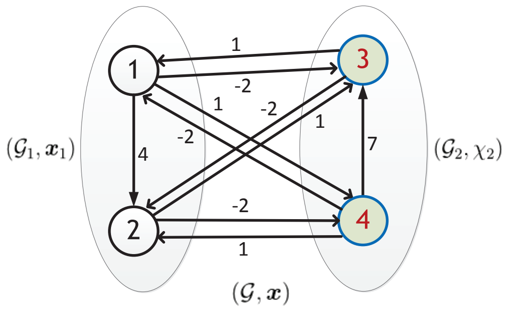


Figure 1: The Communication Topology of NMAS.

is as follows:

$$\mathcal{A}(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 1 & -2 \\ 4 & 0 & -2 & 1 \\ -2 & 1 & 0 & 7 \\ 1 & -2 & 0 & 0 \end{bmatrix}.$$

The Laplacian matrices of the sub-network $(\mathcal{G}_1, \chi_1)$ and $(\mathcal{G}_2, \chi_2)$ are

$$\mathcal{L}(\mathcal{G}_1) = \begin{bmatrix} 0 & 0 \\ -4 & 4 \end{bmatrix}, \qquad \mathcal{L}(\mathcal{G}_2) = \begin{bmatrix} 7 & -7 \\ 0 & 0 \end{bmatrix},$$

$$\Omega(\mathcal{G}_1) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \qquad \Omega(\mathcal{G}_2) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The dynamics of the ith agent is assumed to be:

$$\begin{aligned} \boldsymbol{x}_i(t+1) &= \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{B}\boldsymbol{u}_i(t) \\ \boldsymbol{y}_i(t) &= \boldsymbol{C}\boldsymbol{x}_i(t), \qquad i = 1, \dots, 4, \end{aligned}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0.8869 & 0 \\ 0 & 0.9418 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0.0377 \\ 0.0388 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The initial state of the system (1) is $\boldsymbol{x}_1(0) = [\ 12\ \ 10\ ]^T$, $\boldsymbol{x}_2(0) = [\ -5\ \ -1.4\ ]^T$, $\boldsymbol{x}_3(0) = [\ 15\ \ -3\ ]^T$, $\boldsymbol{x}_4(0) = [\ 8\ \ -15\ ]^T$. The initial state of the observer

(5) is $\hat{\boldsymbol{x}}_1(0) = [\ 0\ 0\ ]^T,\ \hat{\boldsymbol{x}}_2(0) = [\ 0\ 0\ ]^T,\ \hat{\boldsymbol{x}}_3(0) = [\ 0\ 0\ ]^T,\ \hat{\boldsymbol{x}}_4(0) = [\ 0\ 0\ ]^T$. Using the pole-placement technique, the observer gain matrix $\boldsymbol{L}$ is determined as

$$\boldsymbol{L} = \begin{bmatrix} -2.5412 \\ 3.0899 \end{bmatrix},$$

where the desired poles of the observer are set to be $[\ 0.64 + 0.28i\ 0.64 - 0.28i\ ]$, Choose the control gain matrix $\boldsymbol{K}$ as

$$\boldsymbol{K} = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

and calculate Υ as follows:

$$\Upsilon = \begin{bmatrix} 0.7361 & -0.1508 & 0.1131 & 0.1131 \\ -0.1553 & 0.7865 & 0.1165 & 0.1165 \\ -0.1131 & -0.1131 & 0.6231 & -0.2639 \\ -0.1165 & -0.1165 & -0.2718 & 0.6700 \end{bmatrix}.$$

All the eigenvalues of Υ lie in the unit circle, i.e. $\lambda_1 = 0.4924 + 0.1981i$, $\lambda_2 = 0.4924 - 0.1981i$, $\lambda_3 = 0.9154 + 0.0007i$, $\lambda_4 = 0.9154 - 0.0007i$, respectively. The above analysis shows that the NMAS (1) can satisfy time-delay group consensus conditions.

In order to describe the effect of network delay on group consensus of the NMAS and verify the performance of the proposed group consensus protocol, two simulation cases are presented as follows:

(1) Group consensus control for the NMASs (1) without time delay compensation. In this simulation case, the network parameters are set to be $\tau = 5$, that is, there exists five-step communication delays between NMASs (1). By using delayed information state observer directly, the group consensus controller is designed as

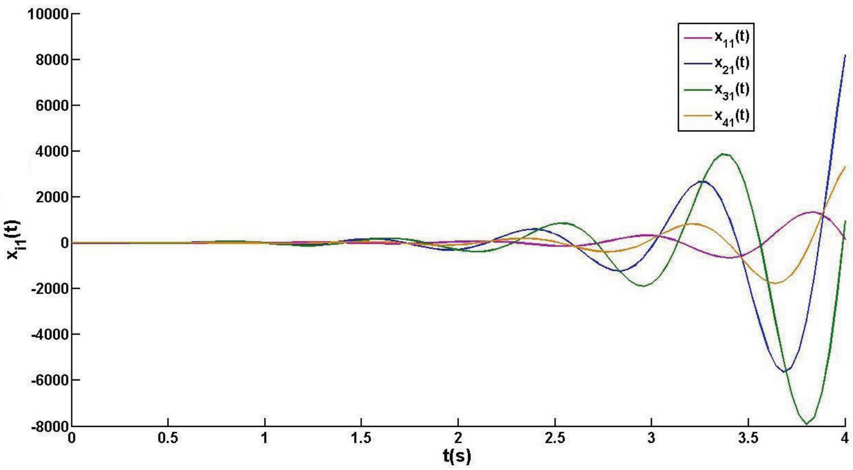
$$
\boldsymbol{u}_i(t)=\begin{cases}
\boldsymbol{K}\Bigg(\displaystyle\sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\big(\hat{\boldsymbol{x}}_j(t-5|t-6)-\hat{\boldsymbol{x}}_i(t-5|t-6)\big) \\
\qquad +\displaystyle\sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\hat{\boldsymbol{x}}_j(t-5|t-6)\Bigg), \forall\ i\ \in \ell_1, \\
\boldsymbol{K}\Bigg(\displaystyle\sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\big(\hat{\boldsymbol{x}}_j(t-5|t-6)-\hat{\boldsymbol{x}}_i(t-5|t-6)\big) \\
\qquad +\displaystyle\sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\hat{\boldsymbol{x}}_j(t-5|t-6)\Bigg), \forall\ i\ \in \ell_2.
\end{cases}
$$

The experimental results from Figure 2 show that state vectors $\boldsymbol{x}_{i1}(t)$ and $\boldsymbol{x}_{i2}(t)$ tend to diverge, that is, the NMAS can not reach group consensus.

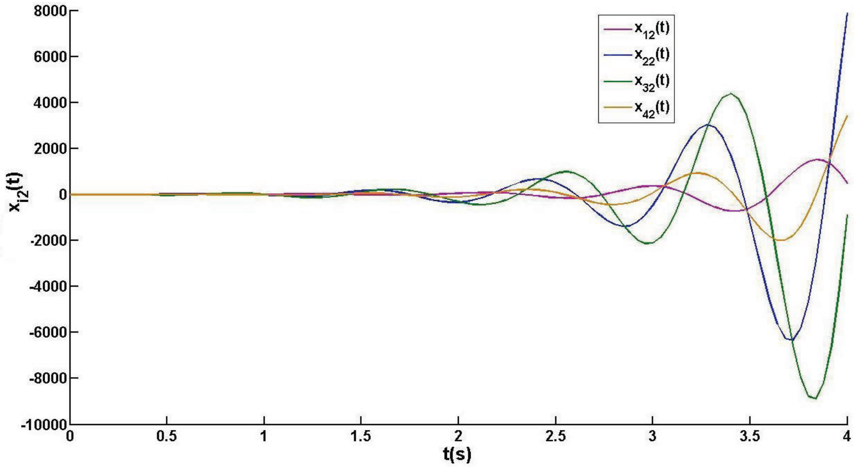
(2) Group consensus control of the NMAS using time delay compensation scheme. In this simulation, the parameter settings of network characteristics are the same to the case (1). To overcome the time delay, the group consensus protocol (7) based on time delay compensation scheme is adopted. Figure 3 shows the state trajectory curve of networked multi-agent systems, which demonstrates that the NMAS can achieve group consensus asymptotically on the basis of time delay compensation.

5. Conclusion

This paper has addressed group consensus problems in networked multi-agent systems (NMAS) with communication delays. Based on the proposed time delay scheme, the group consensus control protocol is designed to compensate the communication delay actively. In light of algebraic graph theories and matrix theories, necessary and(or) sufficient conditions of group consensus with respect to a given admissible control set are obtained for the NMAS with communication delays under mild assumptions. Finally, simulations are worked out to demonstrate the efficacy of the theoretical results.

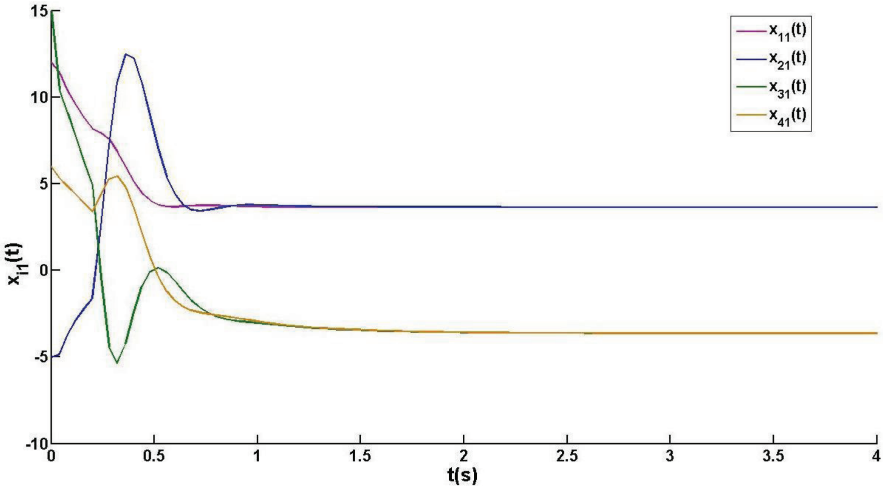


(a) $\boldsymbol{x}_{i1}(t)$

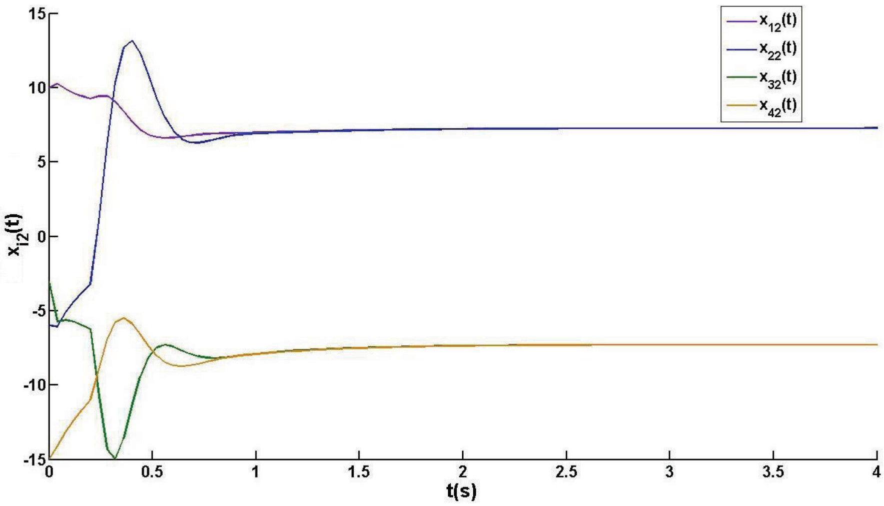


(b) $\boldsymbol{x}_{i2}(t)$

Figure 2: state trajectory



(a) $x_{i1}(t)$



(b) $x_{i2}(t)$

Figure 3: state trajectory

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