European Journal of Operational Research 237 (2014) 898–908

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Production, Manufacturing and Logistics

Optimal credit period and lot size for deteriorating items with expiration dates under two-level trade credit financing

Jiang Wu [a], Liang-Yuh Ouyang [b,*], Leopoldo Eduardo Cárdenas-Barrón [c], Suresh Kumar Goyal [d]

[a] School of Statistics, Southwestern University of Finance & Economics, Chengdu 611130, China
[b] Department of Management Sciences, Tamkang University, Tamshui Dist., New Taipei City, Taiwan, ROC
[c] Department of Industrial and Systems Engineering, School of Engineering, Tecnológico de Monterrey, E. Garza Sada 2501 Sur, C.P. 64849, Monterrey, NL, Mexico
[d] Department of Supply Chain and Business Technology Management, Concordia University, Montreal, Quebec H3G 1M8, Canada

ARTICLE INFO

Article history:
Received 11 September 2013
Accepted 6 March 2014
Available online 15 March 2014

Keywords:
Supply chain management
Deteriorating items
Expiration dates
Trade credit
Inventory

ABSTRACT

In a supplier-retailer-buyer supply chain, the supplier frequently offers the retailer a trade credit of *S* periods, and the retailer in turn provides a trade credit of *R* periods to her/his buyer to stimulate sales and reduce inventory. From the seller's perspective, granting trade credit increases sales and revenue but also increases opportunity cost (i.e., the capital opportunity loss during credit period) and default risk (i.e., the percentage that the buyer will not be able to pay off her/his debt obligations). Hence, how to determine credit period is increasingly recognized as an important strategy to increase seller's profitability. Also, many products such as fruits, vegetables, high-tech products, pharmaceuticals, and volatile liquids not only deteriorate continuously due to evaporation, obsolescence and spoilage but also have their expiration dates. However, only a few researchers take the expiration date of a deteriorating item into consideration. This paper proposes an economic order quantity model for the retailer where: (a) the supplier provides an up-stream trade credit and the retailer also offers a down-stream trade credit, (b) the retailer's down-stream trade credit to the buyer not only increases sales and revenue but also opportunity cost and default risk, and (c) deteriorating items not only deteriorate continuously but also have their expiration dates. We then show that the retailer's optimal credit period and cycle time not only exist but also are unique. Furthermore, we discuss several special cases including for non-deteriorating items. Finally, we run some numerical examples to illustrate the problem and provide managerial insights.

Introduction

In practice, the seller usually provides to her/his buyer a permissible delay in payments to stimulate sales and reduce inventory. During the credit period, the buyer can accumulate the revenue and earn interest on the accumulative revenue. However, if the buyer cannot pay off the purchase amount during the credit period then the seller charges to the buyer interest on the unpaid balance. One of the first works along this line of research is Goyal (1985). He established the retailer's optimal economic order quantity (EOQ) when the supplier offers a permissible delay in payments. On the other hand, Shah (1993) then considered a stochastic inventory model for deteriorating items when delays in payments are permissible. Later, Aggarwal and Jaggi (1995) extended the EOQ model from non-deteriorating items to deteriorating items. Jamal, Sarker, and Wang (1997) further generalized the EOQ model with trade credit financing to allow shortages. After, Teng (2002) provided an easy analytical closed-form solution to this type of problem. Afterwards, Huang (2003) extended the trade credit problem to the case in which a supplier offers its retailer a credit period, and the retailer in turn provides another credit period to its customers. Furthermore, Liao (2008) extended Huang's model to an economic production quantity (EPQ) model for deteriorating items. Subsequently, Teng (2009) provided the optimal ordering policies for a retailer to deal with bad credit customers as well as good credit customers. Conversely, Min, Zhou, and Zhao (2010) proposed an EPQ model under stock-dependent demand and two-level trade credit. Later, Kreng and Tan (2011) obtained the optimal replenishment decision in an EPQ model with defective items under trade credit policy. After, Teng, Krommyda, Skouri, and Lou (2011) obtained the optimal ordering policy for stock-dependent demand under progressive payment scheme. Further, Teng, Min, and Pan (2012) extended the demand pattern from

* Corresponding author. Address: Department of Management Sciences, Tamkang University, No.151, Yingzhuan Road, New Taipei City 251, Taiwan, ROC. Tel.: +886 2 2621 5656x2075; fax: +886 2 321 7843.
E-mail address: liangyuh@mail.tku.edu.tw (L.-Y. Ouyang).

http://dx.doi.org/10.1016/j.ejor.2014.03.009
0377-2217/

constant to increasing in time. Recently, Ouyang and Chang (2013) built up an EPQ model with imperfect production process and complete backlogging. Concurrently, Chen, Cárdenas-Barrón, and Teng (in press) established the retailer's optimal EOQ when the supplier offers conditionally permissible delay in payments link to order quantity. In all articles described above, the EOQ/EPQ models with trade credit financing were studied only from the perspective of the buyer. How to determine the optimal credit period for the seller has received only a few attentions by the researchers such as Chern, Pan, Teng, Chan, and Chen (2013) and Teng and Lou (2012). Currently, Seifert, Seifert, and Protopappa-Sieke (2013) organized the trade credit literature and derived a detailed agenda for future research in trade credit area.

It is well know that many products such as vegetables, fruits, volatile liquids, blood banks, fashion merchandises and high-tech products deteriorate continuously due to several reasons such as evaporation, spoilage, obsolescence among others. In this course, Ghare and Schrader (1963) proposed an EOQ model by assuming an exponentially decaying inventory. Then Covert and Philip (1973) generalized the constant exponential deterioration rate to a two-parameter Weibull distribution. Later, Dave and Patel (1981) established an EOQ model for deteriorating items with linearly increasing demand and no shortages. Then Sachan (1984) further extended the EOQ model to allow for shortages. Conversely, Goswami and Chaudhuri (1991) generalized an EOQ model for deteriorating items from a constant demand pattern to a linear trend in demand. Concurrently, Raafat (1991) provided a survey of literature on continuously deteriorating inventory model. On the other hand, Hariga (1996) studied optimal EOQ models for deteriorating items with time-varying demand. Afterwards, Teng, Chern, Yang, and Wang (1999) generalized EOQ models with shortages and fluctuating demand. Later, Goyal and Giri (2001) wrote a survey on the recent trends in modeling of deteriorating inventory. Teng, Chang, Dye, and Hung (2002) further extended the model to allow for partial backlogging. Skouri, Konstantaras, Papachristos, and Ganas (2009) established inventory EOQ models with ramp-type demand rate and Weibull deterioration rate. In a subsequent paper, Skouri, Konstantaras, Papachristos, and Teng (2011) further generalized the model for deteriorating items with ramp-type demand and permissible delay in payments. Mahata (2012) proposed an EPQ model for deteriorating items under retailer partial trade credit policy. Recently, Dye (2013) studied the effect of technology investment on deteriorating items. Wee and Widyadana (2013) developed a production model for deteriorating items with stochastic preventive maintenance time and rework. Although a deteriorating item has its own expiration date (a.k.a., maximum lifetime), none of the above mentioned papers take the maximum lifetime into consideration. Currently, Bakker, Riezebos, and Teunter (2012) wrote a review of inventory systems with deterioration since 2001.

In this paper, we propose an EOQ model for the retailer to obtain her/his optimal credit period and cycle time when: (a) the supplier grants to the retailer an up-stream trade credit of S years while the retailer offers a down-stream trade credit of R years to the buyer, (b) the retailer's down-stream trade credit to the buyer not only increases sales and revenue but also opportunity cost and default risk, and (c) a deteriorating item not only deteriorates continuously but also has its maximum lifetime. We then formulate the retailer's objective functions under different possible cases. In fact, the proposed inventory model forms a general framework that includes many previous models as special cases such as Goyal (1985),eng (2002), Teng and Goyal (2007), Teng and Lou (2012), Lou and Wang (2013), Wang, Teng, and Lou (2014), and others. By applying concave fractional programming, we prove that there exists a unique global optimal solution to the retailer's replenishment cycle time. Similarly, using Calculus we show that the retailer's optimal down-stream credit period not only exists but also is unique. Furthermore, we discuss a special case for non-deteriorating items. Finally, we run several numerical examples to illustrate the problem and provide some managerial insights.

The rest of the paper is designed as follows. To establish the proposed EOQ model, we define the notation and assumptions in section 'Notation and assumptions'. Then we derive mathematical expressions of the relevant factors and the retailer's annual total profit function under each distinct possible case in section 'Mathematical model'. In section 'Theoretical results and optimal solution', we show that both the optimal cycle time and the optimal trade credit exist uniquely by applying concave fractional programming and Calculus, respectively. In section 'Some special cases', several previous EOQ models with trade credit financing are shown to be special cases of the proposed model including those non-deteriorating items. Numerical examples and sensitivity analysis are presented to illustrate the model and provide managerial insights in section 'Numerical examples'. Finally, the conclusion and future extensions of the proposed model are established in section 'Conclusions and future research'.

Notation and assumptions

The following notation and assumptions are used in the entire paper.

Notation

For the retailer

o	ordering cost per order in dollars.
c	purchase cost per unit in dollars.
p	selling price per unit in dollars, with $p > c$.
h	unit holding cost per year in dollars excluding interest charge.
r	annual compound interest paid per dollar per year.
I_e	interest earned per dollar per year.
I_c	interest charged per dollar per year.
t	the time in years.
$I(t)$	inventory level in units at time t.
$\theta(t)$	the time-varying deterioration rate at time t, where $0 \leqslant \theta(t) < 1$.
m	the expiration date or maximum lifetime in years of the deteriorating item.
S	up-stream credit period in years offered by the supplier.
R	down-stream trade credit period in years offered by the retailer (a decision variable).
$D = D(R)$	the market annual demand rate in units which is a concave and increasing function of R.
T	replenishment cycle time in years (a decision variable).
Q	order quantity.
$TP(R,T)$	total annual profit, which is a function of R and T.
R^*	optimal down-stream credit period in years.
T^*	optimal replenishment cycle time in years.
TP^*	optimal annual total profit in dollars.

Assumptions

Next, the following assumptions are made to establish the mathematical inventory model.

1. All deteriorating items have their expiration dates. Hence, the deterioration rate must be closed to 1 when time is approaching to the expiration date m. We may assume that the deteriorating

rate is $\theta(t) = \lambda/(\lambda + m - t)$, or $\theta(t) = e^{\lambda(t-m)}$, where λ is a constant. However, to make the problem tractable, we assume that the deterioration rate is the same as that in Sarkar (2012) and Wang et al. (2014) as follow:

$$\theta(t) = \frac{1}{1+m-t}, \ 0 \leqslant t \leqslant T \leqslant m. \tag{1}$$

Note that it is clear from (1) that the replenishment cycle time T must be less than or equal to m, and the proposed deterioration rate is a general case for non-deteriorating items, in which $m \to \infty$ and $\theta(t) \to 0$.

2. Similar to the assumption in Chern et al. (2013) and Teng and Lou (2012), we assume that the demand rate $D(R)$ is a positive exponential function of the retailer's down-stream credit period R as

$$D(R) = Ke^{aR}, \tag{2}$$

where K and a are positive constants with $0 < a < 1$. For convenience, $D(R)$ and D will be used interchangeably.

3. The longer the retailer's down-stream credit period, the higher the default risk to the retailer. For simplicity, we may assume that the rate of default risk giving the retailer's down-stream credit period R is assumed as

$$F(R) = 1 - e^{-bR}, \tag{3}$$

where b is the coefficient of the default risk, which is a positive constant.

4. If the annual compound interest rate is r, then a dollar received at time t is equivalent to e^{-rt} dollars received now. The retailer offers the buyer a credit period of R. Hence, the retailer's net revenue received after default risk and opportunity cost is:

$$pD(R)[1 - F(R)]e^{-rR} = pKe^{aR}e^{-bR}e^{-rR} = pKe^{[a-(b+r)]R}. \tag{4}$$

5. If $T \geqslant S$, then the retailer settles the account at time S and pays for the interest charges on items in stock with rate I_c over the interval $[S, T]$. If $T \leqslant S$, then the retailer settles the account at time S and there is not interest charge in stock during the whole cycle. On the other hand, if $S > R$, the retailer can accumulate revenue and earn interest during the period from R to S with rate I_e under the up-stream and down-stream trade credit conditions.
6. Replenishment rate is instantaneous.
7. In today's time-based competition, we may assume that shortages are not allowed to occur. Given the above notation and assumptions, it is possible to formulate the retailer's annual total profit as a function of the down-stream trade credit R and the replenishment cycle time T for deteriorating items with maximum lifetime into a mathematical model.

Mathematical model

During the replenishment cycle $[0, T]$, the inventory level is depleted by demand and deterioration, and hence governed by the following differential equation:

$$\frac{dI(t)}{dt} = -D - \theta(t)I(t), \ 0 \leqslant t \leqslant T, \tag{5}$$

with the boundary condition $I(T) = 0$. Solving the differential Eq. (5), we get

$$I(t) = e^{-\delta(t)} \int_t^T e^{\delta(u)} D du, \ 0 \leqslant t \leqslant T, \tag{6}$$

where

$$\delta(t) = \int_0^t \theta(u)du = \int_0^t \frac{1}{1+m-u} du$$
$$= \ln(1+m) - \ln(1+m-t) = \ln\left(\frac{1+m}{1+m-t}\right). \tag{7}$$

Substituting (7) into (6), we obtain the inventory level at time t as

$$I(t) = D\left(\frac{1+m-t}{1+m}\right) \int_t^T \frac{1+m}{1+m-u} du$$
$$= D(1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right), \ 0 \leqslant t \leqslant T. \tag{8}$$

As a result, the retailer's order quantity is

$$Q = I(0) = D(1+m)\ln\left(\frac{1+m}{1+m-T}\right). \tag{9}$$

Therefore, the retailer's holding cost excluding interest cost per cycle is

$$h\int_0^T I(t)dt = hD\int_0^T (1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt$$
$$= hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]. \tag{10}$$

From the values of R and S, we have two potential cases: (1) $R \leqslant S$, and (2) $R \geqslant S$. Let us discuss them separately.

Case 1. $R \leqslant S$

Based on the values of S (i.e., the time at which the retailer must pay off the purchase amount to the supplier to avoid interest charge) and $T + R$ (i.e., the time at which the retailer receives the payment from the last customer), we have two possible sub-cases. If $T + R > S$ (i.e., there is an interest charge), then the retailer pays off all units sold by $S - R$ at time S, keeps the profits, and starts paying for the interest charges on the items sold after $S - R$, which is shown in Fig. 1. If $T + R \leqslant S$ (i.e., there is no interest charge), then the retailer receives the total revenue at time $T + R$, and will pay off the total purchase cost at time S. The graphical representation of this case is shown in Fig. 2. Now, let us discuss the detailed formulation in each sub-case.

Sub-case 1-1: $S \leqslant T + R$

In this sub-case, the supplier's up-stream credit period S is shorter than or equal to the customer last payment time $T + R$. Hence, the retailer cannot pay off the purchase amount at time S, and must finance all items sold after time $S - R$ at an interest charged I_c per dollar per year. As a result, the interest charged per cycle is $(c/p)I_c$ times the area of the triangle BCD as shown in

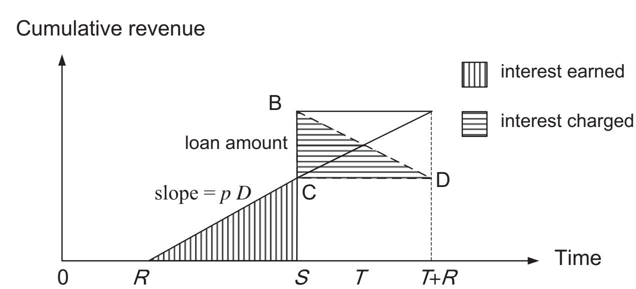

Fig. 1. $R \leqslant S$ and $S \leqslant T + R$.

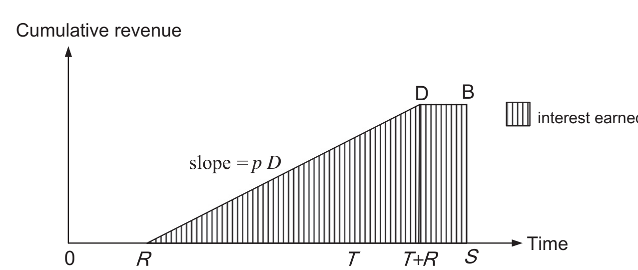

Fig. 2. $R \leqslant S$ and $T + R \leqslant S$.

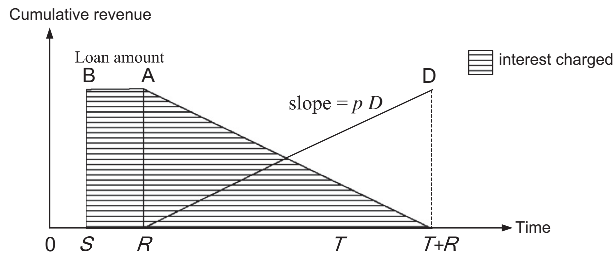

Fig. 3. $R \geqslant S$.

Fig. 1. Notice that (i) the vertical axis in Figs. 1–3 represents the cumulative revenue, not cumulative sale volume, and (ii) the slope of the increasing line in Figs. 1–3 is pD. Therefore, the interest charged per year is given by

$$\frac{cI_cD}{2T}(T+R-S)^2, \tag{11}$$

which is similar to Eq. (3) in Teng and Lou (2012).

On the other hand, the retailer sells deteriorating items at time 0, but receives the money at time R. Thus, the retailer accumulates revenue in an account that earns I_e per dollar per year from R through S. Therefore, the interest earned per cycle is I_e multiplied by the area of the triangle RSC as shown in Fig. 1. Hence, the interest earned per year is similar to Eq. (4) of Teng and Lou (2012) as

$$\frac{pI_eD(S-R)^2}{2T}. \tag{12}$$

The retailer's ordering cost per cycle is o dollars, and the purchase cost per cycle is $c\,I(0)$ dollars. Hence, the retailer's annual total profit can be expressed as follows

$TP_1(R,T)$ = net annual revenue after default risk and opportunity cost − annual purchase cost − annual ordering cost − annual holding cost excluding interest cost − interest charged + interest earned

$$\begin{aligned}&= pKe^{[a-(b+r)]R} - \frac{c(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{o}{T}\\ &\quad - \frac{h}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]\\ &\quad - \frac{cI_c}{2T}Ke^{aR}(T+R-S)^2 + \frac{pI_e}{2T}Ke^{aR}(S-R)^2.\end{aligned} \tag{13}$$

Next, we discuss the other sub-case in which $S \geqslant T+R$.

Sub-case 1-2: $S \geqslant T+R$

In this sub-case, the retailer receives the total revenue at time $T+R$, and is able to pay off the total purchase cost at time S. Hence, there is no interest charge while the interest earned per cycle is I_e multiplied by the area of the trapezoid on the interval $[R,S]$ as shown in Fig. 2. Consequently, the retailer's annual interest earned is

$$\frac{pI_eDT^2}{2T} + \frac{pI_eDT(S-T-R)}{T} = pI_eKe^{aR}\left(S-R-\frac{T}{2}\right). \tag{14}$$

Hence, similar to (13), we know that the retailer's annual total profit is

$$\begin{aligned}TP_2(R,T) &= pKe^{[a-(b+r)]R} - \frac{c(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{o}{T}\\ &\quad - \frac{h}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]\\ &\quad + pI_eKe^{aR}\left(S-R-\frac{T}{2}\right).\end{aligned} \tag{15}$$

We know from (13) and (15) that

$$TP_1(R,S-R) = TP_2(R,S-R). \tag{16}$$

Finally, we formulate the retailer's annual total profit for the case of $R \geqslant S$ below.

Case 2. $R \geqslant S$

Since $R \geqslant S$, there is no interest earned for the retailer. In addition, the retailer must finance the entire purchase cost at time S, and pay off the loan from time R to time $T+R$. Consequently, the interest charged per cycle is $(c/p)I_c$ multiplied by the area of the trapezoid on the interval $[S,T+R]$, as shown in Fig. 3. Thus, the interest charged per year is given by

$$\frac{cI_cD}{2}[2(R-S)+T]. \tag{17}$$

Hence, the retailer's annual total profit is

$$\begin{aligned}TP_3(R,T) &= pKe^{[a-(b+r)]R} - \frac{c(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{o}{T}\\ &\quad - \frac{h}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]\\ &\quad - cI_cKe^{aR}\left(R-S+\frac{T}{2}\right).\end{aligned} \tag{18}$$

Therefore, the retailer's objective is to determine the optimal credit period R^* and cycle time T^* such that the annual total profit $TP_i(R,T)$ for i = 1, 2, and 3 is maximized. In the next section, we characterize the retailer's optimal credit period and cycle time in each case, and then obtain the conditions in which the optimal T^* is in either $T+R \leqslant S$ or $T+R \geqslant S$.

Theoretical results and optimal solution

To solve the problem, we apply the existing theoretical results in concave fractional programming. We know from Cambini and Martein (2009) that the real-value function

$$q(x) = \frac{f(x)}{g(x)} \tag{19}$$

is (strictly) pseudo-concave, if $f(x)$ is non-negative, differentiable and (strictly) concave, and $g(x)$ is positive, differentiable and convex. For any given R, by applying (19), we can prove that the retailer's annual total profit $TP_i(R,T)$ for i = 1, 2, and 3 is strictly pseudo-concave in T. As a result, for any given R, there exists a unique global optimal solution T_i^* such that $TP_i(R,T)$ is maximized. Similar to section 'Mathematical model', we discuss the case of $R \leqslant S$ first, and then the case of $R \geqslant S$.

Optimal solution for the case of $R \leqslant S$

By applying the concave fractional programming as in (19), we can prove that the retailer's annual total profit $TP_i(R,T)$ for i = 1, and 2 is strictly pseudo-concave in T. Consequently, we have the following theoretical results.

Theorem 1. *For any given R,*

(a) *$TP_1(R,T)$ is a strictly pseudo-concave function in T, and hence exists a unique maximum solution T_1^*.*
(b) *If $S \leqslant T_1^* + R$, then $TP_1(R,T)$ subject to $S \leqslant T+R$ is maximized at T_1^*.*
(c) *If $S \geqslant T_1^* + R$, then $TP_1(R,T)$ subject to $S \leqslant T+R$ is maximized at $S-R$.*

Proof. See Appendix A. □

To find T_1^*, taking the first-order partial derivative of $TP_1(R,T)$, setting the result to zero, and re-arranging terms, we get

$$\frac{\partial TP_1(R,T)}{\partial T}=\frac{1}{T^2}\left\{o+(1+m)Ke^{aR}\left[c+\frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-T}\right)-\frac{T}{1+m-T}\right]\right\}+\frac{Ke^{aR}}{2T^2}(S-R)^2(cI_c-pI_e)-\frac{Ke^{aR}}{4}(h+2cI_c)=0. \quad (20)$$

For any given T, taking the first-order partial derivative of $TP_1(R,T)$ with respect to R, setting the result to zero, and re-arranging terms, we have

$$\begin{aligned}\frac{\partial TP_1(R,T)}{\partial R}&=[a-(b+r)]pKe^{[a-(b+r)]R}-\frac{ac(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right)\\&-\frac{ah}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]\\&-\frac{aKe^{aR}}{2T}[cI_c(T+R-S)^2-pI_e(S-R)^2]\\&-\frac{Ke^{aR}}{T}[cI_c(T+R-S)+pI_e(S-R)]=0.\end{aligned} \quad (21)$$

Taking the second-order partial derivative of $TP_1(R,T)$ with respect to R, and re-arranging terms, we obtain

$$\begin{aligned}\frac{\partial^2 TP_1(R,T)}{\partial R^2}&=[a-(b+r)]^2pKe^{[a-(b+r)]R}-\frac{a^2c(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right)\\&-\frac{a^2h}{2T}Ke^{aR}(1+m)\left[(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-T\right]\\&-\frac{1}{4}a^2hTKe^{aR}-\frac{a^2Ke^{aR}}{2T}\left[cI_c(T+R-S)^2-pI_e(S-R)^2\right]\\&-\frac{2aKe^{aR}}{T}[cI_c(T+R-S)+pI_e(S-R)]-\frac{Ke^{aR}}{T}(cI_c-pI_e).\end{aligned} \quad (22)$$

To identify whether R_1^* is 0 or positive, let's use (21) to define the discrimination term

$$\begin{aligned}\Delta_{R1}&=[a-(b+r)]p-\frac{ac(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right)\\&-\frac{ah}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]\\&-\frac{a}{2T}[cI_c(T-S)^2-pI_eS^2]-\frac{1}{T}[cI_c(T-S)+pI_eS]\end{aligned} \quad (23)$$

Then applying the following lemma, we can prove Theorem 2 below.

Lemma 1. *For all $T>0$,*

$$\frac{1+m}{T}\ln\left(\frac{1+m}{1+m-T}\right)>1. \quad (24)$$

Proof. The proof is the same as that in Wang et al. (2014). Let's set

$$L(T)=(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-T.$$

It is clear that $L(0)=0$. Taking the first-order derivative of $L(T)$, we yield $L\prime(T)=\frac{1+m}{1+m-T}-1>0$, for all $T>0$. Therefore, $L(T)>0$, for all $T>0$. This completes the proof. □

Theorem 2. *For any given $T>0$, if $[a-(b+r)]^2p-a^2c\leqslant 0$, and $\frac{h}{2}(aT)^2+cI_c\ \{[a(T+R-S)+2]^2-2\}-pI_e\left\{2-[2-a(S-R)]^2\right\}\geqslant 0$. then we obtain*

(a) *$TP_1(R,T)$ is a strictly concave function in R, and hence exists a unique maximum solution R_1^*.*
(b) *If $\Delta_{R1}\leqslant 0$, then $TP_1(R,T)$ is maximized at $R_1^*=0$.*
(c) *If $\Delta_{R1}>0$, then there exists a unique $R_1^*>0$ such that $TP_1(R,T)$ is maximized.*

Proof. See Appendix B. □

Likewise, applying the concave fractional programming to $TP_2(R,T)$, we obtain the following results:

Theorem 3. *For any given R,*

(a) *$TP_2(R,T)$ is a strictly pseudo-concave function in T, and hence exists a unique maximum solution T_2^*.*
(b) *If $S\geqslant T_2^*+R$, then $TP_2(R,T)$ subject to $S\geqslant T+R$ is maximized at T_2^*.*
(c) *If $S\leqslant T_2^*+R$, then $TP_2(R,T)$ subject to $S\geqslant T+R$ is maximized at $S-R$.*

Proof. See Appendix C. □

To find T_2^*, taking the first-order partial derivative of $TP_2(R,T)$, setting the result to zero, and re-arranging terms, we get

$$\frac{\partial TP_2(R,T)}{\partial T}=\frac{1}{T^2}\left\{o+(1+m)Ke^{aR}\left[c+\frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-T}\right)-\frac{T}{1+m-T}\right]\right\}-\frac{1}{4}(h+2pI_e)Ke^{aR}=0. \quad (25)$$

To identify which one is the optimal solution (i.e., either T_1^* or T_2^*), let's define the discrimination term

$$\Delta_T=o+(1+m)Ke^{aR}\left[c+\frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-S+R}\right)-\frac{S-R}{1+m-S+R}\right]-\frac{1}{4}(h+2pI_e)Ke^{aR}(S-R)^2. \quad (26)$$

Combining Theorems 1 and 3, and Eq. (16), we can prove the following theoretical results:

Theorem 4. *For any given R,*

(a) *If $\Delta_T>0$, then the retailer's optimal cycle time is T_2^*.*
(b) *If $\Delta_T=0$, then the retailer's optimal cycle time is $S-R$.*
(c) *If $\Delta_T<0$, then the retailer's optimal cycle time is T_1^*.*

Proof. See Appendix D. □

Next, we discuss the optimal trade credit for $TP_2(R,T)$. For any given T, taking the first-order partial derivative of $TP_2(R,T)$ with respect to R, setting the result to zero, and re-arranging terms, we have

$$\begin{aligned}\frac{\partial TP_2(R,T)}{\partial R}&=[a-(b+r)]pKe^{[a-(b+r)]R}-\frac{ac(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right)\\&-\frac{ah}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]\\&+apI_eKe^{aR}\left(S-R-\frac{T}{2}\right)-pI_eKe^{aR}=0.\end{aligned} \quad (27)$$

Taking the second-order partial derivative of $TP_2(R,T)$ with respect to R, and re-arranging terms, we obtain

$$\begin{aligned}\frac{\partial^2 TP_2(R,T)}{\partial R^2} &= [a-(b+r)]^2 pKe^{[a-(b+r)]R} \\ &\quad - \frac{a^2 c(1+m)}{T} Ke^{aR} \ln\left(\frac{1+m}{1+m-T}\right) \\ &\quad - \frac{a^2 h}{2T} Ke^{aR}(1+m)\left[(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - T\right] \\ &\quad - \frac{1}{4}a^2 hTKe^{aR} + a^2 pI_e Ke^{aR}\left(S-R-\frac{T}{2}\right) - 2apI_e Ke^{aR}. \end{aligned} \tag{28}$$

To identify whether R_2^* is 0 or positive, let's use (27) to define the discrimination term

$$\begin{aligned}\Delta_{R2} &= [a-(b+r)]p - \frac{ac(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) \\ &\quad - \frac{ah}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \\ &\quad + apI_e\left(S-\frac{T}{2}\right) - pI_e. \end{aligned} \tag{29}$$

By applying Lemma 1, we have the following result.

Theorem 5. *For any given $T > 0$, if $[a-(b+r)]^2 p - a^2 c \leqslant 0$, and $a(S-R-T/2) \leqslant 2$, then we obtain*

(a) *$TP_2(R,T)$ is a strictly concave function in R, and hence exists a unique maximum solution R_2^*.*
(b) *If $\Delta_{R2} \leqslant 0$, then $TP_2(R,T)$ is maximized at $R_2^* = 0$.*
(c) *If $\Delta_{R2} > 0$, then there exists a unique $R_2^* > 0$ such that $TP_2(R,T)$ is maximized.*

Proof. See Appendix E. □

4.2. Optimal solution for the case of $R \geqslant S$

Again, applying the concave fractional programming, one can obtain that the retailer's annual total profit $TP_3(R,T)$ is strictly pseudo-concave in T. Consequently, we have the following theoretical results.

Theorem 6. *For any given R, $TP_3(R,T)$ is a strictly pseudo-concave function in T, and hence exists a unique maximum solution, T_3^*.*

Proof. See Appendix F. □

To find T_3^*, taking the first-order partial derivative of $TP_3(R,T)$ with respect to T, setting the result to zero, and re-arranging terms, we get

$$\begin{aligned}\frac{\partial TP_3(R,T)}{\partial T} &= \frac{1}{T^2}\left\{o + (1+m)Ke^{aR}\left[c + \frac{h(1+m)}{2}\right]\right. \\ &\quad \left.\left[\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right]\right\} \\ &\quad - \frac{1}{4}(h+2cI_c)Ke^{aR} = 0. \end{aligned} \tag{30}$$

For any given T, taking the first-order partial derivative of $TP_3(R,T)$ with respect to R, setting the result to zero, and re-arranging terms, we yield

$$\begin{aligned}\frac{\partial TP_3(R,T)}{\partial R} &= [a-(b+r)]pKe^{[a-(b+r)]R} \\ &\quad - \frac{ac(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) \\ &\quad - \frac{ah}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \\ &\quad - cI_c Ke^{aR}\left[a\left(R-S+\frac{T}{2}\right)+1\right] = 0. \end{aligned} \tag{31}$$

Taking the second-order partial derivative of $TP_3(R,T)$ with respect to R, and re-arranging terms, we obtain

$$\begin{aligned}\frac{\partial^2 TP_3(R,T)}{\partial R^2} &= [a-(b+r)]^2 pKe^{[a-(b+r)]R} \\ &\quad - \frac{a^2 c(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) \\ &\quad - \frac{a^2 h}{2T}Ke^{aR}(1+m)\left[(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - T\right] \\ &\quad - a^2 Ke^{aR}\left[cI_c\left(R-S+\frac{T}{2}\right) + \frac{hT}{4}\right] - acI_c Ke^{aR}. \end{aligned} \tag{32}$$

For simplicity, let's define another discrimination term

$$\begin{aligned}\Delta_{R3} &= [a-(b+r)]p - \frac{ac(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) \\ &\quad - \frac{ah}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \\ &\quad - cI_c\left[a\left(\frac{T}{2}-S\right)+1\right]. \end{aligned} \tag{33}$$

Theorem 7. *For any given $T > 0$, if $[a-(b+r)]^2 p - a^2 c \leqslant 0$, then we get:*

(a) *$TP_3(R,T)$ is a strictly concave function in R, and hence exists a unique maximum solution R_3^*.*
(b) *If $\Delta_{R3} \leqslant 0$, then $TP_3(R,T)$ is maximized at $R_3^* = 0$.*
(c) *If $\Delta_{R3} > 0$, then there exists a unique $R_3^* > 0$ such that $TP_3(R,T)$ is maximized.*

Proof. See Appendix G. □

Next, we show that the proposed model is a general case of many previous models such as Goyal (1985), Teng (2002), Teng and Goyal (2007), Teng and Lou (2012), Lou and Wang (2013), Wang et al. (2014), and others.

Some special cases

Firstly, if there is no expiration date (i.e., the maximum lifetime is approaching to infinity), then the proposed model becomes for non-deteriorating items. From Calculus, we get

$$\begin{aligned}\lim_{m\to\infty}\left[\frac{1+m}{T}\ln\left(\frac{1+m}{1+m-T}\right)\right] &= \lim_{m\to\infty}\left[\frac{\frac{d}{dm}\ln\left(\frac{1+m}{1+m-T}\right)}{\frac{d}{dm}\frac{T}{1+m}}\right] \\ &= \lim_{m\to\infty}\frac{1+m}{1+m-T} = 1. \end{aligned} \tag{34}$$

Consequently, the retailer's order quantity per cycle in (9) becomes

$$Q = I(0) = D(1+m)\ln\left(\frac{1+m}{1+m-T}\right) = DT \text{ when } m \to \infty. \tag{35}$$

Similarly, we can obtain

$$\begin{aligned}&\lim_{m\to\infty}\left[\tfrac{(1+m)^2}{2}\ln\left(\tfrac{1+m}{1+m-T}\right) - \tfrac{(1+m)T}{2}\right] \\ &= \tfrac{1}{2}\lim_{m\to\infty}\left[\frac{\ln\left(\frac{1+m}{1+m-T}\right)-\frac{T}{1+m}}{\frac{1}{(1+m)^2}}\right] = \tfrac{1}{2}\lim_{m\to\infty}\left[\frac{\frac{-T}{(1+m)(1+m-T)}+\frac{T}{(1+m)^2}}{\frac{-2}{(1+m)^3}}\right] \\ &= \tfrac{1}{2}\lim_{m\to\infty}\left[\frac{T^2(1+m)}{2(1+m-T)}\right] = \tfrac{1}{4}\lim_{m\to\infty}T^2 = \tfrac{T^2}{4}. \end{aligned} \tag{36}$$

As a result, we know that the retailer's holding cost excluding interest charge per cycle in (10) is simplified to

$$\lim_{m\to\infty} hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]=\frac{hDT^2}{2}. \quad (37)$$

Hence, the retailer's annual total profit in (13) is reduced to:

$$TP_1(R,T)=pKe^{[a-(b+r)]R}-cKe^{aR}-\frac{o}{T}-\frac{h}{2}Ke^{aR}T -\frac{cI_c}{2T}Ke^{aR}(T+R-S)^2+\frac{pI_e}{2T}Ke^{aR}(S-R)^2. \quad (38)$$

Similarly, if there is no expiration date, then we get

$$TP_2(R,T)=pKe^{[a-(b+r)]R}-cKe^{aR}-\frac{o}{T}-\frac{h}{2}Ke^{aR}T +pI_eKe^{aR}\left(S-R-\frac{T}{2}\right), \quad (39)$$

and

$$TP_3(R,T)=pKe^{[a-(b+r)]R}-cKe^{aR}-\frac{o}{T}-\frac{h}{2}Ke^{aR}T -cI_cKe^{aR}\left(R-S+\frac{T}{2}\right). \quad (40)$$

This simplified problem with $r = 0$ has been solved by Teng and Lou (2012).

In fact, several previous models are indeed special cases of the proposed inventory model here.

(i) When $S = 0$, and $r = 0$, then the proposed model is simplified to that in Wang et al. (2014).
(ii) When $m \to \infty$ and $r = 0$, then the proposed model is reduced to that in Teng and Lou (2012).
(iii) When $m \to \infty$, $S = 0$, and $r = 0$, then the proposed model is the same as that in Lou and Wang (2013).
(iv) When $m \to \infty$, $a = 0$, $b = 0$, and $r = 0$, then the proposed model is simplified to that in Teng and Goyal (2007).
(v) When $m \to \infty$, $R = 0$, $a = 0$, $b = 0$, and $r = 0$, then the proposed model is similar to that in Teng (2002).
(vi) When $m \to \infty$, $R = 0$, $p = c$, $a = 0$, $b = 0$, and $r = 0$, then the proposed model is reduced to that in Goyal (1985).

Numerical examples

In this section, we use LINGO 12.0 to run several numerical examples in order to illustrate theoretical results as well as to gain some managerial insights.

Example 1. Let's assume $a = 0.2$/year, $b = 0.1$/year, $r = 0.05$/ year, $K = 1000$ units/year, $p = \$15$/unit, $c = \$10$/ unit, $o = \$20$/order, $h = \$2$/unit/year, $S = 0.25$ years (i.e., 3 months), and $m = 1$ year. We check the following common condition first:

$$[a-(b+r)]^2p-a^2c=0.0375-0.4\leqslant 0.$$

By using software LINGO 12.0, we have the maximum solution to $TP_i(R,T)$ for $i = 1,2$, and 3. as follow:

$$R_1^*=0.1516\text{ years},\ T_1^*=0.0985\text{ years},\text{ and } TP_1^*=\$4,275.71;$$
$$R_2^*=0.0000\text{ years},\ T_2^*=0.0706\text{ years},\text{ and } TP_2^*=\$4,625.52;$$

and

$$R_3^*=0.2500\text{ years},\ T_3^*=0.0699\text{ years},\text{ and } TP_3^*=\$4,109.00.$$

Consequently, the retailer's optimal solution is:

$$R^*=0.0000\text{ years},\ T^*=0.0706\text{ years},\text{ and } TP^*=\$4,625.52.$$

Example 2. Using the same data as those in Example 1 except $S = 0.0548$ years (i.e., 20 days), we obtain the following results:

$$R_1^*=0.0000\text{ years},\ T_1^*=0.0710\text{ years},\text{ and } TP_1^*=\$4,479.56;$$
$$R_2^*=0.0000\text{ years},\ T_2^*=0.0548\text{ years},\text{ and } TP_2^*=\$4,460.70;$$

and

$$R_3^*=0.0548\text{ years},\ T_3^*=0.0712\text{ years},\text{ and } TP_3^*=\$4,374.85.$$

Therefore, the retailer's optimal solution is:

$$R^*=0.0000\text{ years},\ T^*=0.0710\text{ years},\text{ and } TP^*=\$4,479.56.$$

Example 3. Using the same data as those in Example 1 except $b = 0.01$/year, $p = \$20$/unit, and $S = 0.0548$ years (i.e., 20 days), we get the following results:

$$[a-(b+r)]^2p-a^2c=0.392-0.4\leqslant 0;$$
$$R_1^*=0.0000\text{ years},\ T_1^*=0.0703\text{ years},\text{ and } TP_1^*=\$9,484.88;$$
$$R_2^*=0.0000\text{ years},\ T_2^*=0.0548\text{ years},\text{ and } TP_2^*=\$9,467.55;$$

and

$$R_3^*=0.9718\text{ years},\ T_3^*=0.0651\text{ years},\text{ and } TP_3^*=\$9,603.63.$$

Thus, the retailer's optimal solution is:

$$R^*=0.9718\text{ years},\ T^*=0.0651\text{ years},\text{ and } TP^*=\$9,603.63.$$

Example 4. Using the same data as those in Example 3, we study the sensitivity analysis on the optimal solution with respect to each parameter in appropriate unit. The computational results are shown in Table 1.

The sensitivity analysis reveals that: (i) if the value of a, K, p, or o increases, then the values of R^* and $TP^*(R^*,T^*)$ increase while the value of T^* decreases; (ii) by contrast, if the value of b increases, then the values of R^* and $TP^*(R^*, T^*)$ decrease while the value of T^* increases; (iii) a higher value of c or h causes lower values of R^*, T^*,

Table 1
Sensitivity analysis on parameters.

Parameter	R^*	T^*	$TP^*(R^*,T^*)$
$a = 0.20$	0.9718	0.0651	\$9,603.63
$a = 0.25$	2.2313	0.0544	\$10,460.60
$a = 0.30$	3.0396	0.0457	\$12,006.99
$b = 0.01$	0.9718	0.0651	\$9,603.63
$b = 0.02$	0.1841	0.0703	\$9,479.02
$b = 0.03$	0.0548	0.0712	\$9,465.64
$K = 1000$	0.9718	0.0651	\$9,603.63
$K = 2000$	1.0290	0.0406	\$19,567.26
$K = 3000$	1.0541	0.0376	\$29,590.19
$p = 20$	0.9718	0.0651	\$9,603.63
$p = 25$	3.3276	0.0516	\$16,426.78
$p = 30$	5.2970	0.0425	\$25,634.53
$c = 10$	0.9718	0.0651	\$9,603.63
$c = 12$	0.0000	0.0660	\$7,449.93
$c = 14$	0.0000	0.0624	\$5,417.10
$o = 20$	0.9718	0.0651	\$9,603.63
$o = 15$	0.9980	0.0564	\$9,685.96
$o = 10$	1.0290	0.0460	\$9,783.63
$h = 2$	0.9718	0.0651	\$9,603.63
$h = 4$	0.9472	0.0582	\$9,528.42
$h = 8$	0.9045	0.0492	\$9,399.00
$m = 1.0$	0.9718	0.0651	\$9,603.63
$m = 1.5$	0.9854	0.0701	\$9,646.58
$m = 2.0$	0.9949	0.0740	\$9,677.04

and $TP^*(R^*,T^*)$; and (iv) conversely, a higher value of m causes higher values of R^*, T^*, and $TP^*(R^*,T^*)$. A simple economic interpretation of (i) is as follows: if a is higher, then the effect of trade credit R to demand (as well as annual profit) gets higher. Hence, a higher value of a causes higher values of trade credit R^* and annual total profit $TP^*(R^*,T^*)$ while a lower value of T^* to reduce holding cost. Similarly, a simple economic interpretation of (iv) is as follows: if the expiration date of the deteriorating item m is longer, then it is worth to increase the trade credit R^* as well as the cycle time T^* in order to increase the sales and the annual total profit $TP^*(R^*,T^*)$. Likewise, one can easily interpret the rest of the managerial insights by using the analogous argument.

Conclusions and future research

Taking care of both up-stream and down-stream trade credits simultaneously for deteriorating items with expiration dates has received relatively little attention from the researchers. In this paper, we have built an EOQ model for the retailer to obtain its optimal credit period and cycle time in a supplier-retailer-buyer supply chain in which (a) the retailer receives an up-stream trade credit from the supplier while offers a down-stream trade credit to the buyer, (b) deteriorating items not only deteriorate continuously but also have their expiration dates, and (c) down-stream credit period increases not only demand but also opportunity cost and default risk. Then we have proved that the optimal trade credit and cycle time exist uniquely. Moreover, we have shown that the proposed model is a generalized case for non-deteriorating items and several previous EOQ models. Finally, we have used software LINGO 12.0 to study the sensitivity analysis on the optimal solution with respect to each parameter to illustrate the inventory model and provide some managerial insights.

For future research, one can study the recent review paper of trade credit literature by Seifert et al. (2013) who have derived a detailed agenda for future research in trade credit financing. In addition, we can extend the mathematical inventory model in several ways. For example, one immediate possible extension could be allowable shortages, cash discounts, etc. Also, one may generalize a single player local optimal solution to an integrated cooperative solution for both players, or a non-cooperative Nash or Stackelberg equilibrium solution for each player. Finally, one can extend the fully trade credit policy to the partial trade credit policy in which a seller requests its credit-risk customers to pay a fraction of the purchase amount at the time of placing an order as a collateral deposit, and then grants a permissible delay on the rest of the purchase amount.

Acknowledgements

The authors would like to thank Editor Robert Dyson and three anonymous referees for their positive and constructive comments, and Ms. Lin Feng for helping us completed the references.

Appendix A. Proof of Theorem 1

Let's use (13) to define

$$f_1(T) = pKe^{[a-(b+r)]R}T - c(1+m)Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) - o - hKe^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] - \frac{cI_c}{2}Ke^{aR}(T+R-S)^2 + \frac{pI_e}{2}Ke^{aR}(S-R)^2, \tag{A1}$$

and

$$g_1(T) = T. \tag{A2}$$

Taking the first-order and second-order derivatives of $f_1(T)$, we have

$$f_1'(T) = pKe^{[a-(b+r)]R} - \frac{c(1+m)Ke^{aR}}{1+m-T} - hKe^{aR}\left[\frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2} - \frac{1+m}{2}\right] - cI_cKe^{aR}(T+R-S), \tag{A3}$$

and

$$f_1''(T) = -\frac{c(1+m)Ke^{aR}}{(1+m-T)^2} - hKe^{aR}\left[\frac{(1+m)^2}{2(1+m-T)^2} + \frac{1}{2}\right] - cI_cKe^{aR} < 0. \tag{A4}$$

Therefore, $TP_1(R,T) = \frac{f_1(T)}{g_1(T)}$ is a strictly pseudo-concave function in T, which completes the proof of Part (a) of Theorem 1. The proof of Parts (b) and (c) immediately follows from Part (a) of Theorem 1. This completes the proof of Theorem 1.

Appendix B. Proof of Theorem 2

From (21) let's define

$$B(R) = [a-(b+r)]pKe^{[a-(b+r)]R} - \frac{ac(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{ah}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] - \frac{aKe^{aR}}{2T}\left[cI_c(T+R-S)^2 - pI_e(S-R)^2\right] - \frac{Ke^{aR}}{T}[cI_c(T+R-S) + pI_e(S-R)]. \tag{B1}$$

Applying (23) and Lemma 1, and simplifying (21), we get

$$B(0) = K\Delta_{R1}, \tag{B2}$$

and

$$\lim_{R\to\infty}B(R) = \lim_{R\to\infty}Ke^{aR}\left\{[a-(b+r)]pe^{-(b+r)R} - \frac{ac(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{ah(1+m)}{2}\left[\frac{(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - 1\right] - \frac{ahT}{4} - \frac{pI_e}{2T}(S-R)[2-a(S-R)] - \frac{cI_c}{2T}(T+R-S)[2+a(T+R-S)]\right\} = -\infty. \tag{B3}$$

Notice that in general both up-stream and down-stream credit periods are less than a year. Hence, we may assume without loss of generality that $1-(S-R-1)^2 \geqslant 0$.

Using Lemma 1, re-arranging (22), and the fact that $T+R \geqslant S$, we have

$$\frac{dB(R)}{dR} = [a-(b+r)]^2pKe^{[a-(b+r)]R} - a^2cKe^{aR}\frac{(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{a^2h}{2}Ke^{aR}(1+m)\left[\frac{1+m}{T}\ln\left(\frac{1+m}{1+m-T}\right) - 1\right] - \frac{1}{4}a^2hTKe^{aR} - \frac{cI_cKe^{aR}}{2T}\left\{[a(T+R-S)+2]^2 - 2\right\} - \frac{pI_eKe^{aR}}{2T}\left\{2-[a(S-R)-2]^2\right\}\frac{dB(R)}{dR} < [a-(b+r)]^2pKe^{aR} - a^2cKe^{aR} \leqslant 0,\ \text{if}\ [a-(b+r)]^2p - a^2c \leqslant 0,\ \text{and}\ \frac{h}{2}(aT)^2 + cI_c\left\{[a(T+R-S)+2]^2 - 2\right\} - pI_e\left\{2-[a(S-R)-2]^2\right\} \geqslant 0. \tag{B4}$$

This completes the proof of Part (a) of Theorem 2.

If $\varDelta_{R1} \leqslant 0$, then $B(0) \leqslant 0$, $B(R) < 0$ for all $R > 0$, and $TP_1(R,T)$ is a decreasing function in R. Hence, the retailer's optimal down-stream credit period is $R_1^* = 0$, which completes the proof of Part (b).

Finally, if $\varDelta_{R1} > 0$, then $B(0) > 0$, and $\lim_{R\to\infty} B(R) = -\infty$. By applying the Mean-value Theorem and Part (a) of Theorem 2, we know that there exists a unique $R_1^* > 0$ such that $B(R_1^*) = 0$. Consequently, $TP_1(R,T)$ is maximized at the unique point $R_1^* > 0$, which satisfies (21). This completes the proof of Part (c) of Theorem 2.

Appendix C. Proof of Theorem 3

From (15), let's define

$$f_2(T) = pKe^{[a-(b+r)]R}T - c(\lambda + m)Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) - o - hKe^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + pI_eKe^{aR}\left(ST - RT - \frac{T^2}{2}\right), \tag{C1}$$

and

$$g_2(T) = T. \tag{C2}$$

Taking the first-order and second-order derivatives of $f_2(T)$, we have

$$f_2'(T) = pKe^{[a-(b+r)]R} - \frac{c(1+m)Ke^{aR}}{1+m-T} - hKe^{aR}\left[\frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2} - \frac{1+m}{2}\right] + pI_eKe^{aR}(S-R-T), \tag{C3}$$

and

$$f_2''(T) = -\frac{c(1+m)Ke^{aR}}{(1+m-T)^2} - hKe^{aR}\left[\frac{(1+m)^2}{2(1+m-T)^2} + \frac{1}{2}\right] - pI_eKe^{aR} < 0. \tag{C4}$$

Therefore, $TP_2(R,T) = \frac{f_2(T)}{g_2(T)}$ is a strictly pseudo-concave function in T, which completes the proof of Part (a) of Theorem 3. The proof of Parts (b) and (c) immediately follows from Part (a) of Theorem 3. This completes the proof of Theorem 3.

Appendix D. Proof of Theorem 4

Let's use (25) to define

$$G(T) = \frac{\partial TP_2(R,T)}{\partial T} = \frac{1}{T^2}\left\{o + (1+m)Ke^{aR}\left[c + \frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right]\right\} - \frac{1}{4}(h+2pI_e)Ke^{aR}. \tag{D1}$$

Then we know from (26) that

$$G(S-R) = \frac{\varDelta_T}{(S-R)^2}. \tag{D2}$$

Using L'Hospital's Rule, we obtain:

$$\lim_{T\to 0}\left[\tfrac{1}{T^2}\ln\left(\tfrac{1+m}{1+m-T}\right) - \tfrac{1}{(1+m-T)T}\right] = \lim_{T\to 0}\left[\frac{(1+m-T)\ln\left(\frac{1+m}{1+m-T}\right)-T}{(1+m-T)T^2}\right] = \lim_{T\to 0}\left[\frac{-\ln\left(\frac{1+m}{1+m-T}\right)}{2T+2mT-3T^2}\right] = \lim_{T\to 0}\left[\frac{-1}{(2+2m-6T)(1+m-T)}\right] = \frac{-1}{2(1+m)^2}. \tag{D3}$$

By using (D1) and (D3), we get

$$\lim_{T\to 0} G(T) = \lim_{T\to 0}\left\{\frac{-D}{2(1+m)}\left[c + \frac{h(1+m)}{2}\right] + \frac{o}{T^2}\right\} - \frac{D(h+2pI_e)}{4} = \infty. \tag{D4}$$

If $\varDelta_T < 0$, then $G(S-R) = \varDelta_T/(S-R)^2 < 0$. By applying the Mean-value Theorem and Theorem 2, we know that there exists a unique $T_2^* \in (0, S-R)$ such that $G(T_2^*) = 0$.

$$TP_2(T) \text{ is maximized at the unique point } T_2^*, \text{ which satisfies (25).} \tag{D5}$$

By using the analogous argument, let's use (20) to define

$$J(T) = \frac{\partial TP_1(R,T)}{\partial T} = \frac{1}{T^2}\left\{o + (1+m)Ke^{aR}\left[c + \frac{h(1+m)}{2}\right] \times \left[\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right]\right\} + \frac{Ke^{aR}}{2T^2}(S-R)^2(cI_c - pI_e) - \frac{Ke^{aR}}{4}(h+2cI_c). \tag{D6}$$

From (26) we get

$$J(S-R) = \varDelta_T/(S-R)^2 < 0, \text{ if } \varDelta_T < 0. \tag{D7}$$

From Theorem 1 and (D7), we know that $J(T) < 0$ for all $T \geqslant S - R$. Hence,

$$\text{for all } T \geqslant S-R, TP_1(T) \text{ is decreasing and maximized at } S-R. \tag{D8}$$

By using (16), (D5), and (D8), we obtain that if $\varDelta_T < 0$, then

$$TP_2(T_2^*) \geqslant TP_2(S-R) = TP_1(S-R) \geqslant TP_1(T), \text{ for all } T \geqslant S-R. \tag{D9}$$

As a result, if $\varDelta_T < 0$, then $TP(T)$ is maximized at T_2^*. Thus, we complete the proof of Part (a) of Theorem 4. By using the analogous argument, one can prove the rest of Theorem 4. This completes the proof of Theorem 4.

Appendix E. Proof of Theorem 5

By using (27) we define

$$E(R) = [a-(b+r)]pKe^{[a-(b+r)]R} - \frac{ac(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{ah}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + apI_eKe^{aR}\left(S - R - \frac{T}{2}\right) - pI_eKe^{aR}. \tag{E1}$$

Applying (29) and Lemma 1, and simplifying (27), we get

$$E(0) = K\varDelta_{R2}, \tag{E2}$$

and

$$\lim_{R\to\infty} E(R) = \lim_{R\to\infty} Ke^{aR}\left\{[a-(b+r)]pe^{-(b+r)R} - \frac{ac(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{ah(1+m)}{2}\left[\frac{(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - 1\right] - \frac{ahT}{4} - pI_e\left[1 - a\left(S - R - \frac{T}{2}\right)\right]\right\} = -\infty. \tag{E3}$$

Notice that in general both up-stream and down-stream credit periods are less than a year. Hence, we may assume without loss of generality that $1 - a(S - R - T/2) \geqslant 0$.

Using Lemma 1, re-arranging (28), and the fact that $T + R \leqslant S$, we have

$$\begin{aligned}\frac{dE(R)}{dR} &= [a-(b+r)]^2 pKe^{[a-(b+r)]R} - a^2cKe^{aR}\frac{(1+m)}{T}\\ &\quad\times \ln\left(\frac{1+m}{1+m-T}\right) - \frac{a^2h}{2}Ke^{aR}(1\\ &\quad+m)\left[\frac{1+m}{T}\ln\left(\frac{1+m}{1+m-T}\right)-1\right]-\frac{1}{4}a^2hTKe^{aR}\\ &\quad- apI_eKe^{aR}\left[2-a\left(S-R-\frac{T}{2}\right)\right]\frac{dE(R)}{dR}\\ &< [a-(b+r)]^2pKe^{aR} - a^2cKe^{aR}\\ &\leqslant 0,\ \text{if}\ [a-(b+r)]^2p - a^2c \leqslant 0. \end{aligned} \tag{E4}$$

This completes the proof of Part (a) of Theorem 5.

If $\Delta_{R2} \leqslant 0$, then $E(0) \leqslant 0$, $E(R) < 0$ for all $R > 0$, and $TP_2(R,T)$ is a decreasing function in R. Hence, the retailer's optimal downstream credit period is $R_2^* = 0$, which completes the proof of Part (b).

Finally, if $\Delta_{R2} > 0$, then $E(0) > 0$, and $\lim_{R\to\infty}E(R) = -\infty$. By applying the Mean-value Theorem and Part (a) of Theorem 5, we know that there exists a unique $R_2^* > 0$ such that $E(R_1^*) = 0$. Consequently, $TP_2(R,T)$ is maximized at the unique point $R_2^* > 0$, which satisfies (27). This completes the proof of Part (c) of Theorem 5.

Appendix F. Proof of Theorem 6

From (18), let's define

$$\begin{aligned} f_3(T) &= pKe^{[a-(b+r)]R}T - c(1+m)Ke^{aR}\ln\left(\frac{1}{1+m-T}\right) - o\\ &\quad - hKe^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]\\ &\quad - cI_cKe^{aR}\left(RT-ST+\frac{T^2}{2}\right), \end{aligned} \tag{F1}$$

and

$$g_3(T) = T. \tag{F2}$$

Taking the first-order and second-order derivatives of $f_3(T)$, we have

$$\begin{aligned} f_3'(T) &= pKe^{[a-(b+r)]R} - \frac{1}{1+m-T} - hKe^{aR}\left[\frac{(1+m)^2}{2(1+m-T)}+\frac{T}{2}-\frac{1+m}{2}\right]\\ &\quad - cI_cKe^{aR}(R-S+T), \end{aligned} \tag{F3}$$

and

$$f_3''(T) = -\frac{c(1+m)Ke^{aR}}{(1+m-T)^2} - hKe^{aR}\left[\frac{(1+m)^2}{2(1+m-T)^2}+\frac{1}{2}\right] - cI_cKe^{aR} < 0. \tag{F4}$$

Therefore, $TP_3(R,T) = \frac{f_3(T)}{g_3(T)}$ is a strictly pseudo-concave function in T, which completes the proof of Theorem 6.

Appendix G. Proof of Theorem 7

From (31) let's define

$$\begin{aligned} N(R) &= [a-(b+r)]pKe^{[a-(b+r)]R} - \frac{ac(1+m)}{T}Ke^{aR}\ln\left(\frac{1+m}{1+m-T}\right)\\ &\quad - \frac{ah}{T}Ke^{aR}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]\\ &\quad - cI_cKe^{aR}\left[a\left(R-S+\frac{T}{2}\right)+1\right]. \end{aligned} \tag{G1}$$

By using (33) and Lemma 1, and re-arranging (31), we get

$$N(0) = K\Delta_{R3}, \tag{G2}$$

and

$$\begin{aligned} \lim_{R\to\infty}N(R) &= \lim_{R\to\infty}Ke^{aR}\Big\{[a-(b+r)]pe^{-(b+r)R} - \tfrac{ac(1+m)}{T}\ln\left(\tfrac{1+m}{1+m-T}\right)\\ &- \tfrac{ah(1+m)}{2}\left[\tfrac{(1+m)}{T}\ln\left(\tfrac{1+m}{1+m-T}\right)-1\right] - \tfrac{ahT}{4}\\ &- cI_c\left[a\left(R-S+\tfrac{T}{2}\right)+1\right]\Big\} = -\infty. \end{aligned} \tag{G3}$$

By applying Lemma 1, re-arranging (32), and the fact that $R \geqslant S$, we have

$$\begin{aligned} \frac{dN(R)}{dR} &= [a-(b+r)]^2pKe^{[a-(b+r)]R} - a^2cKe^{aR}\frac{(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right)\\ &- \frac{a^2h}{2}Ke^{aR}(1+m)\left[\frac{1+m}{T}\ln\left(\frac{1+m}{1+m-T}\right)-1\right]\\ &- a^2Ke^{aR}\left[cI_c\left(R-S+\frac{T}{2}\right)+\frac{hT}{4}\right]\\ &- acI_cKe^{aR}\frac{dN(R)}{dR} < [a-(b+r)]^2pKe^{aR}\\ &- a^2cKe^{aR} = Ke^{aR}\{[a-(b+r)]^2p - a^2c\} \leqslant 0,\\ &\text{if}\ [a-(b+r)]^2p - a^2c \leqslant 0. \end{aligned} \tag{G4}$$

This completes the proof of Part (a) of Theorem 7.

If $\Delta_{R3} \leqslant 0$, then $N(0) \leqslant 0$, $N(R) < 0$ for all $R > 0$, and $TP_3(R,T)$ is a decreasing function in R. Hence, the retailer's optimal downstream credit period is $R_3^* = 0$, which completes the proof of Part (b).

Finally, if $\Delta_{R3} > 0$, then $N(0) > 0$, and $\lim_{R\to\infty}N(R) = -\infty$. By applying the Mean-value Theorem and Part (a) of Theorem 7, we know that there exists a unique $R_3^* > 0$ such that $N(R_3^*) = 0$. Consequently, $TP_3(R,T)$ is maximized at the unique point R_3^*, which satisfies (31). This completes the proof of Part (c) of Theorem 7.

References

Aggarwal, S. P., & Jaggi, C. K. (1995). Ordering policies of deteriorating items under permissible delay in payments. *Journal of the Operational Research Society, 46*(5), 658–662.

Bakker, M., Riezebos, J., & Teunter, R. H. (2012). Review of inventory systems with deterioration since 2001. *European Journal of Operational Research, 221*(2), 275–284.

Cambini, A., & Martein, L. (2009). *Generalized convexity and optimization: Theory and application*. Berlin, Heidelberg, USA: Springer-Verlag.

Chen, S. C., Cárdenas-Barrón, L. E., & Teng, J. T. (in press). Retailer's economic order quantity when the supplier offers conditionally permissible delay in payments link to order quantity. *International Journal of Production Economics*. in press.

Chern, M. S., Pan, Q., Teng, J. T., Chan, Y. L., & Chen, S. C. (2013). Stackelberg solution in a vendor-buyer supply chain model with permissible delay in payments. *International Journal of Production Economics, 144*(1), 397–404.

Covert, R. B., & Philip, G. S. (1973). An EOQ model for items with Weibull distribution deterioration. *AIIE Transactions, 5*(4), 323–326.

Dave, U., & Patel, L. K. (1981). (T, S_i) policy inventory model for deteriorating items with time proportional demand. *Journal of the Operational Research Society, 32*(2), 137–142.

Dye, C. Y. (2013). The effect of preservation technology investment on a non-instantaneous deteriorating inventory model. *Omega, 41*(5), 872–880.

Ghare, P. M., & Schrader, G. P. (1963). A model for an exponentially decaying inventory. *Journal of Industrial Engineering, 14*, 238–243.

Goswami, A., & Chaudhuri, K. S. (1991). An EOQ model for deteriorating items with shortages and a linear trend in demand. *Journal of the Operational Research Society, 42*(12), 1105–1110.

Goyal, S. K. (1985). Economic order quantity under conditions of permissible delay in payments. *Journal of the Operational Research Society, 36*(4), 335–338.

Goyal, S. K., & Giri, B. C. (2001). Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research, 134*(1), 1–16.

Hariga, M. A. (1996). Optimal EOQ models for deteriorating items with time-varying demand. *Journal of the Operational Research Society, 47*(10), 1228–1246.

Huang, Y. F. (2003). Optimal retailer's ordering policies in the EOQ model under trade credit financing. *Journal of the Operational Research Society, 54*(9), 1011–1015.

Jamal, A. M., Sarker, B. R., & Wang, S. (1997). An ordering policy for deteriorating items with allowable shortage and permissible delay in payment. *Journal of the Operational Research Society, 48*(8), 826–833.

Kreng, V. B., & Tan, S. J. (2011). Optimal replenishment decision in an EPQ model with defective items under supply chain trade credit policy. *Expert Systems with Applications, 38*(8), 9888–9899.

Liao, J. J. (2008). An EOQ model with noninstantaneous receipt and exponentially deteriorating items under two-level trade credit. *International Journal of Production Economics, 113*(2), 852–861.

Lou, K. L., & Wang, W. C. (2013). Optimal trade credit and order quantity when trade credit impacts on both demand rate and default risk. *Journal of the Operational Research Society, 64*(10), 1551–1556.

Mahata, G. C. (2012). An EPQ-based inventory model for exponentially deteriorating items under retailer partial trade credit policy in supply chain. *Expert Systems with Applications, 39*(3), 3537–3550.

Min, J., Zhou, Y. W., & Zhao, J. (2010). An inventory model for deteriorating items under stock-dependent demand and two-level trade credit. *Applied Mathematical Modelling, 34*(11), 3273–3285.

Ouyang, L. Y., & Chang, C. T. (2013). Optimal production lot with imperfect production process under permissible delay in payments and complete backlogging. *International Journal of Production Economics, 144*(2), 610–617.

Raafat, F. (1991). Survey of literature on continuously deteriorating inventory model. *Journal of the Operational Research Society, 42*(1), 27–37.

Sachan, R. S. (1984). On (T, S_i) policy inventory model for deteriorating items with time proportional demand. *Journal of the Operational Research Society, 35*(11), 1013–1019.

Sarkar, B. (2012). An EOQ model with delay in payments and time varying deterioration rate. *Mathematical and Computer Modelling, 55*(3–4), 367–377.

Seifert, D., Seifert, R. W., & Protopappa-Sieke, M. (2013). A review of trade credit literature: Opportunity for research in operations. *European Journal of Operational Research, 231*(2), 245–256.

Shah, N. H. (1993). Probabilistic time-scheduling model for an exponentially decaying inventory when delay in payment is permissible. *International Journal of Production Economics, 32*(1), 77–82.

Skouri, K., Konstantaras, I., Papachristos, S., & Ganas, I. (2009). Inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate. *European Journal of Operational Research, 192*(1), 79–92.

Skouri, K., Konstantaras, I., Papachristos, S., & Teng, J. T. (2011). Supply chain models for deteriorating products with ramp type demand rate under permissible delay in payments. *Expert Systems with Applications, 38*(12), 14861–14869.

Teng, J. T. (2002). On the economic order quantity under conditions of permissible delay in payments. *Journal of the Operational Research Society, 53*(8), 915–918.

Teng, J. T. (2009). Optimal ordering policies for a retailer who offers distinct trade credits to its good and bad credit customers. *International Journal of Production Economics, 119*(2), 415–423.

Teng, J. T., Chang, H. J., Dye, C. Y., & Hung, C. H. (2002). An optimal replenishment policy for deteriorating items with time-varying demand and partial backlogging. *Operations Research Letters, 30*(6), 387–393.

Teng, J. T., Chern, M. S., Yang, H. L., & Wang, Y. J. (1999). Deterministic lot-size inventory models with shortages and deterioration for fluctuating demand. *Operations Research Letters, 24*(1–2), 65–72.

Teng, J. T., & Goyal, S. K. (2007). Optimal ordering policies for a retailer in a supply chain with up-stream and down-stream trade credits. *Journal of the Operational Research Society, 58*(9), 1252–1255.

Teng, J. T., Krommyda, I. P., Skouri, K., & Lou, K. R. (2011). A comprehensive extension of optimal ordering policy for stock-dependent demand under progressive payment scheme. *European Journal of Operational Research, 215*(1), 97–104.

Teng, J. T., & Lou, K. R. (2012). Seller's optimal credit period and replenishment time in a supply chain with up-stream and down-stream trade credits. *Journal of Global Optimization, 53*(3), 417–430.

Teng, J. T., Min, J., & Pan, Q. (2012). Economic order quantity model with trade credit financing for non-decreasing demand. *Omega, 40*(3), 328–335.

Wang, W. C., Teng, J. T., & Lou, K. R. (2014). Seller's optimal credit period and cycle time in a supply chain for deteriorating items with maximum lifetime. *European Journal of Operational Research, 232*(2), 315–321.

Wee, H. M., & Widyadana, G. A. (2013). A production model for deteriorating items with stochastic preventive maintenance time and rework process with FIFO rule. *Omega, 41*(6), 941–954.