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On stability of time-varying linear differential-algebraic equations

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Abstract

We develop a stability theory for time-varying linear differential algebraic equations (DAEs). Standard stability concepts for ODEs are formulated for DAEs and characterized. Lyapunov's direct method is derived as well as the converse of the stability theorems. Stronger results are achieved for DAEs which are transferable into standard canonical form; in this case the existence of the generalized transition matrix is exploited.

Keywords: Time-varying linear differential algebraic equations, exponential stability, Lyapunov's direct method, Lyapunov equation, Lyapunov function, Lyapunov transformation, standard canonical form, analytic solvability, generalized transition matrix

1 Introduction

We study stability of solutions of time-varying linear differential-algebraic equations (DAEs) of the form

$$E(t)\dot{x} = A(t)x + f(t), \tag{1.1}$$

where $(E, A, f) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2 \times \mathcal{C}((\tau,\infty);\mathbb{R}^n)$, $n \in \mathbb{N}$, $\tau \in [-\infty,\infty)$. For brevity, we identify the tuple

$$(E, A, f) \qquad \text{or} \qquad (E, A) := (E, A, 0)$$

with the inhomogeneous or homogeneous DAE (1.1), resp.

Time-invariant linear DAEs are well studied, see the textbooks by [Cam80, Cam82, Dai89, KM06]. However, for the stability theory of time-varying linear DAEs only a few contributions are available: [DVP07] treat DAEs with constant E and time-varying A; [SC04] use the ansatz of "regularizing operators" to obtain Lyapunov stability criteria; in [HMT03a, HMT03b, LMW96, MRS01] results for DAEs with index 1 or 2 are obtained; in [KM07] some stability results for a special subclass of time-varying DAEs are obtained and in [LM09] Lyapunov, Bohl and Sacker-Sell spectral intervals for DAEs of this class are investigated. However, results on arbitrary index and arbitrary E (particularly of variable rank) and a Lyapunov theory is not available.

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The paper is organized as follows: In Section 2 we show the relationships and consequences of different solution concepts for DAEs; the considerable difference to ODEs becomes clear. In Section 3 we introduce the subclass of DAEs (E, A) which are transferable into standard canonical from (SCF) and recall its basic properties relevant for the present paper. Different stability concepts are introduced and characterized in Section 4. In Section 5.1 we present Lyapunov's direct method for DAEs and develop a theory of Lyapunov functions and Lyapunov equations on the set of all pairs of consistent initial values. We stress that in Section 2 and Section 5.1 as well as in Theorem 4.3 only continuity of E, A, f is required.

Nomenclature

$\mathbb{N},\ \mathbb{N}_0$		the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\ker A$		the kernel of the matrix $A \in \mathbb{R}^{m\times n}$
$\operatorname{im} A$		the image of the matrix $A \in \mathbb{R}^{m\times n}$
$\mathbf{Gl}_n(\mathbb{R})$		the general linear group of degree n, i.e. the set of all invertible $n \times n$ matrices over $\mathbb{R}$
$\lVert x\rVert$	$:=$	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\mathcal{B}_\delta(x^0)$	$:=$	$\left\{\ x \in \mathbb{R}^n \ \middle|\ \lVert x - x^0\rVert < \delta\ \right\}$, the open ball of radius $\delta > 0$ around $x^0 \in \mathbb{R}^n$
$\lVert A\rVert$	$:=$	$\sup\{\ \lVert Ax\rVert \mid \lVert x\rVert = 1\ \}$, induced matrix norm of $A \in \mathbb{R}^{n\times m}$
$\mathcal{C}(\mathcal{I};\mathcal{S})$		the set of continuous functions $f : \mathcal{I} \to \mathcal{S}$ from an open set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\mathcal{C}^k(\mathcal{I};\mathcal{S})$		the set of k-times continuously differentiable functions $f : \mathcal{I} \to \mathcal{S}$ from an open set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\operatorname{dom} f$		the domain of the function f
$f\mid_{\mathcal{M}}$		the restriction of the function f on a set $\mathcal{M} \subseteq \operatorname{dom} f$
$A \le B$	$:\Leftrightarrow$	$\forall\, x \in \mathbb{R}^n :\ x^\top A x \le x^\top B x;\ \ A, B \in \mathbb{R}^{n\times n}$
$A(\cdot) \le_{\mathcal{U}} B(\cdot)$	$:\Leftrightarrow$	$\forall\, (t, x) \in \mathcal{U} :\ x^\top A(t)x \le x^\top B(t)x;\ \ A, B : (\tau,\infty) \to \mathbb{R}^{n\times n}$, $\tau \in [-\infty,\infty)$, $\mathcal{U} \subseteq (\tau,\infty) \times \mathbb{R}^n$
$A(\cdot) =_{\mathcal{U}} B(\cdot)$	$:\Leftrightarrow$	replace $\le$ by $=$ in the definition of $A(\cdot) \le_{\mathcal{U}} B(\cdot)$
$\mathcal{P}_{\mathcal{U}}$	$:=$	$\left\{ M : (\tau,\infty) \to \mathbb{R}^{n\times n} \ \middle|\ \begin{array}{l} M \text{ is continuous and symmetric,} \\ \exists\, m_1, m_2 > 0 :\ m_1 I_n \le_{\mathcal{U}} M(\cdot) \le_{\mathcal{U}} m_2 I_n \end{array} \right\}$ for $\mathcal{U} \subseteq (\tau,\infty) \times \mathbb{R}^n$

2 Solutions and singular behaviour

The concept of a solution and its extendability is introduced similarly to ODEs, see for example [Ama90, Sec. 5].

Definition 2.1 (Solutions)**.** Let $(E, A, f) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2 \times \mathcal{C}((\tau,\infty);\mathbb{R}^n)$ and $(a, b) \subseteq (\tau,\infty)$. A function $x : (a, b) \to \mathbb{R}^n$ is called

$$\textit{solution of } (E, A, f) \quad :\Longleftrightarrow \quad x \in \mathcal{C}^1((a, b);\mathbb{R}^n) \text{ and } x \text{ satisfies (1.1) for all } t \in (a, b).$$

A solution $\tilde{x} : (a, \tilde{b}) \to \mathbb{R}^n$ of (E, A, f) is called a

$$\textit{(right) extension of } x \quad :\Longleftrightarrow \quad \tilde{b} \geq b \text{ and } x = \tilde{x}\,|_{(a,b)}.$$

x is called

$$\begin{aligned}
\textit{right maximal} \quad &:\Longleftrightarrow \quad b = \tilde{b} \text{ for every extension } \tilde{x} : (a, \tilde{b}) \to \mathbb{R}^n \text{ of } x,\\
\textit{right global} \quad &:\Longleftrightarrow \quad b = \infty,\\
\textit{global} \quad &:\Longleftrightarrow \quad (a, b) = (\tau, \infty).
\end{aligned}$$

A right maximal solution $x : (a, b) \to \mathbb{R}^n$ of (E, A, f) which is not right global, i.e. $b < \infty$, is said to

$$\begin{aligned}
\text{have a } \textit{finite escape time} \quad &:\Longleftrightarrow \quad \limsup_{t \nearrow b} \|x(t)\| = \infty,\\
\text{be } \textit{non-extendable} \quad &:\Longleftrightarrow \quad x \text{ has no finite escape time.}
\end{aligned}$$

◇

To avoid confusion, note that the notion "non-extendable" is often used for solutions which are right maximal in our terms, see e.g. [Ama90, Har82].

Let $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$; then the set of all *right maximal solutions* of the initial value problem (E, A, f), $x(t^0) = x^0$ is denoted by

$$\begin{aligned}
\mathcal{S}_{E,A,f}(t^0, x^0) \quad &:= \quad \left\{ x : \mathcal{J} \to \mathbb{R}^n \;\middle|\; \begin{array}{l} \mathcal{J} \text{ open interval, } t^0 \in \mathcal{J},\ x(t^0) = x^0, \\ x(\cdot) \text{ is a right maximal solution of } (E, A, f) \end{array} \right\},\\
\mathcal{S}_{E,A}(t^0, x^0) \quad &:= \quad \mathcal{S}_{E,A,0}(t^0, x^0),
\end{aligned}$$

and the set of all *right global solutions* of (E, A, f), $x(t^0) = x^0$ by

$$\begin{aligned}
\mathcal{G}_{E,A,f}(t^0, x^0) \quad &:= \quad \{x(\cdot) \in \mathcal{S}_{E,A,f}(t^0, x^0) \mid x(\cdot) \text{ is right global solution of } (E, A, f)\},\\
\mathcal{G}_{E,A}(t^0, x^0) \quad &:= \quad \mathcal{G}_{E,A,0}(t^0, x^0).
\end{aligned}$$

The *set of all pairs of consistent initial values* of $(E, A) \in \mathcal{C}((\tau, \infty); \mathbb{R}^{n \times n})^2$ and the linear subspace of *initial values which are consistent at time* $t^0 \in (\tau, \infty)$, resp., is denoted by

$$\begin{aligned}
\mathcal{V}_{E,A} \quad &:= \quad \left\{ (t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n \;\middle|\; \exists \text{ (local) sln. } x(\cdot) \text{ of } (E, A):\ t^0 \in \operatorname{dom} x,\ x(t^0) = x^0 \right\},\\
\mathcal{V}_{E,A}(t^0) \quad &:= \quad \left\{ x^0 \in \mathbb{R}^n \;\middle|\; (t^0, x^0) \in \mathcal{V}_{E,A} \right\}.
\end{aligned}$$

Note that if $x : \mathcal{J} \to \mathbb{R}^n$ is a solution of (E, A), then $x(t) \in \mathcal{V}_{E,A}(t)$ for all $t \in \mathcal{J}$.

In the case of an ODE $\dot{x} = f(t, x)$, $f \in \mathcal{C}((\tau, \infty) \times \mathbb{R}^n; \mathbb{R}^n)$, there is only one possibility for the behaviour of a right maximal, but not right global, solution $x : (a, b) \to \mathbb{R}^n$ at its right endpoint b (see [Wal98, p. 68] for the case $n = 1$ and [Wal98, § 10, Thm. VI] for $n > 1$):

$$x \text{ has a finite escape time, i.e. } \limsup_{t \nearrow b} \|x(t)\| = \infty.$$

DAEs are very different in this respect; this is illustrated by the following example (from [KM06, Ex. 3.1] tailored for our purposes):

Example 2.2. Consider the real analytic initial value problem

$$E(t)\dot{x} = A(t)x + f(t),\ x(t^0) = 0,$$
$$\text{where}\quad E(t) := \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix},\ A(t) := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ f(t) := \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ t \in \mathbb{R},\ t^0 \in \mathbb{R}. \quad (2.1)$$

Note that the matrix pencil $\lambda E(t) - A(t)$ is regular for every $t \in \mathbb{R}$; recall (see, e.g., [KM06]) that a matrix pencil $sE - A \in \mathbb{R}^{n\times n}[s]$ is called *regular* if, and only if, $0 \neq \det(sE - A) \in \mathbb{R}[s]$.

Then $x : \mathcal{J} \to \mathbb{R}^n$ is a solution of (2.1) if, and only if, $\mathcal{J} \subseteq \mathbb{R}$ is an open interval and $x(t) = c(t)\begin{pmatrix} t \\ 1 \end{pmatrix}$, $t \in \mathcal{J}$, for some $c(\cdot) \in \mathcal{C}^1(\mathcal{J};\mathbb{R})$ with $c(t^0) = 0$. Therefore, (2.1) has uncountable many solutions which allow for the following scenario:

(i) (2.1) *has a global solution.* For example the trivial solution is a global solution of (2.1).

(ii) (2.1) *has a right maximal solution with finite escape time.* Choose $\omega \in (t^0, \infty)$ and let $c(t) = -\frac{1}{t-\omega} + \frac{1}{t^0-\omega}$, $t < \omega$. Then $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)(t,1)^\top$ is a solution of (2.1) and $\limsup_{t\nearrow\omega} \|x(t)\| = \infty$.

(iii) (2.1) *has a right maximal solution which has no finite escape time at* $\omega \in (t^0, \infty)$ *and is not continuous at* ω. Choose $c(t) = \sin\frac{a}{t-\omega}$, $t < \omega$, $a = \pi(t^0 - \omega)$. Then $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)(t,1)^\top$ is a solution of (2.1) and the limit $\lim_{t\nearrow\omega} x(t)$ does not exist.

(iv) (2.1) *has a right maximal solution which is continuous but not differentiable at a finite time* $\omega \in (t^0, \infty)$. Choose $c(t) = (t-\omega)\sin\frac{a}{t-\omega}$, $t < \omega$, $a = \pi(t^0 - \omega)$. Then $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)(t,1)^\top$ is a solution of (2.1) and the limit of the difference quotient $\lim_{t\nearrow\omega}\frac{x(t)-\tilde{x}}{t-\omega}$, where $\tilde{x} = \lim_{t\nearrow\omega} x(t)$, does not exist.

(v) (2.1) *has a right maximal solution which is continuous and differentiable at a finite time* $\omega \in (t^0, \infty)$, *but its derivative is not continuous at* ω. Choose $c(t) = (t-\omega)^2\sin\frac{a}{t-\omega}$, $t < \omega$, $a = \pi(t^0 - \omega)$. Then $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)(t,1)^\top$ is a solution of (2.1) and the limit $\lim_{t\nearrow\omega} \dot{x}(t)$ does not exist.

In (iii)-(v) there does not exist any extension of the solution over ω; this cannot occur in the case of an ODE. In particular, (iii)-(v) represent all (distinct) possibilities for the behaviour of any non-extendable solution of a system (E, A, f) at its right endpoint. $\diamond$

Example 2.3. The following example will show in particular that the property

$$x_1(\cdot) \in \mathcal{S}_{E,A,f}(t^0, x^1),\ x_2(\cdot) \in \mathcal{S}_{E,A,f}(t^0, x^2)$$
$$\Longrightarrow \quad \Big((x_1 - x_2) : \operatorname{dom} x_1 \cap \operatorname{dom} x_2 \to \mathbb{R}^n\Big) \in \mathcal{S}_{E,A}(t^0, x^1 - x^2), \quad (2.2)$$

which is trivial for ODEs, does in general not hold for DAEs (E, A, f). Property (2.2) means that the difference of two right maximal solutions of (E, A, f), defined on the intersection of their domains, is a right maximal solution of (E, A).
Consider the scalar equation

$$t\dot{x} = -tx + 1, \quad t \in \mathbb{R} \quad (2.3)$$

and the associated homogeneous equation

$$t\dot{x} = -tx, \quad t \in \mathbb{R}. \quad (2.4)$$

Then

$$\mathcal{S}_{(2.4)}(t^0,x^0) = \left\{ \ x:(t^-,\infty)\to\mathbb{R} \ \middle| \ t^- \in [-\infty,t^0),\ x(t) = \mathrm{e}^{-(t-t^0)}x^0 \ \right\}, \quad (t^0,x^0)\in\mathbb{R}^2. \tag{2.5}$$

To show that (2.2) does not hold, first observe that (2.3) reads $0=1$ for $t=0$, and hence a solution of (2.3) cannot be defined for $t=0$. System (2.3) becomes, in the regions $(-\infty,0)$ and $(0,\infty)$, the ODE

$$\dot{x} = -x + \frac{1}{t}, \tag{2.6}$$

and we find, for $x^0\in\mathbb{R}$,

$$\begin{aligned}&\mathcal{S}_{(2.3)}(t^0,x^0)\\ &\quad = \begin{cases} \left\{ \ x:(t^-,\ 0)\to\mathbb{R} \ \middle| \ t^-\in[-\infty,t^0),\ x(t) = \mathrm{e}^{-(t-t^0)}x^0 + \int_{t^0}^t \mathrm{e}^{-(t-s)}s^{-1}\,\mathrm{d}s \ \right\}, & t^0<0,\\ \left\{ \ x:(t^-,\infty)\to\mathbb{R} \ \middle| \ t^-\in[0,t^0),\ x(t) = \mathrm{e}^{-(t-t^0)}x^0 + \int_{t^0}^t \mathrm{e}^{-(t-s)}s^{-1}\,\mathrm{d}s \ \right\}, & t^0>0.\end{cases}\end{aligned}$$

Let

$$\begin{aligned} x_1 &: (-\infty,0)\to\mathbb{R}, \quad t\mapsto \mathrm{e}^{-(t-t^0)}x^1 + \int_{t^0}^t \mathrm{e}^{-(t-s)}s^{-1}\,\mathrm{d}s \qquad \text{for } t^0<0,\ x^1\in\mathbb{R},\\ x_2 &: (-\infty,0)\to\mathbb{R}, \quad t\mapsto \mathrm{e}^{-(t-t^0)}x^2 + \int_{t^0}^t \mathrm{e}^{-(t-s)}s^{-1}\,\mathrm{d}s \qquad \text{for } x^2\in\mathbb{R}\setminus\{x^1\}.\end{aligned}$$

Then $x(\cdot) := x_1(\cdot) - x_2(\cdot)$ with $\operatorname{dom} x = (-\infty,0) = \operatorname{dom} x_1 \cap \operatorname{dom} x_2$ is not a right maximal solution of (2.4), since it could be extended to all of $\mathbb{R}$, however $x_1(\cdot)$ and $x_2(\cdot)$ are right maximal solutions of (2.3).

Note that the solutions of any linear ODE cannot have a finite escape time for any continuous inhomogeneity. If we consider (2.3) in the region $(-\infty,0)$, then we obtain the ODE (2.6) and find that any global solution $x(\cdot)$ of (2.6) (on $(-\infty,0)$) fulfills $\lim_{t\nearrow 0}|x(t)|=\infty$. Hence system (2.6) has a singular point at $t=0$ where the inhomogeneity is not continuous. If we multiply the system by t and obtain the DAE (2.3), we keep the singular point but the inhomogeneity becomes continuous. $\diamond$

The following proposition shows that the shortcoming explained in Example 2.3 can be resolved by the mild assumption that $x_1(\cdot)$ or $x_2(\cdot)$ is right global; this is also important for stability results proved in Theorem 4.3.

Proposition 2.4 (Right maximal solutions)**.** *Consider the DAE $(E,A,f)\in\mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2\times\mathcal{C}((\tau,\infty);\mathbb{R}^n)$ and its associated homogeneous DAE (E,A). Then we have, for any $x^0,y^0\in\mathbb{R}^n$, $t^0>\tau$:*

(i) *If $x(\cdot)\in\mathcal{S}_{E,A,f}(t^0,x^0)$ is right global and $y(\cdot)\in\mathcal{S}_{E,A,f}(t^0,y^0)$, then $(x-y:\operatorname{dom}x\cap\operatorname{dom}y\to\mathbb{R}^n)\in\mathcal{S}_{E,A}(t^0,x^0-y^0)$.*

(ii) *If $x(\cdot)\in\mathcal{S}_{E,A,f}(t^0,x^0)$ is right global and $y(\cdot)\in\mathcal{S}_{E,A}(t^0,y^0)$, then $(x+y:\operatorname{dom}x\cap\operatorname{dom}y\to\mathbb{R}^n)\in\mathcal{S}_{E,A,f}(t^0,x^0+y^0)$.*

Proof: (i): Note that $z = x-y:\operatorname{dom}x\cap\operatorname{dom}y\to\mathbb{R}^n$ is a solution of the initial value problem

$$E(t)\dot{z} = A(t)z, \quad z(t^0) = x^0 - y^0\,.$$

Let $(\alpha,\omega) := \operatorname{dom} z(\cdot)$. If $\omega=\infty$, then the claim holds. Let $\omega<\infty$. Since $y(\cdot)$ is right maximal and $\omega = \sup\operatorname{dom} y(\cdot)$, there are 4 distinct possibilities for the behaviour of $y(\cdot)$ at ω (see also Example 2.2):

(a) $y(\cdot)$ has a finite escape time, i.e. $\limsup_{t\nearrow\omega}\|y(t)\| = \infty$,

(b) $y(\cdot)$ has no finite escape time and the limit $\lim_{t\nearrow\omega} y(t)$ does not exist,

(c) $y(\cdot)$ is continuous at ω ($\lim_{t\nearrow\omega} y(t)$ exists), but $\lim_{t\nearrow\omega} \frac{y(t)-\tilde{y}}{t-\omega}$, where $\tilde{y} = \lim_{t\nearrow\omega} y(t)$, does not exist,

(d) $y(\cdot)$ is continuous and differentiable at ω ($\tilde{y} = \lim_{t\nearrow\omega} y(t)$ and $\lim_{t\nearrow\omega} \frac{y(t)-\tilde{y}}{t-\omega}$ exist), but $\lim_{t\nearrow\omega} \dot{y}(t)$ does not exist.

Since $x(\cdot)$ is right global and therefore has no such singular behaviour at ω, the difference $z(\cdot)$ inherits the behaviour from $y(\cdot)$. Since the cases (a)-(d) are distinct it is easy to see that if $y(\cdot)$ fulfills one of them, then $z(\cdot)$ fulfills the same.
We show that $z(\cdot)$ is right maximal. Let $\mu : (\alpha, \tilde{\omega}) \to \mathbb{R}^n$ be an extension of $z(\cdot)$, i.e.

$$\omega \le \tilde{\omega} \quad \text{and} \quad z = \mu\mid_{(\alpha,\omega)} .$$

Then $\mu(\cdot)$ has the same singular behaviour as $z(\cdot)$ at ω and since $\mu(\cdot)$ is continuously differentiable (as a solution of (E, A)) it follows that $\tilde{\omega} \le \omega$ and hence $\omega = \tilde{\omega}$.
(ii): The proof is analogous and omitted. $\square$

3 Standard canonical form

In this section we introduce the subclass of DAEs (E, A) which are transferable into standard canonical from (SCF). We give a short summary and recall properties needed in the subsequent sections; for a detailed analysis and motivation of this class see [BI10] and the references therein.

Definition 3.1 (Equivalence of DAEs [KM06, Def. 3.3])**.** The DAEs $(E_1, A_1), (E_2, A_2) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2$ are called *equivalent* if, and only if, there exists $(S, T) \in \mathcal{C}((\tau,\infty); \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1((\tau,\infty); \mathbf{Gl}_n(\mathbb{R}))$ such that

$$E_2 = SE_1T\,, \quad A_2 = SA_1T - SE_1\dot{T}\,; \qquad \text{we write} \quad (E_1,\, A_1) \overset{S,T}{\sim} (E_2,\, A_2)\,. \tag{3.1}$$

$\diamond$

Definition 3.2 (Standard canonical form (SCF) [Cam83, CP83])**.** A system (E, A) is called *transferable into standard canonical form (SCF)* if, and only if, there exist $(S, T) \in \mathcal{C}((\tau,\infty); \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1((\tau,\infty); \mathbf{Gl}_n(\mathbb{R}))$ and $n_1, n_2 \in \mathbb{N}$ such that

$$(E,\, A) \overset{S,T}{\sim} \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right), \tag{3.2}$$

where $N : (\tau,\infty) \to \mathbb{R}^{n_2\times n_2}$ is pointwise strictly lower triangular and $J : (\tau,\infty) \to \mathbb{R}^{n_1\times n_1}$; a matrix N is called *pointwise strictly lower triangular* if, and only if, all entries of $N(t)$ on the diagonal and above are zero for all $t \in \mathcal{I}$. $\diamond$

Equivalence of DAEs is in fact an equivalence relation (see e.g. [KM06, Lem. 3.4]) and transferability into SCF as well as the constants n_1, n_2 are invariant under equivalence of DAEs (see [BI10, Thm. 2.1]).

In [BI10] we have shown that DAEs which are transferable into SCF allow for a generalized transition matrix; the main properties needed in the following sections are recalled:

Proposition 3.3 (Generalized transition matrix $U(\cdot\,,\cdot)$)**.** *Let $(E, A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ be transferable into SCF for (S, T) as in Definition 3.2. Then any solution of the initial value problem (E, A), $x(t^0) = x^0$, where $(t^0, x^0) \in \mathcal{V}_{E,A}$, extends uniquely to a global solution $x(\cdot)$; this solution satisfies*

$$x(t) = U(t,t^0)x^0, \quad \textit{where} \quad U(t,t^0) := T(t)\begin{bmatrix}\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix}T(t^0)^{-1}, \quad t\in(\tau,\infty), \tag{3.3}$$

and $\Phi_J(\cdot,\cdot)$ denotes the transition matrix of $\dot z = J(t)z$; $U(\cdot,\cdot)$ is called the generalized transition matrix *of (E, A) and does not depend on the choice of (S, T) in* (3.2)*; it satisfies, for all $t, r, s \in (\tau,\infty)$,*

(i) $E(t)\frac{\mathrm{d}}{\mathrm{d}t}U(t,s) = A(t)\,U(t,s)$,

(ii) $\operatorname{im} U(t,s) = \mathcal{V}_{E,A}(t)$,

(iii) $U(t,r)U(r,s) = U(t,s)$,

(iv) $U(t,t)^2 = U(t,t)$,

(v) $\forall\, x \in \mathcal{V}_{E,A}(t):\ U(t,t)x = x$,

(vi) $\frac{\mathrm{d}}{\mathrm{d}t}U(s,t) = -U(s,t)\,T(t)\,S(t)\,A(t)$.

Proof: It remains to show Property (vi), all others are shown in [BI10, Sect. 3]. We exploit

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(T^{-1}) = -T^{-1}\dot T T^{-1}, \tag{3.4}$$

which follows from differentiation of the identity $I = T^{-1}T$, and that

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}U(s,t) &\overset{(3.3)}{=} T(s)\begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_J(s,t) & 0\\ 0 & 0\end{bmatrix}T(t)^{-1} + T(s)\begin{bmatrix}\Phi_J(s,t) & 0\\ 0 & 0\end{bmatrix}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(T(t)^{-1}\right)\\
&\overset{(3.3)}{=} T(s)\begin{bmatrix}-\Phi_J(s,t)\,J(t) & 0\\ 0 & 0\end{bmatrix}T(t)^{-1} + U(s,t)\,T(t)\,\tfrac{\mathrm{d}}{\mathrm{d}t}\left(T(t)^{-1}\right)\\
&\overset{(3.3)}{=} U(s,t)\,T(t)\begin{bmatrix}-J(t) & 0\\ 0 & -I_{n_2}\end{bmatrix}T(t)^{-1} + U(s,t)\,T(t)\,\tfrac{\mathrm{d}}{\mathrm{d}t}\left(T(t)^{-1}\right)\\
&\overset{(3.1)}{=} U(s,t)\,T(t)\left\{-S(t)\,[A(t) - E(t)\,\dot T(t)\,T(t)^{-1}] + \tfrac{\mathrm{d}}{\mathrm{d}t}\left(T(t)^{-1}\right)\right\}\\
&\underset{(3.4)}{\overset{(3.1)}{=}} -U(s,t)\,T(t)\,S(t)\,A(t) - U(s,t)\,T(t)\,[S(t)E(t)T(t) - I_n]\,\tfrac{\mathrm{d}}{\mathrm{d}t}\left(T(t)^{-1}\right)\\
&\underset{(3.3)}{\overset{(3.2)}{=}} -U(s,t)\,T(t)\,S(t)\,A(t) - T(s)\begin{bmatrix}\Phi_J(s,t) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & N(t)-I\end{bmatrix}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(T(t)^{-1}\right)\\
&= -U(s,t)\,T(t)\,S(t)\,A(t)\,.
\end{aligned}$$

□

For later use we also record the following elementary properties.

Proposition 3.4. *Let $(E, A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ be transferable into SCF for (S, T) as in Definition 3.2. Then*

(i) $(t, x^0) \in \mathcal{V}_{E,A} \iff x^0 \in \operatorname{im} T(t)\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix}$,

(ii) $(t,x^0)\in\mathcal{V}_{E,A} \iff T(t)S(t)E(t)x^0 = x^0$,

(iii) $\forall\, t>\tau:\ \mathcal{V}_{E,A}(t)\cap\ker E(t)=\{0\}$,

(iv) $\forall\,(t^0,x^0)\in\mathcal{V}_{E,A}\ \forall\, t>\tau\ :\ \big[E(t)U(t,t^0)x^0=0 \quad\iff\quad U(t,t^0)x^0=0\big]$.

Proof: (i): See [BI10, Prop. 3.2(i)].
(ii): To show "$\Rightarrow$" let $(t,x^0)\in\mathcal{V}_{E,A}$. Then by (i) there exists $v^0\in\mathbb{R}^{n_1}$ such that $x^0=T(t)\,\left(\begin{smallmatrix}v^0\\0\end{smallmatrix}\right)$, and therefore

$$T(t)S(t)E(t)x^0 = T(t)S(t)E(t)T(t)\begin{pmatrix}v^0\\0\end{pmatrix} \overset{(3.2)}{\underset{(3.1)}{=}} T(t)\begin{bmatrix}I_{n_1} & 0\\ 0 & N(t)\end{bmatrix}\begin{pmatrix}v^0\\0\end{pmatrix} = x^0 .$$

To show "$\Leftarrow$", let $T(t^0)S(t^0)E(t^0)x^0=x^0$. Then

$$T(t^0)^{-1}x^0 = S(t^0)E(t^0)T(t^0)T(t^0)^{-1}x^0 = \begin{bmatrix}I & 0\\ 0 & N(t^0)\end{bmatrix}T(t^0)^{-1}x^0,$$

thus having $\begin{bmatrix}0 & 0\\ 0 & I-N(t^0)\end{bmatrix}T(t^0)^{-1}x^0=0$, which gives $x^0\in\operatorname{im}T(t^0)\begin{bmatrix}I_{n_1}\\0\end{bmatrix}\overset{\text{(i)}}{=}\mathcal{V}_{E,A}(t^0)$.
(iii) is a consequence of (ii) and (iv) finally follows from (iii) and Proposition 3.3(ii). $\square$

4 Stability

In this section we introduce a stability theory for DAEs $(E,A,f)\in\mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2\times\mathcal{C}((\tau,\infty);\mathbb{R}^n)$. Since the system is linear, it suffices – analogous to ODEs – to consider the stability behaviour of the zero solution of the homogeneous part (E,A); this is proved in Theorem 4.3. Further characterizations of stability are shown for the subclass of DAEs transferable into standard canonical form.

Definition 4.1 (Stability)**.** A right global solution $x:(a,\infty)\to\mathbb{R}^n$ of $(E,A,f)\in\mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2\times\mathcal{C}((\tau,\infty);\mathbb{R}^n)$, $a\geq\tau$, is said to be

$$\begin{array}{rcl} \textit{stable} & :\iff & \forall\,\varepsilon>0\ \forall\, t^0>a\ \exists\,\delta>0\ \forall\, y^0\in\mathcal{B}_\delta(x(t^0))\ \forall\, y(\cdot)\in\mathcal{S}_{E,A,f}(t^0,y^0):\\ & & [t^0,\infty)\subseteq\operatorname{dom}y \quad\wedge\quad \forall\, t\geq t^0: y(t)\in\mathcal{B}_\varepsilon(x(t)).\\ \textit{attractive} & :\iff & \forall\, t^0>a\ \exists\,\eta>0\ \forall\, y^0\in\mathcal{B}_\eta(x(t^0))\ \forall\, y(\cdot)\in\mathcal{S}_{E,A,f}(t^0,y^0):\\ & & [t^0,\infty)\subseteq\operatorname{dom}y \quad\wedge\quad \lim_{t\to\infty}(y(t)-x(t))=0.\\ \textit{asymptotically stable} & :\iff & x(\cdot)\text{ is stable and attractive.}\\ \textit{exponentially stable} & :\iff & \exists\,\alpha,\beta>0\ \forall\, t^0>a\ \exists\,\eta>0\ \forall\, y^0\in\mathcal{B}_\eta(x(t^0))\ \forall\, y(\cdot)\in\mathcal{S}_{E,A,f}(t^0,y^0):\\ & & [t^0,\infty)\subseteq\operatorname{dom}y\ \wedge\ \forall\, t\geq t^0: \|y(t)-x(t)\|\leq\alpha e^{-\beta(t-t^0)}\|y(t^0)-x(t^0)\|. \end{array}$$

Remark 4.2.

(i) Note that stability does neither imply that every initial value problem is solvable in the neighborhood of the considered solution nor does it mean that a possibly existing solution has to be unique; the only requirement is that every existing solution in a neighborhood of the considered one stays in an ε-neighborhood of it.

(ii) If the trivial solution of the homogeneous DAE (E,A) is stable, then – opposed to linear ODEs – a solution of the inhomogeneous system (E,A,f) is not necessarily stable. To see this, revisit

Example 2.3. By (2.5), the trivial solution of (2.4) is exponentially stable. Since

$$\lim_{t \nearrow 0} \int_{-1}^{t} s^{-1} \mathrm{e}^{-(t-s)} \, \mathrm{d}s = -\infty,$$

it follows that

$$\left(x : (-1,0) \to \mathbb{R}^n, \quad t \mapsto \mathrm{e}^{-(t-1)} + \int_{-1}^{t} s^{-1} \mathrm{e}^{-(t-s)} \, \mathrm{d}s \right) \in \mathcal{S}_{(2.3)}(-1,1)$$

has a finite escape time; therefore it cannot be exponentially stable.
However, an inspection of $\mathcal{S}_{(2.3)}(t^0, x^0)$ for $t^0 > 0$ reveals that every right global solution of (2.3) is exponentially stable.

(iii) If (E, A) is transferable into SCF and the J-block in the SCF does not exist, i.e. $n_1 = 0$, then

$$\forall\, t^0 > \tau : \; U(\,\cdot\,, t^0) \equiv 0,$$

and Proposition 3.3 yields that (E, A) is exponentially stable. ⋄

It is well known (see, for example, [Aul04, Satz 7.5.1]) that for ODEs it suffices to consider the stability behaviour of the zero solution. For time-varying DAEs one has to be, due to the difference between maximal and global solutions, more careful. However, we show that the analogous result also holds true and stress that no extra assumptions are made on (E, A, f) and its solutions.

Theorem 4.3 (Uniform stability behaviour of all right global solutions)**.** *Consider the inhomogeneous DAE $(E, A, f) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2 \times \mathcal{C}((\tau,\infty); \mathbb{R}^n)$ and the associated homogeneous DAE (E, A).*

(i) *If the trivial solution of (E, A), restricted to (α, ∞) for some $\alpha \geq \tau$, has one of the properties {stable, attractive, asymptotically stable, exponentially stable}, then every right global solution $x : (\beta, \infty) \to \mathbb{R}^n$ of (E, A, f) with $\beta \geq \alpha$ has the respective property.*

(ii) *If there exists a right global solution $x(\cdot)$ of (E, A, f) with one of the properties {stable, attractive, asymptotically stable, exponentially stable}, then the trivial solution of (E, A), restricted to* $\operatorname{dom} x(\cdot)$*, has the respective property.*

Proof: We prove the claim for stability, the other concepts are proved similarly.
(i): Let the trivial solution of (E, A), restricted to (α, ∞) for some $\alpha \geq \tau$, be stable and let $\mu : (\beta, \infty) \to \mathbb{R}^n$ be a right global solution of (E, A, f) , $\beta \geq \alpha$. We show that $\mu(\cdot)$ is stable.
Let $\varepsilon > 0$ and $t^0 > \beta$. Since the trivial solution of (E, A), restricted to (α, ∞), is stable, Definition 4.1 yields

$$\exists\, \delta > 0 \; \forall\, y^0 \in \mathcal{B}_\delta(0) \; \forall\, y(\cdot) \in \mathcal{S}_{E,A}(t^0, y^0) \;:\; y(\cdot) \text{ is right global } \wedge \left[\forall\, t \geq t^0 : \; y(t) \in \mathcal{B}_\varepsilon(0) \right]. \tag{4.1}$$

Let $\eta \in \mathcal{B}_\delta(\mu(t^0))$. If $\mathcal{S}_{E,A,f}(t^0, \eta) = \emptyset$, then the claim holds. Let $\lambda(\cdot) \in \mathcal{S}_{E,A,f}(t^0, \eta)$. By Proposition 2.4 (i) and since $t^0 \in \operatorname{dom} \lambda \cap \operatorname{dom} \mu$, we have

$$(\mu - \lambda : \operatorname{dom} \lambda \cap \operatorname{dom} \mu \to \mathbb{R}^n) \in \mathcal{S}_{E,A}(t^0, \mu(t^0) - \eta)\,.$$

Then $\mu(t^0) - \eta \in \mathcal{B}_\delta(0)$ and (4.1) yield that $(\mu - \lambda)(\cdot)$ is right global, and hence $\lambda(\cdot)$ must be right global, and

$$\left[\forall\, t \geq t^0 : \; \lambda(t) - \mu(t) \in \mathcal{B}_\varepsilon(0) \right] \quad \Longrightarrow \quad \left[\forall\, t \geq t^0 : \; \lambda(t) \in \mathcal{B}_\varepsilon(\mu(t)) \right]$$

and therefore $\mu(\cdot)$ is stable.
(ii): Let $\mu : \mathcal{J} \to \mathbb{R}^n$ be a right global and stable solution of (E, A, f) . We show that the trivial solution of (E, A), restricted to $\mathcal{J}$, is stable.
Let $\varepsilon > 0$ and $t^0 \in \mathcal{J}$. Since $\mu(\cdot)$ is stable, Definition 4.1 yields

$$\exists \delta > 0\, \forall y^0 \in \mathcal{B}_\delta(\mu(t^0))\, \forall y(\cdot) \in \mathcal{S}_{E,A,f}(t^0, y^0) \;:\; y(\cdot) \text{ is right global} \wedge \forall t \geq t^0 \;:\; y(t) \in \mathcal{B}_\varepsilon(\mu(t)). \quad (4.2)$$

Let $\eta \in \mathcal{B}_\delta(0)$. If $\mathcal{S}_{E,A}(t^0, \eta) = \emptyset$, then the claim holds. Let $\lambda(\cdot) \in \mathcal{S}_{E,A}(t^0, \eta)$. By Proposition 2.4 (ii) and since $t^0 \in \operatorname{dom} \lambda \cap \operatorname{dom} \mu$ we have

$$(\mu + \lambda : \operatorname{dom} \lambda \cap \operatorname{dom} \mu \to \mathbb{R}^n) \in \mathcal{S}_{E,A,f}(t^0, \mu(t^0) + \eta)\,.$$

Then $\mu(t^0) + \eta \in \mathcal{B}_\delta(\mu(t^0))$ and (4.2) yield that $(\mu + \lambda)(\cdot)$ is right global, and hence $\lambda(\cdot)$ must be right global, and

$$\big[\forall\, t \geq t^0 :\; \mu(t) + \lambda(t) \in \mathcal{B}_\varepsilon(\mu(t))\big] \quad \Longrightarrow \quad \big[\forall\, t \geq t^0 :\; \lambda(t) \in \mathcal{B}_\varepsilon(0)\big]$$

and therefore the trivial solution of (E, A), restricted to $\mathcal{J}$, is stable. □

Theorem 4.3 justifies (similar to linear ODEs) the following definition.

Definition 4.4. The DAE $(E, A, f) \in \mathcal{C}((\tau, \infty); \mathbb{R}^{n\times n})^2 \times \mathcal{C}((\tau, \infty); \mathbb{R}^n)$ is called *stable*, *attractive*, *asymptotically stable* or *exponentially stable* if, and only if, the global trivial solution of (E, A) has the respective property. ◇

We will show that previous stability concepts can be characterized similar to ODEs if (E, A) is transferable into SCF; first, the latter is discussed in the following remark.

Remark 4.5 (Transferable into SCF)**.**

(i) If the DAE (E, A) is time-invariant, i.e. $(E, A) \in (\mathbb{R}^{n\times n})^2$, then

$$(E, A) \text{ is exp. stable} \implies sE - A \in \mathbb{R}^{n\times n}[s] \text{ is regular} \implies (E, A) \text{ is transferable into SCF.}$$

To see this, assume that $sE - A$ is not regular, then there exist $\lambda > 0$ and $x^0 \in \mathbb{R}^n \setminus \{0\}$ such that $(\lambda E - A)x^0 = 0$ and hence the unstable function $t \mapsto e^{\lambda t} x^0$ solves (E, A), a contradiction. The second implication is Weierstraß' result, see [KM06, Thm. 2.7].

(ii) If $(E, A) \in \mathcal{C}((\tau, \infty); \mathbb{R}^{n\times n})^2$ is exponentially stable, then it is not necessarily transferable into SCF. Consider the analytic DAE

$$0 \cdot \dot{x} = t\, x\,, \qquad t \in \mathbb{R} \quad (4.3)$$

which is exponentially stable: any solution $x : \mathcal{J} \to \mathbb{R}$ of (4.3) fulfills $x(t) = 0$ for all $t \in \mathcal{J} \setminus \{0\}$ and since the solutions must be continuous it follows that $x \equiv 0$. We also have $\mathcal{G} = \mathbb{R} \times \{0\}$. However, if (4.3) were transferable into SCF, then

$$SET = 0 \quad \text{and} \quad SAT - SE\dot{T} = 1 \qquad \text{for some } S, T : \mathbb{R} \to \mathbb{R} \setminus \{0\}.$$

But evaluation at $t = 0$ gives $S(0)A(0)T(0) - S(0)E(0)\dot{T}(0) = 0$, a contradiction. ◇

In the following theorem we consider DAEs which are transferable into SCF and characterize, exploiting the existence of a generalized transition matrix, the different stability concepts. The proofs use similar arguments as for ODEs; see, for example, [Aul04, Satz 7.5.3].

Theorem 4.6 (Stability). *Suppose system $(E,A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ is transferable into SCF and let $U(\cdot,\cdot)$ denote the generalized transition matrix of (E,A). Then the following characterizations hold:*

(i) *(E,A) is stable* $\iff$ $\forall\, t^0 > \tau\ \exists\, M \geq 0\ \forall\, x^0 \in \mathcal{V}_{E,A}(t^0)\ \forall\, t \geq t^0 :\ \|U(t,t^0)x^0\| \leq M\|x^0\|$.

(ii) *The following are equivalent:*

(a) *(E,A) is attractive.*

(b) *(E,A) is asymptotically stable.*

(c) *Every global solution $x : (\tau,\infty) \to \mathbb{R}^n$ of (E,A) satisfies* $\lim_{t\to\infty} x(t) = 0$.

(iii) *(E,A) is exp. stable* $\iff$ $\exists\, \alpha,\beta > 0\ \forall\, (t^0,x^0) \in \mathcal{V}_{E,A}\ \forall\, t \geq t^0 :\ \|U(t,t^0)x^0\| \leq \alpha\, \mathrm{e}^{-\beta(t-t^0)}\|x^0\|$.

Proof: By Remark 4.2(iii), we may assume $n_1 > 0$.
(i): Let (E,A) be stable, $t^0 > \tau$, and $\varepsilon = 1$. By Definition 4.1 and Proposition 3.3, there exists $\delta = \delta(t^0) > 0$ such that

$$\forall\, x^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}_{E,A}(t^0)\ \forall\, t \geq t^0 :\ \|U(t,t^0)x^0\| \leq 1. \tag{4.4}$$

Define $M := 2/\delta$ and let $x^0 \in \mathcal{V}_{E,A}(t^0)$. If $x^0 = 0$, then $U(t,t^0)x^0 = 0$ for all $t \geq t^0$. If $x^0 \neq 0$, then

$$\forall\, t \geq t^0 :\ \left\| U(t,t^0)\frac{\delta x^0}{2\|x^0\|} \right\| \overset{(4.4)}{\leq} \frac{2}{\delta}\cdot\frac{\delta}{2} = M \left\| \frac{\delta x^0}{2\|x^0\|} \right\|,$$

which is equivalent to the right hand side of the equivalence. The converse is immediate from the definition of stability.
(ii): "(a)$\Rightarrow$(b)": Let $\varepsilon > 0$ and $t^0 > \tau$. Attractivity of (E,A) gives

$$\exists\, \delta = \delta(t^0) > 0\ \forall\, x^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}_{E,A}(t^0)\ \forall\, x(\cdot) \in \mathcal{S}_{E,A}(t^0,x^0) :\ 0 = \lim_{t\to\infty} x(t) = \lim_{t\to\infty} U(t,t^0)x^0.$$

For

$$X^0 := \frac{\delta}{2\|T(t^0)\|} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$$

we have, in view of Proposition 3.4(i), $X_i^0 \in \mathcal{V}_{E,A}(t^0)$ for all $i = 1,\ldots,n_1$, and, since $\|X^0\| < \delta$, we obtain $X_i^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}_{E,A}(t^0)$ for all $i = 1,\ldots,n_1$. Therefore,

$$0 = \lim_{t\to\infty} U(t,t^0)X^0 = \frac{\delta}{2\|T(t^0)\|} \lim_{t\to\infty} T(t) \begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}.$$

From this it follows that $\lim_{t\to\infty} U(t,t^0) = 0$ and hence there exists $\lambda = \lambda(t^0) > 0$ such that

$$\forall\, t \geq t^0 :\ \|U(t,t^0)\| \leq \lambda.$$

Define $\eta = \eta(\varepsilon,t^0) := \varepsilon/\lambda$. Then

$$\forall\, x^0 \in \mathcal{B}_\eta(0) \cap \mathcal{V}_{E,A}(t^0)\ \forall\, x(\cdot) \in \mathcal{S}_{E,A}(t^0,x^0)\ \forall\, t \geq t^0 :\ \|x(t)\| = \|U(t,t^0)x^0\| \leq \|U(t,t^0)\|\|x^0\| < \lambda\frac{\varepsilon}{\lambda} = \varepsilon.$$

Therefore (E,A) is stable.
"(b)$\Rightarrow$(c)": Let $(t^0,x^0) \in \mathcal{V}_{E,A}$ and $x(\cdot)$ be the global solution of (E,A), $x(t^0) = x^0$. Since (E,A) is attractive in particular, it follows, as in the proof of "(a)$\Rightarrow$(b)", that

$$\lim_{t\to\infty} U(t,t^0) = 0, \qquad \text{and thus} \qquad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} U(t,t^0)x^0 = 0.$$

"(c)⇒(a)": By Proposition 3.3 every local solution of (E,A) extends uniquely to a global solution, thus every right maximal solution is right global. Then attractivity of (E,A) follows immediately.
(iii): Let (E,A) be exponentially stable and let $(t^0,x^0)\in\mathcal{V}_{E,A}$. We use Proposition 3.3. If $x^0=0$, then $U(t,t^0)x^0=0$ for all $t\geq t^0$ and by Definition 4.1 we have

$$\exists\,\alpha,\beta>0\;\exists\,\delta=\delta(t^0)>0\;\forall\,y^0\in\mathcal{B}_\delta(0)\cap\mathcal{V}_{E,A}(t^0)\;\forall\,t\geq t^0:\;\|U(t,t^0)y^0\|\leq\alpha\mathrm{e}^{-\beta(t-t^0)}\|y^0\|. \tag{4.5}$$

If $x^0\neq 0$ then (4.5) gives

$$\forall\,t\geq t^0\;:\;\left\|U(t,t^0)\frac{\delta x^0}{2\|x^0\|}\right\|\leq\alpha\mathrm{e}^{-\beta(t-t^0)}\left\|\frac{\delta x^0}{2\|x^0\|}\right\|$$

which is equivalent to the right hand side of the equivalence. The converse follows immediately.
This completes the proof of the theorem. □

Remark 4.7. Theorem 4.6 does not hold true for systems which are not transferable into SCF: Consider the initial value problem

$$t\dot{x}=(1-t)x,\quad x(t^0)=x^0,\qquad t\in\mathbb{R}\,, \tag{4.6}$$

for $(t^0,x^0)\in\mathbb{R}^2$. In passing, note that $t\mapsto(E(t),A(t))=(t,t-1)$ is real analytic. For $t^0\neq 0$, $x^0\in\mathbb{R}$, the unique global solution $x(\cdot)$ of (4.6) is

$$x:\mathbb{R}\to\mathbb{R},\;t\mapsto\frac{te^{-t}}{t^0e^{-t^0}}x^0.$$

For $t^0=x^0=0$ the problem (4.6) has infinitely many global solutions and every (local) solution $x:\mathcal{J}\to\mathbb{R}$ extends uniquely to a global solution

$$x_c:\mathbb{R}\to\mathbb{R},\;t\mapsto cte^{-t},\qquad\text{where } c=\frac{e^T}{T}\,x(T)\text{ for some } T\in\mathcal{J}\setminus\{0\}.$$

The solutions $x_c(\cdot)$ are the only global solutions of the initial value problem (4.6), $t^0=x^0=0$. Furthermore, any initial value problem (4.6), $t^0=0$, $x^0\neq 0$ does not have a solution. Therefore, the zero solution is attractive, but not asymptotically stable. ◇

In the remainder of this section we give sufficient conditions so that the stability behaviour of the DAE (E,A) is not changed under equivalence of DAEs. We introduce Lyapunov transformations (see for example [Rug96, Def. 6.14] for ODEs) on the set of all pairs of consistent initial values.

Definition 4.8 (Lyapunov transformation)**.** Let $(E,A)\in\mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$. Then $T\in\mathcal{C}^1((\tau,\infty);\mathbf{Gl}_n(\mathbb{R}))$ is called a *Lyapunov transformation on* $\mathcal{V}_{E,A}$ if, and only if,

$$\boxed{T(\cdot)^{-\top}T(\cdot)^{-1}\in\mathcal{P}_{\mathcal{V}_{E,A}}}\,. \tag{4.7}$$

◇

A state space transformation T is a Lyapunov transformation on $\mathcal{V}_{E,A}$ if, and only if,

$$\exists\,p_1,p_2>0\;\forall\,(t,x)\in\mathcal{V}_{E,A}\;:\;p_1\,\|x\|^2\leq\|T(t)^{-1}x\|^2\leq p_2\,\|x\|^2\,. \tag{4.8}$$

If

$$(E,A)\overset{S,T}{\sim}(\tilde{E},\tilde{A})\,,\qquad\text{for } (S,T)\in\mathcal{C}((\tau,\infty);\mathbf{Gl}_n(\mathbb{R}))\times\mathcal{C}^1((\tau,\infty);\mathbf{Gl}_n(\mathbb{R})),$$

and T is a Lyapunov transformation on $\mathcal{V}_{E,A}$, then in particular $x(\cdot)$ solves (E,A) if, and only if, $z(t)=T(t)^{-1}x(t)$ satisfies $(\tilde{E},\tilde{A})$. In view of $T(t)^{-1}\mathcal{V}_{E,A}(t)=\mathcal{V}_{\tilde{E},\tilde{A}}(t)$ for $t>\tau$, we see that (4.8) is equivalent to

$$\exists\, p_1,p_2>0\ \forall\,(t,z)\in\mathcal{V}_{\tilde{E},\tilde{A}}\ :\ p_2^{-1}\,\|z\|^2\le\|T(t)\,z\|^2\le p_1^{-1}\,\|z\|^2. \tag{4.9}$$

If (E,A) is an ODE, then $\mathcal{V}_{E,A}=(\tau,\infty)\times\mathbb{R}^n$. Therefore, in this case the boundedness condition (4.7) on the subspace of consistent initial values is equivalent to boundedness of $T(\cdot)$ and $T(\cdot)^{-1}$; the latter is called Lyapunov transformation in [Rug96, Def. 6.14].

We are now ready to state the proposition.

Proposition 4.9 (Stability behaviour is preserved under Lyapunov transformation)**.** *Suppose system $(E,A)\in\mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ is transferable into SCF as in Definition 3.2. If*

$$(E,\,A)\ \overset{S,T}{\sim}\ (\tilde{E},\,\tilde{A})\qquad\textit{for some } S\in\mathcal{C}((\tau,\infty);\mathbf{Gl}_n(\mathbb{R})),\ T\in\mathcal{C}^1((\tau,\infty);\mathbf{Gl}_n(\mathbb{R}))$$

and T is a Lyapunov transformation on $\mathcal{V}_{E,A}$, then

(i) *(E,A) is stable $\iff$ $(\tilde{E},\tilde{A})$ is stable.*

(ii) *(E,A) is attractive $\iff$ $(\tilde{E},\tilde{A})$ is attractive.*

(iii) *(E,A) is asymptotically stable $\iff$ $(\tilde{E},\tilde{A})$ is asymptotically stable.*

(iv) *(E,A) is exponentially stable $\iff$ $(\tilde{E},\tilde{A})$ is exponentially stable.*

Proof: Let p_1,p_2 be as in (4.8) and (4.9).
(i): To see "$\Rightarrow$" set, for given $\varepsilon>0$ and $t^0>\tau$, $\tilde{\varepsilon}:=\frac{\varepsilon}{\sqrt{p_2}}$ and then obtain, due to stability of (E,A), $\tilde{\delta}=\tilde{\delta}(\tilde{\varepsilon},t^0)$. Then $\delta:=\sqrt{p_1}\tilde{\delta}$ gives stability of $(\tilde{E},\tilde{A})$. The proof for "$\Leftarrow$" is analogous and omitted.
By Theorem 4.6(ii) attractivity and asymptotic stability of (E,A) are equivalent to

$$\forall \text{ global sln. } x:(\tau,\infty)\to\mathbb{R}^n \text{ of } (E,A):\ \lim_{t\to\infty}x(t)=0.$$

Then, invoking the boundedness condition (4.7), assertions (ii) and (iii) follow immediately.
To prove (iv), let $\tilde{U}(\cdot,\cdot)$ denote the generalized transition matrix of $(\tilde{E},\tilde{A})$. As a consequence of the uniqueness of the generalized transition matrix (see Proposition 3.3) we find

$$\forall\, s,t>\tau:\ \tilde{U}(t,s)=T(t)^{-1}U(t,s)T(s). \tag{4.10}$$

Then in view of Theorem 4.6(iii) we may conclude

$$\begin{aligned}
&\forall\,(t^0,z^0)\in\mathcal{V}_{\tilde{E},\tilde{A}}\ \forall\, t\ge t^0:\\
&\qquad\|\tilde{U}(t,t^0)z^0\|^2 \overset{(4.10)}{=} \|T(t)^{-1}U(t,t^0)T(t^0)z^0\|^2 \overset{(4.8)}{\le} p_2\,\|U(t,t^0)T(t^0)z^0\|^2\\
&\qquad\qquad\le p_2\,\alpha^2\,\mathrm{e}^{-2\beta(t-t^0)}\|T(t^0)z^0\|^2 \overset{(4.9)}{\le} \frac{p_2}{p_1}\,\alpha^2\,\mathrm{e}^{-2\beta(t-t^0)}\|z^0\|^2\,,
\end{aligned}$$

and hence (iv) follows. □

As an immediate consequence of Proposition 4.9 we obtain that the stability behaviour of (E,A) is inherited from the stability behaviour of the underlying ODE in the SCF.

Corollary 4.10 (Stability behaviour is inherited from subsystem)**.** *Let $(E,A)\in\mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ be transferable into SCF as in Definition 3.2 and suppose T is a Lyapunov transformation on $\mathcal{V}_{E,A}$. Then (E,A) has one of the properties {stable, attractive, asymptotically stable, exponentially stable} if, and only if, either $n_1=0$ or the ODE $\dot{z}=J(t)z$ has the respective property.*

5 Lyapunov equations and Lyapunov functions

In this section we develop a version of Lyapunov's direct method for DAEs as well as the converse of the stability theorems; stronger results are achieved if the considered DAE is transferable into SCF, in this case the existence of the generalized transition matrix is exploited. All results are generalizations of the corresponding results for time-varying ODEs (see for example [HP05, Sec. 3]) and time-invariant DAEs: see e.g. [OD85] and [Sty02] (confer also Remark 5.13); a good overview is given in [DVP07].

5.1 General results

We start with introducing Lyapunov functions for time-varying DAEs $(E, A) \in \mathcal{C}((\tau, \infty); \mathbb{R}^{n\times n})^2$; these functions are defined on the set of all initial values (t, x) for which (E, A) has a right global solution:

$$\mathcal{G}(E, A) \ := \ \{ \ (t, x) \in (\tau, \infty) \times \mathbb{R}^n \ \mid \mathcal{G}_{E,A}(t, x) \neq \emptyset \ \} \, .$$

Definition 5.1 (Lyapunov function)**.** Let $(E, A) \in \mathcal{C}((\tau, \infty); \mathbb{R}^{n\times n})^2$. A function $V : \mathcal{G}(E, A) \to \mathbb{R}$ is called *Lyapunov function for* (E, A) if, and only if,

$$\exists\, \ell_1, \ell_2 > 0 \ \forall\, (t, x) \in \mathcal{G}(E, A) : \ \ell_1 \|x\|^2 \le V(t, x) \le \ell_2 \|x\|^2 \tag{5.1}$$

and

$$\exists\, \lambda > 0 \ \forall\, (t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n \ \forall\, x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0) \ \forall\, t \ge t^0 : \tfrac{\mathrm{d}}{\mathrm{d}t} V(t, x(t)) \le -\lambda V(t, x(t)). \tag{5.2}$$

$\diamond$

We stress that we consider Lyapunov functions for (E, A) on $\mathcal{G}(E, A)$, not on $(\tau, \infty) \times \mathbb{R}^n$. The reason is that the set

$$\mathcal{G}(E, A)(t) \ := \ \{ \ x \in \mathbb{R}^n \ \mid (t, x) \in \mathcal{G}(E, A) \ \} \, , \qquad t > \tau,$$

is a linear subspace of $\mathbb{R}^n$ and if $x : (a, \infty) \to \mathbb{R}^n$ is a right global solution of (E, A), then $x(t) \in \mathcal{G}(E, A)(t)$ for all $t > a$.

The next theorem shows that the existence of a Lyapunov function for (E, A) yields a sufficient condition for "almost" exponential stability of the trivial solution of (E, A). "Almost" in the sense that we cannot guarantee that every existing right maximal solution in a neighborhood of the trivial solution is right global. But we can guarantee that all right global solutions tend exponentially to zero. In this sense it is a DAE-version of Lyapunov's direct method (cf. [HP05, Cor. 3.2.20] in the case of ODEs).

Theorem 5.2 (Lyapunov's direct method)**.** *Let* $(E, A) \in \mathcal{C}((\tau, \infty); \mathbb{R}^{n\times n})^2$*. If there exists a Lyapunov function for* (E, A)*, then*

$$\exists\, \alpha, \beta > 0 \ \forall\, (t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n \ \forall\, x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0) \ \forall\, t \ge t^0 : \ \|x(t)\| \le \alpha\, \mathrm{e}^{-\beta(t-t^0)} \|x^0\|.$$

Proof: Let $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ be arbitrary. If $\mathcal{G}_{E,A}(t^0, x^0) = \emptyset$ there is nothing to show. Hence let $x^0 \in \mathcal{G}(E, A)(t^0)$ and $x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0)$. Let $V : \mathcal{G}(E, A) \to \mathbb{R}$ denote a Lyapunov function for (E, A) as in Definition 5.1. Separation of variables applied to equation (5.2) gives

$$\forall\, t \ge t^0 \ \ : \ V(t, x(t)) \le \mathrm{e}^{-\lambda(t-t^0)} V(t^0, x^0). \tag{5.3}$$

Then, since $(t, x(t)) \in \mathcal{G}(E, A)$ for all $t \ge t^0$, we find

$$\forall\, t \ge t^0 : \ \|x(t)\|^2 \overset{(5.1)}{\le} \frac{1}{\ell_1} V(t, x(t)) \overset{(5.3)}{\le} \frac{1}{\ell_1} \mathrm{e}^{-\lambda(t-t^0)} V(t^0, x^0) \overset{(5.1)}{\le} \frac{\ell_2}{\ell_1} \mathrm{e}^{-\lambda(t-t^0)} \|x^0\|^2,$$

which proves the claim. $\square$

Next we seek for Lyapunov functions for (E, A) by determining solutions to a generalized time-varying Lyapunov equation.
For time-invariant DAEs $(E, A) \in (\mathbb{R}^{n\times n})^2$ it is well known that one seeks for (positive) solutions $(P, Q) \in (\mathbb{R}^{n\times n})^2$ of the Lyapunov equation

$$A^\top PE + E^\top PA = -Q, \tag{5.4}$$

and the corresponding Lyapunov function candidate is

$$V : \mathcal{V}^*_{E,A} \to \mathbb{R},\ x \mapsto x^\top \Big(E^\top PE\Big)\, x,$$

where $\mathcal{V}^*_{E,A} = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}$; see e.g. [OD85, Thm. 2.2].
For time-varying DAEs $(E, A) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2$, the analogous Lyapunov function candidate is

$$\boxed{V : \mathcal{G}(E, A) \to \mathbb{R}, \quad (t, x) \mapsto x^\top \big(E(t)^\top P(t)E(t)\big)\, x}. \tag{5.5}$$

We will show that differentiation of $V(t, x(t))$ along any solution $x(\cdot)$ of (E, A) forces $P(\cdot)$ to satisfy the *generalized time-varying Lyapunov equation*

$$\boxed{A(\cdot)^\top P(\cdot)E(\cdot) + E(\cdot)^\top P(\cdot)A(\cdot) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(\cdot)^\top P(\cdot)E(\cdot)\big) =_{\mathcal{G}(E,A)} -Q(\cdot)}. \tag{5.6}$$

The next theorem shows that the existence of a solution to the generalized time-varying Lyapunov equation yields a Lyapunov function for (E, A). Theorem 5.3 shows also that symmetry, differentiability and the boundedness conditions are only required for $E^\top PE$, not for P; therefore, $E^\top PE$ is the object of interest.

Theorem 5.3 (Sufficient conditions for existence of a Lyapunov function)**.** *Let $(E, A) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2$ and write $\mathcal{G} := \mathcal{G}(E, A)$, $\mathcal{G}(t) := \mathcal{G}(E, A)(t)$ for brevity. If $(P, Q) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n}) \times \mathcal{P}_\mathcal{G}$ is a solution to* (5.6) *such that $E^\top PE \in \mathcal{P}_\mathcal{G} \cap\ \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})$, then V as in* (5.5) *is a Lyapunov function for (E, A).*

Proof: Choose $q_1, q_2, p_1, p_2 > 0$ such that

$$q_1 I_n \leq_\mathcal{G} Q(\cdot) \leq_\mathcal{G} q_2 I_n \quad \text{and} \quad p_1 I_n \leq_\mathcal{G} E(\cdot)^\top P(\cdot)E(\cdot) \leq_\mathcal{G} p_2 I_n. \tag{5.7}$$

Then V as in (5.5) satisfies (5.1) for $\ell_1 = p_1$ and $\ell_2 = p_2$. We show that V satisfies (5.2). Let $(t^0, x^0) \in (\tau,\infty) \times \mathbb{R}^n$ be arbitrary. If $\mathcal{G}_{E,A}(t^0, x^0) = \emptyset$, then there is nothing to show. Hence let $x^0 \in \mathcal{G}(t^0)$ and $x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0)$. Since $(t, x(t)) \in \mathcal{G}$ for all $t \geq t^0$, differentiation of V along $x(\cdot)$ yields

$$\forall\, t \geq t^0 :\ \tfrac{\mathrm{d}}{\mathrm{d}t} V(t, x(t)) \overset{(5.6)}{=} -x(t)^\top Q(t)x(t) \overset{(5.7)}{\leq} -q_1 x(t)^\top x(t) \overset{(5.7)}{\leq} -\frac{q_1}{p_2} V(t, x(t)).$$

This completes the proof of the theorem. □

An alternative to Theorem 5.3, in terms of

$$\mathcal{E}\mathcal{G}(E, A) \ := \ \{\ (t, x) \in (\tau,\infty) \times \mathbb{R}^n \ \mid x \in E(t)\mathcal{G}(E, A)(t)\ \}\,, \qquad (E, A) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2,$$

is the following.

Theorem 5.4 (Alternative to Theorem 5.3)**.** *Let $(E, A) \in \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n}) \times \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})$ such that $E^\top E \in \mathcal{P}_\mathcal{G}$ and write $\mathcal{G} = \mathcal{G}(E, A)$, $\mathcal{E}\mathcal{G} = \mathcal{E}\mathcal{G}(E, A)$ for brevity. If $(P, Q) \in \big(\mathcal{P}_{\mathcal{E}\mathcal{G}} \cap\ \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})\big) \times \mathcal{P}_\mathcal{G}$ is a solution to* (5.6)*, then V as in* (5.5) *is a Lyapunov function for (E, A).*

The proof of Theorem 5.4 is an immediate consequence of Theorem 5.3 together with the following lemma.

Lemma 5.5 (Relationship between P and $E^\top PE$)**.** *For any DAE $(E,A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ such that $E^\top E \in \mathcal{P}_\mathcal{G}$ and $P \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})$ is symmetric we have (write $\mathcal{G} = \mathcal{G}(E,A)$ and $\mathcal{EG} = \mathcal{EG}(E,A)$ for brevity) that*

$$P \in \mathcal{P}_{\mathcal{EG}} \quad \iff \quad E^\top PE \in \mathcal{P}_\mathcal{G}\,.$$

Proof: $E^\top E \in \mathcal{P}_\mathcal{G}$ means

$$\exists\, \alpha,\beta > 0:\ \alpha I_n \le_\mathcal{G} E(\cdot)^\top E(\cdot) \le_\mathcal{G} \beta I_n. \tag{5.8}$$

We have to show that

$$\exists\, p_1,p_2 > 0:\ p_1 I_n \le_{\mathcal{EG}} P(\cdot) \le_{\mathcal{EG}} p_2 I_n \tag{5.9}$$

is equivalent to

$$\exists\, q_1,q_2 > 0:\ q_1 I_n \le_\mathcal{G} E(\cdot)^\top P(\cdot) E(\cdot) \le_\mathcal{G} q_2 I_n\,. \tag{5.10}$$

"$\Rightarrow$": If (5.9) holds, then for any $(t,x) \in \mathcal{G}$ we have $(t,E(t)x) \in \mathcal{EG}$ and thus

$$p_1\,\alpha\,\|x\|^2 \overset{(5.8)}{\le} p_1\,\|E(t)x\|^2 \overset{(5.9)}{\le} x^\top E(t)^\top P(t)E(t)x \overset{(5.9)}{\le} p_2\,\|E(t)x\|^2 \overset{(5.8)}{\le} p_2\,\beta\,\|x\|^2\,,$$

whence (5.10).
"$\Leftarrow$": If (5.10) holds, then for $(t,x) \in \mathcal{EG}$ we may choose $y \in \mathbb{R}^n$ such that $(t,y) \in \mathcal{G}$ and $x = E(t)y$. Then

$$\begin{aligned}\frac{q_1}{\beta}\,\|x\|^2 = \frac{q_1}{\beta}\,(E(t)y)^\top(E(t)y) &\overset{(5.8)}{\le} q_1\|y\|^2 \overset{(5.10)}{\le} y^\top E(t)^\top P(t)E(t)y \\ &= x^\top P(t)x \overset{(5.10)}{\le} q_2\,\|y\|^2 \overset{(5.8)}{\le} \frac{q_2}{\alpha}\,(E(t)y)^\top(E(t)y) = \frac{q_2}{\alpha}\,\|x\|^2\,. \quad \square\end{aligned}$$

Remark 5.6.

(i) By Remark 4.5(i), any exponentially stable time-invariant DAE $(E,A) \in \left(\mathbb{R}^{n\times n}\right)^2$ is transferable into SCF, i.e. any time-invariant DAE which satisfies the assumptions of Theorem 5.3 or Theorem 5.4 (particularly the existence of a solution (P,Q) to (5.6)) is already transferable into SCF.

(ii) If $(E,A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ satisfies the assumptions of Theorem 5.3 or Theorem 5.4, then (E,A) is not necessarily transferable into SCF. To see this, consider system (4.3) discussed in Remark 4.5(ii). $\diamond$

Remark 5.7. Consider the simple DAE

$$h(t)\dot{x} = -h(t)x, \tag{5.11}$$

where $h \in \mathcal{C}(\mathbb{R};\mathbb{R})$ such that $h(t) \neq 0$ for all $t \in \mathbb{R}\setminus\{0\}$ and $h(0) = 0$. (5.11) is not transferable into SCF which can be seen by applying the same argument as in Remark 4.5(ii). The only global solution to (5.11), $x(t^0) = x^0 \in \mathbb{R}, t^0 \in \mathbb{R}$, is $t \mapsto \mathrm{e}^{-(t-t^0)}x^0$. Therefore (5.11) is exponentially stable. However, (5.11) does not satisfy the assumptions of Theorem 5.3 since for any $P \in \mathcal{C}(\mathbb{R};\mathbb{R})$ we have $h(0)^2P(0) = 0$. $\diamond$

To overcome the shortcoming described in Remark 5.7, we may generalize Theorem 5.2 and Theorem 5.3 on a discrete set $\mathcal{I} \subseteq (\tau,\infty)$, i.e. $\mathcal{I} \cap K$ contains only finitely many points for every compact set $K \subseteq (\tau,\infty)$. To keep the formulation close to Theorem 5.2 and Theorem 5.3, we introduce the (rather technical) notation for $(E,A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ and $k \in \mathbb{N}_0$:

$$\begin{array}{lcl}
V \text{ is an } \textit{almost} & & V : \mathcal{G}(E,A) \to \mathbb{R} \text{ and there exists as discrete set } \mathcal{I} \subseteq (\tau,\infty)\text{: } V \\
\textit{Lyapunov function} & :\Longleftrightarrow & \text{satisfies (5.2) and} \\
& & \exists\, \ell_1, \ell_2 > 0\ \forall\, t \in (\tau,\infty) \setminus \mathcal{I}\ \forall\, x \in \mathcal{G}(E,A)(t) : \\
& & \qquad \ell_1 \|x\|^2 \le V(t,x) \le \ell_2 \|x\|^2 \\
Q \overset{ae}{\in} \mathcal{P}_{\mathcal{G}(E,A)} & :\Longleftrightarrow & \text{there exists as discrete set } \mathcal{I} \subseteq (\tau,\infty)\text{: } \operatorname{dom} Q = (\tau,\infty)\setminus\mathcal{I},\ Q \in \\
& & \mathcal{C}((\tau,\infty)\setminus\mathcal{I};\mathbb{R}^{n\times n}),\ Q = Q^\top, \\
& & \exists\, q_1, q_2 > 0\ \forall\, t \in (\tau,\infty)\setminus\mathcal{I}\ \forall\, x \in \mathcal{G}(E,A)(t) : \\
& & \qquad q_1\|x\|^2 \le x^\top Q(t) x \le q_2 \|x\|^2 \\
P \overset{ae}{\in} \mathcal{C}^k((\tau,\infty);\mathbb{R}^{n\times n}) & :\Longleftrightarrow & \text{there exists as discrete set } \mathcal{I} \subseteq (\tau,\infty)\text{: } \operatorname{dom} P = (\tau,\infty)\setminus\mathcal{I} \text{ and} \\
& & P \in \mathcal{C}^k((\tau,\infty)\setminus\mathcal{I};\mathbb{R}^{n\times n})
\end{array}$$

Theorem 5.8 (Sufficient conditions for exponential stability)**.** *The following implications hold for any DAE $(E,A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2$ (write $\mathcal{G} := \mathcal{G}(E,A)$ for brevity):*

(i) *If $Q \overset{ae}{\in} \mathcal{P}_{\mathcal{G}}$, $P \overset{ae}{\in} \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})$ such that $E^\top PE \overset{ae}{\in} \big(\mathcal{P}_{\mathcal{G}} \cap\ \mathcal{C}^1((\tau,\infty);\mathbb{R}^{n\times n})\big)$ and $E^\top PE$ is extendable to a continuously differentiable function on (τ,∞) and (5.6) is satisfied in all points in the joint domain of all functions involved, then V as in (5.5) is an almost Lyapunov function for (E,A).*

(ii) *If $V : \mathcal{G}(E,A) \to \mathbb{R}$ is any almost Lyapunov function for (E,A), then*

$$\exists\, \alpha,\beta > 0\ \forall\, (t^0,x^0) \in (\tau,\infty)\times\mathbb{R}^n\ \forall\, x(\cdot) \in \mathcal{G}_{E,A}(t^0,x^0)\ \forall\, t \ge t^0 :\ \|x(t)\| \le \alpha\, \mathrm{e}^{-\beta(t-t^0)}\|x^0\|.$$

Proof: The proof is very similar to the proofs of Theorem 5.2 and Theorem 5.3: Some care must be exercised on the discrete set, so the inequalities must be derived on the open set $\operatorname{dom} x \supsetneq [t^0,\infty)$ (to avoid problems in the case $t^0 \in \mathcal{I}$) and most of them hold only almost everywhere; however, in case of (i), the assumption yields that $V(\cdot, x(\cdot))$ is continuously differentiable on $\operatorname{dom} x$, and thus the final inequality can be extended to all of $[t^0,\infty)$. The details are omitted for brevity. ☐

Theorem 5.8 generalizes the results of Theorem 5.2, Theorem 5.3 and Theorem 5.4 considerably; isolated singular points as in Example (5.11) are resolved.

Example 5.9. Revisit Example (5.11). Define $\mathcal{I} := \{0\}$ and $P : \mathbb{R}\setminus\{0\} \to \mathbb{R},\ t \mapsto \frac{1}{2h(t)^2}$ and $Q = 1$. Then $h(t)^2 P(t) = \frac{1}{2}$ for all $t \in \mathbb{R}\setminus\{0\}$ and hence $h(\cdot)^2P(\cdot)$ is extendable to a continuously differentiable function on $\mathbb{R}$. Furthermore, invoking $\mathcal{G}(E,A) = \mathbb{R}\times\mathbb{R}$,

$$\forall\, t \in \mathbb{R}\setminus\{0\} :\ -2h(t)^2P(t) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(h(t)^2P(t)\big) = -1 = -Q(t).$$

Now all assumptions of Theorem 5.8(i) are satisfied and exponential stability of (5.11) may be deduced. ◇

5.2 Stability for systems transferable into SCF

In this section we derive, for systems (E,A) which are transferable into SCF, a variant of Theorem 5.3 (and Theorem 5.4) and also give the converse of the stability theorem.
Some notation is convenient:

$$\mathcal{EV}_{E,A} := \{\ (t,x) \in (\tau,\infty)\times\mathbb{R}^n\ \mid x \in E(t)\mathcal{V}_{E,A}(t)\ \},\qquad (E,A) \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})^2\,.$$

Proposition 3.3 yields, for DAEs (E, A) transferable into SCF, that

$$\mathcal{V}_{E,A} = \mathcal{G}(E, A) \qquad \text{and} \qquad \mathcal{EV}_{E,A} = \mathcal{EG}(E, A)\,.$$

If the DAE (E, A) is transferable into SCF as in (3.2), then the Lyapunov equation (5.6) may be generalized to

$$\boxed{A(\cdot)^\top P(\cdot)E(\cdot) + E(\cdot)^\top P(\cdot)A(\cdot) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(\cdot)^\top P(\cdot)E(\cdot)\big) =_{\mathcal{V}_{E,A}} -Q(\cdot)} \tag{5.12}$$

and the candidate for the solution P is

$$\boxed{P : (\tau, \infty) \to \mathbb{R}^{n\times n}, \quad t \mapsto S(t)^\top T(t)^\top \int_t^\infty U(s,t)^\top Q(s)U(s,t)\ \mathrm{d}s\ \ T(t)S(t)} \tag{5.13}$$

where $U(\cdot,\cdot)$ denotes the generalized transition matrix of (E, A), see (3.3).

We are now in the position to state the main result of this section.

Theorem 5.10 (Necessary and sufficient conditions for exponential stability of systems transferable into SCF)**.** *For any $(E, A) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2$ transferable into SCF as in* (3.2) *(write $\mathcal{V} = \mathcal{V}_{E,A}$ and $\mathcal{EV} = \mathcal{EV}_{E,A}$ for brevity) we have:*

(i) *If $(P, Q) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n}) \times \mathcal{P}_\mathcal{V}$ solves* (5.12) *and $E^\top PE \in \mathcal{P}_\mathcal{V} \cap \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})$, then (E, A) is exponentially stable.*

(ii) *Let E be continuously differentiable and $E^\top E \in \mathcal{P}_\mathcal{V}$. If $(P, Q) \in \big(\mathcal{P}_{\mathcal{EV}} \cap\ \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})\big) \times \mathcal{P}_\mathcal{V}$ solves* (5.12)*, then (E, A) is exponentially stable.*

(iii) *Let $E,\ N \in \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})$, $E^\top E \in \mathcal{P}_\mathcal{V}$, and E and $\dot{E}{+}A$ be bounded. If (E, A) is exponentially stable, then for any $Q \in \mathcal{P}_\mathcal{V}$ the function P as in* (5.13) *is a solution to* (5.12)*, furthermore $E^\top PE \in \mathcal{P}_\mathcal{V} \cap \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})$.*

(iv) *Let $E,\ S \in \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})$, $E^\top E \in \mathcal{P}_\mathcal{V}$, and E and $\dot{E}{+}A$ be bounded. If (E, A) is exponentially stable, then for any $Q \in \mathcal{P}_\mathcal{V}$ the function P as in* (5.13) *is a continuously differentiable solution to* (5.12)*, furthermore $P \in \mathcal{P}_{\mathcal{EV}}$.*

Proof: (i): This follows from Theorem 4.6(iii), Theorem 5.2, Theorem 5.3 and $\mathcal{V}_{E,A} = \mathcal{G}(E, A)$.
(ii): This follows from Theorem 4.6(iii), Theorem 5.2, Theorem 5.4 and $\mathcal{V}_{E,A} = \mathcal{G}(E, A)$.
(iii): The assumption $Q, E^\top E \in \mathcal{P}_\mathcal{V}$ means

$$\exists\, q_1, q_2 > 0 :\ q_1 I_n \le_\mathcal{V} Q(\cdot) \le_\mathcal{V} q_2 I_n, \qquad \exists\, e_1, e_2 > 0 :\ e_1 I_n \le_\mathcal{V} E(\cdot)^\top E(\cdot) \le_\mathcal{V} e_2 I_n. \tag{5.14}$$

Step 1: Let $(t^0, x^0) \in (\tau,\infty) \times \mathbb{R}^n$ be arbitrary and $T > t^0$. Set

$$\begin{bmatrix} v \\ w \end{bmatrix} := S(t^0)x^0,\ v \in \mathbb{R}^{n_1},\ w \in \mathbb{R}^{n_2}\,, \quad \text{and} \quad y^0 := T(t^0)\begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathcal{V}_{E,A}(t^0).$$

Then

$$\forall\, s > \tau :\ U(s, t^0)T(t^0)\begin{bmatrix} 0 \\ w \end{bmatrix} = T(t^0)\begin{bmatrix} \Phi_J(s, t^0) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ w \end{bmatrix} = 0, \tag{5.15}$$

and since $U(s,t^0)y^0 \in \mathcal{V}_{E,A}(s)$, Theorem 4.6(iii) yields

$$
\begin{aligned}
&(x^0)^\top S(t^0)^\top T(t^0)^\top \int_{t^0}^T U(s,t^0)^\top Q(s) U(s,t^0) \text{ d}s \; T(t^0) S(t^0) x^0 \\
&\quad \overset{(5.15)}{=} \int_{t^0}^T (U(s,t^0)y^0)^\top Q(s) (U(s,t^0)y^0) \text{ d}s \overset{(5.14)}{\leq} \int_{t^0}^T q_2 (U(s,t^0)y^0)^\top (U(s,t^0)y^0) \text{ d}s \\
&\quad \overset{\text{Thm. 4.6(iii)}}{\leq} q_2 \int_{t^0}^T \alpha^2 \mathrm{e}^{-2\beta(s-t^0)} \|y^0\|^2 \text{ d}s = \frac{q_2\alpha^2}{2\beta}\|y^0\|^2 \left(1 - \mathrm{e}^{-2\beta(T-t^0)}\right).
\end{aligned}
$$

Taking the limit for $T \to \infty$ yields existence of $P(t^0)$.
Step 2: We show that $E(\cdot)^\top P(\cdot) E(\cdot) \leq_{\mathcal{V}} c I_n$ for some $c > 0$.
Let $(t,x) \in \mathcal{V}$. Then $x = T(t)\begin{bmatrix} v \\ 0 \end{bmatrix}$ for some $v \in \mathbb{R}^{n_1}$ and therefore

$$
\begin{aligned}
&x^\top E(t)^\top P(t) E(t) x \\
&\quad \overset{(3.2)}{=} [v^\top, 0] T(t)^\top T(t)^{-\top} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)^\top \end{bmatrix} S(t)^{-\top} P(t) S(t)^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix} T(t)^{-1} T(t) \begin{bmatrix} v \\ 0 \end{bmatrix} \\
&\quad = [v^\top, 0] T(t)^\top \int_t^\infty U(s,t)^\top Q(s) U(s,t) \text{ d}s \; T(t) \begin{bmatrix} v \\ 0 \end{bmatrix} = \int_t^\infty (U(s,t)x)^\top Q(s) (U(s,t)x) \text{ d}s \,.
\end{aligned}
$$

We may conclude, similar to Step 1,

$$
x^\top E(t)^\top P(t) E(t) x \leq \frac{q_2\alpha^2}{2\beta}\|x\|^2,
$$

and since $(t,x) \in \mathcal{V}$ the claim follows.
Step 3: We may write, for all $t > \tau$,

$$
\begin{aligned}
&E(t)^\top P(t) E(t) \\
&\quad \overset{(3.2)}{=} T(t)^{-\top} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)^\top \end{bmatrix} T(t)^\top \int_t^\infty U(s,t)^\top Q(s) U(s,t) \text{ d}s \; T(t) \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix} T(t)^{-1}, \quad (5.16)
\end{aligned}
$$

and since Q and $U(\cdot,\cdot)$ are continuous and T and N are continuously differentiable, $E^\top P E$ is continuously differentiable.
Furthermore, P is symmetric due to symmetry of Q, and therefore $E^\top P E$ is symmetric.
Step 4: We show that $c I_n \leq_{\mathcal{V}} E(\cdot)^\top P(\cdot) E(\cdot)$ for some $c > 0$. Boundedness of E and $\dot{E} + A$ means

$$
\exists\, c_E, c_A > 0 \;\forall\, t > \tau : \; \|E(t)\| \leq c_E \;\wedge\; \left\|\dot{E}(t) + A(t)\right\| \leq c_A.
$$

For arbitrary $(t,x^0) \in \mathcal{V}$ and $x(\cdot) := U(\,\cdot,t)x^0$, we find

$$
\forall\, s > \tau : \; \tfrac{\mathrm{d}}{\mathrm{d}s}\left(E(s)x(s)\right) = \dot{E}(s)x(s) + E(s)\dot{x}(s) = \big(\dot{E}(s) + A(s)\big)x(s), \quad (5.17)
$$

and

$$
0 \leq \|E(s)x(s)\| \leq c_E \|U(s,t)x^0\| \xrightarrow[s\to\infty]{\text{Thm. 4.6(iii)}} 0. \quad (5.18)
$$

Therefore

$$
\begin{aligned}
(x^0)^\top E(t)^\top P(t)E(t)x^0 &= \int_t^\infty x(s)^\top Q(s)x(s) \,\mathrm{d}s \geq \int_t^\infty q_1 x(s)^\top x(s) \,\mathrm{d}s \\
&\geq q_1 \int_t^\infty \frac{\|E(s)\| \, \|\dot E(s)+A(s)\|}{c_E c_A} x(s)^\top x(s) \,\mathrm{d}s \geq \frac{q_1}{c_E c_A} \int_t^\infty \left| (E(s)x(s))^\top (\dot E(s)+A(s))x(s) \right| \,\mathrm{d}s \\
&\overset{(5.17)}{\geq} \frac{q_1}{c_E c_A} \left| \int_t^\infty (E(s)x(s))^\top \left(\tfrac{\mathrm{d}}{\mathrm{d}s}(E(s)x(s))\right) \,\mathrm{d}s \right| = \frac{q_1}{c_E c_A} \left| \int_t^\infty \frac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}s} \Big((E(s)x(s))^\top (E(s)x(s)) \Big) \,\mathrm{d}s \right| \\
&= \left| \frac{q_1}{2c_E c_A} \, \|E(s)x(s)\|^2 \Big|_t^\infty \right| \overset{(5.18)}{=} \frac{q_1}{2c_E c_A} \|E(t)U(t,t)x^0\|^2 \\
&\overset{\text{Prop. 3.3 (v)}}{=} \frac{q_1}{2c_E c_A} \|E(t)x^0\|^2 \overset{(5.14)}{\geq} \frac{q_1 e_1}{2c_E c_A} \|x^0\|^2 ,
\end{aligned}
$$

and the claim follows.
Step 5: The statement (5.12) follows from

$$
\begin{aligned}
&(x^0)^\top \tfrac{\mathrm{d}}{\mathrm{d}t} \left(E(t)^\top P(t)E(t) \right) x^0 \\
&\quad = \tfrac{\mathrm{d}}{\mathrm{d}t} \Big((T(t)S(t)E(t)x^0)^\top \textstyle\int_t^\infty U(s,t)^\top Q(s)\, U(s,t) \,\mathrm{d}s \; (T(t)S(t)E(t)x^0) \Big) \\
&\overset{\text{Prop. 3.4(ii)}}{=} (x^0)^\top \left[\textstyle\int_t^\infty \tfrac{\mathrm{d}}{\mathrm{d}t} \left(U(s,t)^\top Q(s)\, U(s,t) \right) \,\mathrm{d}s - U(t,t)^\top Q(t)\, U(t,t) \right] x^0 \\
&\underset{\text{Prop. 3.3(vi)}}{\overset{\text{Prop. 3.3(v)}}{=}} -(x^0)^\top \textstyle\int_t^\infty \left[U(s,t)^\top Q(s)\, U(s,t) T(t)S(t)A(t) + (U(s,t)T(t)S(t)A(t))^\top Q(s)\, U(s,t) \right] \mathrm{d}s \; x^0 \\
&\qquad -(x^0)^\top Q(t)\, x^0 \\
&\overset{\text{Prop. 3.4(ii)}}{=} -(x^0)^\top E(t)^\top P(t)A(t)x^0 - (x^0)^\top A(t)^\top P(t)E(t)x^0 - (x^0)^\top Q(t)\, x^0
\end{aligned}
$$

for all $(t, x^0) \in \mathcal{V}$. This proves the claim.
(iv): Since S is continuously differentiable by assumption it follows that P is continuously differentiable. Symmetry of P is obvious. As shown in (iii) it holds $E^\top P E \in \mathcal{P}_\mathcal{V}$ and therefore Lemma 5.5 yields $P \in \mathcal{P}_{\mathcal{E}\mathcal{V}}$. That (5.12) is satisfied has also been proved in (iii). ☐

A careful inspection of the proof of Theorem 5.10 yields the following corollary.

Corollary 5.11. *For any exponentially stable $(E, A) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2$ transferable into SCF as in (3.2) (write $\mathcal{V} = \mathcal{V}_{E,A}$ for brevity), $Q \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})$ such that $Q(\cdot) \leq_\mathcal{V} q_2 I_n$ for some $q_2 > 0$, and E, N continuously differentiable, the following statements hold true:*

(i) *P as in (5.13) is well-defined and solves (5.12), $E^\top P E$ is continuously differentiable and $E(\cdot)^\top P(\cdot)E(\cdot) \leq_\mathcal{V} r_2 I_n$ for some $r_2 > 0$.*

(ii) *If Q is symmetric, then P is symmetric.*

(iii) *If S is continuously differentiable, then P is continuously differentiable.*

(iv) *If E and $\dot E + A$ are bounded and there exist $e_1, q_1 > 0$ such that $E(\cdot)^\top E(\cdot) \geq_\mathcal{V} e_1 I_n$ and $Q(\cdot) \geq_\mathcal{V} q_1 I_n$, then $E(\cdot)^\top P(\cdot)E(\cdot) \geq_\mathcal{V} r_1 I_n$ for some $r_1 > 0$.*

Remark 5.12 (Positivity of $E^\top E$)**.** The positivity assumption $E^\top E \in \mathcal{P}_{\mathcal{V}_{E,A}}$ in Theorem 5.10 does not automatically hold for DAEs transferable into SCF – as it may be expected in view of Proposition 3.4(iii) which implies that $E^\top E \in \mathcal{P}_{\mathcal{V}_{E,A}}$ holds true for time-invariant DAEs. We give a counterexample: Consider the DAE (E, A) given by

$$E(t) = \begin{bmatrix} \frac{1}{t^2} & 0 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} \frac{1}{t^2} + \frac{1}{t^3} & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{for } \ t > \tau := 0,$$

which is transferable into SCF

$$(E, A) \overset{S,T}{\sim} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \qquad \text{for } S(t) = T(t) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{C}^1((0,\infty); \mathbf{Gl}_2(\mathbb{R})).$$

Let $t^0 > \tau$ and $x^0 \in \mathcal{V}_{E,A}(t^0) = \operatorname{im} \begin{bmatrix} t^0 \\ 0 \end{bmatrix}$. Then $x^0 = \begin{bmatrix} \alpha t^0 \\ 0 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$ and

$$\|E(t^0)x^0\| = \left\| \begin{bmatrix} \frac{\alpha}{t^0} \\ 0 \end{bmatrix} \right\| = \frac{|\alpha|}{t^0} \underset{t^0 \to \infty}{\longrightarrow} 0\,.$$

Therefore,

$$\exists\, e_1 > 0 : \ e_1 I_n \leq_{\mathcal{V}_{E,A}} E(\cdot)^\top E(\cdot)$$

does not hold true. ⋄

Remark 5.13 (Time-invariant case)**.** Consider time-invariant DAEs $(E, A) \in (\mathbb{R}^{n\times n})^2$ which are transferable into SCF. Then by [BI10, Prop. 2.3], the pencil $sE - A$ is regular and $t \mapsto \mathcal{V}_{E,A}(t) =: \mathcal{V}^*_{E,A}$ is constant. In view of Proposition 3.4(iii), the assumption $E^\top E \in \mathcal{P}_{\mathcal{V}_{E,A}}$ is always fulfilled; and Lemma 5.5 yields

$$P \in \mathcal{P}_{\mathcal{E}\mathcal{V}_{E,A}} \quad \iff \quad E^\top P E \in \mathcal{P}_{\mathcal{V}_{E,A}}.$$

Hence in the time-invariant case, Theorem 5.10 (i) and (ii) say the same and so do Theorem 5.10 (iii) and (iv).
Theorem 5.10 (ii) considered for time-invariant systems is an improvement of [Sty02, Thm. 4.6], since Stykel does not consider the restriction of the generalized Lyapunov equation to the set $\mathcal{V}_{E,A}$. Although [Sty02, Thm. 4.15 & Rem. 4.16] shows uniqueness of the solution, Corollary 5.11 is still a generalization of these results: the matrix P_r (notation from [Sty02]) is a projector onto $\mathcal{V}^*_{E,A}$, and hence "G positive definite" means $P_r^\top G P_r \in \mathcal{P}_{\mathcal{V}_{E,A}}$. The uniqueness condition for the solution of the generalized Lyapunov equation given in [Sty02, Thm. 4.15] is generalized in Corollary 5.17. ⋄

We now show that the solution P of the Lyapunov equation (5.12) is, under appropriate assumptions, unique on $\mathcal{E}\mathcal{V}_{E,A}$. Note that symmetry of P or Q are not required and asymptotic stability of (E, A) is sufficient. However, to ensure existence of a solution, exponential stability is necessary: see Corollary 5.17.

Proposition 5.14 (Unique solution of the Lyapunov equation)**.** *For any asymptotically stable* $(E, A) \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})^2$ *which is transferable into SCF as in* (3.2) *we have: If* $Q \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})$ *and* $P_1, P_2 \in \mathcal{C}((\tau,\infty); \mathbb{R}^{n\times n})$ *solve* (5.12) *such that* $E^\top P_i E \in \mathcal{C}^1((\tau,\infty); \mathbb{R}^{n\times n})$ *for* $i = 1, 2$ *and*

$$\forall\, i \in \{1, 2\}\ \exists\, \alpha_i, \beta_i > 0 \ : \ \alpha_i I_n \leq_{\mathcal{V}_{E,A}} E(\cdot)^\top P_i(\cdot) E(\cdot) \leq_{\mathcal{V}_{E,A}} \beta_i I_n\,, \tag{5.19}$$

then $P_1(\cdot) =_{\mathcal{E}\mathcal{V}_{E,A}} P_2(\cdot)$.

Proof: Differentiation of

$$\Delta(t) := U(t,s)^\top E(t)^\top [P_1(t) - P_2(t)] E(t) U(t,s), \quad t \geq s > \tau$$

yields

$$\begin{aligned}
\dot{\Delta}(t) \;&=\; (E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,s))^\top [P_1(t) - P_2(t)] E(t) U(t,s) + U(t,s)^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top [P_1(t) - P_2(t)] E(t)\big) U(t,s) \\
&\quad + U(t,s)^\top E(t)^\top [P_1(t) - P_2(t)] E(t) \tfrac{\mathrm{d}}{\mathrm{d}t} U(t,s) \\
&\overset{\text{Prop. 3.3 (i)}}{=} (A(t)U(t,s))^\top [P_1(t) - P_2(t)] E(t) U(t,s) + U(t,s)^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top [P_1(t) - P_2(t)] E(t)\big) U(t,s) \\
&\quad + U(t,s)^\top E(t)^\top [P_1(t) - P_2(t)] A(t) U(t,s) \\
&\overset{(5.12)}{=} 0\,,
\end{aligned}$$

where for the bottom equality we have used that $U(t,s)x \in \mathcal{V}_{E,A}(t)$ for all $x \in \mathbb{R}^n$ by Proposition 3.3 (ii). Hence $\Delta(\cdot)$ must be constant. Proposition 3.3 (ii) yields

$$\begin{aligned}
\forall\, t \geq s \;:\; & \alpha_1 U(t,s)^\top U(t,s) - \beta_2 U(t,s)^\top U(t,s) \\
&\overset{(5.19)}{\leq} U(t,s)^\top E(t)^\top P_1(t) E(t) U(t,s) - U(t,s)^\top E(t)^\top P_2(t) E(t) U(t,s) \\
&\qquad = \Delta(t) \overset{(5.19)}{\leq} \beta_1 U(t,s)^\top U(t,s) - \alpha_2 U(t,s)^\top U(t,s)\,.
\end{aligned}$$

Since (E,A) is asymptotically stable we find, as in the proof of Theorem **??**(ii),

$$\lim_{t\to\infty} U(t,s) = 0, \quad \text{and so} \quad \lim_{t\to\infty} \Delta(t) = 0.$$

Hence we get $\Delta(\cdot) = 0$, i.e. $(E(s)U(s,s)x)^\top [P_1(s) - P_2(s)](E(s)U(s,s)x) = 0$ for all $x \in \mathbb{R}^n$, or equivalently,

$$\forall\, x \in \mathcal{V}_{E,A}(s) :\; x^\top E(s)^\top [P_1(s) - P_2(s)] E(s) x = 0. \qquad \square$$

Remark 5.15 (Non-uniqueness of P)**.** We show that the solution of (5.12) is in general not unique: Let

$$E(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & 0 \\ 0 & e^t \end{bmatrix}, \quad t \in \mathbb{R}.$$

Then (E,A) is transferable into SCF by $S(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix}$, $t \in \mathbb{R}$, and $T = I$. Hence $n_1 = n_2 = 1$ and $\mathcal{V}_{E,A} = \mathbb{R} \times \operatorname{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathcal{EV}_{E,A}$. Then, for $Q = I$ and any $p \in \mathcal{C}(\mathbb{R};\mathbb{R})$ the continuous function $P : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto \left[\begin{smallmatrix} 1/2 & 0 \\ 0 & p(t) \end{smallmatrix}\right]$ solves (5.12) and fulfills $E^\top P E \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{2\times 2}) \cap \mathcal{P}_{\mathcal{G}}$. ⋄

Remark 5.16 (Uniqueness condition)**.** By Proposition 5.14, the uniformly bounded solution of the Lyapunov equation (5.12) is unique on $\mathcal{EV}_{E,A}$. To obtain a unique solution on all of $(\tau,\infty) \times \mathbb{R}^n$, we are somehow free to choose the behaviour of P on $(\tau,\infty) \times \mathbb{R}^n \setminus \mathcal{EV}_{E,A}$. Choose, for instance, $V : (\tau,\infty) \to \mathbb{R}^{n\times n}$ such that $\operatorname{im} V(t) = \mathcal{V}_{E,A}(t)$ for all $t > \tau$, and let Q, P_1, P_2 be as in Proposition 5.14 and (E,A) be asymptotically stable. Then we have:

$$\big[\forall\, i \in \{1,2\}\ \forall\, t > \tau :\; P_i(t) = (E(t)V(t))^\top P_i(t)(E(t)V(t))\big] \quad \Longrightarrow \quad \big[\forall\, t > \tau :\; P_1(t) = P_2(t)\big]\,.$$

The implication is a consequence of Proposition 5.14 which gives $P_1 =_{\mathcal{EV}} P_2$, i.e. $(E(t)V(t))^\top [P_1(t) - P_2(t)](E(t)V(t)) = 0$ for any $t > \tau$. ⋄

However, the following corollary shows that uniqueness of P is guaranteed under additional assumptions. Note that symmetry of P or Q is not required.

Corollary 5.17. *Let (E,A) be exponentially stable, transferable into SCF as in (3.2), and satisfy: E, N are continuously differentiable, E, $\dot{E}+A$ are bounded, $E^\top E \in \mathcal{P}_{\mathcal{V}_{E,A}}$. Then, for any $Q \in \mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n})$ such that $q_1 I_n \le_{\mathcal{V}_{E,A}} Q(\cdot) \le_{\mathcal{V}_{E,A}} q_2 I_n$ for some $q_1,q_2>0$, P as in (5.13) is the unique solution of*

$$\left.\begin{aligned}
&A(\cdot)^\top P(\cdot)E(\cdot) + E(\cdot)^\top P(\cdot)A(\cdot) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(\cdot)^\top P(\cdot)E(\cdot)\big) =_{\mathcal{V}_{E,A}} -Q(\cdot),\\
&\forall\, t>\tau: \ \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} S(t)\right)^\top P(t) \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} S(t)\right) = P(t),\\
&\exists\, p_1,p_2>0: \ p_1 I_n \le_{\mathcal{V}_{E,A}} E(\cdot)^\top P(\cdot)E(\cdot) \le_{\mathcal{V}_{E,A}} p_2 I_n,\\
&P\in\mathcal{C}((\tau,\infty);\mathbb{R}^{n\times n}), \ E^\top P E \in \mathcal{C}^1((\tau,\infty);\mathbb{R}^{n\times n}).
\end{aligned}\right\} \tag{5.20}$$

Proof: Similar to the proof of Theorem 5.10 (iii) it follows that $P(t)$ exists for all $t>\tau$, $E^\top PE$ is continuously differentiable, P solves (5.12) and $p_1 I_n \le_{\mathcal{V}_{E,A}} E(\cdot)^\top P(\cdot)E(\cdot) \le_{\mathcal{V}_{E,A}} p_2 I_n$ for some $p_1,p_2>0$. Furthermore, since

$$\begin{aligned}
&U(s,t)T(t)S(t)\left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} S(t)\right)\\
&\quad = T(s)\begin{bmatrix} \Phi_J(s,t) & 0\\ 0 & 0\end{bmatrix} T(t)^{-1}T(t)\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} S(t) = T(s)\begin{bmatrix} \Phi_J(s,t) & 0\\ 0 & 0\end{bmatrix} S(t) = U(s,t)T(t)S(t)
\end{aligned}$$

for all $s,t>\tau$, the second condition in (5.20) is satisfied and therefore P solves (5.20).
It remains to show that P is unique. Choose $V(t)=U(t,t)$ for $t>\tau$ and observe that $\operatorname{im} V(t) = \mathcal{V}_{E,A}(t)$, $t>\tau$, and

$$\begin{aligned}
\forall\, t>\tau: \ E(t)V(t) &= \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & N(t)\end{bmatrix} T(t)^{-1}T(t)\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} T(t)^{-1}\right)\\
&= \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} T(t)^{-1}\right) = S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} S(t),
\end{aligned}$$

and thus Proposition 5.14 together with Remark 5.16 yield that P is the unique solution of (5.20). $\square$

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