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# Formalizability and Knowledge Ascriptions in Mathematical Practice*

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**Résumé :** Nous examinons les conditions de vérité pour des attributions de savoir dans le cas des connaissances mathématiques. La disposition d'une démonstration formalisable semble être un critère naturel :

> (∗) $X$ sait que $p$ est vrai si et seulement si $X$ en principe dispose d'une démonstration formalisable pour $p$.

La formalisabilité pourtant ne joue pas un grand rôle dans la pratique mathématique effective. Nous présentons des résultats d'une recherche empirique qui indiquent que les mathématiciens *n'employent pas* certaines spécifications de (∗) quand ils attribuent du savoir.

De plus, nous montrons que le concept de savoir mathématique qui est à la base de l'emploi effectif du mot « savoir » de la pratique mathématique est tout à fait compatible avec certaines intuitions philosophiques mais apparaît comme *différent* des concepts philosophiques formant la base de (∗).

**Abstract:** We investigate the truth conditions of knowledge ascriptions for the case of mathematical knowledge. The availability of a formalizable mathematical proof appears to be a natural criterion:

> (∗) $X$ knows that $p$ is true iff $X$ has available a formalizable proof of $p$.

Yet, formalizability plays no major role in actual mathematical practice. We present results of an empirical study, which suggest that certain readings of (∗) are *not* necessarily employed by mathematicians when ascribing knowledge. Further, we argue that the concept of mathematical knowledge underlying the actual use of "to know" in mathematical practice is compatible with certain philosophical intuitions, but seems to *differ* from philosophical knowledge conceptions underlying (∗).

∗. We thank the audience at the $17^{th}$ Novembertagung on the History and Philosophy of Mathematics in Edinburgh and one of the anonymous referees of *Philosophia Scientiae* for helpful comments on earlier versions of this paper.

# 1 Introduction

In this paper, we will be concerned with the role of formalizability in an epistemology of mathematics.

Formalizability is a feature of informal mathematical proofs: A formalizable proof is a proof that can be transformed into a formal proof, that means it can be transformed into a formal derivation with respect to a formal axiomatic system with consistent axioms. However, the notion of formalizability is not a fixed notion like the notion of formal proof. What has still to be specified is the meaning of "can be transformed" in this context. We may choose the semantics of the phrase "formalizable proof" from a spectrum spread between two extremes. One extreme in the definition spectrum would be the weak reading "a proof of $p$ is formalizable iff the informally proven mathematical theorem is also formally derivable in a consistent formal axiomatic system"; the other extreme is the strong reading "a proof of $p$ is formalizable iff it can be translated step by step into a formal proof", which may refer to, for example, proofs that are written in some semi-formal language.[1]

Formalizability is an important feature of mathematical proofs, regarding foundational issues in the philosophy of mathematics: [Rav 1999, 11] suggested referring to the thesis that every informal mathematical proof can be transformed into a formal proof as *Hilbert's thesis*, as it is linked to the so-called *Hilbert programme*.[2] We will not be concerned with *Hilbert's thesis* generally, but with an epistemological application of the thesis:

> Is formalizability also essential for a philosophical understanding of *mathematical knowledge*?

This question is open, and several answers have been proposed. For one, Rav emphasizes that according to the *popular* opinion about mathematics (which he argues against), a mathematician's job consists more or less in manipulating formulas to render a decision "true" or "false" about certain theorems. Facing the increasing use of computer tools, such as computer algebra systems, simulation software or proof checkers, up to the famous examples of computer proofs (e.g., the proof of the *Four Colour Theorem*), the question arises whether mathematicians, in so far as they are working on *proofs*, might even be completely replaceable by computers without loss.

1. Note that the transformation relations include the case that a formalizable proof is already a formal proof. Therefore, all formal proofs are formalizable.

2. In its final version, *Hilbert's programme* towards a new foundation of mathematics was proposed by David Hilbert in 1921. It aims at a formalization of mathematics *and* at a proof that the axioms of the formal system that is used are consistent [Zach 2003]. *Hilbert's thesis* is actually weaker than *Hilbert's programme* itself, which, as is known, failed as a result of Gödel's theorems about the completeness and incompleteness of formal systems. Whether *Hilbert's thesis* is true or not may strongly depend on which reading of "can be transformed" is chosen, and therefore, on which meaning of "formalizable proof" is used.

Consider taking *Hilbert's thesis* as an exhaustive characterization of the epistemic role of mathematical proof. What effect would this assumption have on an epistemology of mathematics? As one aspect of this question, we will examine the epistemological impact of criteria for "$X$ knows that the mathematical statement $p$ is true" that are based on the availability of formalizable proof.

The paper is also a first report on work in progress concerning a *socio-empirical* investigation of actual mathematical practice. Besides a purely analytical treatment of the epistemic role of formalizability in mathematics, we will take into account first results of our recently conducted empirical research on formalizability and knowledge ascriptions in mathematical practice.

# 2 Preliminaries

A *formalizability criterion* for "$X$ knows that $p$ is true" is a criterion of the following form:

($*$) $X$ has available a formalizable proof of $p$,

where the notion of availability may be weaker than the notion of current cognitive access.

Our main research question is:

> Is there any specification of "formalizable" and "available" such that a formalizability criterion provides an adequate criterion for the truth of "$X$ knows that the mathematical statement $p$ is true"?

As we will discuss below (Section 3.2), *prima facie* there seems to be a natural way to answer 'yes' to this question from the philosophical point of view.

However, it turns out that in actual mathematical practice, formalizability seems to play no major role. Employing the results of an empirical study, we will suggest that, at least for common readings of ($*$), mathematicians do not necessarily use formalizability criteria when ascribing knowledge. Knowledge ascriptions in mathematical practice work in a way that differs systematically from how they should work if they were based on these readings of formalizability criteria. This points to the conclusion that, in spite of their *prima facie* attractiveness, formalizability criteria may be inadequate for capturing conceptions of mathematical knowledge as employed in actual mathematical practice.

We hold the view that for an adequate investigation of the epistemic role of formalizability, one needs to distinguish between a purely analytical account of the concept of mathematical knowledge, and an approach towards conceptions of knowledge that are actually employed in mathematical practice. Both should contribute to an adequate philosophical understanding of mathematical knowledge. In our paper, we will focus on the following aspect of this requested mutual contribution: The truth conditions of "$X$ knows that the mathematical statement $p$ is true" which a philosophical theory of mathematical knowledge yields should somehow *fit*, or should at least *not systematically contradict*, the meaning of knowledge ascriptions employed in actual mathematical practice.

Our investigation will proceed (iteratively) in three main steps:

- Identify those philosophical conceptions of knowledge that are consistent with a formalizability criterion—this presupposes an appropriate classification of knowledge conceptions, due to the different possible specifications of "formalizable" and "available".
- Examine and analyze conceptions of mathematical knowledge, especially the meaning of knowledge ascriptions, employed in actual mathematical practice. Can they be captured by particular philosophical conceptions of knowledge?
- Combine the results of both analyses. Are there differences between the results of Step 1 and Step 2? Are these differences due to different aspects of knowledge? Can we develop a conceptual epistemological framework which is able to cope with these different aspects?

The approach is also meant to shed light on the nature and epistemic role of mathematical *proof*, which is of great interest for modern philosophy of mathematics (see for example [Detlefsen 1992]).

The paper will sketch a showcase part of the analysis from Step 1 of our agenda. We will then turn to Step 2 with two main concerns: On the one hand, we will present first results of our recently conducted empirical study about the conception of knowledge in mathematical practice. On the other hand, we will discuss a reasonable embedding of these empirical results into the overall philosophical framework. This will provide an orientation for further elaboration of Step 1, and is also necessary for developing Step 3, a long term goal. While we will give a quite detailed examination of a part of our empirical results (Step 2), we will not be able to perform an equally detailed discussion of Step 1 in full generality here.

# 3 Step 1: Knowledge conceptions behind formalizability criteria—a showcase analysis

Mathematical knowledge is alleged to have a special epistemic status. On the one hand, this is due to the fact that its theorems can be *proven*, which is exceptional among the sciences. On the other hand, it is due to the historical stability of the body of mathematical knowledge, and the high degree of systematic unity and uniformity even across different branches of mathematics.

We stated above that, *prima facie*, there seems to be an obvious specification of an adequate formalizability criterion, as there seems to be a particular reading of formalizability criteria that can easily account for the exceptional epistemic status of mathematical knowledge. In this section, we will first examine this claim by discussing the specification of formalizability criteria involved and the way it accounts for the exceptional epistemic status. We will then investigate the question whether this specification really yields an adequate semantics for "$X$ knows that the mathematical statement $p$ is true" in view of Step 2 and Step 3 of our agenda. This question will lead us to the empirical part of the paper.

## 3.1 The classical conception of knowledge

Classical epistemology defines knowledge as *justified true belief* via the following three individually necessary and jointly sufficient conditions for "$X$ knows that $p$ is true":

- $X$ believes that $p$ is true
- $p$ is true
- $X$ has good reasons to believe that $p$ is true with respect to some fixed epistemic standards.

This definition of knowledge has high intuitive appeal. Yet, it has been seriously challenged by the famous *Gettier examples*, and by scepticism.[3] The Gettier examples, as well as the sceptical challenge, are essentially based on the fact that there is a gap between justification and truth. As the classical conception of knowledge cannot fill this gap for the general case, this conception has to be regarded as inadequate for a general theory of knowledge, despite its intuitive appeal.

By employing the distinction between *internalistic* and *externalistic* conceptions, and the distinction between *strict invariantist* and *context sensitive* conceptions of knowledge, the classical conception of knowledge finds a systematical place among the different conceptions of knowledge that have been proposed as a reaction to Gettier and scepticism. We will understand strict invariantism and context sensitivism as two competing views, concerning the epistemic standards a subject $X$ must meet in order to know that $p$ is true: A strict invariantist demands that these epistemic standards are fixed, whereas a context sensitivist allows the epistemic standards to vary situationally.[4] Each

3. The Gettier examples show that in certain cases, justified true belief is not sufficient for knowledge. Gettier published his examples in 1963 [Gettier 1963]. The so-called *sceptical challenge* derives from the possibility of sceptical scenarios like Descartes' *Evil Demon* or Putnam's *brain in a vat* scenario [Putnam 1992].

4. Note that when using the term "context sensitivism", we do not distinguish between contextualism and a sensitive form of invariantism, namely *subject sensitive invariantism* [MacFarlane 2005]. Our main argument in this section hits *strict* invariantist readings of formalizability criteria, and will not be in favor of either sensitive invariantism or contextualism. We will touch this issue again in Section 5.1 as part of the outlook on Step 3 of our agenda.

of these views is compatible with both internalistic and externalistic theories of knowledge. An internalist claims that $X$ has to meet the relevant epistemic standards *and* has to have (actual) cognitive access to this fact, whereas an externalist claims that $X$ has to meet the standards objectively, but does not need to have access to that fact. Accordingly, the classical conception of knowledge falls into the category of *internalistic strict invariantism.*

## 3.2 Internalistic strict invariantism and formalizability criteria

A criterion for "$X$ knows that $p$ is true" that satisfies the following conditions:

(Int) It yields an internalistic and

(Inv) strict invariantist conception of knowledge, *and*

(T) links epistemic justification reliably to truth

apparently characterizes some exceptional epistemic status, following the line of thought developed in Section 3.1.

The following specification of $(*)$ satisfies the conditions (Int) and (Inv), due to the employed adjustments of the two parameters "available" and "formalizable":

$(*')$ $X$ has current cognitive access to a formal proof of $p$.

Moreover, having access to a formal proof *guarantees* the existence of a formal derivation in a consistent axiomatic system, and thus the truth of the derived theorem relative to the truth of the axioms. In that sense, one might say that $(*')$ exemplifies a *perfectly reliable* relation between truth and justification, and therefore fulfills (T). So, "$X$ has current cognitive access to a formal proof of $p$" as a criterion for "$X$ knows that the mathematical statement $p$ is true" may seem to account for the desired exceptional epistemic status of mathematical knowledge. But is $(*')$ also adequate with regard to our agenda, respecting both the alleged exceptional nature of mathematical knowledge *and* actual mathematical practice?

In a recent paper on a context sensitive account of mathematical knowledge [Löwe & Müller 2008], Benedikt Löwe and Thomas Müller argue that *any* attempt to specify the notion of availability in "$X$ has available a formal proof of $p$" as a criterion for knowing that $p$ is true without violating (Inv) or (Int) either leads to the conclusion that mathematicians actually have nearly zero non-trivial mathematical knowledge, or leads to counter-intuitive examples of true knowledge ascriptions. In the first case, "available" is understood in the narrowest sense as current cognitive access, such as in $(*')$. Here, Löwe and Müller argue that facing the *empirical fact* that a formal derivation of some non-trivial mathematical statement can easily take up to thousands of logical steps, current cognitive access to formal proofs of theorems of higher mathematics is not a realistic option for a working mathematician. In the second case, the reading of "available" is severely weakened. As a paradigmatic example, Löwe and Müller discuss the possibility of introducing a certain time frame to distinguish availability from actual possession. We may formulate the corresponding specification of $(*)$ as follows:

> $(*'')$ Within a time period of length $t$, $X$ will have current cognitive access to a formal proof of $p$.

Here, their objections are based on more philosophical grounds: If we take $(*'')$ as a criterion for knowing that $p$ is true, which amount of time is reasonable, regarding the enormous complexity of formal proofs of the theorems of higher mathematics?

Following the argumentation of Löwe and Müller, and somehow in anticipation of Step 2 and 3 for the particular case, $(*')$ (and $(*'')$ as well) should thus be rejected as candidates for an adequate criterion for "$X$ knows that $p$ is true" in the spirit of our agenda.

Still, there may be other specifications of $(*)$ that satisfy (Inv), (Int), and (T), and there may also be different ways to account for the exceptional epistemic status of mathematical knowledge *instead* of an analysis via (Inv), (Int) and (T). A question of special interest in this context is whether a strictly invariantist *and at the same time* internalist reading of formalizability criteria could be adequate for the purpose at hand at all, or whether an externalist reading could be better suited. Yet, we will have to leave this as an open question for now, as it is a question of a much more general kind than questioning the adequateness of special readings of formalizability criteria which we discuss in the scope of this paper. Anyway, abstracting from the special case discussed in [Löwe & Müller 2008], we may state as a conjecture that an adequate reading of $(*)$ that strictly fulfills (Inv) and (T) cannot fulfill (Int), as it is hard to imagine how "available" could be appropriately specified under any strong (due to (T)), invariantist reading of "formalizable".

For the purpose of this paper, we will postpone further philosophical analysis of possible readings of $(*)$ to Section 5, and will now turn to first results of a socio-empirical study on the conditions for ascribing knowledge in mathematical practice. The results already point to the conclusion that a particular family of possible readings of $(*)$ is incompatible with conceptions of knowledge that appear to be employed in actual mathematical practice. On the other hand, the results also suggest an alternative way of specifying $(*)$ that appears to be a promising candidate for Step 3 of our agenda.

# 4 Step 2: A socio-empirical study on the meaning of knowledge ascriptions in mathematical practice

How can we investigate the meaning of knowledge ascriptions employed in actual mathematical practice? Emphasizing the fact that the mathematical community is, like any other scientific community, first and foremost a *social* collective, we make the following explicit assumptions regarding our approach:

- An important part of the meaning (including truth-conditional semantics) of knowledge ascriptions in mathematical practice is determined by the *actual use* of knowledge ascriptions among working mathematicians.
- A working mathematician affirms a knowledge ascription if and only if she takes it to be true.[5]

If we are interested in the meaning of knowledge ascriptions in mathematical practice, we should thus shift our attention to how mathematicians use knowledge ascriptions in practice. We should investigate the conditions under which working mathematicians use sentences of the form "$X$ knows that $p$ is true".[6]

This shift of attention provides a natural starting point for an empirical investigation of mathematical practice, particularly by means of empirical sociology. While it might not be reasonable to investigate empirically whether some particular knowledge ascription *is actually true or not*, the use of knowledge ascriptions seems to be a natural domain for empirical research on mathematical practice. In focussing on the latter, we regard knowledge ascriptions as *actions* that can be more or less appropriate relative to the mathematical community. Some ascriber $Y$ acts in a certain way by saying "$X$ knows that the mathematical statement $p$ is true" [Kompa 2001, 16–17]. Whether an action counts as appropriate (or justified) in a certain community is determined by social mechanisms rather than by explicit verbal rules. Tools for an empirical investigation of social actions and social mechanisms are provided by sociology in the form of quantitative and qualitative research techniques. Under the methodological assumption that mathematicians ascribe knowledge systematically, the results of a socio-empirical investigation of knowledge ascriptions in

5. The "only if" part of this assumption may be seen as a version of one of the Gricean conversational maxims: "Do not say what you believe to be false", falling under the category of Quality of his cooperative principle of conversation [Grice 1975]. The "if" part may not be valid in general, but reduces to a plausible methodological premise in the setting of our empirical research project, a questionnaire study.

6. Referring to the well-known semantics-pragmatics distinction, our first assumption emphasizes that this shift does *not* imply a restriction to "pure pragmatics" of knowledge ascriptions, leaving the semantics as an autonomous constituent of their meaning aside. At least truth-conditional semantics is not taken to be autonomous with respect to pragmatics in general, e.g. [Carston 1991, 47–48].

mathematical practice will point to certain standards for knowledge ascriptions that are taken to be appropriate by mathematicians.

In what follows, we will present a mainly quantitative web-based survey on standards for knowledge ascriptions in actual mathematical practice. Recalling that our overall interest is in the role of available formalizable proof for mathematical knowledge, we have restricted the range of situational, concrete case studies to knowledge ascriptions based on claimed proof.[7]

## 4.1 What is out there?

Many authors emphasize the role of sociological considerations for an epistemology of mathematics, for instance the later Wittgenstein, Philip Kitcher [Kitcher 1984], Paul Ernest [Ernest 1998], or David Bloor [Bloor 1996, 2004].

Yet, there has hardly been done any *socio-empirical* research about actual mathematical research practice: The first socio-empirical study published was Bettina Heintz's work about the culture and practice of mathematics as a scientific discipline [Heintz 2000]. In her study, Heintz used qualitative methods. Her work is based on a detailed field study (at the Max-Planck-Institute for Mathematics in Bonn, Germany) and a series of qualitative interviews with mathematicians.[8]

In our survey on the conditions for knowledge ascriptions in mathematical practice, we employed mainly quantitative methods. We used an online questionnaire which contained mostly multiple choice questions, and some space for free-text comments. The survey was announced together with a link to the online questionnaire via postings in different scientific newsgroups. It was opened in August 2006 and closed in October 2006.

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7. It is important to note that the restriction to knowledge ascriptions based on claimed proof does *not* coincide with a restriction to an internalist epistemology, neither regarding knowledge nor regarding justification. One might be tempted to bring this up as an objection, as knowledge ascriptions based on claimed proof seem to focus too much on *the epistemic subject's argumentative practice* [Brendel 1999, 236], entailing that all justifying factors are part of what is actually accessible to the epistemic subject. However, this depends essentially on the particular specification of "available" and "formalizable". As we have already seen, there are a lot more notions of availability than just the narrow notion of immediate, conscious cognitive access. Another obvious way to incorporate external factors in formalizability criteria is to exploit the possibility that an epistemic subject may have available a formalizable proof *without* having (conscious) cognitive access to the fact that the proof *is* formalizable (cf. Section 5.1).

8. In his PhD Thesis, Jörg Markowitsch also refers to results from interviews with mathematicians [Markowitsch 1997].

## 4.2 Methodology and project data

### 4.2.1 Quantitative vs. qualitative methods

Our study may be seen as a methodological attempt to employ quantitative methods for a socio-empirical investigation of actual mathematical research practice, though we do not aim at using all state-of-the-art statistical evaluation tools from empirical sociology. In the first place, we try to explore if and how the empirical results may generally be brought to bear on an epistemology of mathematics. The use of quantitative methods is due to the assumption that the way mathematicians *think* about mathematics and mathematical knowledge is heavily influenced by individual factors and might even be a matter of personal style. In contrast, mathematical *practice* appears to be very homogenous and uniform. It is a well-known phenomenon in sociology that the way people think about certain issues does not necessarily coincide with how they act, it may only point into the right direction [Klammer 2005, 220]. By the use of quantitative methods, we hope to get more significant results about the conditions for appropriate knowledge ascriptions on which mathematicians *agree*.

The qualitative results from the free-text part are supposed to deepen the hypotheses that were tested in the quantitative part. These results were also used to develop an interview guideline for a follow-up qualitative interview study with international research mathematicians.

### 4.2.2 Target group and adequacy of the sample

A great methodological obstacle for anyone who wants to investigate mathematical practice by surveys is the apparent unwillingness of its protagonists to participate. To circumvent this obstacle, we used an online questionnaire in our project which was posted in three scientific Internet newsgroups, as newsgroup readers are supposed to be less averse to surveys. We do not regard the resulting restriction of the sample as a serious limitation: A significant correlation between the habit of reading newsgroups and a certain attitude towards formalizable proof does not seem very likely.[9]

The link to the online questionnaire was posted two times. The first posting was a test-run in the newsgroup `sci.math`. For the second time, the link was posted in `sci.math.research` and `de.sci.mathematik`, with 214 responses in total.

The target group of our survey consisted of participants with research or teaching experience in any branch of mathematics. A preliminary personal data section of the questionnaire served to decide whether a participant belonged to the target group or not. A response was considered as valid if the personal data part and at least one question in one of the three main parts

9. The follow-up interview study mentioned above shall also serve as a control mechanism in this regard.

of the questionnaire (cf. Section 4.3) was completed. We received 76 valid responses from the target group.[10] 13.2% of these 76 participants received a B.A. (or an equivalent degree), 19.7% a M.Sc. (or an equivalent degree), and 46.1% a Ph.D. (or an equivalent degree) in mathematics. Most of the valid responses from the target group came from the United States of America, Germany, and the Netherlands, from mathematicians with both research and teaching experiences at university level for at least one year.

What we will present below are the results from the second posting, exclusively based on valid answers from the target group.

## 4.3 Empirical hypotheses

The questionnaire we used fell into three main parts:

- **Part I** dealt with the abstract concept of knowledge and proof mathematicians develop,
- **Part II** focussed on knowledge ascriptions "in action", and
- **Part III** was on mathematical beauty.

In what follows, we will only consider results from the first and the second part, because the results from Part III do not bear on the hypotheses we want to discuss within the scope of this paper.

The results from Part I and II were supposed to complement each other. The questions in Part I also served as control questions for Part II.

The empirical hypotheses that were tested in Part II derive from two sources. The first source is the analysis of the concept of knowledge underlying formalizability criteria, which means, the results of Step 1. The other source are experiences from mathematical practice itself. We will discuss three of these hypotheses here:

(H1) *Whether mathematicians ascribe knowledge based on claimed proof depends essentially and systematically on contextual factors.*

The claim that in mathematical practice, knowledge ascriptions based on claimed proof are context sensitive has been made in [Löwe & Müller 2008]. As mentioned in Section 3.2, we believe that this claim might be true due to an essential incompatibility of an account of the epistemic role of mathematical proof that strictly preserves (T) and (Inv), and an internalist view of epistemic justification. Hence, it must not be reduced to a statement about purely *pragmatic* features of knowledge ascriptions. The apparent uniformity of mathematical practice suggests that the claimed context sensitivity should be systematic, and (therefore) accessible by empirical means.

10. In the presentation of the results in Section 4.4, we will give the total count of valid answers from the target group for each reported multiple choice question.

(H2) *The actual possession of a formal proof by the epistemic subject $X$ is sufficient for ascribing knowledge to $X$.*

In Section 3.2, it is argued that truth criteria for "$X$ knows that $p$ is true" which demand the actual possession of a formal proof of $p$ are too restrictive, because as a consequence nearly no mathematician would have much more than non-trivial mathematical knowledge. Yet, for any current philosophical conception of knowledge, the actual possession of a formal proof of $p$ is *sufficient* to know that $p$ is true. Therefore, we also expect to find in mathematical practice that the possession of a formal proof of $p$ is sufficient to count as knowing that $p$ is true. Otherwise, it would appear questionable whether the endeavour of combining the analytical, philosophical conception of mathematical knowledge with the concept of knowledge used in mathematical practice might be successful.

(H3) *Mathematicians do not necessarily demand a formal proof in order to ascribe knowledge.*

Formal proofs are rarely used in mathematical practice. Based on this, the authors of [Löwe & Müller 2008] argue that the actual possession of a formal proof of $p$ is not a reasonable criterion for knowing that $p$ is true. Accordingly, the standards for knowledge ascriptions actually used in mathematical practice should not include the constraint that $X$ must actually possess a formal proof of $p$ in order to know that $p$ is true.

## 4.4 Selected results

In the following, we will present selected results of Part I and II by giving an excerpt of the questions together with the corresponding data. As the questionnaire had 74 questions in total, we will only present those questions and results that have been significant for the interpretation regarding our hypotheses (H1), (H2) and (H3).

**Part I** These are two examples of questions participants were asked in the first part of the questionnaire:

"*Is mathematical knowledge objective?*"

| **response** | **frequency** | **count** ($\Sigma$ 74) |
|---|---|---|
| yes | 82.4% | 61 |
| no | 17.6% | 13 |

"*Please select to which degree you accept the following statement: 'One can precisely define what a mathematical proof is.'*"

| **response** | **frequency** | **count** (Σ 74) |
|---|---|---|
| strongly agree | 28.4% | 21 |
| agree | 60.8% | 45 |
| disagree | 9.5% | 7 |
| strongly disagree | 1.4% | 1 |

Participants who gave a positive answer to the latter question, either "strongly agree" or "agree", were then asked in a free-text question to give a definition of "mathematical proof". We'll give selected quotes from the answers. The majority of answers gave a formal definition of mathematical proof. The quotes below show some less frequently given answers. Note that the second and fourth answer still state that the informally proven theorem has to be formally derivable:[11]

- "Formally: from a given set of deductive rules, and a set of axioms, a sequence of statements (machine-verifiable in the correctness of application of the rules), starting with hypothesis and ending with conclusion. Ideally, anyway."
- "I would defined a proof fundamentally as argument that convinces mathematicians, less fundamentally as an argument that can be formalised and proven mechanically."
- "A convincing argument that instills belief that it is possible to construct a sequence of formal logical steps leading from generally accepted axioms to the given assertion."
- "A finite sequence of statements following logically from each other, that begin with a given set of axioms and have the statement to be proved as a conclusion. In actual mathematical practice, this sequence tends to be shortened and written in some human language rather than pure symbolic logic, so the only difficulty that may arise in defining what a 'real-life' mathematical proof is, is to decide what constitutes an acceptable abbreviation of the hypothetical, full-length logical proof."

**Part II** In the second part, people were led through four scenarios. Each screen of the online questionnaire contained a piece of the story, and at the end of each screen the participants were repeatedly asked whether they would ascribe knowledge to the protagonist of the scenario or not. In the following, we will give excerpts from Scenario 1 and Scenario 3. The ongoing story of each scenario will be shortly summarized when parts are left out.

**In Scenario 1,** the protagonist was a PhD student named John:

"***Scenario 1***

*John is a graduate student, and Jane Jones, a world famous expert on holomorphic functions, is his supervisor. One evening, John is working on the Jones conjecture and seems to have made a break-through. He produces scribbled notes on yellow sheets of paper and convinces himself that these notes constitute a proof of his theorem.*"

11. The orthography of all quotations has been corrected, but the grammar has been left as it was in the original responses.

Then, participants were asked to answer the following question:

> "*Does John know that the Jones conjecture is true?*"

On the following screens, the story continued, and the question "Does John know that the Jones conjecture is true?" was repeated several times. In the final part of the story, John and his supervisor jointly prove the Jones Conjecture, and publish their proof in a mathematical journal of high reputation:

> "*Eighteen months later, the editor accepts the paper for publication, based on a positive referee report.*"

At this point of the story, the majority (84.9%) of the participants gave a positive answer to the question:

> "*Does John know that the Jones conjecture is true?*"

| **response** | **frequency** | **count** ($\Sigma$ 66) |
|---|---|---|
| yes | 28.8% | 19 |
| almost surely yes | 56.1% | 37 |
| almost surely no | 3.0% | 2 |
| no | 4.5% | 3 |
| can't tell | 7.6% | 5 |

On the next screen, which was the last of Scenario 1, the story ended with:

> "*After his Ph.D., John continues his mathematical career. Five years after the paper was published, he listens to a talk on anti-Jones functions. That evening, he discovers that based on these functions, one can construct a counterexample to the Jones conjecture. He is shocked, and so is professor Jones.*"

Now, 61.3% of the participants gave a positive answer on the question whether John knows that the Jones conjecture is *false*:

> "*Does John know that the Jones conjecture is false?*"

| **response** | **frequency** | **count** ($\Sigma$ 62) |
|---|---|---|
| yes | 14.5% | 9 |
| almost surely yes | 46.8% | 29 |
| almost surely no | 6.5% | 4 |
| no | 8.1% | 5 |
| can't tell | 24.2% | 15 |

*On the same screen*, the participants were also queried:[12]

12. Participants were able, though not requested, to switch back and forth between the different screens.

"*Did John know that the Jones conjecture was true on the morning before the talk?*"

The result is somehow astonishing, because still, 71% gave a positive answer to this question:

| **response** | **frequency** | **count** ($\Sigma$ 62) |
|---|---|---|
| yes | 24.2% | 15 |
| almost surely yes | 46.8% | 29 |
| almost surely no | 4.8% | 3 |
| no | 14.5% | 9 |
| can't tell | 9.7% | 6 |

Note that when answering these last two questions, participants *already had the information* that John has discovered a counterexample to the Jones conjecture.

After finishing Scenario 1, we asked the participants to give comments on the scenario in a free-text field. These are some selected quotes, emphasizing different factors that influenced the answers in Scenario 1:

- "How important is the Jones conjecture? How large is the community?"
- "My answers would have been very different with different time frames mentioned."
- "I don't know John or Bob, so I don't have a good feel for how rigorously they work."

**In Scenario 3,** the protagonist was a student of mathematics named Tom, and the setting was an oral examination at the end of the semester:

"***Scenario 3:***
*Tom Jenkins is a student of mathematics and has to pass an oral exam at the end of the algebra lecture held by his professor Robin Smith. Tom did some oral exams before, so he is not too nervous, and is able to pay concentrated attention to the professor's questions during the whole exam. At some point of the exam, Smith asks Tom for the proof of a certain algebraic theorem T1. The proof consists mainly of a tricky application of the fundamental theorem on homomorphisms and was conducted in one lecture on the blackboard. Tom is able to give a rather technical, but absolutely correct step-by-step proof in full detail.*"

83.3% of the participants gave a positive answer to the question:

"*Does Tom know that T1 is true?*"

| **response** | **frequency** | **count** ($\Sigma$ 54) |
|---|---|---|
| yes | 46.3% | 25 |
| almost surely yes | 37% | 20 |
| almost surely no | 1.9% | 1 |
| no | 1.9% | 1 |
| can't tell | 13% | 7 |

In the proceeding story, professor Smith challenges Tom with questions concerning the ideas behind the proof of T1, and further applications of these ideas. Tom fails at all these questions:

> "*The exam continues with some questions about definitions, and after some minutes Smith asks Tom to explain why the general idea of how to apply the fundamental theorem on homomorphisms in the proof of the former theorem is also fruitful to prove a second algebraic theorem T2. Tom completely fails in his answer.*"

Still, 83.3% gave a positive answer to the question whether Tom knows that T1 is true. There is only a slight adjustment from "yes" towards "almost surely yes":

> "*Does Tom know that T1 is true?*"

| **response** | **frequency** | **count** ($\Sigma$ 54) |
|---|---|---|
| yes | 40.7% | 22 |
| almost surely yes | 42.6% | 23 |
| almost surely no | 3.7% | 2 |
| no | 1.9% | 1 |
| can't tell | 11.1% | 6 |

On the same screen, participants were asked:

> "*Did Tom know that the first theorem T1 was true before he failed in answering Professor Smith's last question correctly?*"

Again, there is nearly no quantitative effect on the positive knowledge ascriptions, 83% gave a positive answer. Note the slight adjustment from "almost surely yes" towards "yes" compared with the answers corresponding to the preceding screen:

| **response** | **frequency** | **count** ($\Sigma$ 53) |
|---|---|---|
| yes | 47.2% | 25 |
| almost surely yes | 35.8% | 19 |
| almost surely no | 1.9% | 1 |
| no | 1.9% | 1 |
| can't tell | 13.2% | 7 |

The story continued:

> "*Smith asks Tom to formulate the general idea behind the proof of the first theorem T1. Tom fails in his answer.*"

Note that although there is a clear shift from positive to negative answers (13.2%) to the next question, there is no effect on the results concerning the qualitative behavior of positive and negative knowledge ascriptions:

> "*Does Tom know that the first theorem T1 is true?*"

| **response** | **frequency** | **count** ($\Sigma$ 53) |
|---|---|---|
| yes | 32.1% | 17 |
| almost surely yes | 37.7% | 20 |
| almost surely no | 11.3% | 6 |
| no | 5.7% | 3 |
| can't tell | 13.2% | 7 |

After finishing Scenario 3, we asked the participants again to give comments on the scenario in a free-text field. These are some selected quotes. The quotes support the quantitative result that a correct step-by-step proof, at least when it is given by the epistemic subject, is sufficient for knowledge ascriptions or knowledge claims:[13]

- "He knows the truth of the theorems, because these are well known and proved theorems."
- "Once you are sure that you gave a correct proof of theorem T1, you need not revise your opinion on the truth of T1."
- "Tom knew all the time that T1 was true. He could give a complete and correct proof, after all!"
- "Well you don't need to understand the idea behind the proof of some theorem [...] and to some point the idea is not important at all—just as I said before: symbol processing."

## 4.5 Summary and interpretation of the selected results regarding (H1), (H2) and (H3)

In the following, we will propose some interpretation of the above presented results.

First of all, there appears to be a certain tension between the results of Part I and II:

13. Concerning orthography and grammar of questionnaire quotations, cf. Footnote 11.

- In Part I, the majority of the free-text answers to the question on the definition of "mathematical proof" refer to the definition of formal proof. In contrast, the quantitative results from Scenario 1 in Part II affirm (H3): Formal proof is not necessary for knowledge ascriptions in mathematical practice (cf. below).
- In Part I, 82.4% of the participants answered that mathematical knowledge is objective. In contrast, the free-text comments on the scenarios in Part II show that the standards for knowledge ascriptions that are taken to be appropriate seem to depend on less objective factors like the size of the community, the importance of the proven theorems, or on time frames.

Regarding the three hypotheses (H1), (H2) and (H3) formulated in Section 4.3 the following can be observed:

**(H1)** *Whether mathematicians ascribe knowledge based on claimed proof depends essentially and systematically on contextual factors.*

The free-text comments in Part II show that whether subjects ascribe knowledge or not may depend on:

- the size of the community of the corresponding branch of mathematics,
- the number of referees,
- what is at stake,
- how important the proven theorem is,
- the epistemic subject (e.g., working habits),
- time frames.

This suggests a *systematic* context sensitivity of standards for knowledge ascriptions, as it seems to be possible to categorize the relevant contextual factors beyond the actual, concrete context.

The quantitative results from Scenario 1 point to the conclusion that this sensitivity is not contingent, but rather essential. *After the participants had received the information* that John has discovered a counterexample to the Jones conjecture, they were asked both if he does *now* know that the Jones conjecture is *false*, and if he did know *before* his discovery that the Jones conjecture was *true*. Assuming that the appropriateness of the ascription "John knows that the Jones conjecture is false" implies the appropriateness of the ascription "John does not know that the Jones conjecture is true", the results suggest that the standards for knowledge ascriptions that are taken to be appropriate in mathematical practice are not strictly invariant, but may for example depend on the epistemic standards in play at the context of the epistemic subject (John in this case):

- *Does John know that the Jones conjecture is false?*
  61.3% 'yes' or 'almost surely yes'
- *Did John know that the Jones conjecture was true on the morning before the talk?*
  71% 'yes' or 'almost surely yes'

This answering profile was the most frequently represented one, but is incompatible with strictly invariantist standards for knowledge ascriptions – according to the latter, the results for "Did John know that the Jones conjecture was true on the morning before the talk?" should have been a majority of negative answers. Our interpretation is backed up by the free-text comments on Scenario 1 given by participants who answered "can't tell" on the last two questions of the scenario. The following quotes may serve as paradigmatic examples here:

- "[...] There is no objective truth, even in mathematics. There is only mathematicians who may be convinced one way or another, and a 'proof' (or a 'counterexample') may be a means to convince some/many/most of them."
- "I have consistently interpreted the word 'know' in a rather narrow sense. Had I been John, I would have claimed for myself knowledge once I was reasonably confident that I had inspected every detail of a proof—but I might have been wrong in believing that I knew. [...]"

Though knowledge ascriptions seem to work quite well in mathematical practice, participants reflecting on the requirements of a theoretical, rather invariantist notion of "to know" weren't able to decide whether to ascribe knowledge or not in the given scenario anymore.

**(H2)** *The actual possession of a formal proof by the epistemic subject $X$ is sufficient for ascribing knowledge to $X$.*

The quantitative results as well as the free-text comments from Scenario 3 affirm this hypothesis.

**(H3)** *Mathematicians do not necessarily demand a formal proof in order to ascribe knowledge.*

This hypothesis appears to be affirmed, again by the results from Scenario 1 in Part II: 61.3% of the participants gave a positive answer on the question whether John knows that the Jones conjecture is false, *after they had received the information that he has discovered the counterexample*. 71% also gave a positive answer to the question whether he still knew that the Jones conjecture was true before he discovered the counterexample. If a formal proof had been demanded by these participants, they would have committed themselves rather consciously to the claim that a formal proof of both the Jones conjecture and its negation would be possible, and thus to an apparent contradiction: Formal derivability of both the Jones conjecture and its negation yields the inconsistency of the axioms of the formal system that is used, but the notion of formal proof includes the consistency of the axioms.

The results suggest an even stronger conclusion: For readings of formalizability criteria that necessarily entail the logical possibility to prove $p$ formally (with respect to a fixed formal system), the same contradiction holds. So, it seems that at least these formalizability criteria for knowledge ascriptions are not necessarily employed by mathematicians.

# 5 Conclusions

We saw that the effectively employed conditions for knowledge ascriptions in mathematical practice seem to depend on contextual factors, and even that formal provability of a mathematical statement $p$ appears to be not necessary for ascribing knowledge of $p$. On the other hand, actually available formal proof seems to be sufficient for ascribing mathematical knowledge, and in spite of the relative use of knowledge ascriptions, there is still an emphasis on the objectivity of mathematical knowledge and proof in the back of working mathematicians' minds. What conclusions can already be drawn from these observations towards Step 3 of our agenda, and what lessons can be learned regarding a possible iteration of Step 1 and 2?

## 5.1 Towards Step 3

The considerations we have undertaken in Section 4.5 point to some *general* incompatibility of the empirical results and purely internalistic, strictly invariantist readings of formalizability criteria (recall that in Section 3.2, we have only discussed particular variants of such readings of $(*)$). The use of knowledge ascriptions in mathematical practice seems to depend on both *internal and external contextual factors*. Factors related to the epistemic subject, such as working habits, are internal factors, whereas time frames, size of the community, number of referees checking a claimed proof, or importance of the proven theorem are external factors. Therefore, a specification of $(*)$ that can still account for our empirical results should allow for some weaker standards for knowledge ascriptions than strict invariantist standards, and it should also be capable of incorporating external factors.

We would like to remind the reader of our conjecture about formalizability criteria fulfilling (T), (Inv), and (Int), stated in Section 3.2. Employing the terminology used in that section, an incorporation of the external factors may yield a relative notion of formalizability (*i.e.*, a weakening of (T)), and a quite externalist understanding of "having available a formalizable proof" (*i.e.*, a weakening of (Int)) in turn: Whether a given proof is formalizable could be specified relative to time frames involved, size of the community, number of referees checking the proof, or importance of the proven theorem. Hence, an epistemic subject $X$ could have available a formalizable proof without having cognitive access to that fact. Factors related to the epistemic subject $X$, like working habits, or maybe certain mathematical skills, can be modeled by an appropriate specification of "available". Depending on the particular role and significance of these contextual factors (which has to be investigated further), their incorporation may also have a weakening of (Inv) as a result.[14] As mentioned in the interpretation of the empirical results regarding (H1) (cf. Section 4.5), shifts of $X$'s context seem to be more relevant than shifts in the ascriber's context concerning the epistemic standards for knowledge ascriptions in this regard.[15]

Therefore, what we end up with in some kind of first order approximation is that an externalistic, subject sensitive reading of $(*)$ might indeed be better suited than the traditional, internalistic strict invariantist readings discussed in Section 3.2.

## 5.2 Outlook—starting points for an iteration of Step 1 and 2

Regarding Step 1, the next iteration of our three-step research programme will thus be devoted to a theoretical discussion of possible candidates for an externalistic, subject sensitive reading of $(*)$. As a starting point, we will investigate Azzouni's *derivation indicator view* of proof developed in [Azzouni 2004], and Manin's notion of *degrees of proofness* [Manin 1977, 1981] as candidates for the respective specification of "formalizable proof".

Regarding Step 2, it should then be investigated whether and how these candidates for readings of $(*)$ fit our empirical results. To this end, some of these results have to be refined in additional empirical studies. For example, the tension between the empirical results concerning mathematicians' abstract concepts of mathematical knowledge and proof, and the empirical results concerning knowledge ascriptions in mathematical practice, has to be investigated further. Are the conceptions of knowledge and proof used in mathematical practice inconsistent, or is the tension between the results due to different, but compatible, aspects? Two other issues for further empirical investigation are to clarify the role of contextual factors for the epistemic standards employed in mathematical practice, and the role of different factors determining

14. Note that even if the context dependency of epistemic standards might turn out to be an empirical phenomenon of knowledge ascriptions in general, the crucial point would be that mathematics is no exception.

15. A general epistemology that could account for these empirical results might thus be subject sensitive invariantism rather than contextualism in the narrow sense [MacFarlane 2005, 198–199]. Cf. also Footnote 4.

a possible relative notion of formalizability. Are there other or additional factors than those mentioned by the participants of our first study? Part of these questions are being pursued empirically in our ongoing qualitative interview study mentioned in Section 4.2.1.

# References

AZZOUNI, JODY

2004 The Derivation Indicator View of Mathematical Practice, *Philosophia Mathematica*, 12 (2), 81–106.

BLOOR, DAVID

1996 *Scientific Knowledge: A Sociological Analysis*, Chicago: Chicago University Press.

2004 Sociology of Scientific Knowledge, *Handbook of Epistemology*, I. Niiniluoto et al. (eds.), Dordrecht: Kluwer, 919–962.

BRENDEL, ELKE

1999 *Wahrheit und Wissen*, Paderborn: Mentis.

CARSTON, ROBYN

1991 Implicature, Explicature, and Truth-Theoretic Semantics, *Pragmatics. A Reader*, Steven Davis (ed.), New York: Oxford University Press, 33–51.

DETLEFSEN, MICHAEL (ED.)

1992 *Proof, Logic and Formalization*, London: Routledge.

ERNEST, PAUL

1998 *Social Constructivism as a Philosophy of Mathematics*, Albany, New York: SUNY Press.

GETTIER, EDMUND

1963 Is Justified True Belief Knowledge?, *Analysis*, 23, 121–123.

GRICE, HERBERT P.

1975 Logic and Conversation, *Syntax and Semantics*. Vol 3: *Speech Acts*, P. Cole & J. Morgan (eds.), New York: Academic Press.

HEINTZ, BETTINA

2000 *Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin*, Wien: Springer.

KITCHER, PHILIP

1984 *The Nature of Mathematical Knowledge*, New York: Oxford University Press.

KLAMMER, BERND

2005 *Empirische Sozialforschung*, Konstanz: UVK.

KOMPA, NIKOLA

2001 *Wissen und Kontext*, Paderborn: Mentis.

LÖWE, BENEDIKT & MÜLLER, THOMAS

2008 Mathematical Knowledge is Context-Dependent, *Grazer Philosophische Studien*, 76, 91–107.

MACFARLANE, JOHN

2005 The Assessment Sensitivity of Knowledge Attributions, *Oxford Studies in Epistemology*, 1, 197–233.

MANIN, YURI I.

1977 *A Course in Mathematical Logic*, New York, Heidelberg, Berlin: Springer.

1981 A Digression on Proof, *The Two-Year College Mathematics Journal*, Vol. 12(2), 104–107.

MARKOWITSCH, JÖRG

1997 *Metaphysik und Mathematik. Über implizites Wissen, Verstehen und die Praxis in der Mathematik*, PhD Thesis, Universität Wien.

PUTNAM, HILARY

1992 Brains in a Vat, *Skepticism: A Contemporary Reader*, K. DeRose & T.A. Warfield (eds.), Oxford: Oxford University Press.

RAV, YEHUDA

1999 Why Do We Prove Theorems?, *Philosophia Mathematica*, Vol. 7(3), 5–41.

ZACH, RICHARD

2003 Hilbert's Program, *Stanford Encyclopedia of Philosophy* (Fall 2003 Edition), Edward N. Zalta (ed.).
http://plato.stanford.edu/entries/hilbert-program/