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Take-it-or-leave-it contracts in many-to-many matching markets*

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Abstract

We study a class of sequential non-revelation mechanisms through which hospitals make simultaneous take-it-or-leave-it offers to doctors. The inclusion of contracts shrinks the set of equilibrium outcomes and harms doctors more than hospitals, but it maintains the stability of the set of equilibrium outcomes. Our analysis reveals the existence of a first-mover advantage that is absent from the model without contracts. The mechanisms of this class are outcome equivalent and implement the set of stable allocations in subgame perfect equilibrium when enough competitive pressure is present. Equilibrium outcomes form a lattice when preferences are substitutable.

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1 Introduction

In this paper, we study decentralized markets involving many-to-many contractual relationships. Our objective is to understand how the inclusion of contracts affects the ability of agents to negotiate the terms of their relationships. This issue is particularly sensitive, as no stable and strategy-proof revelation mechanism (even for one side of the market) exists in this context, rendering many-to-many markets a more complex subject of study than many-to-one markets.[1]

A classical example of a many-to-many market refers to the allocation of specialty training slots for junior doctors in the UK (see Roth, 1991). The introduction of contracts allows us to model other real word markets where the terms of relationships are not established beforehand. Relevant examples of many-to-many markets with contracts are markets that involve part-time workers (e.g., consultants, lecturers or teachers) and those involving non-exclusive dealings between down-stream firms and up-stream providers.

We analyze a class of mechanisms that we refer to as **take-it-or-leave-it offer mechanisms** or TOM. In the first stage of these mechanisms, hospitals make simultaneous offers to doctors, and then, groups of doctors sequentially accept or reject the offers that they have received. The order that doctors adopt can be arbitrary and even history dependent. The simplest of these mechanisms is one through which acceptances are made simultaneously. We refer to this as the **simultaneous acceptance mechanism**. Versions of this mechanism have been analyzed by Alcalde and Romero-Medina (2000), Sotomayor (2004) and Echenique and Oviedo (2006) without reference to contracts and by Alcalde et al. (1998) based on the many-to-one model with money developed by Kelso and Crawford (1982). In all of these cases, the simultaneous acceptance mechanism fully implements the set of stable

[1] Strategy-proofness imposes very restrictive conditions on the preferences of agents, as proven by Kojima (2013) and Romero-Medina and Triossi (2017) for a model without contracts.

allocations in subgame perfect equilibrium under substitutability. On the other extreme of the class of take-it-or-leave-it offer mechanisms, we find mechanisms through which doctors make selections in a given order. This mechanism has been analyzed by Romero-Medina and Triossi (2014) in a many-to-one market without contracts. The College Sequentially Choose mechanism, as they called it, fully implements the set of stable allocations in subgame perfect equilibrium under substitutability. Klaus and Kljin (2017) describe a related class of mechanisms in many-to-many matching markets without contracts. In their model, hospitals make simultaneous offers, and doctors make selections in a given order. However, offers are not fully binding. Indeed, a doctor cannot join a hospital that has offered her a position if at her turn, that hospital does not have an open position. Thus, hospitals do not pay for the entire cost of their decisions, and unstable equilibrium allocations can arise unless hospitals have enough positions to satisfy all demand. This negative result relates to those of Triossi (2009), Haeringer and Wooders (2011) and Romero-Medina and Triossi (2014), who find that allowing the choices of agents to be reversible or renegotiable can lead to the occurrence of unstable equilibrium allocations.

Although simple, take-it-or-leave-it offer mechanisms mimic decentralized procedures used in labor markets and in college or school admissions systems. They capture relevant interactions among hospitals and doctors and identify the basic forces at work in these settings.

We first prove that at her turn, each doctor has a unique best response that involves accepting the best offers she receives. This result allows us to show that all mechanisms of the TOM class are outcome equivalent. This finding reveals the common structure of mechanisms previously analyzed in the literature. We then prove that every subgame perfect equilibrium (SPE hereinafter) outcome of any game induced by a take-it-or-leave-it offer mechanism is a pairwise stable allocation when doctors' preferences are substitutable. Thus, the structure of the mechanism is simple enough to prevent the manifestation of the above-mentioned coordination problems that can arise in such a setting.

Contracts introduce a new strategic consideration. Without them, matching markets simply assign agents to agents. When contracts are available, each hospital negotiates the details of relationships with its counterparts. Thus, ceteris paribus, every hospital will offer only its preferred contracts from those each doctor is willing to accept. Therefore, unlike that for the model without contracts (see Echenique and Oviedo, 2006), the set of SPE outcomes can be a strict subset of the set of stable allocations even when preferences are substitutable.

The characterization of the best response correspondence offers an insight on the inabilities of the take-it-or-leave-it offer mechanisms to implement all stable allocations. The best response includes not only the best contracts for a hospital in response to those offered by others but also the contracts that will never be accepted but that can spur the rejection of offers from other hospitals outside the equilibrium path. These additional contracts amplify competition between hospitals, forcing them to offer better terms to doctors. It follows that when there is enough competitive pressure, any mechanism of the take-it-or-leave-it offer class fully implements the set of stable allocations, generalizing the results of Echenique and Oviedo (2006).

Given that the set of SPE outcomes can be a strict subset of the set of stable allocations, the existence of stable allocations does not guarantee that the game has an SPE under pure strategies. We prove that when all agents have substitutable preferences and the preferences of hospitals satisfy the law of aggregate demand (see Hatfield and Milgrom, 2005), the hospital-optimal stable allocation is an SPE outcome. This reveals the existence of a first-mover advantage that is absent from the model without contracts, as the doctor-optimal stable allocation may fail to be an SPE outcome.

Next, we generalize the existence result and characterize the set of SPE outcomes. We show that when both sides of the market have substitutable preferences, the maximal best response is increasing, implying that an SPE exists in the case of pure strategies. We next investigate the structure of the set of equilibrium outcomes and show that when both sides of the market have substitutable preferences, the SPE outcomes of the game form a lattice. The lattice structure of equilibrium outcomes reflects the opposition of interests between the two sides of the market.

Finally, we relax the assumption of substitutable preferences and study the existence and stability of SPE. We prove that when hospitals have substitutable preferences doctors have unilaterally substitutable preferences, a pure strategy SPE exists and every SPE outcome is pairwise stable. However, when substitutability fails, the set of SPE allocations fails to form a lattice. Such results relate to those of Yenmez (2015). He proves the existence of stable allocations when agents on one side of the market have substitutably completable preferences and when agents on the other side of the market have substitutable preferences.

This paper is organized as follows. Section 2 introduces the model and notation proposed. Section 3 presents take-it-or-leave-it offer mechanisms and the implementation results. Section 4 characterizes the structure of SPE outcomes and proves the possibility of relaxing the substitutability assumption. Finally, Section 5 concludes. Proofs are given in the appendix.

2 The Model

In our model, there is a group of doctors who seek positions at different hospitals. We denote H and D as (finite) sets of hospitals and doctors, respectively. The set of agents is denoted by $N = H \cup D$. There exists a finite set X of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and with one hospital $x_H \in H$. Each agent can sign multiple contracts. A null contract, whereby the agent signs no contract, is denoted by $\emptyset$. An allocation is a set of contracts $Y \subseteq X$. Let Y be an allocation and let $N' \subseteq N$. Let $Y_{N'} = \{y \in Y \mid \{y_H, y_D\} \cap N' \neq \emptyset\}$ be the set of contracts that belong to Y and that involve a member of N'. By abuse of notation, for all $n \in N$ we use Y_n rather than $Y_{\{n\}}$.

For each $h \in H$, $\succ_h$ is a strict preference relation on X_h. Preference relations are extended to allocations in a natural manner: for all allocations Y, Z, $Y \succ_h Z$ means $Y_h \succ_h Z_h$. A preference profile $\succ_h$ defines a choice function $C_h(\cdot, \succ_h)$. Formally, for each $h \in H$ and $Y \subseteq X$, we define the chosen set in Y as $C_h(Y, \succ_h) = \max_{\succ_h} \{Z \mid Z \subseteq Y_h\}$. When there is no ambiguity surrounding $\succ_h$, we write $C_h(Y)$ rather than $C_h(Y, \succ_h)$. Let $C_H(Y) = \bigcup_{h \in H} C_h(Y)$ be the set of contracts chosen by hospitals when the set of available contracts is Y. For each $d \in D$, $\succ_d$, C_d, Y_d, and C_D are defined in the same manner.

As each choice function is derived from a strict preference relation, for each $n \in N$ $\succ_n$ satisfies the **irrelevance of rejected contracts** condition (IRC from this point on).[2] Formally, for each $Y \subseteq X$ and for each $z \in X \setminus Y$, $z \notin C_n(Y \cup \{z\}) \implies C_n(Y \cup \{z\}) = C_n(Y)$.

We use the notation $\succ_H = (\succ_h)_{h \in H}$, $\succ_D = (\succ_d)_{d \in D}$ and $\succ = (\succ_H, \succ_D)$. We then define two partial orders $\succ_{HB}$ and $\succ_{DB}$ for the set of allocations. We assume that allocation Y is preferred to allocation Z according to **hospital Blair's order** (see Blair, 1988), and we write $Y \succ_{HB} Z$ when for each $h \in H$, $C_h(Y_h \cup Z_h) = Y_h$. We assume that allocation Y is preferred to allocation Z according to **doctor Blair's order**, and we write $Y \succ_{DB} Z$ when $C_d(Y_d \cup Z_d) = Y_d$ for all $d \in D$. In other words, Y is preferred to Z according to hospital Blair's order when each hospital h chooses contracts from Y when selecting among all contracts available in $Y \cup Z$.

[2] Sönmez and Aygün (2013) present a detailed analysis of this condition and of its implications.

2.1 Stability and substitutability

Stability is a key concept of market design. Theoretical and empirical findings suggest that markets that achieve stable outcomes are more successful than markets that do not achieve stable outcomes (see Roth and Sotomayor, 1990, and Abdulkadiroğlu and Sönmez, 2013). Stable allocations are identified by two requirements. The first requirement is individual rationality. An allocation is individually rational when no agent wants to unilaterally cancel any of the contracts assigned. Allocation Y is **individually rational** for agent $n \in N$ if $C_n(Y) = Y_n$. The second requirement is that the allocation must not be "blocked." Intuitively, a coalition blocks an allocation when the members of a coalition can profitably renegotiate the contracts of this allocation. A coalition of agents can block a given allocation in a variety of ways. A hospital-doctor pair (h, d) **pairwise blocks** allocation Y when there exists $x \in X \setminus Y$ such that $x_D = d$, $x_H = h$ and $x \in C_h(Y \cup \{x\}) \cap C_d(Y \cup \{x\})$. A coalition of agents $N' = H' \cup D'$, where $H' \subseteq H$ and $D' \subseteq D$, **blocks** allocation Y if there exists a set of contracts $Z \neq \emptyset$ such that (i) $Z \cap Y = \emptyset$; (ii) $Z_{N'} = N'$; and (iii) for all $j \in N'$, $Z_j \subseteq C_j(Z \cup Y)$. These blocking conditions define the different stability concepts. We assume that allocation Y is **pairwise stable** when it is individually rational and when no coalition pairwise blocks it. The set of pairwise stable allocations is denoted by $\mathcal{PS}(M)$. We assume that allocation Y is **stable** when it is individually rational and when no coalition blocks it. Note that any stable allocation is pairwise stable but that the reverse is not generally true (see Hatfield and Kominers, 2017).

Sets of pairwise stable and stable allocations may be empty. The literature has focused on preference restrictions that guarantee the existence of stable allocations by avoiding complementarities among contracts. Substitutability is a key condition for the existence of stable allocations. The preferences of hospital h, $\succ_h$ are **substitutable** when $x, z \in X$ and $Y \subseteq X$ do not exist such that $z \notin C_h(Y \cup \{z\})$ and $z \in C_h(Y \cup \{x, z\})$. The preferences of hospital h are substitutable when the addition of a contract to the choice set never causes a hospital to accept a contract it previously rejected. Substitutable preferences are defined similarly for doctors.

We also employ an additional condition called the law of aggregate demand. The preferences of agent $n \in N$, $\succ_n$ satisfy the **law of aggregate demand** when for all $Z \subseteq Y \subseteq X$, $|C_n(Z)| \leq |C_n(Y)|$.[3] When the preferences of an agent satisfy the law of aggregate demand and as new contracts become available, the agent will select a (weakly) larger number of contracts.

3 Take-it-or-leave-it offer mechanisms

In the first stage of any take-it-or-leave-it offer mechanism, hospitals make simultaneous offers to doctors. Groups of doctors then sequentially accept or reject the offers they have received. Doctors of the same group choose simultaneously. The order of choice can be arbitrary and/or history dependent.

Let $T > 0$ be a positive integer. **Take-it-or-leave-it offer mechanisms** involve the following procedures:

Stage 0**:** Each hospital h offers contracts to some doctors. Let $X(h) \subseteq X_h$ be the set of contracts offered by hospital h. When h does not make any offer, then $X(h) = \varnothing$. For all $d \in D$, let $X(d) = \left(\bigcup_{h \in H} X(h)\right)_d$, be the set of offers received by doctor d.

Stage t**,** $1 \leq T$**:** A subset of doctors $D' \subseteq D$ who did not move in any stage t', $t' < t$ (D' can depend on doctors' choices made in stages $t' < t$), selects a set of contracts from those offered. Let $Y(d) \subseteq X(d)$ be the set of offers selected by doctor $d \in D'$.

This procedure continues until all doctors have chosen. The outcome of the game is $\bigcup_{d \in D} Y(d)$.

[3] For each set Y, $|Y|$denotes its cardinality.

For a profile of preferences $\succ$, any TOM induces an extensive game of complete and imperfect information (see Maschler et al, 2013). In this paper, we focus on the pure strategy SPE of games induced by TOM.

3.1 Preliminary results

From this point on, $\mathcal{T}$ denotes a mechanism of the TOM class. For each profile of preferences $\succ$, $(\mathcal{T}, \succ)$ denotes the extensive form game with imperfect information induced by $\mathcal{T}$ when the preferences of agents are $\succ$. In the first stage of the game, the strategy space of hospital h is $\mathcal{S}_h = 2^{X_h}$.[4] Note that the only payoff-relevant information for doctor d is the set of contracts she has been offered by hospitals, $X(d)$. For each $d \in D$, let $\mathcal{S}_d$ be the strategy space of doctor d.

We start our analysis by characterizing the equilibrium behaviors of doctors. For each $(X(h))_{h\in H} \in \prod_{h\in H} \mathcal{S}_h$ and for each $(S_{d'})_{d'\neq d} \in \prod_{d'\neq d} \mathcal{S}_{d'}$, define $\hat{S}_d\left((X(h))_{h\in H}, (S_{d'})_{d'\neq d}\right) = C_d\left(\left(\bigcup_{h\in H} X(h)\right)_d\right)$. Strategy $\hat{S}_d$ selects the best contracts offered to doctor d. At her turn, a doctor d has a unique best response and adopts strategy $\hat{S}_d$.

Lemma 1 *Let $\left((S_h^*)_{d\in D}, (S_d^*)_{d\in D}\right) \in \prod_{n\in N} \mathcal{S}_n$ be an SPE of $(\mathcal{T}, \succ)$. Then, for each $d \in D$, $S_d^* = \hat{S}_d$.*

To simplify the analysis, we consider a normal form game Γ where the set of players is H, the strategy space of the hospital h is $\mathcal{S}_h = 2^{X_h}$, and the outcome function is $g\left((S_h)_{h\in H}\right) = C_D\left(\bigcup_{h\in H} S_h\right)$, $\Gamma = \left(H, \succ_H, \left(2^{X_h}\right)_{h\in H}, g\right)$. Lemma 1 implies that there is one-to-one correspondence between the Nash equilibria (NE from this point on) of Γ and the SPE of the game induced by any TOM: determining the SPE of the game induced by any TOM determines the NE of Γ. In particular, the set of SPE allocations of games induced by two different TOM coincide.

[4] For each set Y, 2^Y denotes the set of its subsets.

Proposition 1 *The strategy profile* $\left((S_h^*)_{h\in H}, \left(\hat{S}_d\right)_{d\in D}\right)$ *is an* SPE *of* $(\mathcal{T}, \succ)$ *if and only if* $(S_h^*)_{h\in H}$ *is a Nash equilibrium of* Γ.

We now show that when $\succ_D$ is substitutable, any SPE of $(\mathcal{T}, \succ)$ is pairwise stable regardless of hospitals' preferences.

Proposition 2 *Assume that* $\succ_D$ *is substitutable. Then, all* SPE *outcomes of* $(\mathcal{T}, \succ)$ *are pairwise stable allocations.*

When all agents have substitutable preferences, the set of pairwise stable allocations coincides with the set of stable allocations (see Hatfield and Kominers, 2017). Thus, from Proposition 2, we obtain the following result.

Corollary 1 *Assume that* $\succ$ *is substitutable. Then, all* SPE *outcomes of* $(\mathcal{T}, \succ)$ *are stable.*

Corollary 1 extends to a larger set of mechanisms and to a setup with contracts and presents a weaker version of the implementation results of Alcalde and Romero-Medina (2000), Echenique and Oviedo (2006) and Romero-Medina and Triossi (2014).

3.2 Contracts and competitive pressure

A many-to-many matching market without contracts is a market in which $|X_h \cap X_d| = 1$ for all $h \in H$ and $d \in D$. Echenique and Oviedo (2006) analyze the simultaneous acceptance mechanism of this setup. From Proposition 1 and Theorem 7.1 in Echenique and Oviedo (2006), it follows that in markets without contracts, any TOM implements the set of stable allocations in SPE when all agents have substitutable preferences.

In the model without contracts, hospitals only choose whom to make an offer to. Contracts introduce new strategic considerations. Indeed, they allow each hospital to negotiate terms of collaboration with any doctor they may wish to sign. Intuitively, a hospital can always offer a doctor the worst conditions (e.g., the lowest salary) she is willing to accept based on offers made by other hospitals. Therefore, hospitals benefit from a first-mover advantage. The following example shows that the set of SPE allocations of the game induced by any TOM mechanism does not include all stable allocations.

Example 1 *Assume* $H = \{h\}$ *and* $D = \{d\}$*. Let* $x, x', \tilde{x}$ *denote contracts between hospital h and doctor d. Assume that the preferences of the agents are as follows:*
$\succ_d$: $\{x\}, \{x'\}, \{\tilde{x}\}$;
$\succ_h$: $\{\tilde{x}\}, \{x'\}, \{x\}$.
For example, assume that x, x' *and* $\tilde{x}$ *are contracts that pay salaries of $200,000, $175,000, and $150,000 a year, respectively, and that all other contract terms are identical.*

In Example 1, the hospital prefers to pay less and the doctor prefers to be paid more. Only the $150,000 contract is an SPE outcome of every TOM when the hospital makes the offer. Only the $200,000 contract is an SPE outcome of every TOM when the doctor makes the offer.[5] Unlike in markets without contracts, the set of SPE outcomes of any TOM depends on which side of the market makes offers.

In Example 1, contracts through which the doctor is paid $175,000 and $200,000 are unilaterally renegotiable by hospital h when offers are made. The unilateral renegotiation of contracts plays an important role in shaping the set of SPE outcomes. However, to fully understand the forces

[5] One might conjecture that the set of stable allocations involves the union of SPE outcomes of games through which hospitals make offers and SPE outcomes of games through which doctors make offers. This is not true: in Example 1, the $175,000 contract never offers an SPE outcome for any of these games.

at play, we must characterize the best response correspondence game Γ. Let S_{-h} be a strategy profile for all hospitals but h. Let $R_h(S_h, S_{-h}) = \left\{x \in X_h \mid x \notin C_D\left(\bigcup_{h' \in H} S_{h'} \cup \{x\}\right)\right\}$ be the set of contracts of agent h that would be rejected if they were offered by h jointly with contracts in $\bigcup_{h' \in H} S_{h'}$. Note that $R_h(\emptyset, S_{-h}) = X_h \setminus F_h(S_{-h})$.

Define $br_h(S_{-h})$ $=\max_{\succ_h}\left\{Z \subseteq X_h \mid \left[C_D\left(\bigcup_{h' \neq h} S_{h'} \cup Z\right)\right]_h = Z\right\}$. Define $BR_h(S_{-h}) = br_h(S_{-h}) \cup R_h(br_h(S_{-h}), S_{-h})$. Finally, set $br_H\left((S_h)_{h \in H}\right) = (br_h(S_{-h}))_{h \in H}$, and set $BR_H\left((S_h)_{h \in H}\right) = (BR_h(S_{-h}))_{h \in H}$. We can now present the following lemma, which fully characterizes the best response correspondence.

Lemma 2 *Let $\succ_D$ be substitutable. Then, Y_h is the best response to S_{-h} in Γ if and only if*

$$br_h(S_{-h}) \subseteq Y_h \subseteq BR_h(S_{-h}).$$

Thus, functions $br_h(\cdot)$ and $BR_H(\cdot)$ are the minimal and maximal best responses, respectively. The minimal best response of hospital h includes only its most preferred contracts among those that would cause doctors to take its offer, and thus, it allows for unilateral renegotiation. The maximal best response also involves all contracts that will never be accepted. As these contracts can induce the rejection of other offers, they amplify competition between the hospitals, forcing them to offer better terms to doctors. The trade-off between unilateral renegotiation and such competition shapes the set of SPE allocations. Indeed, there might exist stable allocations that can be unilaterally renegotiated by a hospital but can be sustained as SPE outcomes by apparently redundant offers.

Example 2 *Assume $H = \{h_1, h_2\}$ and $D = \{d_1, d_2\}$. Let x_1 and $\tilde{x}_1$ denote contracts between h_1 and d_1. Let x_2 denote a contract between h_2 and d_1. Let z denote a contract between h_2 and d_2. Assume that the preferences of agents are as follows:*

$\succ_{h_1}$: $\{\tilde{x}_1\}, \{x_1\}$;
$\succ_{h_2}$: $\{x_2\}, \{z_2\}$;
$\succ_{d_1}$: $\{x_1\}, \{x_2\}, \{\tilde{x}_1\}$;
$\succ_{d_2}$: $\{z_2\}$.
A unique stable allocation $\{x_1, z_2\}$ exists. The allocation can be unilaterally renegotiated when hospital h_1 offers $\tilde{x}_1$ rather than x_1. However, strategies $X_1(h_1) = \{x_1, \tilde{x}_1\}$ and $X_1(h_2) = \{x_2, z_2\}$ are an NE of game Γ yielding $\{x_1, z_2\}$.

In Example 1, the offer of contract x_2 that is not accepted at equilibrium prevents hospital h_1 from offering only contract $\tilde{x}_1$. When h_1 only offers contract $\tilde{x}_1$, then doctor d_1 accepts contract x_2 and rejects contract $\tilde{x}_1$. Increased competition from hospital h_2 forces h_1 to offer doctor d_1 a better contract.

Examples 1 and 2 highlight differences that emerge with the introduction of contracts. Each hospital must negotiate the nature of relationships with doctors, and the mechanism provides hospitals with a first-mover advantage. In this case, the threat of counteroffers from other hospitals increases competition and helps sustain stable outcomes, as is shown in Example 2. The notion that the potential entry of new competitors helps sustain efficient outcomes is not new to the field of economics, and it is related to the concept of contestable markets (see Baumol et al., 1982).

Formally, we state that market $(H, D, X, \succ)$ satisfies **contestability** if for each stable allocation Y where $x \in X \setminus Y$ and $y \in Y$ exist such that $x \in C_h(Y \cup \{x\}) \cap C_d(Y \setminus \{y\} \cup \{x\})$, where $h = x_H$ and $d = x_D$, there exists a contract $x' \in X_d \setminus X_h$, $x' \in C_d(Y \setminus \{y\} \cup \{x, x'\})$, $x \notin C_d(Y \setminus \{y\} \cup \{x, x'\})$, $x' \notin C_d(Y \cup \{x, x'\})$. Assume that a stable allocation Y can be unilaterally renegotiated by hospital h by offering contract x rather than contract y to doctor d. When the market satisfies the contestability condition, a contract x' between d and a hospital h' different from h prevents renegotiations from being successful. Thus, the essence of the contestability condition lies in the existence of the threat of a deviation that introduces competitive pressures on hospitals. This condition allows for the full implementation of the set of stable allocations.

Proposition 3 *Assume that the market satisfies contestability and that $\succ$ is substitutable; then, every stable allocation is an SPE outcome of $(\mathcal{T}, \succ)$. Therefore, under contestability, $\mathcal{T}$ implements the set of stable allocations in SPE.*

In the absence of contracts, each allocation represents an agreement on only the identities of counterparts. Thus, the contestability condition holds emptily, and Proposition 3 extends Theorems 7.1 and 7.2 shown in Echenique and Oviedo (2006). Alcalde et al. (1998) proves the implementability of stable allocations in SPE in a many-to-one matching model with money developed by Kelso and Crawford (1982). They use a mechanism that is very similar to the simultaneous acceptance mechanism. Their model satisfies contestability because they assume that at least two firms exist, that each firm finds every worker to be acceptable and that firms can make arbitrarily strong offers. These assumptions allow them to sustain SPE, preventing unilateral deviation with the threat of sufficiently high offers.

3.3 Hospital-optimal stable allocation

Contracts shrink the set of SPE allocations through unilateral deviations and harm doctors more than hospitals. Therefore, the existence of stable allocations cannot guarantee the existence of an SPE. However, when the preferences of hospitals are substitutable and satisfy the law of aggregate demand, the hospital-optimal stable allocation always results as an SPE outcome.

Proposition 4 *Assume that* $\succ$ *is substitutable and that* $\succ_H$ *satisfies the law of aggregate demand. Then, the hospital-optimal stable allocation is an* SPE *outcome of* $(\mathcal{T}, \succ)$.

To prove Proposition 4, we employ the same argument used in Proposition 3. We prove that whenever a contract belonging to the hospital-optimal stable allocation can be unilaterally renegotiated by a hospital, a contract prevents the unilateral renegotiation from being profitable.

Consider a many-to-one situation in which each hospital can hire one doctor at most. If $\succ_D$ does not satisfy the law of aggregate demand, the hospital-optimal stable allocation is not strategy-proof for hospitals (see Hatfield and Milgrom, 2005). However, Proposition 4 implies that it is an NE outcome of any TOM.

Proposition 4 also offers proof of the existence of an SPE and confirms the intuition of hospitals' first-mover advantages: the hospital-optimal stable allocation is always an SPE, while the doctor-optimal one is not necessarily an SPE outcome, as shown in Example 1.

Note that when the law of aggregate demand does not hold, the hospital-optimal stable allocation can fail to forge an equilibrium outcome.

Example 3 *Assume that* $H = \{h_1, h_2\}$ *and* $D = \{d_1, d_2\}$. *Let* x_1 *and* $\tilde{x}_1$ *denote contracts between* h_1 *and* d_1. *Let* y_1 *denote a contract between* h_1 *and* d_2. *Let* x_2 *and* $\tilde{x}_2$ *denote contracts between* h_2 *and* d_1. *Let* y_2 *denote a contract between* h_2 *and* d_1. *Assume that the preferences of agents are as follows:*

$\succ_{h_1}$: $\{\tilde{x}_1\}, \{x_1, y_1\}, \{x_1\}, \{y_1\}$;

$\succ_{h_2}$: $\{\tilde{x}_2\}, \{y_2\}, \{x_2\}$;

$\succ_{d_1}$: $\{x_2\}, \{x_1, \tilde{x}_2\}, \{\tilde{x}_1\}, \{x_1\}, \{\tilde{x}_2\}$;

$\succ_{d_2}$: $\{y_2\}, \{y_1\}$.

Such preferences are substitutable. Allocation $Y = \{x_1, \tilde{x}_2, x_3, y_1\}$ *is the hospital-optimal stable allocation. Next, consider any strategy profile* (S_{h_1}, S_{h_2})

of Γ *yielding* $Y = \{x_1, y_1, \tilde{x}_2\}$ *as an outcome. Note that* $x_2 \notin S_{h_2}$*; otherwise,* $x_2 \in g\left(S_{h_1}, S_{h_2}\right)_{h_1}$*. Then,* $\{\tilde{x}_1\}$ *is a profitable deviation for* h_1*, as it yields* $\{\tilde{x}_1\} = \left(g\left(\{\tilde{x}_1\}, S_{h_2}\right)\right)_{h_1}$*, and thus,* Y *is not an* SPE *outcome of any* TOM*.*

4 The structure of the set of SPE outcomes

In Example 3, even when the hospital-optimal stable allocation does not forge an SPE outcome, any TOM has an SPE as a result of a pure strategy that yields $\{x_1, y_2\}$ as an outcome. The objective of this section is to extend results provided in Proposition 4 to situations in which the law of aggregate demand may fail to hold and to characterize the structure of the set of SPE outcomes.

From this point on, we assume that the entire relationship between a hospital and doctor can be specified by a single contract through a requirement referred to as unitarity (see Kominers, 2012).[6] We model the **unitarity assumption** by assuming that allocations through which an agent $n \in N$ signs more than one contract with the same counterpart are not acceptable to n. Formally, we assume that for each $Y \subseteq X$ and $h \in H$, there exist $y, z \in Y_h$, $y \neq z$, for some $h \in H$ (resp. $y, z \in Y_d$, $y \neq z$, for some $d \in D$) with $y_D = z_D$ (resp. $y_H = z_H$), and then, $\emptyset \succ_h Y$ (resp. $\emptyset \succ_d Y$).

Under unitarity, we generalize the existence result of Proposition 4 and characterize the structure of the set of SPE outcomes. The strategy of the proof involves an increasing selection of the best response correspondence and applying the Tarski's Fixed Point Theorem (see Tarski, 1955). We order the strategy space using the product of the natural set order and prove that the maximal best response $BR_H(\cdot)$ is monotonically increasing.

Lemma 3 *Assume that* $\succ$ *is substitutable. Then, the maximal best response function* $BR_h(\cdot)$ *is increasing: for each* $h \in H$*, if* $S_{-h} \subseteq S'_{-h}$*, then* $BR_h(S_{-h}) \subseteq$

[6]Pepa Risma (2015) and Hatfield and Kominers (2017) depart from such an assumption.

$BR_h(S_{-h})$.

In the proof of Lemma 3, the unitarity assumption is crucial. Let $F_h(S_{-h}) = \left\{x \in X_h \mid x \in C_D\left(\bigcup_{h' \neq h} S_{h'} \cup \{x\}\right)\right\}$ be the set of contracts that would be accepted if they were offered by h when the other hospitals offer contracts in $\bigcup_{h' \neq h} S_{h'}$. When preferences satisfy the unitarity assumption, the minimal best response coincides with $C_h(F_h(S_{-h}))$ (see Lemma 5 in the Appendix). If preferences do not satisfy unitarity, this is no longer true. In this case, all selections of the best response correspondence may fail to be monotonic.

Example 4 *Assume $H = \{h_1, h_2\}$ and $D = \{d_1, d_2\}$. Let x_1 and $\tilde{x}_1$ denote contracts between h_1 and d_1. Let y_1 denote a contract between h_1 and d_2. Let x_2 and $\tilde{x}_2$ denote a contract between h_2 and d_1. Let y_2 denote a contract between h_2 and d_1. Assume that the preferences of agents are as follows:*
$\succ_{h_1}$: $\{\tilde{x}_1, x_1\}, \{\tilde{x}_1, y_1\}, \{x_1, y_1\}, \{y_1\}, \{x_1\}, \{\tilde{x}_1\}$;
$\succ_{h_2}$: $\{x_2\}$;
$\succ_{d_1}$: $\{x_1, x_2\}, \{\tilde{x}_1, x_2\}, \{x_1, \tilde{x}_1\}, \{x_2\}, \{x_1\}, \{\tilde{x}_1\}$;
$\succ_{d_2}$:, $\{y_1\}$.
Preferences are substitutable but do not satisfy unitarity. We have $br_{h_1}(\emptyset) = BR_{h_1}(\emptyset) = \{\tilde{x}_1, x_1\}$ and $br_{h_1}(x_2) = BR_{h_1}(\{x_2\}) = \{\tilde{x}_1, y_1\}$. As $br_{h_1}(\emptyset) \nsubseteq BR_{h_1}(\{x_2\})$, no selection of the best response correspondence is monotonically increasing.

In Example 4, preferences are substitutable. When hospital h_2 does not make an offer, hospital h_1's best response is to offer separate contracts $\tilde{x}_1$ and x_1 to doctor d_1. Note that the minimal best responses to $\{x_2\}$ do not coincide with $C_{h_1}(F_{h_1}(\cdot))$: $Ch_{h_1}(F_h(\{x_2\})) = \{\tilde{x}_1, x_1\}$, but $br_{h_1}(\{x_2\}) = \{\tilde{x}_1, y_1\}$. Thus, if hospital h_2 offers contract x_2, contracts x_1 and $\tilde{x}_1$ are separately but not jointly acceptable to doctor d_1. It follows that hospital h_1's best response is to offer contracts $\tilde{x}_1$ and y_1. Thus, no selection of the best response correspondence is monotonically increasing.

Lemma 3 implies that the set of fixed points of $BR_H(\cdot)$ forms a non-empty lattice. This implies that the game induced by any TOM has a pure strategy SPE, so it generalizes Proposition 4 and provides alternative proof of the existence of stable allocations under substitutability.

To fully exploit this result and to characterize the structure of the set of SPE outcomes, we show that all SPE outcomes of the game induced by any TOM are generated by the set of fixed points of the maximal best response.

Lemma 4 *Let $\succ$ be substitutable. The allocation Y is an NE outcome of Γ if and only if there is a strategy profile $(S_h)_{h\in H}$ such that $BR_H\left((S_h)_{h\in H}\right) = (S_h)_{h\in H}$ and $C_D\left(\bigcup_{h\in H} S_h\right) = Y$.*

Lemma 4 does not extend to every selection of the best response correspondence; for instance, it does not extend to the minimal best response. Indeed, in Example 2, unique stable allocation $\{x_1, z_2\}$ is not a fixed point of the minimal best response, $(br_h(\cdot))_{h\in H}$, as $br_{h_1}(\{z_2\}) = \tilde{x}_1$. Thus, $\{x_1, z_2\}$ cannot be obtained from a fixed point of the minimal best response.

We employ Lemmas 3 and 4 to prove that the set of SPE outcomes of any take-it-or-leave-it mechanism is a lattice according to Blair's orders. More precisely, we prove that the lattice structure of the set of fixed points of BR_H reflects on the set of its outcomes. Furthermore, an opposition of interests within the set of SPE allocations emerges. For SPE outcomes Y and Z, when Y dominates Z according to hospital Blair's order, Z dominates Y according to doctor Blair's order.

Theorem 1 *Let $\succ$ be substitutable.*

(i) The set of SPE outcomes of the game induced by $\mathcal{T}$ is a non-empty lattice with Blair's orders $\succ_{HB}$ and $\succ_{DB}$.

(ii) Let Y, Z be SPE outcomes. Then, $Y \succ_{HB} Z$ if and only if $Z \succ_{DB} Y$.

We provide two algorithms for computing pairwise stable allocations under substitutability.

HO **Algorithm**

Step 0**:**

$(X_0)_h = \emptyset$ for all $h \in H$;

Step $r \geq 1$**:**

$(X_{r+1})_h = BR_h\left((X_r)_{-h}\right)$ for all $h \in H$.

Set $X^{HO} = \bigcup_{h\in H}(X_{\bar{r}})_h$, where $\bar{r} = \min\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\}$.

HP **Algorithm**

Step 0**:**

$(X_0)_h = X_h$ for all $h \in H$;

Step $r \geq 1$**:**

$(X_{r+1})_h = BR_h\left((X_r)_{-h}\right)$ for all $h \in H$.

Set $X^{HP} = \bigcup_{h\in H}(X_{\bar{r}})_h$, where $\bar{r} = \min\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\}$.

Lemmas 3 and 4 imply that the outcome of the HO algorithm, X^{HO}, coincides with the best (resp. worst) SPE for hospitals (resp. doctors) and that the outcome of the HP algorithm, X^{HP}, coincides with the worst (resp. best) SPE for hospitals (resp. doctors). In particular, when hospital preferences satisfy the law of aggregate demand, $C_D\left(X^{HO}\right)$ is the hospital-optimal stable allocation (see Proposition 4). Under contestability, any TOM fully implements the set of stable allocations, and the $C_D\left(X^{HD}\right)$ algorithm denotes doctor-optimal stable allocation. The DO and DP algorithms can be symmetrically defined based on the game through which doctors make the offers.

These algorithms differ from those proposed by Pepa Risma (2015), Yenmez (2015) and Hatfield and Kominers (2017). Future studies should compare their computational efficiency.

4.1 Beyond substitutability

Substitutability is the minimal condition guaranteeing the existence of stable allocations in many-to-many matching markets with contracts, as shown in Hatfield and Kominers, (2017). However, the proof of the result explicitly employs preferences that do not satisfy the unitarity assumption. When the unitarity assumption holds, it is possible to relax the assumption of substitutable preferences and to prove the stability of any SPE outcome and its existence. More precisely, we require the preferences of doctors to be unilaterally substitutable (see Hatfield and Kojima, 2010). Formally, the preferences of doctor d satisfy **unilateral substitutability** when there do not exist $x, z \in X$ or $Y \subseteq X$ such that $z_H \notin Y_H$, $z \notin C_d(Y \cup \{z\})$ and $z \in C_d(Y \cup \{x, z\})$. The preferences of doctor d are unilaterally substitutable if whenever d rejects the contract z that is the only contract with z_H available, it still rejects the contract z when the choice set expands.

When doctors have unilaterally substitutable preferences, the maximal response function BR_H is not monotonic, as shown by the following example.

Example 5 *Let $H = \{h_1, h_2\}$ and $D = \{d_1, d_2\}$. Let x_i, x_i' denote contracts forged between hospital h_j and doctor d_1 for $i = 1, 2$. Let y_i denote contracts forged between hospital h_j and doctor d_2 for $j = 1, 2$. Assume that the preferences of the agents are as follows:*
$\succ_{h_1}$: $\{y_1, x_1\}, \{y_1, x_1'\}, \{x_1\}, \{x_1'\} \{y_1\}$;
$\succ_{h_2}$: $\{y_2\}, \{x_2'\}, \{x_2\}$
$\succ_{d_1}$: $\{x_1', x_2\}, \{x_1, x_2'\}, \{x_1, x_2\}, \{x_1', x_2'\}, \{x_1'\}, \{x_1\}, \{x_2'\}, \{x_2\}$;
$\succ_{d_2}$: $\{y_1\}, \{y_2\}$.
Note that $\succ_{d_1}$ is unilaterally substitutable, but it is not substitutable, as $x_1 \in C_{d_1}(\{x_1, x_1', x_2'\})$ but $x_1 \notin C_{d_1}(\{x_1, x_1'\})$.
Now, consider the following strategies for hospital h_2, $S_{h_2} = \{\emptyset\}$ and $S_{h_2}' = \{x_2'\}$. We have $BR_{h_1}(S_{h_2}) = \{x_1, x_1', y\}$ and $BR_{h_1}\left(S_{h_2}'\right) = \{x_1, y\}$. Then, $BR_{h_1}\left(S_{h_2}'\right) \subseteq BR_{h_1}(S_{h_2})$, but $S_{h_2} \subseteq S_{h_2}'$. Thus, $BR_{h_1}(\cdot)$ is not monotonic.

For each $(S_h)_{h\in H} \in \prod_{h\in H} 2^{X_h}$, define $Br_H\left((S_h)_{h\in H}\right) = (Br_h(S_{-h}))_{h\in H}$, where for each $h \in H$, $Br_h(S_{-h}) = br_h(S_{-h}) \cup (X_h \setminus F_h(S_{-h}))$. To prove the existence of a pure-strategy SPE, we show that Br_H is a monotonic selection of the best response correspondence.

Proposition 5 *Let $\succ_H$ be substitutable, and let $\succ_D$ be unilaterally substitutable.*

(i) Every SPE outcome of $(\mathcal{T}, \succ)$ is a pairwise stable allocation.

(ii) The game $(\mathcal{T}, \succ)$ has an SPE.

Therefore, $\mathcal{T}$ weakly implements the set of pairwise stable allocations in SPE.

Claim (i) depends only on the unitarity condition and does not require the application of any supplementary assumptions on agent preferences (see the proof of the result in the appendix).

Proposition 5 also implies the existence of pairwise stable allocations when agents on one side of the market have unilaterally substitutable preferences and when agents on the other side of the market have substitutable preferences.

Corollary 2 *Let $\succ_H$ be substitutable, and let $\succ_D$ be unilaterally substitutable. Then, the set of pairwise stable allocations is non-empty.*

These results are related to those of Yenmez (2015) based on the cumulative offers algorithm. When agents on one side of the market have unilaterally substitutable preferences, pairwise stability and stability are not equivalent.

Example 6 *Let us assume that $H = \{d\}$ and $D = \{h_1, h_2, h_3\}$. Let x_r be a contract forged between h_1 and d, let y_1 be a contract forged between h_2 and d, and let z_r be a contract forged between h_3 and d, for $r = 1, 2$. Assume that the agents' preferences are as follows:*

$\succ_{h_1}$: $\{x_1\}, \{x_2\}$;

$\succ_{h_2}$: $\{y_1\}$;
$\succ_{h_3}$: $\{z_1\}, \{z_2\}$;
$\succ_d$: $\{x_1, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_2, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_1, y_1, z_1\}, \{x_2, y_1, z_2\},$
$\{x_2, y_1, z_1\}, \{x_2, y_1, z_2\}, \{x_2, y_1\}, \{x_1, y_1\}, \{y_1, z_2\}, \{y_1, z_1\}, \{x_1, z_1\},$
$\{x_2, z_2\}, \{x_1, z_2\}, \{y_1\}, \{x_1\}, \{x_2\}, \{z_1\}, \{z_2\}$.
Preferences $\succ_d$ are not substitutable, as $z_1 \notin C_d(\{y_1, z_1, z_2\})$, though $z_1 \in C_d(\{x_1, x_2, y_1, z_1, z_2\})$. However, $\succ_d$ are unilaterally substitutable. Preferences $\succ_H$ are substitutable. Allocation $Y = \{x_2, y_1, z_2\}$ is pairwise stable but not stable, as it is blocked by $N' = \{h, d_1, d_3\}$ through $Z = \{x_1, z_1\}$. However, $S_{h_1} = \{x_1\}$, $S_{h_2} = \{y_1\}$, and $S_{h_3} = \{z_3\}$ form an NE of Γ yielding Y as an outcome.

In the market of Example 6, SPE outcome Y is pairwise stable but not stable.

Moreover, under the same assumptions, the set of SPE does not even form a lattice with respect to the joint preferences of agents or Blair's orders, as shown by the following example adapted from Hatfield and Kojima (2010).

Example 7 *Assume that $H = \{h_1, h_2\}$ and $D = \{d\}$. Let x_1, x_2, and x_3 denote contracts forged between h_1 and d, and let y_1, y_2, and y_3 denote contracts forged between h_2 and d. Assume that the preferences of the agents are as follows:*
$\succ_{h_1}$: $\{x_2\}, \{x_1\}, \{x_3\}$;
$\succ_{h_2}$: $\{y_2\}, \{y_1\}, \{y_3\}$;
$\succ_d$: $\{x_1, y_3\}, \{x_3, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_3, y_2\}, \{x_2, y_3\}, \{x_2, y_1\}, \{x_1, y_2\},$
$\{x_1, y_1\}, \{x_3\}, \{y_3\}, \{x_2\}, \{y_2\}, \{x_1\}, \{y_1\}$.
The preferences of doctor d are unilaterally substitutable but not substitutable, as $y_3 \in C_d(\{x_1, x_3, y_1, y_3\}) = \{x_1, y_3\}$ but $y_3 \notin C_d(\{x_3, y_1, y_3\}) = \{x_3, y_1\}$.
This market contains three stable allocations: $X_1 = \{x_2, y_2\}$, $X_2 = \{x_3, y_1\}$, and $X_3 = \{x_1, y_3\}$, which are also SPE outcomes. However, $\{X_1, X_2, X_3\}$ is not a lattice with respect to Blair's order.

Consider the following two algorithms:

ho **Algorithm**

Step 0**:**

$(X_0)_h = \emptyset$ for all $h \in H$;

Step $r \geq 1$**:**

$(X_{r+1})_h = Br_h\left((X_r)_{-h}\right)$ for all $h \in H$.

Set $X^{ho} = \bigcup_{h \in H} (X_{\bar{r}})_h$, where $\bar{r} = \min\left\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\right\}$.

hp **Algorithm**

Step 0**:**

$(X_0)_h = X_h$ for all $h \in H$;

Step $r \geq 1$**:**

$(X_{r+1})_h = Br_h\left((X_r)_{-h}\right)$ for all $h \in H$.

Set $X^{hp} = \bigcup_{h \in H} (X_{\bar{r}})_h$, where $\bar{r} = \min\left\{r \mid (X_r)_h = (X_{r+1})_h \text{ for all } h \in H\right\}$.

The monotonicity of $Br_H(\cdot) = (Br_h(\cdot))_{h \in H}$ implies that both algorithms stop after a finite number of steps and yield an NE of Γ. In particular, both $C_D\left(X^{ho}\right)$ and $C_D\left(X^{hp}\right)$ are pairwise stable allocations.

5 Conclusions

In this paper, we study a class of ultimatum games that we call **take-it-or-leave-it offer mechanisms** whereby hospitals make simultaneous offers that are either accepted or rejected by doctors. The mechanisms of this class mimic those of real-world environments and allow us to explore allocative implications of the use of contracts in many-to-many matching markets.

Our results illustrate the strategic effects of introducing contracts into such an environment. Contracts preserve the stability of equilibrium outcomes while reducing the capacity for mechanisms to realize stable allocation. Results are dependent on the fact that contracts allow agents to negotiate the terms of their relationships. This introduces a first-mover advantage that is absent in the model without contracts. Only the introduction of additional competitive pressure formalized in a condition that we call "contestability" allows the mechanism to fully implement the set of stable allocations.

All games induced by TOM share the same strategic structure and are outcome equivalent. In exploiting the common structure of best response correspondences of such games, we prove the existence of SPE in all games induced by TOM when preferences are substitutable. We then characterize the set of SPE allocations as a complete lattice, which preserves the opposition of interests found within the set of stable allocations.

Thus, our analysis reveals the common structure underlying take-it-or-leave-it offer mechanisms. It unifies and generalizes previous results obtained for specific mechanisms of this class for contexts without contracts, revealing that in these environments, the contestability assumption is implicitly satisfied.

Appendix

Proofs of the results in Section 3

Proof of Proposition 2. Assume $S^* = \left((S_h^*)_{h \in H}\right)$ is a NE of Γ, and let $Y = g\left(S^*\right)$. We show by contradiction that Y is a pairwise stable allocation. We first prove that Y is individually rational. The proof of this claim is proven by contradiction. Assume that Y is not an individually rational allocation for agent $n \in N$. Let $n = h \in H$. Then, the substitutability of $\succ_D$ implies that $C_h\left(Y_h\right)$ is a profitable deviation, yielding a contradiction. Let $n = d \in D$; in this case, the contradiction follows from Lemma 1.

We conclude the proof by showing that Y is not pairwise blocked. By contradiction, assume a hospital h, a doctor d, and a contract $x \in X \setminus Y$ exist with

$x_D = d$, $x_H = h$ and $x \in C_h(Y \cup \{x\}) \cap C_d(Y \cup \{x\})$. First, we prove $x \in C_d\left(\bigcup_{s^*_{h'D}=d} \{S^*_{h'}\} \cup \{x\}\right)$. Set $Z = \bigcup_{s^*_{h'D}=d} \{S^*_{h'}\}$. From Lemma 1, $C_d(Z) = Y_d$. By contradiction, assume $x \notin C_d\left(\bigcup_{s^*_{h'D}=d} \{S^*_{h'}\} \cup \{x\}\right) = C_d(Z \cup \{x\})$. From $x \in C_d(Y_d \cup \{x\})$, it follows that $C_d(Z \cup \{x\}) \succ_d C_d(Y_d \cup \{x\})$. However, because $x \notin C_d(Z \cup \{x\})$, $C_d(Z \cup \{x\}) = C_d(Z) = Y_d$, yielding a contradiction.

Consider the following deviation for h, $S_h = C_h(Y \cup \{x\})$. As $\succ_D$ is substitutable, the deviation is profitable to h, yielding a contradiction. ■

Proof of Lemma 2. (i) We first show that $br_h(S_{-h})$ is a best response to S_{-h} and that for each best response Y_h, $br_h(S_{-h}) \subseteq Y_h$. Note that $g(br_h(S_{-h}), S_{-h}) = br_h(S_{-h})$. Let Y_h be a best response to S_{-h}, and set $Y'_h = \left[C_D\left(\bigcup_{h' \neq h} S_{h'} \cup Y_h\right)\right]_h = [g(Y_h, S_{-h})]_h$. The IRC implies that $C_D\left(\bigcup_{h' \neq h} S_{h'} \cup Y'_h\right)_h = Y'_h$. Thus, $br_h(S_{-h}) \succeq_h Y'_h$ from the definition of $br_h(S_{-h})$. As preferences are strict and as Y_h is a best response, $Y'_h = br_h(S_{-h})$. Thus, $br_h(S_{-h})$ is a best response, and $br_h(S_{-h}) = Y' \subseteq Y_h$.

(ii) Now, we show that when Y_h is a best response to S_{-h}, $Y_h \subseteq BR_h(S_{-h})$. Observe that Y_h is a best response to S_{-h} if and only if $\left[C_D\left(\bigcup_{h' \neq h} S_{h'} \cup Y_h\right)\right]_h = br_h(S_{-h})$. Let Y_h be a best response. From part (i) of the proof, we have $Y_h = br_h(S_{-h}) \cup Z_h$ for some $Z_h \subseteq X_h$, $Z_h \cap br_h(S_{-h}) = \emptyset$. From the IRC, it follows that $z \notin \left\{x \in X_h \mid x \in C_D\left(\bigcup_{h' \neq h} S_{h'} \cup br_h(S_{-h}) \cup \{z\}\right)\right\}$ for all $z \in Z_h$; therefore, $Y_h \subseteq BR_h(S_{-h})$.

(iii) Let $br_h(S_{-h}) \subseteq Y_h \subseteq BR_h(S_{-h})$. We can write $Y_h = br_h(S_{-h}) \cup Z_h$, where $Z_h \subseteq \left\{x \in X_h \mid x \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup br_h(S_{-h}) \cup \{x\}\right)\right\}$ and $Z_h \cap br_h(S_{-h}) = \emptyset$. By contradiction, assume that $C_D\left(\bigcup_{h' \neq h} S_{h'} \cup Y_h\right)_h \neq br_h(S_{-h})$. Therefore, there exists $z \in Z_h$ such that $z \in C_{z_D}\left(\bigcup_{h' \neq h} S_{h'} \cup br_h(S_{-h}) \cup Z_h\right)$. Finally, substitutability implies that $z \in C_{z_d}\left(\bigcup_{h' \neq h} S_{h'} \cup br_h(S_{-h}) \cup \{z\}\right)$, yielding a contradiction. ■

Proof of Proposition 3. Let Y be a stable allocation. We construct an NE of Γ yielding Y as an outcome. If $br_h(Y_{-h}) = Y_h$ for all $h \in H$, the proof is complete. Otherwise, let $T_h \neq Y_h$ be such that $T_h = br_h(Y_{-h})$. Then, $T_h = C_h(Y_h \cup T_h)$, and thus, $C_h(Y_h \cup \{t\})$ is a profitable deviation for h for all $t \in T_h \setminus Y_h$. Let $T = \left(\bigcup_{h \in H} T_h\right) \setminus Y$.

Let $t \in T$. As Y is pairwise stable and as preferences are substitutable, $d \in D$ and $y \in Y$ exist such that $y_H = t_H = h$ and $y_D = t_D = d$ such that $t \in C_h(Y \cup \{t\}) \cap C_d((Y \setminus \{y\}) \cup \{t\})$ and $y \notin C_d(Y \cup \{t\})$. As the market is contestable, $t' = t'(t) \in X$ and $h' \neq h$ exist such that $t' \in C_d((Y \setminus \{y\}) \cup \{t, t'\})$, $t \notin C_d((Y \setminus \{y\}) \cup \{t, t'\})$, and $t' \notin C_d((Y) \cup \{t, t'\})$. Set $Y' = Y \cup \bigcup_{t \in T} t'(t)$. Let $S_h = Y'_h$ for all $h \in H$. Observe that by construction, $Y_h = br_h(S_{-h})$. Furthermore, $t'(t) \in R_h\left(Y'_h, Y'_{-h}\right)$, as Y is stable and $\succ_D$ are substitutable. It follows that $(S_h)_{h \in H}$ is an NE of Γ yielding Y as an outcome, completing the proof of the claim. ■

Proof of Proposition 4. Let Y be the hospital-optimal stable allocation. We construct an equilibrium yielding Y as an outcome. If $br_h(Y_{-h}) = Y_h$ for all $h \in H$, the proof is complete. Otherwise, let $T_h \neq Y_h$ be such that $T_h = br_h(Y_{-h})$. Then, $T_h = C_h(Y_h \cup T_h)$, and thus, $C_h(Y_h \cup \{t\})$ is a profitable deviation for h for all $t \in T_h \setminus Y_h$. Let $T = \left(\bigcup_{h \in H} T_h\right) \setminus Y$.

Let $t \in T$. Because Y is pairwise stable and as preferences are substitutable, $d \in D$ and $y \in Y$ exist such that $y_H = t_H = h$ and $y_D = t_D = d$ such that $t \in C_h(Y \cup \{t\}) \cap C_d((Y \setminus \{y\}) \cup \{t\})$ and $y \notin C_d(Y \cup \{t\})$.

Preferences $\succ_H$ satisfy the law of aggregate demand; thus, $|C_h(Y)| \leq |C_h(Y \cup \{t\})|$. As Y is individually rational and as $y \notin C_h(Y \cup \{t\})$, $C_h(Y \cup \{t\}) = (Y_h \setminus \{y\}) \cup \{t\}$. Observe that $C_d((Y \setminus \{y\}) \cup \{t\}) = (Y_d \setminus \{y\}) \cup \{t\}$, as preferences are substitutable and as $C_d(C_d(Y \cup \{t\})) = Y$.

Next consider $Z = (Y \setminus \{y\}) \cup \{t\}$. As preferences are substitutable, allocation Z is individually rational and $Z \succ_H Y$. In particular, Z is not stable. As Y is stable, $t' = t'(t) \in X$ and $h' \neq h$ exist such that $t' \in C_{h'}(Y \cup \{t\}) \cap C_d((Y_d \setminus \{y\}) \cup \{t, t'\})$ and $t \notin C_d((Y_d \setminus \{y\}) \cup \{t, t'\})$.

Set $Y' = Y \cup \bigcup_{t \in T} t'(t)$. Let $S_h = Y'_h$ for all $h \in H$. Observe that by construction, $Y_h = br_h(S_{-h})$. Furthermore, $t'(t) \in R_h\left(Y'_h, Y'_{-h}\right)$, as Y is stable and $\succ_D$ are substitutable. It follows that $(S_h)_{h\in H}$ is an NE of Γ yielding Y as an outcome, completing the proof of the claim. ■

Proofs of the results in Section 4

Lemma 5 *Assume that $\succ$ satisfy the unitarity assumption. Then, for each $h \in H$, $br_h(S_{-h}) = C_h(F_h(S_{-h}))$ for all $S_{-h} \in \Pi_{h\neq H'} S_{h'}$.*

Proof. Let $h \in H$, and let $S_{-h} \in \Pi_{h\neq H'} \mathcal{S}_h$. We prove that $br_h(S_-) = \max_{\succ_h} \left\{Z \subseteq X_h \mid \left[C_D\left(\bigcup_{h'\neq h} S_{h'} \cup Z\right)\right]_h = Z\right\} = C_h(F_h(S_{-h}))$. Let $Y = C_h(F_h(S_{-h}))$.
First, we show that $\left[C_D\left(\bigcup_{h'\neq h} S_{h'} \cup Y\right)\right]_h = Y$. Let $y \in Y$, and let $d = y_D$. The unitarity assumption implies that $Y_d = \{y\}$. It follows that $C_d\left(\bigcup_{h'\neq h} S_{h'} \cup Y\right)_h = Y_d = \{y\}$, as $y \in C_d\left(\bigcup_{h'\neq h} S_{h'} \cup \{y\}\right)$, implying the claim. It follows that $br_h(S_{-h}) R_h C_h(F_h(S_{-h}))$.
Let $Z \in \left\{Z \subseteq X_h \mid \left[C_D\left(\bigcup_{h'\neq h} S_{h'} \cup Z\right)\right]_h = Z\right\}$. We prove that $Z \subseteq F_h(S_{-h})$. Let $z \in Z$, and let $d = z_D$. The unitarity assumption implies that $Y_d = \{z\}$. We have $z \in C_d\left(\bigcup_{h'\neq h} S_{h'} \cup Z_d\right) = C_d\left(\bigcup_{h'\neq h} S_{h'} \cup \{y\}\right)$.
It follows that $C_h(F_h(S_{-h})) R_h Z$ for all $Z \in \left\{Z \subseteq X_h \mid \left[C_D\left(\bigcup_{h'\neq h} S_{h'} \cup Z\right)\right]_h = Z\right\}$, and thus, $C_h(F_h(S_{-h})) R_h br_h(S_{-h})$. As $br_h(S_{-h}) R_h C_h(F_h(S_{-h}))$, we can conclude because preferences are strict. ■

The following Lemma 6 is repeatedly used in the proof of the main results of this section.

Lemma 6 *Assume that $\succ$ satisfies the unitarity assumption. Let $h \in H$, and let $S'_{-h}, S_{-h} \in \Pi_{h'\neq h} X_{h'}$.*

(i) Assuming that $S'_h \subseteq S_h \subseteq X_h$, then $R_h(S'_h, S_{-h}) \subseteq R_h(S_h, S_{-h})$.

(ii) Assume that $\succ_D$ is unilaterally substitutable. $Br_h(S_{-h}) = br_h(S_{-h}) \cup (X_h \setminus F_h(S_{-h}))$ is a best response.

(iii) Assume that $\succ_D$ is unilaterally substitutable. Let $h \in H$. If $S'_{h'} \subseteq S_{h'} \subseteq X_{h'}$ for all $h' \in H \setminus \{h\}$, then $F_h(S_{-h}) \subseteq F_h(S'_{-h})$.

(iv) Assume that $\succ_D$ is substitutable, and let $S_h \subseteq X_h$. Assuming $S'_{-h} \subseteq S_{-h}$, then $R_h(S_h, S'_{-h}) \subseteq R_h(S_h, S_{-h})$.

(v) Assume that $\succ_H$ is substitutable and that $\succ_D$ is unilaterally substitutable. Assuming $S'_{-h} \subseteq S_{-h}$, then $br_h(S'_{-h}) \subseteq Br(S_{-h})$.

Proof. (i) Let $S'_h \subseteq S_h \subseteq X_h$, and let $x \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S'_h \cup \{x\}\right)$. We prove by contradiction that $x \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)$. Assume $x \in C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)$. Unitarity implies $\left[C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)\right]_h = \{x\}$. For all $x' \in S_h \setminus \{h\}$, $x' \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)_h$. As $S'_h \subseteq S_h$, the IRC implies $\left(C_d\left(\bigcup_{h' \neq h} S_{h'} \cup S_h \cup \{x\}\right)\right)_h = \left(C_d\left(\bigcup_{h' \neq h} S_{h'} \cup S'{}_h \cup \{x\}\right)\right)_h$ for all d, creating a contradiction. Therefore, $R_h(S'_h, S_{-h}) \subseteq R_h(S'_h, S_{-h})$.

(ii) To prove this claim, it suffices to show that $C_D(Br_h(S_{-h}), S_{-h}) = br_h(S_{-h})$. Let $x \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup \{x\}\right)$. We prove that $x \notin C_D\left(\bigcup_{h' \neq h} S_{h'} \cup br_h(S_{-h}) \cup X \setminus F_h(S_{-h})\right)$, implying the claim from the IRC. By contradiction, assume that $x \in C_D\left(\bigcup_{h' \neq h} S_{h'} \cup br_h(S_{-h}) \cup X \setminus F_h(S_{-h})\right)$. Let $d = x_D$. The unitarity assumption implies that $\{x\} = \left(C_{x_D}\left(\bigcup_{h' \neq h} S_{h'} \cup br_h(S_{-h}) \cup X \setminus F_h(S_{-h})\right)\right)_h$. Unilateral substitutability implies that $x \in \left(C_{x_D}\left(\bigcup_{h' \neq h} S_{h'} \cup \{x\}\right)\right)$.

(iii) Assuming $x \in F_h(S_{-h})$, then $x \in C_d\left(\bigcup_{h' \neq h} S_{h'} \cup \{x\}\right)$, where $d = x_D$. Note that $h \notin \left[\bigcup_{h' \neq h} S'_{h'}\right]_H$. As $\succ_d$ is unilaterally substitutable, $x \in C_d\left(\bigcup_{h' \neq h} S'_{h'} \cup \{x\}\right)$.

(iv) Let $x \notin C_D\left(\bigcup_{h' \neq h} S'_{h'} \cup S_h \cup \{x\}\right)$. This claim follows directly from the substitutability of $\succ_D$.

(v) We have $br_h(S'_{-h}) = C_h(F_h(S'_{-h})) \cap F_h(S_{-h}) \cup \left(C_h(F_h(S'_{-h})) \setminus F_h(S_{-h})\right)$. From (iii), $F_h(S_{-h}) \subseteq F_h(S'_{-h})$. Therefore, the substitutability of $\succ_h$ implies that $C_h\left(F_h\left(S'_{-h}\right)\right)\cap F_h\left(S_{-h}\right)\subseteq C_h\left(F_h\left(S_{-h}\right)\right)$, concluding the proof of the claim, as from (ii), $br_h\left(S'_{-h}\right)\subseteq Br_h\left(S_{-h}\right)\subseteq BR_h\left(S_{-h}\right)$. ■

Proof of Lemma 3. From Lemma 6, to complete the proof, it suffices to show $R_h\left(br_h\left(S'_{-h}\right),S'_{-h}\right)\subseteq R_h\left(br_h\left(S_{-h}\right),S_{-h}\right)$. Assume $x\notin C_D\left(br_h\left(S'_{-h}\right)\cup\bigcup_{h'\neq h}S'{}_{h'}\cup\{x\}\right)$. From Lemma 6 (v), we have $br_h\left(S'_{-h}\right)\subseteq BR_h\left(S_{-h}\right)=br_h\left(S_{-h}\right)\cup R_h\left(br_h\left(S_{-h}\right),S_{-h}\right)$. From substitutability, $x\notin C_D\left(br_h\left(S_{-h}\right)\cup R_h\left(br_h\left(S_{-h}\right),S_{-h}\right)\cup\bigcup_{h'\neq h}S_{h'}\cup\{x\}\right)$. From the IRC, it follows that $C_D\left(br_h\left(S_{-h}\right)\cup R_h\left(br_h\left(S_{-h}\right),S_{-h}\right)\cup\bigcup_{h'\neq h}S_{h'}\cup\{x\}\right)=$
$=C_D\left(br_h\left(S_{-h}\right)\cup R_h\left(br_h\left(S_{-h}\right),S_{-h}\right)\cup\bigcup_{h'\neq h}S_{h'}\right)=$
$C_D\left(br_h\left(S_{-h}\right)\cup\bigcup_{h'\neq h}S_{h'}\right)$. It follows that $x\notin C_D\left(br_h\left(S_{-h}\right)\cup\bigcup_{h'\neq h}S_{h'}\cup\{x\}\right)$. It follows that $R_h\left(br_h\left(S'_{-h}\right),S'_{-h}\right)\subseteq R_h\left(br_h\left(S_{-h}\right),S_{-h}\right)$. ■

Proof of Lemma 4. The set of fixed points of BR_H are contained in the set of the Nash equilibrium of Γ, so it suffices to show that any NE outcome is the outcome of a fixed point of BR. Let $(S_h^*)_{h\in H}$ be an NE of Γ, and let $Y=g\left((S_h^*)_{h\in H}\right)$.

Assume $h\in H$. Since for each $h\in H$, S_h^* is a best response to S_{-h}^*, we have $br_h\left(S_{-h}^*\right)\subseteq S_h^*\subseteq BR_h\left(S_{-h}^*\right)$. Consider the sequence $T^0=(S_h^*)_{h\in H}$, $T^{k+1}=\left(\left(BR_h\left(T_{-h}^k\right)\right)_{h\in H}\right)$ for all $k\geq 0$. Note that $br_h\left(T_{-h}^k\right)=Y_h$ for each h and $k\geq 0$. As $T^0\subseteq T^1$ and BR are increasing, the sequence $\left(T^k\right)_{k\geq 0}$, $T^k\subseteq T^{k+1}$ for each $t\geq 0$. As X is finite, there exists $K\geq 0$ such that $T^K=T^s$ for all $s\geq K$. It follows that T^K is a fixed point of BR yielding Y as an outcome. ■

Proof of Proposition 1. (i) First, we show two preliminary results.

(a) Let $(A_h)_{h\in H}$ and $(B_h)_{h\in H}$ be fixed points of BR such that $A_h\subseteq B_h$ for all $h\in H$. We show that $X^B=C_D\left(X^A\cup X^B\right)$, where $X^A=g\left((A_h)_{h\in H}\right)$ and $X^B=g\left((B_h)_{h\in H}\right)$. Let $A=\bigcup_{h\in H}A_h$ and $B=\bigcup_{h\in H}B_h$, and note that $A\subseteq B$. We have $X^B=C_D\left(B\right)=C_D\left(A\cup B\right)$. As $X^A\cup X^B\subseteq A\cup B$, we have $X^B=C_D\left(X^A\cup X^B\right)$.

(b) Let $(A_h)_{h\in H}$ and $(B_h)_{h\in H}$ be fixed points of BR, and let $X^A=g\left((A_h)_{h\in H}\right)$ and $X^B=g\left((B_h)_{h\in H}\right)$. Assume that $X^B=C_D\left(X^A\cup X^B\right)$. We show

$A_h \subseteq B_h$ for all $h \in H$. Note that $X^A = C_H\left(X^A \cup X^B\right)$ (see Pepa Risma, 2015). Let $x \in X^A \setminus X^B$. We prove that $x \in BR_h\left(B_{-h}\right)$. Let $h = x_H$, and let $d = x_D$. The substitutability of $\succ_h$ implies that $x \in C_h\left(X^B \cup \{x\}\right)$. The pairwise stability of X^B implies that $x \notin C_d\left(X^B \cup \{x\}\right)$. The substitutability of $\succ_d$ implies that $x \notin C_d\left(X^B \cup \bigcup_{h' \neq h} B_{h'}\right)$. It follows that $x \in R_h\left(X^B, B_{-h}\right) \subseteq BR_h\left(B_{-h}\right)$.

Now, let $x \in R_h\left(X^A, A_{-h}\right) \cap X_h$, and let $d = x_D$. We have $C_d\left(X^A \cup \bigcup_{h' \neq h} A_{h'} \cup \{x\}\right) = X_h^A$ so $X_d^A = C_d\left(X^A \cup \{x\}\right)$. The substitutability of $\succ_d$ implies that $x \notin C_d\left(X^A \cup X^B \cup \{x\}\right) = X^B$, as $C_d\left(X^A \cup X^B \cup \{x\}\right) = X^B$ from the IRC. It follows that $x \notin C_d\left(X^B \cup \{x\}\right)$. Again, the substitutability of $\succ_d$ implies $x \notin C_d\left(X^B \cup \bigcup_{h' \neq h} B_{h'}\right)$; thus, $x \in R_h\left(X^B, B_{-h}\right) \subseteq BR_h\left(B_{-h}\right)$.

It follows that $A_h = X_h^A \cup R_h\left(X_h^A, A_{-h}\right) \subseteq X_h^B \cup R_h\left(X_h^B, B_{-h}\right)$ for all $h \in H$.

The claim follows from (a), (b) and Lemma 4. Note that the set of fixed points of BR forms a non-empty lattice from the Tarski's Fixed Point Theorem.

(ii) The claim follows from (i) and Pepa Risma (2015). ■

Proof of Proposition 5. (i) We prove the claim under unitarity only without making any additional assumptions on $\succ$. Assume $S^* = \left((S_h^*)_{h \in H}\right)$ as the NE of Γ, and let $Y = g\left(S^*\right)$. We show by contradiction that Y is a pairwise stable allocation. We first prove that Y is individually rational. The proof of the claim is determined by contradiction. Assume that Y is not an individually rational allocation for agent $n \in N$. Let $n = h \in H$. Then, unitarity and the IRC imply that $C_h\left(Y_h\right)$ is a profitable deviation, yielding a contradiction. Let $n = d \in D$; in this case, the contradiction follows from Lemma 1.

We conclude the proof by showing that Y is not pairwise blocked. By contradiction, assume that a hospital h, a doctor d, and a contract $x \in X \setminus Y$ exist with $x_D = d$ and $x_H = h$ such that $x \in C_h\left(Y \cup \{x\}\right) \cap C_d\left(Y \cup \{x\}\right)$. First, we prove $x \in C_d\left(\bigcup_{s^*_{h'D} = d} \{S^*_{h'}\} \cup \{x\}\right)$. Set $Z = \bigcup_{s^*_{h'D} = d} \{S^*_{h'}\}$. From Lemma

1, $C_d(Z) = Y_d$. By contradiction, assume $x \notin C_d\left(\bigcup_{s^*_{h'D}=d}\{S^*_{h'}\} \cup \{x\}\right) = C_d(Z \cup \{x\})$. From $x \in C_d(Y_d \cup \{x\})$, it follows that $C_d(Z \cup \{x\}) \succ_d C_d(Y_d \cup \{x\})$. However, as $x \notin C_d(Z \cup \{x\})$, $C_d(Z \cup \{x\}) = C_d(Z) = Y_d$, yielding a contradiction.
Consider the following deviation for h, $S_h = C_h(Y \cup \{x\})$. Unitarity and the IRC imply that the deviation is profitable to h, yielding a contradiction.

(ii) Let $h \in H$ and $S'_{h'} \subseteq S_{h'} \subseteq X_{h'}$ for all $h' \in H \setminus \{h\}$. From Lemma 6, $(Br_h(\cdot))_{h\in H}$ is a selection from the best response correspondence. We prove that for all $h \in H$, $Br_h\left(S'_{-h}\right) \subseteq Br_h(S_{-h})$.

From Lemma 6 (iv), $X \setminus F_h\left(S'_{-h}\right) \subseteq X \setminus F_h(S_{-h})$; from Lemma 6 (v), $br_h\left(S'_{-h}\right) \subseteq br_h(S_{-h}) \cup X \setminus F_h(S_{-h}) = br_h(S_{-h})$. It follows that $(Br_h(\cdot))_{h\in H}$ is an increasing selection from the best response correspondence. Thus, by applying Tarski's fixed point theorem, we obtain the claim. ■

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