# Sharp Gaussian Approximation Bounds for Linear Systems with $\alpha$-stable Noise

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***Abstract*—THIS PAPER IS ELIGIBLE FOR THE STUDENT PAPER AWARD. We report the results of several theoretical studies into the convergence rate for certain infinite Poisson random series representations of $\alpha$-stable random variables, which are motivated by and find application in modelling heavy-tailed noise in time series analysis, inference, and stochastic processes. The use of $\alpha$-stable noise distributions generally leads to analytically intractable inference problems. The special version of the Poisson series representation invoked here shows that the resulting distributions are "conditionally Gaussian", for which inference is relatively straightforward, but an infinite series is still involved. Our approach approximates the residual terms of the series from some point $c > 0$, say, to $\infty$, with a Gaussian random variable. Empirically, this approximation has been found to be very accurate for large $c$. We study the rate of convergence, as $c \to \infty$, of this Gaussian approximation to the exact stable law. This allows the selection of appropriate truncations in order to achieve a required level of accuracy for the approximate model. Convergence is examined and explicit finite-$c$ bounds are obtained for the Kolmogorov distance between distribution functions, through the application of probability-theoretic tools. The theoretical results obtained are found to be in very close agreement with numerical simulations obtained in our previous work.**

## I. Introduction

### A. Background

Time series arising in the natural sciences, engineering and finance [1], [2] are frequently characterised by high data rates and irregular sampling, a situation well-represented by continuous-time state-space models. Perhaps the simplest, most tractable such model is the linear diffusion model with linear observations at discrete times $\{t_i\}$,

$$\mathrm{d}\mathbf{x}_t = \mathbf{A}\mathbf{x}_t\,\mathrm{d}t + \mathbf{h}\,\mathrm{d}\ell_t, \qquad y_{t_i} = \mathbf{b}'\mathbf{x}_{t_i} + v_{t_i}$$

where $\mathbf{x}_t = \left[x_{1,t}, \ldots, x_{P,t}\right]'$ is the state, $\mathbf{A}$ is a $P \times P$ matrix describing the interaction of the components of $\mathbf{x}_t$, $\mathbf{h}$ is a $P$-dimensional vector describing the effects of the noise process $\{\mathrm{d}\ell_t\}$, $\mathbf{b}$ is a $P$-dimensional vector and $\{v_{t_i}\}$ is the observation noise process. A wide range of results have been developed in the literature for the case when $\{\ell_t\}$ is a Brownian motion [3], [4]. However, such models cannot account for the heavy tails and the large "jumps" in the state process often observed in applications. In such cases, the use of linear state-space systems driven by (non-Gaussian) Lévy processes is more appropriate, since these models do exhibit heavy-tailed, discontinuous behaviours [5]–[7]. Despite the simple characterization of Lévy processes [8], specialized probabilistic tools are required in their analysis. The main difficulty stems from lack of closed-form expressions for many relevant quantities of interest. We refer to [9]–[11] for an approximate framework for simulation. The family of $\alpha$-stable Lévy processes is of special importance, in that the class of $\alpha$-stable laws are the natural limiting distributions in the generalized central limit theorem (CLT) with heavy-tailed summands [12]; also see [13], [14] for other relevant background. The self-similarity of $\alpha$-stable Lévy processes [13] implies that transition densities, although still intractable, all come from the same $\alpha$-stable family, and hence they may be considered to be a very natural first approach towards generalising the classical Gaussian process framework to the heavy-tailed case.

In this paper we provide convergence results for the Poisson series representation (PSR) of $\alpha$-stable random variables. These find application in discrete time time-series models with $\alpha$-stable disturbances, and in the above continuous time model when the observation noise $\{v_{t_i}\}$ is $\alpha$-stable. See [15]–[18] for PSR approaches to the full $\alpha$-stable Lévy process[1]. Despite extensive earlier work on properties of $\alpha$-stable systems, there are few results on likelihood or Bayesian parameter inference for such linear models, see e.g. [19]–[21] for some examples.

Our approach to the inference problem involves an auxiliary-variables version of Bayesian Monte Carlo approaches. In our models, for example, where part of the state is conditionally linear and Gaussian, efficient Rao-Blackwellised versions of Sequential Monte Carlo (SMC) can be applied [22], [23], or MCMC in Bayesian parameter inference may be used [16], [24], [25].

Observe that [9] presents an alternative Gaussian approximation to ours, based on simulation only of jumps greater than some magnitude $\epsilon$, including a CLT and convergence rates. We will present comparisons between the two approaches in future work.

### B. The $\alpha$-stable Distribution

We adopt standard notation as in the text [13]. Let $X \sim \mathcal{S}_\alpha(\sigma, \beta, \mu)$ denote an $\alpha$-stable distributed random variable, where $\mu$ is the location parameter, $\sigma > 0$ is the scale parameter, $\beta \in [-1, 1]$ is the skewness parameter, and $\alpha \in (0, 2)$ is the tail parameter. Recall that the probability density function (pdf) of $X$ decays like $1/|x|^{1+\alpha}$ as $|x| \to \infty$, which is a consequence of $\alpha$-stable version of the CLT. The polynomial tails are a consequence of the presence of extreme values, with more extreme values (and hence heavier tails) appearing for smaller values of $\alpha$. When $\beta = 0$ the pdf of $X$ is symmetric, while $\beta = \pm 1$ correspond to the fully left or right skewed cases, respectively.

[1] A modified PSR and corresponding CLT results do apply there, but convergence results are not provided for that case.

Also we recall that the pdf of $\alpha$-stable distributions is not available in closed form, except in few special cases (Gaussian, Cauchy and Lévy distributions), obtained for specific choices of $\alpha$ and $\beta$. Although, as mentioned above, this complicates the analysis of questions related to inference, there is a substantial body of earlier work developing practically applicable methods.

On the other hand, an $\alpha$-stable random variable has characteristic function (CF) $\phi_X(s) := \mathbb{E}\left[\exp\left(isX\right)\right]$, for $s \in \mathbb{R}$, explicitly given by:

$$\log(\phi_X(s)) = \begin{cases} -\sigma^\alpha |s|^\alpha \left\{1 - i\beta\,\mathrm{sgn}(s)\tan\frac{\pi\alpha}{2}\right\} + i\mu s & \text{if } \alpha \neq 1, \\ -\sigma|s|\left\{1 + i\beta\,\mathrm{sgn}(s)\frac{2}{\pi}\log|s|\right\} + i\mu s & \text{if } \alpha = 1. \end{cases}$$

For the sake of simplicity, we concentrate only on the non-singular cases, i.e., on values of $\alpha \neq 1$.

Thus motivated, in this paper we consider the PSR representation of $\alpha$-stable random variables, and adopt the series-truncation approach of the earlier work referenced above. Our main goal is to obtain simple, explicit, quantitative bounds on the quality of approximating the tail of the series by a Gaussian, or, equivalently, of treating the PSR as a conditionally Gaussian representation of the $\alpha$-stable distribution. Our first main contribution is the derivation of closed-form expressions for the relevant CFs, and our second main contribution is the development of non-asymptotic bounds on the distance between the PSR residual and an appropriately defined Gaussian. Only brief outlines of the proofs are given here; complete details can be found in the full paper [26].

## II. PSR and the Conditionally Gaussian Representation

The PSR of the $\alpha$-stable random variable $X \sim \mathcal{S}_\alpha(\sigma, \beta, 0)$, $\alpha \in (0,2)$, $\alpha \neq 1$, has the following random series representation,

$$X \overset{\mathcal{D}}{=} \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} W_j - \mathbb{E}[W_1] b_j^{(\alpha)},$$

where $\overset{\mathcal{D}}{=}$ denotes equality in distribution, $\mathbb{E}[\cdot]$ is the expectation operator, $\{\Gamma_j\}_{j=1}^{\infty}$ are the arrival times of a unit rate Poisson process, and $\left\{W_j\right\}_{j=1}^{\infty}$ are independent and identically distributed (i.i.d.) random variables independent of $\{\Gamma_j\}_{j=1}^{\infty}$, with $\mathbb{E}[|W_1|^\alpha] < \infty$; see [13, p.28] for a detailed exposition. The coefficients $b_j^{(\alpha)}$ are non-zero only if $\alpha \in [1,2)$ and for this case they are readily computed and have a telescoping structure. We further refer to [13] for the non-linear transformations that map the moments of $W_j$ and $\alpha$ to the parameters $\beta, \sigma$.

From the PSR it follows that, if we choose the distribution of the $\{W_j\}$ to be i.i.d. with $W_j \sim \mathcal{N}(\mu_W, \sigma_W^2)$, we can write a conditionally Gaussian model for $X$ as,

$$X|\{\Gamma_j\}_{j=1}^{\infty} \sim \mathcal{N}\left(\mu_W m, \sigma_W^2 S^2\right), \tag{1}$$

where $m := \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} - b_j^{(\alpha)}$ and $S^2 := \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}$ are treated as auxiliary random variables. Figure 1 shows the first 100 terms of sample PSR realizations, when changing the parameters $\alpha$, $\mu_W$, $\sigma_W$.

### A. Truncation of the PSR and the Approximate Conditionally Gaussian Representation

While the exact representation of the stable law (1) is theoretically very appealing, in practice it is computationally intractable because of the infinite summations involved in the definitions of $m$ and $S$. Given that the summands of the PSR are stochastically decaying, the approach we adopt is to truncate the series to values of $\Gamma_j \leq c$, where $c \geq 0$ is a truncation constant, and to approximate the distribution of the residual term of the series by an appropriately chosen Gaussian. Then, the PSR can be split as,

$$X = X_{(0,c)} + R_{(c,\infty)}, \tag{2}$$

where $X_{(0,c)}$ is the truncated PSR,

$$X_{(0,c)} := \sum_{j:\Gamma_j \in [0,c]} W_j \Gamma_j^{-1/\alpha}, \tag{3}$$

and $R_{(c,\infty)}$ is the obvious PSR residual term.

The residual $R_{(c,\infty)}$ is not Gaussian. However, it can be proved that a CLT holds for its normalized counterpart, namely that, under mild conditions

$$Z_{(c,\infty)} := (R_{(c,\infty)} - m_{(c,\infty)})/S_{(c,\infty)}$$

converges to the standard Gaussian distribution, asymptotically as $c \to \infty$. A first result [16] studies the deterministic case $\sigma_W^2 = 0$ and our more recent results give the general case [27]. Such a CLT served to justify our adoption of the following Gaussian approximation of the PSR residual, $\hat{R}_{(c,\infty)}$, for practical inference procedures,

$$R_{(c,\infty)} \overset{\text{approx}}{\sim} \hat{R}_{(c,\infty)} \sim \mathcal{N}\big(m_{(c,\infty)}, S^2_{(c,\infty)}\big),$$

where $\overset{\text{approx}}{\sim}$ means that the distribution of $R_{(c,\infty)}$ converges to the Gaussian on the right-hand side, as $c \to \infty$, and $m_{(c,\infty)}$ and $S^2_{(c,\infty)}$ are, respectively, the mean and the variance of $R_{(c,\infty)}$, with explicit expressions provided in [27]. It follows that, by analogy with (2), we can introduce the random variable,

$$\hat{X} := X_{(0,c)} + \hat{R}_{(c,\infty)},$$

that converges in distribution to $X$, as $c \to \infty$ and it is this approximating random variable that is used in our inference procedures.

In fact, our CLT result for the PSR residual does not assume Gaussianity of the variables $W_j$. Therefore, even in this case we have the following *approximate* conditionally Gaussian representation for the $\alpha$-stable model,

$$X|\{\Gamma_j \leq c\} \overset{\text{approx}}{\sim} \mathcal{N}\left(m_{(0,c)} + m_{(c,\infty)},\ S^2_{(0,c)} + S^2_{(c,\infty)}\right),$$

where we view $m_{(0,c)} := \mu_W \sum_{j:\Gamma_j \in [0,c]} \Gamma_j^{-1/\alpha}$, and $S^2_{(0,c)} := \sigma_W^2 \sum_{j:\Gamma_j \in [0,c]} \Gamma_j^{-2/\alpha}$ as auxiliary random variables, which can now be generated exactly by direct sampling of the truncated Poisson process.

This approximate conditionally Gaussian structure illustrates the power of the proposed approximation: The inference methods that are valid for the exact PSR can be used for such an approximation, with the quality of the approximation controlled directly by the truncation parameter $c$. The inference schemes [15]–[18], [24], [25]

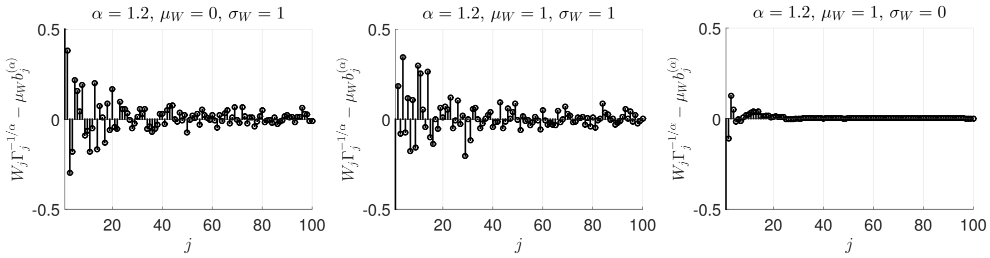


Fig. 1. First 100 terms of PSR realizations, with $W_j \sim \mathcal{N}(\mu_W, \sigma_W^2)$ for $\alpha = 1.2$ and three different scenarios for $\mu_W$ and $\sigma_W^2$.

referenced in the introduction were based on this approximation; however, we had no numerical measure of *how good* the approximation was and how it might vary with $c, \alpha, \beta$.

In this paper we provide explicit probability-theoretic results that can guide the practical choice of the truncation parameter $c$. Specifically, we provide non-asymptotic bounds on the approximation error between the corresponding cumulative distribution functions (cdfs) for finite values of $c$, by making use of Fourier-inversion theorems, summarized in the following.

## III. Distances between Distributions: The Smoothing Lemma

As before, let $c \geq 0$ be the value of the truncation parameter. Suppose $S_c$ and $T$ are random variables with CFs $\phi_{S_c}(s)$ and $\phi_T(s)$, $s \in \mathbb{R}$, respectively, let $F_{S_c}(x)$ and $F_T(x)$, $x \in \mathbb{R}$, be the corresponding cdfs, and assume that $\mathbb{E}[S_c] = \mathbb{E}[T] = 0$. Furthermore, assume that $F_T(x)$ has derivative $p_T(x)$ such that $|p_T(x)| \leq m < \infty$, $\forall x \in \mathbb{R}$. Finally, denote by,

$$\Delta(S_c, T) := \sup_{x\in\mathbb{R}} \left| F_{S_c}(x) - F_T(x) \right|,$$

the Kolmogorov distance between the distributions of $S_c$ and $T$, see e.g. [28]. Then, Berry's smoothing lemma [29, Lemma 2, p.538] states that, for any $\Theta > 0$,

$$\Delta(S_c, T) \leq \frac{1}{\pi}\int_{-\Theta}^{\Theta} \frac{|\phi_{S_c}(s) - \phi_T(s)|}{|s|}\,\mathrm{d}s + \frac{24m}{\pi\Theta} := I(S_c, T) \tag{4}$$

$$\overset{\Theta\to\infty}{\longrightarrow} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\phi_{S_c}(s) - \phi_T(s)|}{|s|}\,\mathrm{d}s := \bar{I}(S_c, T), \tag{5}$$

where (5) is meaningful only if the improper integral converges. Note that the assumption $\mathbb{E}[S_c] = \mathbb{E}[T] = 0$ can be relaxed, if either (4) or (5) are finite.

We first use the smoothing lemma to investigate convergence of the PSR standardized residual to the Gaussian distribution, by deriving an upper bound for the distance,

$$\Delta(Z_{(c,\infty)}, Z) := \sup_{x\in\mathbb{R}} \left| F_{Z_{(c,\infty)}}(x) - F_Z(x) \right|, \tag{6}$$

where $F_{Z_{(c,\infty)}}(x)$ and $F_Z(x)$ denote the cdf of $Z_{(c,\infty)}$ and the standard normal cdf, respectively.

We then use the resulting bound to further bound the Kolmogorov distance between the approximated stable law with Gaussian approximation of the PSR residual, $\hat{X}$, and the 'exact' stable law, $X$,

$$\Delta(X, \hat{X}) := \sup_{x\in\mathbb{R}} \left| F_X(x) - F_{\hat{X}}(x) \right|, \tag{7}$$

where $F_X(x)$ and $F_{\hat{X}}(x)$ are the cdfs of $X$ and $\hat{X}$, respectively.

From this point on, we restrict attention to the case when the $\{W_j\}$ are normally distributed. We provide results for the symmetric stable law (corresponding to $\mu_W = 0$), for which we are able to obtain closed-form expressions for the CF of the residual, as reported in the following lemma, proved in [26]; we leave to future work the extension of these theoretical results to the asymmetric case.

*Lemma 1:* Suppose $W_1 \sim \mathcal{N}(0, \sigma_W^2)$, and denote,

$$a := \frac{\alpha}{2}, \quad \eta := \frac{1-a}{a}, \quad w := \frac{\eta s^2}{2c}, \quad u := w S^2_{(c,\infty)}, \tag{8}$$

for $\alpha \in (0, 2), \alpha \neq 1$. Then,

$$\phi_{Z_{(c,\infty)}}(s) = \psi_{Z_{(c,\infty)}}(w) = \exp\left(c(1 - e^{-w} - w^a \gamma(1-a, w))\right), \tag{9}$$

where $\gamma(s, x) := \int_0^x t^{s-1} e^{-t}\,\mathrm{d}t$, $\mathrm{Re}(s) > 0$, is the lower incomplete gamma function. Moreover,

$$\phi_{X_{(0,c)}}(s) = \omega_{X_{(0,c)}}(u) = \exp(-c(1 - e^{-u} + u^a \Gamma(1-a, u))), \tag{10}$$

where $\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t}\,\mathrm{d}t$, $\mathrm{Re}(s) > 0$, is the upper incomplete gamma function.

As a consequence of Lemma 1, and through the independence of the random variables $X_{(0,c)}$ and $\hat{R}_{(c,\infty)}$, it follows that, when $\mu_W = 0$ the CF of $\hat{X}$, the approximated stable distribution is $\phi_{\hat{X}}(s) = \omega_{\hat{X}}(u)$, such that

$$\log \omega_{\hat{X}}(u) = -c(1 - e^{-u} + u^a \Gamma(1-a, u) + u/\eta).$$

## IV. Nonasymptotic Bounds on the Convergence of the PSR Residual

In this Section we apply the smoothing lemma to derive explicit bounds on the distance $\Delta(Z_{(c,\infty)}, Z)$, defined in (6). When $\mu_W = 0$, the closed-form expression in (9) for $\phi_{Z_{(c,\infty)}}(s)$ can be used to further bound above the term $\bar{I}(Z_{(c,\infty)}, Z)$ in (5). This results in the following result, whose proof, together with those of the subsequent theorems, can be found in [26].

*Theorem 1:* Let $W_j \sim \mathcal{N}(0, \sigma_W^2)$ and let $\Delta(Z_{(c,\infty)}, Z)$ be the Kolmogorov distance between $Z_{(c,\infty)}$ and $Z$, as in (6). Let $a = a(\alpha)$ and $\eta = \eta(\alpha)$ as in (8), and $g(w) := 1 - e^{-w} - w^a \gamma(1-a, w) < 0$. Let $\gamma(s, x)$ and $\Gamma(s, x)$ be lower and upper incomplete gamma functions,

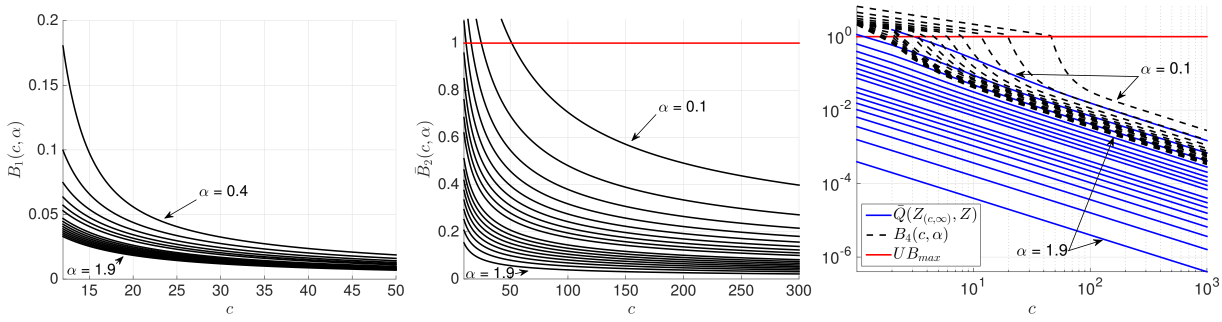


Fig. 2. Bounds on $\Delta(Z_{(c,\infty)}, Z)$. Left: each curve represents the values of the bound $B_1(c,\alpha)$ for $\alpha = 0.4, 0.9, \ldots, 1.9$, $\alpha \neq 1$, plotted for $12 \leq c \leq 50$. Centre: each curve represents the values of the bound $\bar{B}_2(c,\alpha)$ for $\alpha = 0.1, 0.2, \ldots, 1.9$, $\alpha \neq 1$, plotted for $10 \leq c \leq 300$. Right: comparison between $\bar{Q}(Z_{(c,\infty)}, Z)$ and $B_4(c,\alpha)$, for $\alpha = 0.1, 0.2, \ldots, 1.9$, $\alpha \neq 1$, plotted for $3 \leq c \leq 300$. The red horizontal line is simply equal to 1, the maximum possible value of the Kolmogorov distance.

respectively, and let $\bar{\gamma}(a) := \gamma(1-a, 1)$. Then, for any $c > 1$, $\Delta(Z_{(c,\infty)}, Z)$ is bounded above by:

$$B_1(c,\alpha) := \frac{1}{\pi}\left(\frac{c}{c-1}\right)\left(\frac{a}{2(2-a)} + \frac{1}{\eta^2}\right)\left[\frac{1}{(c-1)g^2(1)} + \right.$$
$$+\left(\frac{1}{g(1)} - \frac{1}{(c-1)g^2(1)}\right)\exp\left((c-1)g(1)\right) +$$
$$\left. + \frac{(c-1)\exp\left\{(c-1)(1-e^{-1})\right\}}{a[(c-1)\bar{\gamma}(a)]^{2/a}}\Gamma\left(\frac{2}{a}, (c-1)\bar{\gamma}(a)\right)\right].$$

From Theorem 1 it is easy to deduce the following.

*Corollary 2:* Under the same assumptions and notation of Theorem 1, $\Delta(Z_{(c,\infty)}, Z) = O(1/c)$.

For values of $\alpha$ greater than 0.4, $B_1(c,\alpha)$ gives very good bounds, as shown on the left-hand side of Figure 2. But for $\alpha$ below 0.4, the results deteriorate significantly; for example, for $\alpha = 0.2$, $B_1(c,\alpha)$ is below 1 (the maximum possible Kolmogorov distances) only for $c > 115$. The following result, obtained by bounding $I(Z_{(c,\infty)}, Z)$, gives an $O(1/\sqrt{c})$ bound which is, of course, asymptotically inferior to that in Theorem 1, but which gives sharper results for small $c$ and $\alpha < 0.4$.

*Theorem 3:* Under the same assumptions and notation as in Theorem 1, for any $\delta \in (0,2)$, $\Delta(Z_{(c,\infty)}, Z)$ is bounded above by,

$$B_2(c,\alpha,\delta) := \frac{9.6\sqrt{\eta}}{\pi\sqrt{2(2-\delta)c}} + B_3(c,\alpha,\delta),$$

where $B_3(c,\alpha,\delta)$ is the following $O(1/c)$ term:

$$B_3(c,\alpha,\delta) := \frac{1}{\pi c}\left[\frac{a}{2(2-a)} + \frac{1}{\eta^2}\right]\left(\frac{c(2-\delta)}{(c-1)g(2-\delta)}\right)^2 \times$$
$$\times\left\{1 - \left[1 - g(2-\delta)(c-1)\right]\exp\left(g(2-\delta)(c-1)\right)\right\}.$$

Numerically minimizing the bound $B_2(c,\alpha,\delta)$ over $\delta$ yields $\bar{B}_2(c,\alpha)$, shown in the central part of Figure 2.

Finally, we combine the results of Theorem 1 and Theorem 3, to obtain useful bounds essentially for all values of $\alpha \in (0,2)$, $\alpha \neq 1$, and $c > 1$ as,

$$B_4(c,\alpha) := \min\left\{B_1(c,\alpha), \bar{B}_2(c,\alpha)\right\}.$$

Figure 2 shows a comparison between the theoretical bound $B_4(c,\alpha)$ and the numerical estimates $\bar{Q}(Z_{(c,\infty)}, Z)$ of $\bar{I}(Z_{(c,\infty)}, Z)$ reported in [30]. The numerical values are produced through the Matlab routine `quadgk`; we do not show here the numerical error intervals because they are negligibly small for $c \geq 3$. Observe that $B_4(c,\alpha)$ has the same asymptotic rate as $\bar{Q}(Z_{(c,\infty)}, Z)$.

## V. Convergence Rate of the Approximated $\alpha$-stable Distribution

Finally, we examine the distance $\Delta(X, \hat{X})$; note that in terms of inference, ultimately, it is $\Delta(X, \hat{X})$ that we wish to make "small", by appropriately choosing the value of the parameter $c$. Using the smoothing lemma (5), and the bound in Theorem 1, we can establish the following.

*Theorem 4:* Let $\Delta(X, \hat{X})$ be the Kolmogorov distance between $X$ and $\hat{X}$, as in (7), under the same assumptions and notation as in Theorem 1. Let $N \geq 1$, and introduce arbitrary abscissae $0 =: u_0 < u_1 < \cdots < u_N := 1$ together with the corresponding ordinates $f_0 := 0$ and $f_i := \log(\omega_{X_{(0,c)}}(u_i))$, for $i = 1, 2, \ldots, N$. Also let, $m_i := (f_{i+1} - f_i)/(u_{i+1} - u_i)$, and $q_i := -m_i u_i + f_i$, for $i = 0, 1, \ldots, N-1$. Then, for any $c > 1$, $\Delta(X, \hat{X})$ is bounded above by:

$$B_5(c,\alpha,N) := \frac{1}{\pi}c\left(\frac{a}{2(2-a)} + \frac{1}{\eta^2}\right) \times$$
$$\times\left\{\sum_{i=0}^{N-1}\frac{e^{q_i}}{\tilde{m}_i}\left[e^{\tilde{m}_i u_{i+1}}\left(u_{i+1} - \frac{1}{\tilde{m}_i}\right) - e^{\tilde{m}_i u_i}\left(u_i - \frac{1}{\tilde{m}_i}\right)\right]\right.$$
$$\left. + \frac{e^{\tilde{k}_{(1,\infty)}}}{a\left(\tilde{l}_{(1,\infty)}\right)^{2/a}}\Gamma\left(\frac{2}{a}, \tilde{l}_{(1,\infty)}\right)\right\},$$

where $\tilde{m}_i := m_i + (c-1)g(1)$,

$$k_{(1,\infty)} := -c((1 - \exp(-1)) + \Gamma(1-a, 1)) < 0,$$

$\tilde{k}_{(1,\infty)} := k_{(1,\infty)} - (c-1)(e^{-1} - 1)$ and $\tilde{l}_{(1,\infty)} := (c-1)\gamma(1-a, 1)$.

The $N$ abscissae $u_i$ and ordinates $f_i$ serve to define a piece-wise linear envelope on $\omega_{X_{(0,c)}}(u)$ for $u \in [0,1]$, that is used in the proof. Increasing $N$ improves (i.e., decreases) $B_5(c,\alpha,N)$, but the changes are minimal for $N \geq 10$ and logarithmically spaced points.

In Figure 3 we compare the numerical estimates of $\bar{I}(X, \hat{X})$ obtained in [30] (denoted $\bar{Q}(X, \hat{X})$) with the bound $B_5(c,\alpha,N)$ for $N = 10$. Note that the bound of Theorem 4 correctly captures the dependence on $\alpha$. The approximation error is lower for smaller values of $\alpha$, a

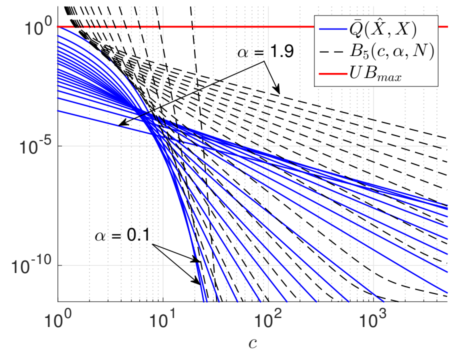


Fig. 3. Bounds on $\Delta(X, \hat{X})$. The blue solid lines represent $\bar{Q}(X, \hat{X})$; the black dashed lines represent $B_5(c, \alpha, N)$ with $N = 10$ and logarithmically spaced points. The values are plotted for $\alpha = 0.1, 0.2, \ldots, 1.9$, $\alpha \neq 1$, and for $1 \leq c \leq 5000$. The red horizontal line is simply equal to 1, the maximum possible value of the Kolmogorov distance.

reversal of the trend shown in Fig. 2. We believe this is because, as $\alpha$ decreases, the relative significance of the residual term is much smaller, when compared with the heavy-tailed initial terms in the PSR. We also observe that the rate of convergence is dramatically better for smaller $\alpha$, again in contrast with the analysis of the residual approximation in Fig. 2.

## VI. Conclusion

In this paper we have provided explicit bounds on Kolmogorov distances of interest, when approximating the PSR of symmetric $\alpha$-stable random variables. The theoretical results are in agreement with our previous numerically computed convergence rates. These results form then a collection of tools that can be used in future to automatically select the PSR truncation parameter $c$, so as to control the quality of the resulting approximation. We expect that the present bounds, and our current work extending them to the case of continuous-time stochastic process, will be of value in quantifying the accuracy of inference algorithms employing the PSR.

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