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Timing Estimation based on Higher-Order Cyclostationarity for Faster-than-Nyquist Signals

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Abstract—Faster-than-Nyquist (FTN) transmission is a promising technique to increase spectral efficiency at fixed constellation size. However, traditional timing synchronization algorithms mostly rely on the eye diagram opening after matched-filtering (e.g., Gardner, zero-crossing). Such approaches are thus unsuitable in presence of FTN-induced intersymbol interference.

In this paper, we first show that FTN signals exhibit cyclostationarity at the \( l \)-th order \((l > 2)\) according to their bandwidth and symbol period. Then, we derive non-data-aided timing offset estimators from the cyclic temporal cumulant function. Besides its relevance for strong FTN scenarios, the proposed solution is also robust to frequency and phase offsets.

Index Terms—Faster-than-Nyquist, timing synchronization, higher-order cyclostationarity, non-data-aided.

I. INTRODUCTION

FASTER-than-Nyquist (FTN) transmission targets a spectral efficiency increase at fixed constellation size [1] while generating intersymbol interference (ISI) at the output of any linear receiver. Efficient equalizers have been proposed to make FTN transmission reliable [2], [3] but few contributions address the timing synchronization problem [4, Sec. IV].

In particular, usual timing error detectors require an ISI-free observation (e.g., Gardner, zero-crossing, Mueller and Müller) [5, Ch. 8]. A possible solution consists in adding pilot sequences at the Nyquist rate [6]. However, a non-data-aided (NDA) approach based on a complete FTN observation pilot sequences at the Nyquist rate [6]. Nevertheless, one may use higher-order cyclostationary (HOCS) properties of FTN signals to overcome this issue.

In this paper, we first give the conditions under which a single-carrier linearly modulated signal exhibits \( l \)-th order cyclostationarity depending on its bandwidth and its symbol period. To address the FTN case, we propose HOCS-based NDA timing offset estimators as a generalization of the Oerder and Meyr algorithm [7] while keeping a similar computational complexity. Simulations assess the performance of the proposed estimators in the case of frequency/phase impaired receiver over an additive white Gaussian noise (AWGN) channel.

II. SYSTEM MODEL: FASTER-THAN-NYQUIST SIGNALS

We consider complex symbols \( c_k \), zero-mean, independent and uniformly distributed (i.u.d.) in a constellation of size \( N_c \). The received single-carrier linearly modulated signal is

\[
r_c(t) = e^{j(2\pi F_c t + \theta)} \sum_{k \in \mathbb{Z}} c_k g^{(ix)}(t - kT - \Gamma) + z_c(t), \quad t \in \mathbb{R}
\]

with \( T \) the symbol period, \( g^{(ix)}(t) \) a real pulse shape bandlimited to \( B \) (bilateral bandwidth), \( z_c(t) \) a zero-mean circularly symmetric white Gaussian noise with power spectral density \( 2N_0 \), \( \Gamma = \gamma T \in [0,T] \) a timing offset, \( F_c \) and \( \theta \) frequency and phase offsets, respectively.

The spectral efficiency is defined as \( \eta = \rho \log_2(N_c) \) with \( \rho = 1/(BT) \) the signaling density.

A matched filter \( g^{(ix)}(t) = g^{(ix)}(-t) \) is considered at the receiver side to maximize the signal-to-noise ratio (SNR). Then, the received signal \( r(t) = (r_c * g^{(ix)})(t) \) is sampled at a period \( T_s = T/N_s \) where \( N_s \) is a strictly positive integer chosen such that \( 1/T_s > B \). The noise-free received signal is given by

\[
x[n] = e^{j(2\pi f_c n + \theta)} \sum_{k \in \mathbb{Z}} c_k g[n - kN_s - \gamma N_s], \quad n \in \mathbb{Z}
\]

with \( f_c = F_c T_s \) and \( x[n] = x(nT_s) \). The equivalent pulse shape is defined as \( g[n] = \left( g^{(ix)} * g^{(rx)} \right)(nT_s) \) with \( g^{(rx)}(t) = g^{(rx)}(t) \exp(-j2\pi F_r t) \). The observed samples are \( r[n] = x[n] + z[n] \) with \( z[n] = (z_c * g^{(ix)})(nT_s) \) the noise samples, possibly correlated but independent of the symbols.

In current literature, FTN systems are generally defined as follows [4]: considering a reference orthogonal system at symbol period \( T_{Nyq} \), namely \( \left( g^{(ix)} * g^{(rx)} \right)(kT_{Nyq}) = \delta_{0,k} \) with \( \delta \) the Kronecker symbol and \( k \in \mathbb{Z} \), we set an actual symbol period \( T < T_{Nyq} \) to increase spectral efficiency while introducing ISI. However, if \( \rho \leq 1 \), it should be noted that there still exists a linear processing to cancel ISI at the receiver side [9, Ch. 7].

Consequently, we use in this work a more strict definition of FTN, only based on the transmitter’s signaling density.

Definition 1. A linearly modulated signal as defined in (1) is said Faster-than-Nyquist if \( \rho > 1 \).
III. NDA TIMING ESTIMATION BASED ON HOCS

In this Section, we first determine the conditions under which a linearly modulated signal exhibits \(l\)th-order cyclostationarity. Then, we propose NDA timing estimators based on higher-order time-domain cumulants.

A. On the existence of HOCS features for FTN signals

Cyclic cumulants provide a convenient statistical representation of linear modulations [10, Sec. V]. Generalizing this work to the case of phase and frequency shifted signals (2), the \(l\)th-order temporal cumulant function (TCF) is given by

\[
C_x(n, \tau)_l = \sum_{k \in \mathbb{Z}} \left( -1 \right)^{u-1} (u - 1)! \left( \sum_{i=1}^{u} R_{c,u_i} \right)
\]

where \(\tau = [\tau_1, \ldots, \tau_l] \in \mathbb{Z}^l\) is a lag vector and \((-)\) means optional conjugation of the \(i\)th factor. \(C_{c,l}\) is the \(l\)th-order cumulant of a symbol \(c\):

\[
C_{c,l} = \sum_{\{U_l\}} \left( -1 \right)^{u-1} (u - 1)! \left( \sum_{i=1}^{u} R_{c,u_i} \right)
\]

with optional conjugations \((\cdot)^{\ast}\) chosen in accordance with aforementioned optional negative symbols. Here, we focus on even values of \(l\) with half optional negative symbols in (3) to discard frequency and phase offsets. Consequently, the \(N_s\)-periodicity of (3) enables a Fourier series expansion, yielding the cyclic TCF (CTCF), as established in [10]:

\[
C_x^{(p/N_s)}(\tau)_l = \frac{C_{c,l}}{N_s} e^{-j2\pi p N_s} \sum_{q \in \mathbb{Z}} g[n] e^{-j2\pi N_s q}. \tag{6}
\]

Definition 2. \(x[n]\) is said \(l\)th-order cyclostationary (or exhibiting \(l\)th-order cyclic features) if there exists at least one value of \(p \neq 0\) and \(\tau\) such that \(C_x^{(p/N_s)}(\tau)_l \neq 0\).

We add the following usual assumptions to \(x[n]\):

AS1. \(g(t)\) is real and symmetric (i.e., \(g(t) = g(-t)\)); this requires a negligible frequency offset [11];

AS2. \(g(t)\) has a frequency support \([-B/2; B/2]\) with a decreasing amplitude spectrum.

Regarding the noise signal \(z[n]\), we recall that Gaussian random variables have null higher-order cumulants [12]. Furthermore, wide sense stationary processes do not exhibit cyclic features. Consequently, since \(x[n]\) and \(z[n]\) are independent, \(C_r^{(p/N_s)}(\tau)_l = C_r^{(0/N_s)}(\tau)_l + C_z^{(p/N_s)}(\tau)_l\). Either one of the conditions \(p \neq 0\) or \(l > 2\) is sufficient to obtain \(C_r^{(p/N_s)}(\tau)_l = 0\), thus discarding the noise contribution.

**Lemma 1.** Consider a linearly modulated signal \(x[n]\) (2) characterized by a transmission density \(\rho\), a constellation enabling \(C_{c,l} \neq 0\) (\(l\) even) and a pulse shaping scenario fulfilling (AS1)-(AS2). Such a signal exhibits \(l\)th-order cyclostationarity if there exists a non-zero integer \(p\) such that \(|2p| \leq l/\rho\) with \(l \geq 2\).

**Corollary 1.** Since \(\rho > 1\), an FTN signal \(x[n]\) cannot exhibit second-order cyclostationarity.

**Corollary 2.** If \(x[n]\) is \(l\)th-order cyclostationary, it is also \((l + 1)\)th-order cyclostationary, provided \(C_{c,l+1} \neq 0\).

**Proof of Lemma 1:** We express the reduced dimension CTCF of \(x[n]\) by setting \(\tau_0 = 0\) in (6). From this function, the cyclic polyspectrum is obtained through a \((l - 1)\) fold Fourier transform over \(\tau' = [\tau_1, \ldots, \tau_{l-1}]\):

\[
p_x^{(l/N_s)}(f)_l = \frac{C_{c,l}}{N_s} e^{-j2\pi p N_s} \sum_{q \in \mathbb{Z}} g[n] e^{-j2\pi q} \sum_{i=1}^{l-1} i \prod_{i=1}^{l-1} G(f_i)
\]

with

\[
G(f) = \sum_{q \in \mathbb{Z}} g[q] e^{-j2\pi f q}. \tag{8}
\]

Restricting our analysis to \(f \in [-1/2, 1/2]\) bandlimited assumption implies \(G(f) = 0\) if \(f \notin [-BT_s/2; BT_s/2]\). Assuming \(C_{c,l} \neq 0\) we have from (7):

\[
p_x^{(l/N_s)}(f)_l \neq 0 \iff \begin{cases}
-\frac{BT_s}{2} \leq f_l \leq \frac{BT_s}{2} & (9a) \\
-\frac{BT_s}{2} \leq f_1 \leq \frac{BT_s}{2} & (9b) \\
\vdots \\
-\frac{BT_s}{2} \leq f_{l-1} \leq \frac{BT_s}{2} & (9c)
\end{cases}
\]

By injecting (9b)-(9c) into (9a), the inequality is satisfied with \(|2p| \leq l/\rho\), which is obtained by recalling \(\rho = 1/(BT_s)\) and \(T_s = T/N_s\). \(\square\)

B. Proposed timing offset estimators

A linear and non-dispersive channel motivates a restriction to the zero-delay vector \(\tau = [0, \ldots, 0]\) in (6), as already suggested in [13]; we obtain

\[
C_r^{(p/N_s)}(0)_l = \frac{C_{c,l}}{N_s} e^{-j2\pi p N_s} \sum_{q \in \mathbb{Z}} q^l[q] e^{-j2\pi q N_s}. \tag{10}
\]

Furthermore, (AS2) and (9a)-(9c) justify the use of only the first cycle frequency, namely \(p = 1\).

Considering the \(l\)th-order cycloergodicity of \(r[n]\), the CTCF can be estimated through a finite length signal: \(r[n] = 0\) for \(n \notin [0, \ldots, N - 1]\) by applying:

\[
C_r^{(1/N_s)}(0)_l = \sum_{\{U_l\}} \left( -1 \right)^{u-1} (u - 1)! \left( \sum_{i=1}^{u} R_{c,u_i}^{(1/N_s)}(0)_{u_i} \right) \tag{11}
\]
where $\hat{R}_{C}^{[l/N_4]}(0)_l$ is the estimate of the co-called cyclic temporal moment function (CTMF) at the first cycle frequency, given by

$$\hat{R}_{C}^{[l/N_4]}(0)_l = \frac{1}{N} \sum_{n=0}^{N-1} |r[n]|^2 e^{-j2\pi n\frac{c_l}{N}} .$$ (12)

**Proposition 1.** Consider a linearly modulated signal $x[n]$ with density $\rho$, observed over an AWGN channel. There exists a consistent timing offset estimator based on the 1st-order estimated CTCF with $l \geq [2\rho]$ (Lemma 1):

$$\hat{\gamma}_l = -\frac{1}{2\pi} \arg \left( \frac{C_{\rho}^{[l/N_4]}(0)_l}{C_{c.l}} \right).$$ (13)

with $C_{\rho}^{[l/N_4]}(0)_l$ a simplified version of $C_{\rho}^{[l/N_4]}(0)_l$ where contributions from CTMF of orders strictly lower than $[2\rho]$ are dropped. Note: the constellation (thus $C_{c,l}$) is assumed known by the receiver.

**Proof of Proposition 1:** Recalling (AS1), the sum in (10) is real and positive. Since $N_c$ is a real integer, taking the argument of $C_{\rho}^{[l/N_4]}(0)_l$ captures the timing offset parameter and justifies (13). $\square$

**Example 1 (4th-order estimator with QPSK symbols).** Let us assume QPSK symbols with possible values $(\pm 1, \pm j)/\sqrt{2}$. We start by computing $C_{4,4}$ and choose to conjugate second and fourth elements in the lag product (5). Since symbols are independent and zero-mean, we discard partitions including elements of odd size. Consequently, relevant partitions to be considered in (5) are:

- $U_1 = \{1; 2\}; \{3; 4\}$ such that $R_{c,2} = 1$ in each case;
- $U_2 = \{1; 4\}; \{2; 3\}$ such that $R_{c,2} = 1$ in each case;
- $U_3 = \{1; 3\}; \{2; 4\}$ such that $R_{c,2} = 0$ in each case;
- $U_4 = \{1; 2; 3; 4\}$ such that $R_{c,4} = 1$.

We obtain from (4) that $C_{4,4} = -1$. Finally, $\hat{\gamma}_4$ is computed from $C_{\rho}^{[1/N_4]}(0)_4$. According to Lemma 1, $\hat{\gamma}_4$ can be used for FTN transmission up to $\rho = 2$.

**C. Particular cases and computational complexity**

Interestingly, the Oerder and Meyr estimator [7] can be seen as a particular case of $\hat{\gamma}_2$ where $C_{c,2} = 1$. Both share a linear computational complexity with the received sequence length $N$ as observed in (12). As an extension to higher-orders (with $l$ even), one needs to (i) compute $l/2$ estimates of the CTMF according to (12) and (ii) combine the results as prescribed in (11). While the latter operation has a factorial complexity with $l$, zero-mean and independent symbols assumptions yield various simplifications [10]. In practice, for reasonable orders ($l \leq 10$) we witnessed a quasi-linear complexity with $Nl/2$.

**IV. SIMULATIONS**

**A. General parameters of the simulations**

We evaluate the mean-squared-error (MSE) performance of $\hat{\gamma}_4$ and $\hat{\gamma}_6$ given by Proposition 1. For comparison purposes, we also introduce two reference estimators: Oerder and Meyr (OM) [7] and compressed maximum likelihood (CML), as proposed in [8] via a grid-search approach. In both cases, those two estimators only rely on second-order statistics. As a usual limit in synchronization problems, we plot the modified Cramèr–Rao bound (MCRB) as the best achievable result for a Nyquist-rate system based on root-raised-cosine (RRC) filters [5, Ch. 1].

The parameters of the simulations are the following:

- observations are made of $K$ QPSK symbols, $K$ ranging from 10 000 to 100 000 with an oversampling factor $N_v = 4$;
- each MSE measure relies on 500 realizations;
- $g(t)c(t)$ is the impulse response of an RRC filter, creating a transmitted signal of bandwidth $B = (1 + \beta)/(\xi T)$, with roll-off factor $\beta = 0.4$ and compression factor $\xi \in (0; 1]$;
- transmission density $\rho = 1/(1.4\xi)$ varies from 0.714 (Nyquist case, $\xi = 1$) to $\rho = 3$ (severe FTN, $\xi = 0.24$);
- constellation and filters are normalized (i.e., $E\{|c_k|^2\} = 1$ and $\|g\|_2^2 = 1$) such that SNR = $1/E\{|z[n]|^2\}$; it varies between 0 and 40 dB;
- normalized frequency offsets $f_c$ are taken between 0 and 0.1 to illustrate mismatch between transmitter and receiver filters;
- phase offsets $\theta$ are chosen between 0 and $2\pi$.

**B. Pure AWGN channel**

We first consider a pure AWGN channel, without synchronization impairments in phase and in frequency (i.e., $f_c = 0$ and $\theta = 0$).

In Fig. 1, we confirm that second-order estimators such as OM and CML are sufficient to approach the MCRB in the Nyquist case ($\rho = 0.714$). As already discussed in [13], higher-order estimators are more sensitive to AWGN (see SNR region between 0 and 10 dB). Moreover, self-noise increases with the order of the chosen estimator, as observed for $\hat{\gamma}_6$ at high SNR. In the chosen FTN scenario ($\rho = 1.2$), second-order estimators cannot reveal cyclic features, as forecasted by Lemma 1. However, $\hat{\gamma}_4$ and $\hat{\gamma}_6$ may provide a reliable timing estimation even if a self-noise floor appears at high SNR.

Fig. 2 also illustrates the implications of Lemma 1: due to the loss of cyclic features, performance of second-order and fourth-order estimators collapses at a density greater than 1 and 2, respectively. Outside this area, MSE globally increases with the density and decreases with the observation length. Also, for a fixed-length observation $r$, the amplitude of $\hat{R}_{C}^{[l/N_4]}(0)_l$ declines with $l$. This is exemplified for $\hat{\gamma}_6$ for $\rho \in [2; 2.5]$; while the signal is 6th-order cyclostationary, $K = 100 000$ symbols are required to provide a suitable timing estimate. In realistic scenarios (e.g., slowly time-varying $\gamma$, short bursts), a restriction to $l \leq 8$ often provides an acceptable trade-off between computational complexity and performance.

**C. Phase and frequency nuisance parameters**

In Fig. 3, we evaluate the performance of the proposed estimators in presence of a random phase shift ($\theta$ uniformly distributed over $[0; 2\pi]$) as a function of a normalized frequency offset $f_c \in [0; 0.1]$. 
From the modulus operation in (12), simulations confirm that phase offset are irrelevant w.r.t. the MSE performance. For the same reason, the proposed algorithms are globally resilient to residual frequency offsets (e.g., \( f_e < 0.01 \)). However, larger values of \( f_e \) create a mismatch between \( g(\delta_k(t)) \) and \( g(\delta_k(t)) \), overriding (AS1)-(AS2) and thus decreasing the performance. Unsurprisingly, high-density systems are more affected by frequency offsets since the transmitted signal bandwidth decreases with \( \rho \) (at fixed symbol period \( T \)). In the chosen FTN scenarios (\( \rho \in \{1.2; 1.6\} \) at SNR = 20 dB and \( K = 10\,000 \) symbols), \( \hat{\gamma}_6 \) is more robust to frequency shifts than \( \hat{\gamma}_4 \). Indeed, since \( \rho < 2 \), the former estimator combines fourth-order and sixth-order statistics; thus justifying its superiority.

![Fig. 1. MSE performance of timing offset estimators with \( K = 10\,000 \) symbols for two transmission densities: (i) a Nyquist-rate scenario at \( \rho = 0.714 \) (dashed lines) and \( \rho = 1.2 \) (solid lines) require higher-order estimators (i.e., \( \hat{\gamma}_4 \) and \( \hat{\gamma}_6 \)).](image1)

![Fig. 2. MSE performance of timing offset estimators at SNR = 20 dB for two observation lengths: an increase from \( K = 10\,000 \) symbols (solid lines) to \( K = 100\,000 \) symbols (dashed lines) is beneficial to higher-order estimators in FTN scenarios.](image2)

![Fig. 3. MSE performance of timing offset estimators at SNR = 20 dB and \( K = 10\,000 \) symbols in two FTN scenarios: high-order estimator are more robust to frequency mismatch, as exemplified for \( \rho = 1.2 \) (dashed lines) and \( \rho = 1.6 \) (solid lines).](image3)

Fig. 3. MSE performance of timing offset estimators at SNR = 20 dB and \( K = 10\,000 \) symbols in two FTN scenarios: high-order estimator are more robust to frequency mismatch, as exemplified for \( \rho = 1.2 \) (dashed lines) and \( \rho = 1.6 \) (solid lines).

show that low-order cyclic features of FTN signals disappear as the transmission density increases. It is thus appropriate to increase the order of the proposed estimators consequently. Finally, over an AWGN channel, the proposed estimator is not sensitive to phase offsets and quite robust to frequency shifts.

Future work includes performance evaluation of complete FTN receivers, with various symbol detectors (e.g., turbo equalizers). The results should also be extended to multipath (in)variant channels.

V. CONCLUSIONS

Higher-order cyclostationary properties of Faster-than-Nyquist signals have been used to specify non-data-aided timing offset estimators. More precisely, an ad-hoc estimator is built from the \( l \)th-order cyclic temporal cumulant function estimate, assuming a known constellation at the receiver side. We

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