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Operations Research

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1. Introduction

Since the industrial revolutions, there has been a significant growth in the size and complexity of organizations. One of the most notable characteristics of this revolutionary change have been the increase in the division of labor and segmentation of management responsibilities in these organizations. While the results have been tremendous, one of the problem was that the divisions and components of organizations often grew autonomously, losing sight of the overall organizational objectives and benefits. A related problem was that as the complexity of organization increased, it became much more difficult to allocate the resources to various activities in accordance with the most productive, valuable way for the overall organizations. The need to solve these problem let to emergence of Operation Research (often referred to as OR).

The word Operation Research came from the military services early in World War II. As the countries needed to allocate scarce resources to the various military operations and activities in most effective ways, the British and then the U.S. military management called a large number of scientists to come up with the strategies. The scientists were asked to do *research* on (military) *operations.* They were instrumental in winning the Air Battle of Britain as well as the Battle of the North Atlantic. After the war, OR have been applied to other areas outside the military. By 1950s, a variety of organizations in business, industry, and government adopted the use of OR.

As implied in the name, operation research is used in the problems regarding how to conduct and coordinate the *operation*s within an organization. OR has been used extensively in such diverse areas as manufacturing, transportation, construction, telecommunications, financial planning, health care, the military, and public services, etc. The *research* part of the name came as the OR uses an approach that is similar to that of scientific fields. Thus, the OR begins by carefully observing and formulating problems, including gathering data. Then, the scientific/ mathematical model is constructed to abstracts the essence of the real problem and to hypothesize it. Next, suitable experiments are conducted to test this hypothesis. Thus, OR involves creative scientific research. However, as OR is specifically concerned with the practical management of the organizations, a successful OR provides positive, practical and understandable conclusions to the decision makers. Additionally, the OR adopts an organizational point of view – resolve the conflicts of interest among the division of organizations in a way that is best for the organization as a whole. (See reference [2])

Application Vignette 1

Federal Express (FedEx), the world's largest express transportation company, delivers more than 6.5 million documents, packages, and other items throughout the United States and more than220 countries and territories around the world. The logical challenges in this services are staggering: these millions of daily shipments must be individually sorted and routed to the correct general location and then delivered to the exact destination in an extremely short period of time. OR is the technological engine that make this possible. Since its founding in 1973, OR was used to make decisions regarding equipment investment, route structure, scheduling, finances, and location of facilities, etc. As OR was credited with literally saving the company during its early years, it became the custom to have OR represented at the weekly senior management meetings and, indeed, several of the senior corporative vice presidents have come up from the outstanding FedEx OR group. (See reference [2])

Application Vignette 2

Prostate cancer is the most common form of cancer diagnosed in men. Like many other forms of cancer, radiation therapy is a common method to treat this cancer, where the goal is to have enough radiation dosage in the tumor region to kill the malignant cells while minimizing the radiation exposure to the healthy structure near the tumor. This radiation therapy also uses OR: the research process (*optimization* which will be explained soon) is done in a matter of minutes by an automated computerized planning system that can be operated readily by medical personnel when beginning the procedure of inserting the seeds into the patient's prostate. OR thus has had a great impact on both health care costs and quality of life for treated patients because of tis much greater effectiveness and the substantial reduction in side effects. It is estimated that annual cost saving would be about $500 million when all U.S. clinics adopt this procedure, as it will eliminate the need for pretreatment planning meeting and post-operation CT scan, as well as providing a more efficient surgical procedure and reducing the need to treat subsequent side effects. This approach is also anticipated to be applied to treatment of breast, cervix, esophagus, biliary tract, pancreas, head and neck, and eyes. (See reference [2])

2. Linear Programming

The development of linear programming (LP) has been ranked among the most important scientific advances of the mid-20th century. Linear programming, the very basic but powerful tool of OR, involves the general problem of allocating limited resources among competing activities in a best possible way; Linear programming helps to select the level of certain activities that compete for scarce resources to make the best outcome possible for the organization as a whole.

As its name implies, LP deals with mathematical model that uses only linear functions. The word programming indicates planning. Thus, LP is planning of activities to obtain an optimal result – result that reaches the specified goal best among all feasible alternatives.

In more technical term, linear programming may be defined as *maximizing or minimizing a linear function subject to linear constraints.*

2.1 Standard Maximum Problem

We are given an m-vector, b= $(b_{1,\dots,}b_m)^T$, an n-vector, c = $(c_{1,\dots,}c_n)^T$, and an m x n matrix,

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

of real numbers.

Find an n-vector, x =$(x_{1,\dots,}x_n)^T$, to maximize

$$c^T x = c_1x_1 + \cdots + c_nx_n$$

Subject to the constraints ($Ax \leq b$)

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \end{gathered}$$

and ($x \geq 0$)

$$x_1 \geq 0, x_2 \geq 0, \dots x_n \geq 0$$

The Standard Minimum Problem:

Find an m-vector, y =$(y_{1,\dots,}y_m)$, to minimize

$$y^T b = y_1 b_1 + \cdots + y_m b_m$$

Subject to the constrains ($y^T A \leq c^T$)

$$\begin{gathered} y_1 a_{11} + y_2 a_{21} + \cdots + y_m a_{m1} \ \leq c_1 \\ y_1 a_{12} + y_2 a_{22} + \cdots + y_m a_{m2} \leq c_2 \\ \vdots \\ y_1 a_{1n} + y_2 a_{2n} + \cdots + y_m a_{mn} \leq c_n \end{gathered}$$

and ($y \geq 0$)

$$y_1 \geq 0, y_2 \geq 0, \dots y_m \geq 0$$

Terminology

- Objective function: the function to be maximized or minimized.
- Feasible: a vector, x for the standard maximum problem or y for the standard minimum problem if it satisfies the corresponding constraints.
- Constraint set: the set of feasible vector
- A linear programming problem is said to be feasible if the constraint set is not empty; otherwise it is said to be infeasible.
- A feasible maximum/ minimum problem is said to be unbounded if the objective function can assume arbitrarily large positive/ negative values at feasible vectors; otherwise, it is said to be bounded. There are three possibilities for a linear programming problem. It may be bounded feasible, it may be unbounded feasible, and it may be infeasible.
- The value of a bounded feasible maximum/ minimum problem is the maximum/ minimum value of the objective function as the variable range over the constraint set.
- A feasible vector at which the objective function achieves the value is called optimal. (See reference [1])

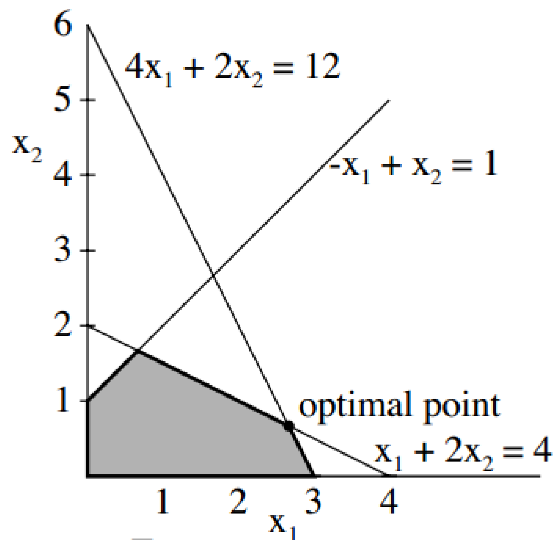
Example 1

Find numbers $x_1 and\ x_2$ that maximize the sum $x_1 + \ x_2$ subject to the constraints $x_1 \geq 0, x_2 \geq 0$, and

$$\begin{aligned} &x_1 + 2x_2 \ \leq 4 \\ &4x_1 + 2x_2 \ \leq 12 \\ &-x_1 + x_2 \ \leq 1 \end{aligned}$$

This standard maximum problem has two unknowns and five constraints. We have the non-negativity constraints, $x_1 \geq 0\ and\ x_2 \geq 0$. The objective function here is $x_1 + \ x_2$.

One way to approach this is to draw graphs.



Drawing graphs helps us see that the maximum occurs along an entire edge or a corner point of the bounded regions.

Without drawing graphs, we can also solve this problem. The function $x_1 + x_2$ is constant on line with slope -1, and we can move this line $x_1 + x_2$=1 further from the origin. Thus, we find a point where the line of slope -1 is farthest from the origin and still lies within the constraints. This is where $x_1 + 2x_2 = 4$ and $4x_1 + 2x_2 = 12$ intersect: $x_1 = \frac{8}{3}$ *and* $x_2 = \frac{2}{3}$. Thus the solution to the objective function is $\frac{10}{3}$ (See reference [1]).

2.2 Duality

Every linear program has a dual linear program.

Definition. The dual of the standard maximum problem,

Maximize $c^T x$
Subject to the constraints $Ax \leq b$ and $x \geq 0$

is defined to be the standard minimum problem

Minimize $y^T b$
Subject to the constraints $y^T A \leq c^T$ and $y \geq 0$

Let's recall the numerical example from above: Find x_1 and x_2 to maximize $x_1 + x_2$ subject to the constraints $x_1 \geq 0, x_2 \geq 0, and$

$$x_1 + 2x_2 \leq 4$$
$$4x_1 + 2x_2 \leq 12$$
$$-x_1 + x_2 \leq 1$$

The dual of this standard maximum problem is the standard minimum problem:

Find $y_1, y_2, and\ y_3$ to minimize $4y_1 + 12y_2 + y_3$ subject to the constraints: $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$, and

$$y_1 + 2y_2 - y_3 \geq 1$$
$$2y_1 + 2y_2 + y_3 \geq 1$$

The general standard maximum problem and the dual standard minimum problem may be simultaneously exhibited in the display:

	x_1	x_2	$\cdots$	x_n	
y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\leq b_1$
y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\leq b_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\leq b_m$
	$\geq c_1$	$\geq c_2$	$\cdots$	$\geq c_n$	

Thus the example could be expressed as

	x_1	x_2	
y_1	1	2	≤ 4
y_2	4	2	≤ 12
y_3	-1	1	≤ 1
	≥ 1	≥ 1	

Example 2

Find the dual to the following standard minimum problem. Find $y_1, y_2, and\ y_3$ to minimize$y_1 + 2y_2 + y_3$, subject to the constraints $y_i \geq 0$ for all i, and

$$y_1 - 2y_2 + y_3 \geq 2$$
$$-y_1 + y_2 + y_3 \geq 4$$
$$2y_1 + y_3 \geq 6$$

$$y_1 + y_2 + y_3 \geq 2$$

We can easily set up the table:

	x_1	x_2	x_3	x_4	
y_1	1	-1	2	1	1
y_2	-2	1	0	1	2
y_3	1	1	1	1	1
	2	4	6	2	

Thus, the dual for the problem is to maximize

$$2x_{1+}4x_2 + 6x_3 + 2x_4$$

subject to the constraints $x_i \geq 0$ for all i, and

$$x_1 - x_2 + 2x_3 + x_4 \leq 1$$
$$-2x_1 + x_2 + x_4 \leq 2$$
$$x_1 + x_2 + x_3 + x_4 \leq 1$$

By solving the dual of the problem, we can make sure that we have the correct answer.

2.3 Simplex Method

Example 3

Maximize $p = 50x_1 + 80x_2$

Subject to $x_1 \geq 0, x_2 \geq 0$ and

$$x_1 + 2x_2 \leq 32$$
$$3x_1 + 4x_2 \leq 84$$

We can re-write these equations - using the slack variables - as:

$$x_1 + 2x_2 + s = 32$$
$$3x_1 + 4x_2 + r = 84$$
$$p - 50x_1 - 80x_2 = 0$$

Here, the non-basic variables are : $x_1, x_2, and\ p$ and the basic variables are: $s\ and\ r$

Then, we create the simplex tableau.

	x_1	x_2	s	r	p	
s	1	2	1	0	0	32
r	3	4	0	1	0	84
p	-50	-80	0	0	1	0

Here, we pick x_2 as the pivot column, which is farthest from the origin and thus has the highest impact. Then among the two non-basic variable rows, we pick s to be the pivot row because;

When $x_1 = 0$,

$$s = 32 - 2x_2;\ x_2 = 16$$
$$r = 84 - 4x_2;\ x_2 = 21$$

We need to pick a smaller number to remove the risk of crossing over the feasible reason. Picking a bigger number has a chance of getting s and r as negative values when doing simplex method.

	x_1	x_2	s	r	p	
s	1	2	1	0	0	32
r	3	4	0	1	0	84
p	-50	-80	0	0	1	0

Now, we can replace the basic variable, s, with non-basic variable, x_2. Then, we need to have the value of $x_2 = 1$ while the value of every other variable in pivot column to be 0. Thus, we first divide the pivot row by 2.

	x_1	x_2	s	r	p	
x_2	1/2	1	1/2	0	0	16
r	3	4	0	1	0	84
p	-50	-80	0	0	1	0

Then, we do the operation:

(- row) — 4 * (pivot row) and
(p - row) + 80*(pivot row)

	x_1	x_2	s	r	p	
x_2	1/2	1	1/2	0	0	16
r	1	0	-2	1	0	20
p	-10	0	40	0	1	1280

Then, we do the same process again: select the remaining x_1 to be pivot column, and r to be the pivot column. We also replacing basic variable, r, with non- basic variable, x_1.

	x_1	x_2	s	r	p	
x_2	1/2	1	1/2	0	0	16
x_1	1	0	-2	1	0	20
p	-10	0	40	0	1	1280

We then reduce value of x_1 to be 1 and make the remaining basic variables in pivot column to be 0 by doing operations:

2*(x_2-row) – (pivot row) and
(p-row) + (pivot row)

	x_1	x_2	s	r	p	
x_2	0	1	3/2	-1/2	0	6
x_1	1	0	-2	1	0	20
p	0	0	60	10	1	1480

We have completed the simplex method. Thus, this simplex tableau tells us that:

$$x_1 = 20$$
$$x_2 = 6$$

and the solution to the objective function to be 1480. And this is correct because our objective function was $p = 50x_1 + 80x_2$; 50 (20) + 80(6) = 1480.

2.4 Excel Solver to solve the standard maximum/minimum problem

One of the many linear programming program, I looked into the Excel solver – which was most accessible to me.

Example 4

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants: aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products. Due to declining earnings, top management has decided to revamp the company's product line: while some unprofitable products are being discontinued, two new products having large sales potential are being launched.

Product 1: A glass door with aluminum framing
Product 2: Wood-framed window

Product 1 requires some of the production capacity in Plant 1 and 3, while Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which mix of the two products would be most profitable.

The OR team develops this problem. Determine what production rates should be for the two products to maximize the company's total profit, subject to the restriction of limited production capacities of the three plants. (Here, the production refers to the number of batches produced per week) (See reference [2])

The data for this problem is given:

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

Using the datag given, we can express this as a standard maximum linear programming.

Let

x_1 = number of batches of product 1 produced per week
x_2= number of batches of product 2 produced per week
Z = total profit per week (in thousand dollars) from producing these two products

Thus, in this problem, x_1 and x_2 are the decision variable and the objective function is

$$Z = 3x_1 + 5x_2.$$

We can also have equations for constraints. The Plant 1 can only available for 4 hours per week and product 1 needs one hour of capacity of the Plant 1; the Plant 2 has 12 available hours of capacity when the product 2 needs two hours per product; and while the Plant 3 has 18 hours of capacity, product 1 and 2 needs respectively three and two hours each. Thus, the constraints are:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

And

$$x_1 \geq 0, x_2 \geq 0$$

We can formulate this model on a spreadsheet using excel (See reference [3])

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Hours Used per Unit Produced				Hours
4			Doors	Windows	Totals		Available
5		Plant 1	1	0	0	≤	4
6		Plant 2	0	2	0	≤	12
7		Plant 3	3	2	0	≤	18
8		Unit Profit ($thousands)	3	5	0		
9		Solution	0	0			

Where

E5 = C5*E9 + D5*E9
E6 = C6*E9 + D6*E9 =>
E7 = C7*E9 + D7*E9

	E
5	=SUMPRODUCT(C5:D5,C9:D9)
6	=SUMPRODUCT(C6:D6,C9:D9)
7	=SUMPRODUCT(C7:D7,C9:D9)
8	=SUMPRODUCT(C8:D8,C9:D9)

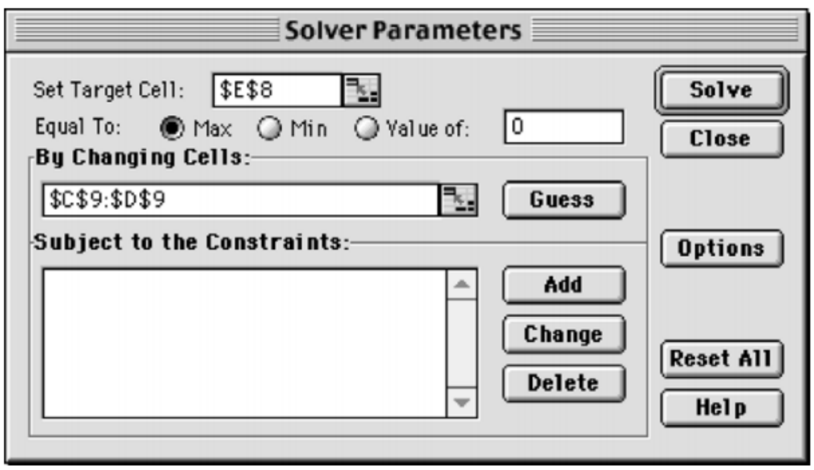
And E8 = C8*C9 + D8*D9 (the objective function). Thus, this spreadsheet is constructed to maximize the value of E8.

EXCEL SOLVER (See reference [3])

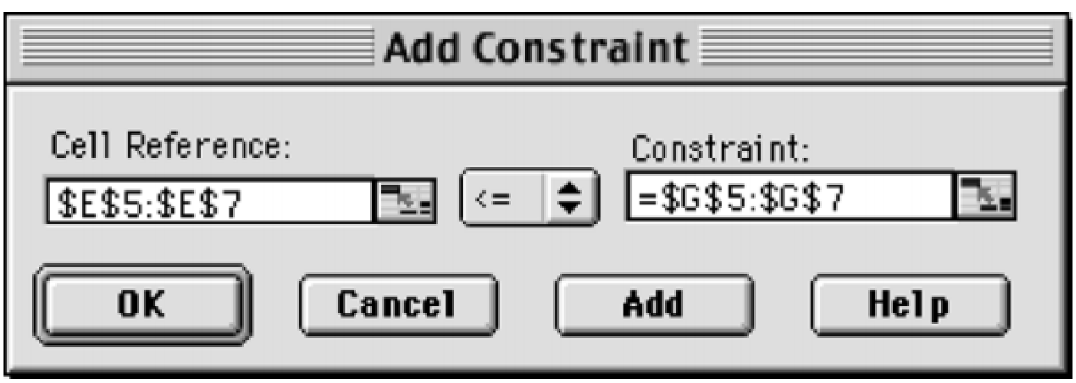
Excel has a powerful tool called Solver that uses the simplex method to find an optimal solution. Before going into solver, we have to make sure we have all the required components:

Decision variable (C9 and D9), the objective function and its value (E8), and each functional constraints (row 5, 6, and 7).

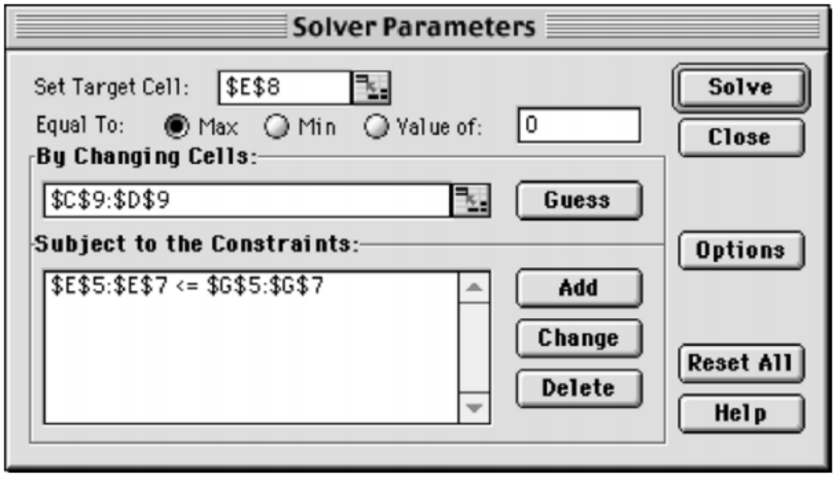
After you press Solver in the Data tab, you will get



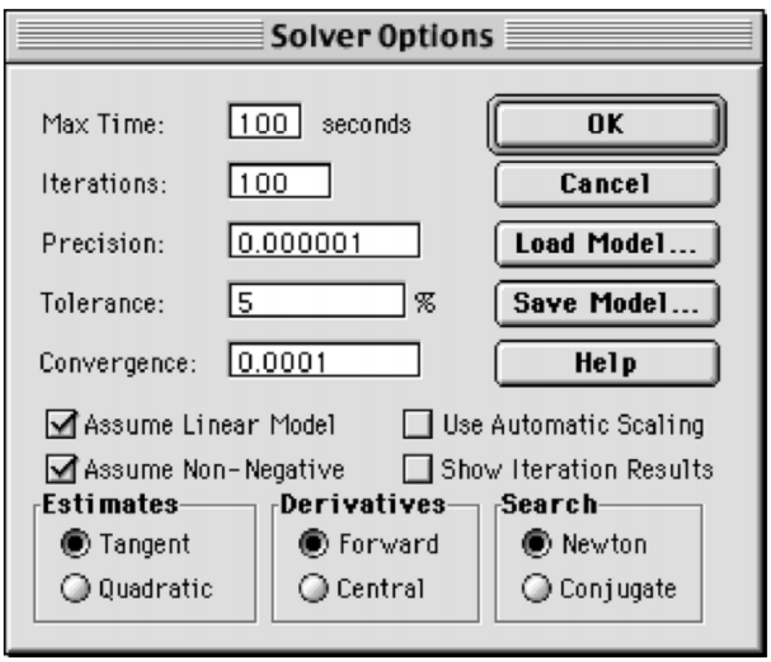
And for the constraints: you click add



And keep adding all the constraints



Then, at last, you select the linear model with non-negative variable.



Then the Solver will provide the solution to the objective function

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Hours Used per Unit Produced				Hours
4			Doors	Windows	Totals		Available
5		Plant 1	1	0	2	≤	4
6		Plant 2	0	2	12	≤	12
7		Plant 3	3	2	18	≤	18
8		Unit Profit ($thousands)	3	5	36		
9		Solution	2	6			

Thus, the solution suggest that to maximize the profit, the Wyndor Glass CO. needs to produce 2 batches of Product 1 and 6 batches of Product 2 in a week, which will lead to the maximum profit of $36,000 (See reference [3]).

Application Vignette 3

Welch's, Inc., is the world's largest processor of Concord and Niagara grapes, with annual sales above $550 million per year. Every September, growers deliver grapes to processing plants that then press the raw grapes into juice. Time must pass before the grape juice is ready for conversion into finished jams, jellies, juices and concentrates. Deciding how to allocate the grape crop is a complex task given changing demand and uncertain crop quality and quantity. Typical decision include what recipes to use for major product groups, the transfer of grape juice between plants, and the mode of transportation for these transfers.

In the beginning, the OR team developed a linear programming model with 8,000 decision variables that focused on the component level of detail. Then, they made the model more useful by aggregating demand by product group rather than by components. Reduced to 324 decision variables and 361 functional constraints, the model then was incorporated into a spreadsheet.

The company had revised the spreadsheet every month since 1994 to provide senior management with information on the optimal logistics plan generated by the Solver. This saved approximately $150,000 in the first year alone. (See reference [2])

3. Dynamic Programming

Application Vignette 4

Six days after Saddam Hussein ordered his Iraqi military forces to invade Kuwait on August 2, 1990, the United States started deploying many of its own military units and cargo to the region for intervention. The military operation called Operation Desert Storm was launched in January 17, 1991 as 35 nations (led by the U.S.) formed a coalition in attempt to expel the Iraqi troops from Kuwait. This led to a decisive victory for the coalition forces, which liberated Kuwait and penetrated Iraq.

The logical challenge was quickly transporting the needed troops and cargo to the war zone. A typical airlift mission carrying troops and cargo from the U.S. to the Persian Gulf required a three-day round-trip, visited seven or more different airfields, burned almost one million pounds of fuel, and cost $280,000. Operation Desert Storm was the largest airlift in history, in which the Military Airlift Command (MAC) commanded more than100 such missions daily.

Operation research – specifically, Dynamic Programming - was applied to meet this challenge in developing the decision support system to schedule and route each airlift mission. The stages in the dynamic programming formulation correspond to the airfields in the network of flight legs relevant to the mission. For a given airfield, the states are characterized by the departure time from the airfield and the remaining available duty for the current crew. The objective function to be minimized is a weighted sum of several measures of performance: the lateness of deliveries, the flushing time of the mission, the ground time, and the number of crew changes. The constraints include a lower bound on the load carried by the mission and upper bounds on the availability of crew and ground-support resources at airfields. This application of dynamic programming, thus, is recognized for having a dramatic impact on the ability to deliver the necessary cargo and personnel to the Persian gulf quickly to support the Operation Desert Storm. (See reference [2])

Characteristics of Dynamic Programming

Dynamic programming is useful method for making a sequence of interrelated decision; systematic procedure for determining the optimal combination of decisions. Dynamic programming does not have a standard mathematical model like the linear programming; it is a general type of problem solving where model has to be developed to fit each situation.

While dynamic programming process is said to be created to fit its own situation, there are some basic features and characteristics:

1) The problem can be divided into stages, with the policy decision required at each stage. It requires making a sequence of interrelated decisions, where each decision corresponds to one stage of the problem.

2) Each stage has number of states associated with the beginning of that stage – the states are the various possible conditions in which the system might be at that stage of the problem

3) The effect of the policy decision at each stage is to transform the current state to a state associated with the beginning of the next stage.

4) The solution procedure is designed to find an optimal policy for the overall problem.

5) An optimal policy for the remaining stages is independent of the policy decisions adopted in previous stages. Thus, the optimal decision depends on the current state and not on how you got there – it is called the principle of optimality for dynamic programming.

6) The solution procedure begins by finding the optimal policy for the last stage.

7) A recursive relationship that identifies the optimal policy for stage n, given the optimal policy for stage n+1, is available.

N = number of stages

n = label for current stage (n= 1,2,…, N)

s_n = current state for stage n

x_n= decision variable for stage n

$x_n^* =$ optimal value of x_n (given s_n)

$f_n(s_n, x_n)$ = contribution of stages n, n+1, …N to objective function if system starts in state s_n at stage n, immediate decision is x_n, and optimal decisions are made thereafter.

$$f_n^*(s_n) = f_n(s_n, x_n^*)$$

The recursive relationship will always be of the form

$$f_n^*(s_n) = \max \{f_n(s_n, x_n^*)\} \qquad \text{or} \qquad f_n^*(s_n) = \min \{f_n(s_n, x_n^*)\}$$

8) When using the recursive relationship, the solution procedure starts at the end and moves backward stage by stage until it finds the optimal policy starting at the initial stage – which yields an optimal solution for the entire problem (See reference [2]).

Example 5

A government space project is conducting research on a certain engineering problem that must be solved before people can fly safely to Mars. Three research teams are currently trying three different approaches for solving this problem. The estimate has been made that, under present circumstances, the probability that the respective teams – call them 1,2 and 3 – will not succeed is 0.40, 0.60, and 0.80, respectively. Thus, the current probability that all three teams will fail is (0.4)(0.6)(0.8) = 0.192. Because the objective is to minimize the probability of failure, two more top scientist have been assigned to the project.

The table is added to give more data about this problem – estimated probability that the respective teams will fail when 0,1, or 2 additional scientists are added to that team.

New Scientists	Probability of Failure		
	Team		
	1	2	3
0	0.40	0.60	0.80
1	0.20	0.40	0.50
2	0.15	0.20	0.30

In this case, stage n (n = 1,2,3) is research team n, and the state $s_n$is the number of new scientist still available for allocation to the remaining teams. The decision variables, x_n are the number of additional scientists sent to team n. We let $p_i(x_i)$ be the probability of failure of team i if it is assigned x_i additional scientists. Thus, the objective function for this problem is to

$$\text{Minimize } \prod_{i=1}^{3} p_i(x_i) = p_1(x_1)\ p_2(x_2)\ p_3(x_3)$$

Subject to

$$\sum_{i=1}^{3} x_i = 2$$

And x_i are nonnegative integers

Consequently, $f_n(s_n, x_n)$ for this problem is

$$f_n(s_n, x_n) = \ p_n(x_n) \cdot \min \prod_{i=n+1}^{3} p_i(x_i),$$

Where the minimum is taken over $x_{n+1,\dots}\ x_3$ such that

$$\sum_{i=n}^{3} x_i = s_n$$

And x_i are nonnegative integers,

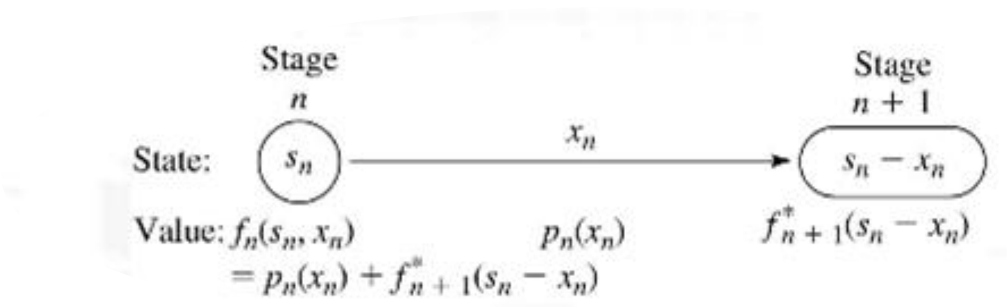
For n = 1,2,3. Thus,

$$f_n^* \ (s_n) = min_{x_n=0,1,,\dots,s_n} f_n(s_n, x_n),$$

Where

$$f_n(s_n, x_n) = p_n(x_n) \cdot f_{n+1}^* \ (s_n - \ x_n)$$

(With f_4^* defined to be 1). The diagram summarizes these relationship



Thus, the recursive relationship relating the $f_1^*, f_2^*, and\ f_3^*$ functions in this case is

$$f_n^* \ (s_n) = min_{x_n=0,1,,\dots,s_n}\{p_n(x_n) \cdot f_{n+1}^* \ (s_n - \ x_n)\} \text{ for n = 1,2,}$$

And when n=3,

$$f_3^* = \ min_{x_n=0,1,,\dots,s_n} p_3(x_3).$$

The resulting dynamic programming calculations are as follows:

n= 3,

s_3	$f_3^* \ (s_3)$	x_3^*
0	.8	0
1	.5	1
2	.3	2

n=2,

x_2 / s_2	$f_2^*(s_2, x_2) = p_2(x_2) \cdot f_3^*(s_2 - x_2)$			$f_2^*(s_2)$	x_2^*
	0	1	2		
0	.48	-	-	.48	0
1	.3	.32	-	.30	0
2	.18	.2	.16	.16	2

x_1 / s_1	$f_2^*(s_1, x_1) = p_1(x_1) \cdot f_2^*(s_1 - x_1)$			$f_1^*(s_1)$	x_1^*
	0	1	2		
2	.064	.06	.072	.060	1

Thus, the optimal solution must have x_1^*=1, which makes s_2= 2-1 =1, so that x_2^* = 0, which makes s_3= 1 – 0 = 1, so that x_3^* = 1. Therefore, teams 1 and 3 should each receive one additional scientist. The new probability that all three teams will fail would then be 0.060 (See reference [2] and [3]).

Reference

[1] Ferguson, Thomas. “LINEAR PROGRAMMING: A Concise Introduction.” *UCLA MATH*. N.p., n.d. Web. 5 May 2014. <http://www.math.ucla.edu/~tom/LP.pdf>.

[2] Lieberman, Gerald J., and Frederick S. Hillier. *Introduction to Operations Research*. New York City: McGraw-Hill Science/Engineering/Math, 2009. Print.

[3] Wang, Jin. “Operation Research II.” *College of Management, NCTU*. N.p., n.d. Web. 5 Mar. 2009. <http://ocw.nctu.edu.tw/upload/classbfs1210015708111683.pdf>.