2016

# Determining Corresponding Artinian Rings to Zero-Divisor Graphs

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April 2016

Submitted in Partial Fulfillment
of the
Prerequisite for Honors
in the Department of Mathematics

Wellesley College

## Acknowledgements

I would like to thank my advisor, Professor Diesl, who helped me throughout the process, guided me on my research, read my drafts, and fueled my enthusiasm for zero-divisor graphs. I would also like to thank my committee members Professor Phyllis McGibbon, Professor Megan Kerr, and Professor Andrew Schultz.

Also, a very special thank you to my friends, who shaped my Wellesley experience and supported me for the past four years, and to my family, for their unwavering love and support.

## Abstract

We introduce Anderson's and Livingston's definition of a zero-divisor graph of a commutative ring. We then redefine their definition to include looped vertices, enabling us to visualize nilpotent elements. With this new definition, we examine the algebraic and graph theoretic properties of different types of Artinian rings, culminating in an algorithm that determines the corresponding Artinian rings to a zero-divisor graph. We also will explore and develop an algorithm for the specific case of Artinian rings of the form $\mathbb{Z}_n$, and we will conclude by examining the uniqueness of zero-divisor graphs.

# Contents

# 1 Introduction

Istvan Beck first introduced the idea of a zero-divisor graph of a commutative ring in 1988 [1], where he mostly focused on colorings. In 1998, Anderson and Livingston further explored the zero-divisor graph of a commutative ring, investigating the interplay between ring-theoretic properties of $R$ and graph theoretic properties of $\Gamma(R)$ [2]. The zero-divisor graph that Anderson and Livingston introduced allows us to visually represent algebraic properties of a commutative, unital ring through graph theoretic properties. This ability to use graph-theoretic properties to visualize underlying algebraic properties is applicable to many different types of rings. Since Anderson's and Livingston's initial paper, variations of zero-divisor graphs of many different types of rings have been studied extensively. DeMeyer, McKenzie, and Scheider defined a zero-divisor graph for a commutative semi-group [3], Akbari created a zero-divisor graph of non-commutative rings [4], Axtell, Coykendall, and Stickles created a zero-divisor graph of polynomials and power series over commutative rings [5], and Redmond created an ideal-based zero-divisor graph [6]. These are just a few of the variations of zero-divisor graphs. Mathematicians are constantly looking at new ways of using zero-divisor graphs to visually represent underlying algebraic properties.

In this paper, we will introduce Anderson's and Livingston's zero-divisor graph. We will then modify their definition in order to visualize nilpotent elements. This will allows us to visualize an element's relationship with itself.

**Definition 1.1. Zero-Divisor Graph of a Commutative Ring** [2]. *Let $R$ be a commutative ring. We associate a simple graph $\Gamma(R)$ to $R$ with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of $R$, and for distinct $x, y \in Z(R)^*$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$.*

Throughout this paper, all rings are assumed to be commutative with unity unless explicitly stated otherwise. A ring $R$ is said to be *reduced* if $R$ has no non-zero nilpotent elements. A ring $R$ is said to be *decomposable* if it can be written as a direct product $R_1 \times R_2$, where $R_1$ and $R_2$ are nonzero rings, otherwise $R$ is said to be *indecomposable*. A ring is said to be a $field$, denoted $F$, if it is a commutative ring with unity in which every nonzero element is a unit. An *ideal* of a ring is a subring $I$ of a ring R such that for every $r \in R$ and $a \in I$ both $ar$ and $ra$ are in $I$. A *maximal ideal* of a ring $R$ is an ideal $I$, not equal to $R$, such that if $J$ is an ideal which contains $I$ as a subset, then either $J = I$ or $J = R$. An $annihilator$, denoted $Ann_R(S)$, of $S$ is the set of all elements $r \in R$ such that for each $a \in S$, $ra = 0$. The *nilradical* of a ring $R$ is the set of all nilpotent elements. The *Jacobson radical* of $R$ is the intersection of the maximal ideals in $R$.

A graph is a *complete graph* with $n$ vertices if each pair of graph vertices is connected by an edge, and we denote it by $K_n$. The *degree* of a vertex $x$, denoted as $\delta(x)$, is the number of edges incident to $x$. This definition does not include the edge when a vertex is adjacent to itself. The *distance* between two vertices $x$ and $y$ is the length of the shortest path between them, declaring the length of each edge to be 1, and is denoted by $d(a, b)$. The *diameter* of a graph $G$ is $sup\{d(x, y) : x$ and $y$ are distinct vertices of $G\}$. A *walk* of a graph $G$ is an alternating sequence of points and lines $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$, beginning and ending with points, in which each line is incident with the two points immediately preceding and following it. If the walk is closed, then it is a *cycle* provided its $n$ points are distinct and $n \geq 3$. The *girth* of a graph $G$, denoted $g(G)$, is defined as the length of the shortest cycle. A graph is a *bipartite graph* if its graph vertices can be decomposed into two disjoint sets, with $m$ and $n$ elements, respectively, such that no two graph vertices within the same set are adjacent. A *complete bipartite graph* is a bipartite graph where every vertex of the first set is connected to every vertex of the second set, and is denoted $K_{m,n}$. The complete bipartite graph $K_{1,n}$, is called a *star*. A graph $G$ is said to be *star-shaped reducible* if and only if there exists a $g \in V(G)$ such that $g$ is adjacent to all other vertices in $G$ and $g^2 = 0$. This means that the vertex $g$ in the graph $G$ is looped once.

In this paper, we will first establish a few realizability properties of zero-divisor graphs. We then will classify zero-divisor graphs of Artinian rings. In approaching this problem, we will first introduce the basic "building blocks" of Artinian rings, which are local rings and fields. We will then take direct products of these rings to build more complex Artinian rings, and we will observe how the structure of the zero-divisor graph reflects these complexities. We will introduce algorithms to identify the different types of Artinian rings corresponding to different zero-divisor graphs. We will also generate a specific algorithm classifying Artinian rings of the form $\mathbb{Z}_n$ to different zero-divisor graphs. To conclude, we will develop an overarching algorithm that combines the algorithms we will develop throughout the paper in order to correspond specific Artinian rings to a zero-divisor graph.

In this algorithm, we first observe the vertices of our zero-divisor graph. If all of the vertices are looped, then our corresponding Artinian ring is a local ring (that is not a field); if all of the vertices are not looped, then our corresponding ring is a direct product of fields; if there exists looped and unlooped vertices, but all of the vertices are either looped or adjacent to a looped vertex, then the corresponding ring is a direct product of local rings (that are not fields); and if there exists looped and unlooped vertices and at least one unlooped vertex is not adjacent to any looped vertex, then the corresponding ring is a direct product of local rings and fields. After observing the vertices and determining the type of ring associated to the zero-divisor graph, we follow the specific algorithm for each type of ring to determine a narrowed set of possible associated rings to the zero-divisor graph. Following this algorithm, we will observe the uniqueness of our zero-divisor graphs.

This final algorithm that we will develop allows us to visually observe any given zero-divisor graph of an Artinian ring and determine a small set of possible associated rings to the zero-divisor graph.

# 2 Looped Zero-Divisor Graphs

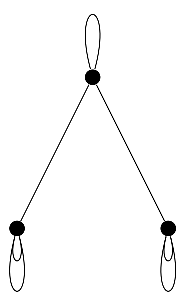
Anderson's and Livingston's zero-divisor graph $\Gamma(R)$ allows us to visualize the zero-divisors of a ring as well as the relationships between the zero-divisors. However, this definition does not show us relationships of zero-divisors with themselves. The zero-divisor graphs of many rings are distinguished by nilpotent elements. Therefore, we will modify Anderson's and Livingston's zero-divisor graph definition so that we can visualize nilpotent elements through looped vertices. We will do this by defining loops in relation to the degree of nilpotence of an element. Before we do this, however, we want to also distinguish between the different types of degrees that an element, $x$, will have in the zero-divisor graph. We defined $\delta(x)$ in the introduction as the number of vertices incident to $x$. We also want to define the degree, or number, of loops a nilpotent element will have in the zero-divisor graph.

**Definition 2.1. Degree of Loops.** *If a vertex, $x$ is a nilpotent element such that $x^n = 0$ and $x^{n-1} \neq 0$, then we draw $n-1$ loops and define $\zeta(x) = n-1$, where $\zeta$ is the number of times that the vertex $x$ loops around itself.*

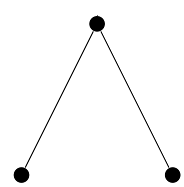
With this new definition of $\zeta(x)$, we can extend Anderson's and Livingston's definition of the zero-divisor graph to include loops.

**Definition 2.2. Zero-Divisor Graph of a Commutative Ring with Loops**. *Let us associate a simple graph $\Gamma^*(R)$ to $R$ with vertices $Z(R)^* = Z(R) - \{0\}$, for distinct $x, y \in Z(R)^*$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$ and a vertex $x$ is adjacent to itself, with $\zeta(x) = n-1$, if $x^n = 0$ and $x^{n-1} \neq 0$ $\forall n \in \mathbb{Z}$ with $n \geq 0$.*

**Example 2.1.** By Anderson's and Livingston's zero-divisor graph definition, $\Gamma(\mathbb{Z}_8)$ and $\Gamma(\mathbb{Z}_6)$ look the same. However, when we use this new definition that includes looped vertices, we can see that $\Gamma^*(\mathbb{Z}_6)$ and $\Gamma^*(\mathbb{Z}_8)$ are actually distinguished by nilpotent elements. Furthermore, we can see that $\Gamma^*(\mathbb{Z}_8)$ contains vertices, $x, y$, and $z$ with $\zeta(x) = 1$, and $\zeta(y) = \zeta(z) = 2$. Whereas in $\Gamma^*(\mathbb{Z}_6)$ the vertices $x, y$, and $z$ have $\zeta(x) = \zeta(y) = \zeta(z) = 0$.



$\Gamma^*(\mathbb{Z}_8)$



$\Gamma^*(\mathbb{Z}_6)$, $\Gamma(\mathbb{Z}_8)$, $\Gamma(\mathbb{Z}_6)$

There are some fundamental properties that Anderson and Livingston established in their definition of a zero-divisor graph that continue to hold in this new definition of a zero-divisor graph that contains loops. This next theorem establishes restrictions on the diameter and girth of zero divisor graphs:

**Theorem 2.1.** ([2], Theorem 2.3.) *Let $R$ be a commutative ring. Then $\Gamma^*(R)$ is connected and diam$(\Gamma^*(R)) \leq 3$. Moreover, if $\Gamma^*(R)$ contains a cycle, then $g(\Gamma^*(R)) \leq 7$.*

Axtell, Stickles, and Trampbachls established an important property about local rings: ([7], Theorem 2.3.)

**Theorem 2.2.** *Let $R$ be a finite commutative ring with identity. Then the following are equivalent:*

*(1) $Z(R)$ is an ideal;*
*(2) $Z(R)$ is a maximal ideal;*
*(3) $R$ is local;*
*(4) Every $x \in Z(R)$ is nilpotent;*
*(5) There exists $b \in Z(R)$ such that $bZ(R) = 0$, and hence $\Gamma(R)$ is star-shaped reducible.*

This leads to the following facts about local rings:

**Fact 2.1.** *If $R$ is a local ring, then $Z(R)$ =Nil(R).*
*Proof.* If $R$ is local, then every $x \in Z(R)$ is nilpotent from Theorem 2.2. The only possible nilpotent elements of a ring are zero-divisors. Therefore, we have no other nilpotent elements, so $Z(R) = Nil(R)$. □

Therefore, we can see that if we have a local ring, then every zero-divisor will be nilpotent. So, if we have a zero-divisor graph where all the vertices are looped, then we can assume that $R$ is a local ring (that is not a field). The following fact establishes that a local ring will have a unique maximal ideal, $M = Z(R)$.

**Fact 2.2.** *If $R$ is a finite local ring, then $Z(R)$ is the unique maximal ideal.*
*Proof.* From Theorem 2.2. we know that if $R$ is a finite local ring then $Z(R)$ forms a maximal ideal. To prove that is must be unique, suppose that we have another ideal, $I$ of $R$. Then either $I \subsetneq Z(R)$, in which case it is not maximal or $I$ contains a unit, and so it is not proper. Therefore, if $R$ is local, then it has one maximal ideal. □

We also want to focus on when certain structures of graphs are realizable. Anderson and Livingston established the following corollary and theorem about complete zero-divisor graphs.

**Corollary 2.1.** ([2], Corollary 2.7.) *Let $R$ be a finite commutative ring. Then there is a vertex of $\Gamma^*(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times F$, where $F$ is a finite field, or $R$ is local.*

**Theorem 2.3.** ([2], Theorem 2.8.) *Let $R$ be a commutative ring. Then $\Gamma^*(R)$ is complete if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

When $\Gamma^*(R)$ is a complete graph, the following theorem shows that all vertices in $\Gamma^*(R)$ will be looped once.

**Theorem 2.4.** *Let $R$ be a finite commutative ring. if $\Gamma^*(R)$ is a complete graph of the form $K_n$ where $n \geq 2$ then every vertex $x \in \Gamma^*(R)$ is looped with $\zeta(x) = 1$. There is an exception when $n = 2$, and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, none of the vertices will be looped.*
*Proof.* Suppose that $R$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. From Corollary 2.1. and Theorem 2.3.,

$\Gamma^*(R)$ is complete if and only if $R$ is local with maximal ideal $M$ and $M^2 = 0$. Since $R$ is a local ring, by Theorem 2.2., $M = Z(R)$. Since $M^2 = 0$, then $x^2 = 0\ \forall x \in M$. Therefore, $x^2 = 0\ \forall x \in Z(R)$. Since $x^2 = 0$, then $\zeta(x) = 1\ \forall x \in Z(R)$. Therefore, every vertex $x \in \Gamma^*(R)$ is looped with $\zeta(x) = 1$.
Now suppose that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $Z(R)^* = \{(0,1),(1,0)\}$. Clearly, neither vertex is looped, since $(0,1)^2 = (0,1)$ and $(1,0)^2 = (1,0)$. Since the $\Gamma(R)^*$ only has 2 adjacent vertices, it is clearly complete. Therefore, when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\Gamma(R)^*$ is complete without loops, and therefore is an exception to Theorem 2.4. □

With our modified definition of a zero-divisor graph, we can establish the following theorem about vertices with $\delta(x) = 1$:

**Theorem 2.5.** *Let $R$ be a finite commutative ring, and let $|\Gamma^*(R)| > 2$. For any end vertex $x$ with $\delta(x) = 1$, $x^2 \neq 0$.*
*Proof.* Suppose that $x$ is an end vertex, and suppose, for a contradiction, that $x^2 = 0$. Since $x$ is an end vertex with $\delta(x) = 1$, then $x$ must be adjacent to another vertex, $y \in Z(R)^*$. Therefore, we have that $xy = 0$ and $x^2 = 0$, thus $x(x + y) = 0$. Since $x \in Z(R)^*$, then $x + y \in Z(R)$. We have 4 possible cases:
(1) Suppose that $x+y = 0$. Then $-x = y$. Since $|\Gamma^*(R)| > 2$ and $\Gamma^*(R)$ is a connected graph, we know that $yz = 0$, where $z \in Z(R)^*/\{x, y\}$. Thus $z(-x) = 0$. This is a contradiction since $\delta(x) = 1$. (2) Suppose that $x + y = y$. Then $x = 0$, a contradiction since $x \in Z(R)^*$. (3) Suppose that $x + y = x$. Similarly to case (2), this is a contradiction. (4) Suppose that $x + y = z$. Then $xz = 0$, a contradiction. Therefore, we have shown that all possible cases are contradictions. Therefore, it must be the case that $x^2 \neq 0$. □

The next theorem states when a complete bipartite graph is realizable as a zero-divisor graph.

**Theorem 2.6.** *Let $R$ be a finite commutative ring and let $\Gamma^*(R)$ be a complete bipartite graph, $K_{m,n}$, where $m, n > 1$. $\Gamma^*(R)$ must contain no looped vertices.*
*Proof.* Suppose that $\Gamma^*(R)$ is a complete bipartite graph, $K_{m,n}$, where $m, n > 1$. We can split the vertices into two distinct sets, $X$ and $Y$.
Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$. By definition, $\forall x_i, x_k \in X$ and $y_j, y_l \in Y$, we have $x_i y_j = 0$, $x_i x_k \neq 0$, and $y_j y_l \neq 0$.
Suppose, for a contradiction, that the $\forall x_k \in X$ and $\forall y_j \in Y$, $x_k^n = 0$ and $y_j^n = 0$ for some $n \in \mathbb{Z}$ and $n > 0$. Therefore, there is vertex $x_i$ has $x_i^2 = 0$, therefore $\zeta(x_i) = 1$. Then $x_i(x_i + y_1) = 0$. Since $x_i \in Z(R)^*$, then $x_i + y_1 \in Z(R)^*$. We have 5 possible cases:
(1) Suppose that $x_i + y_1 = 0$. Then $y_1 = -x_i$. So, $(-x_i)x_k = 0$. This is a contradiction. (2) Suppose that $x_i + y_1 = x_i$. Then $y_1 = 0$, a contradiction since $y_1 \in Z(R)^*$ (3) Suppose that $x_i + y_1 = y_1$. Similarly, to case (2), this is a contradiction. (4) Suppose that $x_i + y_1 = x_k$. Then $x_i x_k = 0$, a contradiction. (5) Suppose that $x_i + y_1 = y_l$. Then $y_1 = y_l - x_i$. So $x_k y_1 = x_k(y_l - x_i) = x_k y_l - x_k x_i$. Since $\Gamma^*(R)$ is complete bipartite, $x_k y_l = 0$, therefore $0 - x_k x_i = 0$. Then $x_k x_i = 0$, which is a contradiction.
Therefore, we have shown that all possible cases are contradictions. Therefore, it must be the case that $\Gamma^*(R)$ contains no looped vertices. □

In Section 8 of this paper, we will prove restrictions on the value of $n$ and $m$ in a zero-divisor graph $\Gamma^*(R)$ of the form $K_{n,m}$.

Now that we have established some fundamental properties of zero-divisor graphs, let us now observe zero-divisor graphs of Artinian rings.

# 3 The Algebraic Structure of Artinian Rings

We first want to establish the algebraic structure of Artinian rings, which are rings that satisfy the descending chain condition [8]. In this section, we will establish the algebraic structure of Artinian rings. This algebraic structure will be reflected in the zero-divisor graph.

The following theorem states how an Artinian ring can be decomposed:

**Theorem 3.1. Structure Theorem for Artinian Rings** (Theorem 8.7, [8])
*An Artinian ring $R$ is uniquely (up to isomorphism) a finite direct product of Artinian local rings.*

This theorem allows us to understand the basic "building blocks" of Artinian rings, which we will distinguish as fields and as local rings (that are not fields). In this paper, we will explore the zero-divisor graphs of both types of local rings. We will then use direct products to build more complicated Artinian rings, and we will examine how the zero-divisor graph reflects the more complicated underlying algebraic structure.

We will next establish certain properties about nilradicals of Artinian rings and the ideals of direct products of rings. The following corollary states a property about the nilradical of an Artinian ring (Atiyah and Macdonald Corollary 8.2. and Proposition 8.4., [8]):

**Corollary 3.1.** *In an Artinian ring $R$, the nilradical $Nil(R)$ is equal to the Jacobson Radical and is nilpotent.*

The following theorem allow us to understand how ideals are formed in direct products of rings.

**Theorem 3.2.** *If $I_i$ is an ideal of $R_i$ for $i = 1, \ldots, n$, then every ideal of $R = \prod_{i=1}^{n} R_i$ is of the form $I_1 \times \cdots \times I_n$.*

*Proof.* We can prove this via induction of $n$. The case of $n = 1$ is trivial. It suffices to prove the assertion for $n = 2$. Let $(x_1, x_2) \in I_1 \times I_2$ and $(r_1, r_2) \in R_1 \times R_2$. Then $(x_1, x_2)(r_1, r_2) = (x_1 r_1, x_2 r_2) \in I_1 \times I_2$ and $(r_1, r_2)(x_1, x_2) = (r_1 x_1, r_2 x_2) \in I_1 \times I_2$. So, $I_1 \times I_2$ is an ideal of $R_1 \times R_2$. Now let $K \subseteq R_1 \times R_2$. Let $I_1 = \{x_1 \in R_1 | (x_1, x_2) \in K$ for some $x_2 \in R_2\}$ and $I_2 = \{x_2 \in R_2 | (x_1, x_2) \in K$ for some $x_1 \in R_1\}$. Clearly, $K \subseteq I_1 \times I_2$. Now let $(x_1, x_2) \in I_1 \times I_2$. Then $(x_1, x_2'), (x_1', x_2) \in K$ for some $x_1'$ and $x_2'$. Then, $(x_1, x_2) = (1, 0)(x_1, x_2') + (0, 1)(x_1', x_2) \in K$. Therefore, $K = I_1 \times I_2$. $\square$

Furthermore, we can establish a theorem about the maximal ideals of a direct product of rings.

**Theorem 3.3.** *If $I = I_1 \times \cdots \times I_n$ is an ideal of $R = R_1 \times \cdots \times R_n$, then the maximal ideals of $R$ have all $I_i' s$ equal to $R_i$ for $i = 1, \ldots, n$, except one $I_j$, with $I_j \neq R_j$ and $I_j$ is maximal.*

*Proof.* From Theorem 3.2., we know that if $I_i$ is an ideal of $R_i$ for $i = 1, \ldots, n$, then $I_1 \times \cdots \times I_n$ is an ideal of $R_1 \times \cdots \times R_n$. Now suppose that $I$ is such that more than one of the $I_i$'s is different from $R_i$. Then we can replace one of these $I_i' s$ with $R_i$ and get an ideal properly containing $I$. Therefore, a maximal ideal has all $I_i' s$ equal to $R_i$ except for one, $I_i$. It is clear that the one $I_i$ with $I_i \neq R_i$ is also maximal. □

Primary Decompositions play an important role in Artinian rings because, as the next propositions and theorems point out, every ideal of an Artinian ring has a primary decomposition.

**Definition 3.1. Primary Ideals** *An ideal $I$ in a ring $R$ is primary if $I \neq R$ and if $xy \in I$, then either $x \in I$ or $y^n \in I$ for some $n > 0$.*

**Definition 3.2. Primary Decompositions** *Let $I$ be an ideal of a ring $R$. A primary decomposition of $I$ is a finite collection $\{Q_1, Q_2, \ldots, Q_m\}$ of primary ideals such that $I = Q_1 \cap Q_2 \cap \ldots \cap Q_n$.*

**Proposition 3.2. Existence of Primary Decomposition** *In a Noetherian ring, every ideal has a primary decomposition.*

**Theorem 3.4.** (Atiyah and Macdonald Theorem 8.5., [8]). *A ring $A$ is Aritnian if and only if $A$ is Noetherian and $dim A = 0$.*

Therefore, in an Artinian ring, every ideal has a primary decomposition. The following theorem allows us to distinguish between primary and non-primary ideals in the ideal lattice of a ring, $R$.

**Theorem 3.5.** *Suppose that $I$ is an ideal of a ring $R$ and suppose that $I$ is not principal. Therefore, $I$ can be decomposed into two or more principal ideals.*

*Proof.* Since $I$ is not principal, then by Proposition 3.2. it has a primary decomposition consisting of two or more primary ideals. So $I = (x_1, \ldots, x_n) = \{a_1x_1 + \ldots + a_nx_n | a_1, \ldots, a_n \in R\}$. Therefore, $(x_1) \neq (x_n)$. This is because if $(x_1) = (x_n)$, then $x_1$ generates $x_n$ and all the elements in $(x_n)$, so we would not have both $x_1$ and $x_n$ written in $I = (x_1, \ldots, x_n)$. Furthermore, we can see that $(x_1) = \{ax_1 | a \in R\}$ and $(x_n) = \{ax_n | a \in R\}$, so $(x_1) \subset I$ and $(x_n) \subset I$. Therefore, we can see that the lattice structure will break into at least two ideals after $I$. □

Now that we have established some properties about the algebraic structure of Artinian rings and their ideals, we will start visualizing how these are reflected in the zero-divisor graph for local Artinian rings and fields.

# 4 Zero-Divisor Graphs of Fields and Other Local Rings

Fields and other local rings will be our basic building blocks in Artinian rings. This is because all Artinian rings can be decomposed into a direct product of local Artinian rings (as stated in the Structure Theorem for Artinian rings). First, let us establish that fields have no zero-divisor graphs.

**Theorem 4.1.** *The field $F$ has no zero-divisor graph.*
*Proof.* By definition, a field $F$ is a ring in which $1 \neq 0$ and every non-zero element is a unit. Therefore, since every non-zero element is a unit, there can be no zero-divisors. Therefore, $\Gamma^*(F)$ will be empty. □

The following theorem allows us to identify the ideals of a field. It is also important to note that the maximal ideal of a field $F$ is $(0)$.

**Theorem 4.2.** *A ring $R$ is a field if and only if $(0)$ and $(1)$ are its ideals.*
*Proof.* $\Rightarrow$ Suppose that the ring $R$ is a field, and let $I$ be an ideal of $R$. If $I$ only contains 0, then $I = (0)$. If $I$ does not only contain 0, then it contains at least one non-zero element $x$ of $R$, so $x \in I$. Since $R$ is a field, then by definition every non zero element has a multiplicative inverse, so $x^{-1} \in R$. By the definition of an ideal $xx^{-1} = 1$ must be in $I$. Therefore, $1 \in I$, so $I = R$. Hence $I = (1)$. Therefore, if a ring $R$ is a field, then $(0)$ and $(1)$ are its ideals.
$\Leftarrow$ Suppose that we have two ideals of a ring $R$, $(0)$ and $(1)$. It suffices to show that every non-zero element, $x \in R$ contains a multiplicative inverse. So let $x \in r$ and consider $(x)$. By assumption, since the only two ideals of $R$ are $(0)$ and $(1)$, then this means that $(x)$ must be $(0)$ or $(1)$. Since we assumed that $x$ is nonzero, we find that $(x) \neq (0)$. So $(x) = (1)$. This implies that 1 is a multiple of $x$, so there exists another nonzero element $y$ such that $xy = 1$. This is the definition of $x$ having a multiplicative inverse. Therefore, $R$ is a field. So we have shown that if $(0)$ and $(1)$ are the only two ideals of $R$, then $R$ is a field. □

Now we can observe the ideal lattice of local Artinian rings, and the relation between the structure of the zero-divisor graph and the ideal lattice. We will start by introducing the following proposition proved by Atiyah and Macdonald (Proposition 8.8, [8])

**Proposition 4.1.** *Let $R$ be an Artinian local ring and $k$ be the residue field $R/M$. Then the following are equivalent:*

*(1) Every ideal in $R$ is principal;*
*(2) The maximal ideal of $M$ is principal;*
*(3) $dim_k(M/M^2) \leq 1$.*

This leads us to the following result for local Artinian rings (that are not fields).

**Proposition 4.2.** *Suppose that $R$ is a local Artinian ring and that $M$ is the unique maximal ideal of $R$ and $M$ is also principal. Then every nonzero ideal of $R$ is a power of $M$.*
*Proof.* Since $R$ is a local Artinian ring, by Corollary 3.1., the nilradical equals the Jacobson radical, so $M$ is nilpotent. Therefore, given a proper non-zero ideal $I$ of $R$, since $R$ is local

and $I \subseteq M$, then there is some $r \geq 1$ such that $I \subseteq M^r$, but $I \nsubseteq M^{r+1}$.
Next, we will prove that $I = M^r$. Let us choose $y \in I$ such that $y \notin M^{r+1}$. We know that $M$ is principal, so suppose that $M = (x)$ for some $x \in R$. Then, for some $a \in I$, we have that $y = ax^r$.
But since $I \nsubseteq (x^{r+1})$, then we have that $a \notin (x)$. Since $R$ is a local ring, then $(x) = Z(R)$. Therefore, if $a \notin (x)$, then $a$ is a unit in $R$. Therefore $y = ax^r$ which implies that $x^r = a^{-1}y \in I$ since $y \in I$ and $a^{-1} \in R$. So $M^r \subseteq I$. □

**Proposition 4.3.** *Suppose that $R$ is a local Artinian ring (not a field) and that $M$ is a unique maximal ideal of $R$ and $M$ is also principal. Then $|M| = p^{m(n-1)}$, therefore $|\Gamma^*(R)| = p^{m(n-1)} - 1$, where $p^m$ is the order of the residue field $k$ and $n$ is the length of the ideal chain.*
*Proof.* Suppose that $R$ is a local ring with maximal principal ideal $M$, and let $k = R/M$ be its residue field. Since $R$ is Artinian, then $M$ is finitely generated, so the images in $M/M^2$ of a set of generators of $M$ will span $M/M^2$ as a vector space, and therefore $\dim_k(M/M^2)$ is finite, where $\dim_k$ is the dimension of $M/M^2$ as a vector space over $k$. From Proposition 4.1., we know that $\dim_k(M/M^2) \leq 1$. So we have two cases. If $\dim_k(M/M^2) = 0$, then $M = M^2$, implying that $M = 0$, and therefore $R$ is a field, which which assumed was not the case. Therefore $\dim_k(M/M^2) = 1$. We also know by ([9]) that a finite field of order $q$ exists if and only if the order of $q$ is a prime power, $p^m$ (where order is the number of elements). Therefore, $|R/M| = p^m$, since it is a finite field, and since $M/M^2$ is a 1-dimensional vector space, then $|M/M^2| = p^m$ as well. In fact $|M^i/M^{i+1}| = p^m$ for all $1 \leq i \leq n$, where $n$ is the length of the chain of ideals. This is because $R/M = \{r + M | r \in R\}$ and $M^i/M^{i+1} = \{m^i + M^{i+1} | m^i \in M^i\}$. Therefore, the $R/M$ scalar multiplication on $M^i/M^{i+1}$ is $(r + M)(m^i + M^{i+1}) = rm^i$, which implies that $M^i/M^{i+1}$ has the structure of a $k-$vector space. Therefore, $\dim_k(M^i/M^{i+1}) \leq 1$. If $\dim_k(M^i/M^{i+1}) = 0$, this implies that $M^i = M^{i+1} = (0)$. This implies that $M^i$ is the last ideal in our chain, so in this case $i = n$. If $i \neq n$, then $\dim_k(M^i/M^{i+1}) = 1$. Since $|R/M| = p^m$, then $|M^i/M^{i+1}| = p^m$. So we have $|M^{n-1}/(0)| = p^m$, implying that $|M^{n-1}| = p^m$. Since $|M/M^2| = p^m$, then $|M| = p^m|M^2|$. Since $|M^2/M^3| = p^m$, then $|M^2| = p^m|M^3|$. This implies that $|M| = p^m p^m |M^3|$. This continues until we get to $|M^{n-2}/M^{n-1}| = p^m$, implying that $|M^{n-2}| = p^m|M^{n-1}| = p^m p^m$. Therefore, $|M| = p^{m(n-1)}$, and so $|\Gamma^*(R)| = p^{m(n-1)} - 1$. □

Next, we can observe the properties of $\delta(x)$ and $\zeta(x)$ of a vertex $x \in \Gamma^*(R)$ where $R$ is a local Artinian ring with a principal maximal ideal.

**Proposition 4.4.** *Suppose that $R$ is a local Artinian ring and that $M$ is a unique principal maximal ideal of $R$. If a vertex, $x$ has the greatest value of $\zeta(x)$, then $\delta(x) = p^m - 1$.*
*Proof.* Suppose that $R$ is a local Artinian ring and that $M$ is a unique maximal ideal of $R$ and $M$ is also principal. Let $M = (a)$, and suppose that the index of $M$ is $m$, so $a^m = 0$ for all $m \in \mathbb{Z}$ where $m > 1$. Let $x \in M$ and suppose that $x$ has the highest value of loop, $\zeta(x)$=$n - 1$, where $n$ is the length of the ideal chain. Let $x^m = 0$ and $x^s \neq 0$, $\forall s \in \mathbb{Z}$ such that $s < m$. So only the zero-divisors in the last ideal in the ideal chain, $M^{n-1}$ are adjacent to $x$. As we saw in the proof of Proposition 4.3., $|M^{n-1}| = p^m$. However this includes 0, so in the zero-divisor graph, we will see all vertices of the highest $\zeta(x)$ adjacent to $p^m - 1$

vertices. Therefore, $\delta(x) = p^m - 1$. $\square$

In the proof of this proposition, we showed that for all vertices in $\Gamma^*(R)$ such that $x$ has the greatest $\zeta(x) = n-1$, only the zero-divisors in the last ideal chain $M^{n-1}$ will be adjacent to $x$. This implies that in our zero-divisor graph of a local Artinian ring with a principal maximal ideal, our vertices with the greatest $\zeta(x)$ must be adjacent only to the vertices in our minimal ideal, which we can distinguish as the vertices adjacent to every vertex in $\Gamma^*(R)$.

Below are examples of the different ideal lattices of local Artinian rings with a principal maximal ideal.

**Example 4.1.** *Ideal Lattice of a Local Artinian Ring with a Principal Maximal Ideal.* On the far left is the general structure, and then to the right of this there are specific examples, such as $\mathbb{F}_8/(x^2)$, $\mathbb{Z}_{81}$, and $\mathbb{Z}_2[x]/(x^3)$.

$$\begin{array}{c} \mathrm{R} \\ | \\ \mathrm{M} \\ | \\ M^2 \\ | \\ \cdots \\ | \\ M^m \\ | \\ (0) \end{array} \qquad \begin{array}{c} \mathbb{F}_8/(x^2) \\ | \\ (x) \\ | \\ (0) \end{array} \qquad \begin{array}{c} \mathbb{Z}_{81} \\ | \\ (3) \\ | \\ (9) \\ | \\ (27) \\ | \\ (0) \end{array} \qquad \begin{array}{c} \mathbb{Z}_2[x]/(x^3) \\ | \\ (x) \\ | \\ (x^2) \\ | \\ (0) \end{array}$$

We can generate the following algorithm that will allow us to identify a zero-divisor graph correlating to a local Artinian ring with a single ideal chain, and then narrow down a set of possible associated ring simply by observing the zero-divisor graph:

**Algorithm 4.1.** *Determining a Local Artinian Ring with a Principal Maximal Ideal*

**1)** Determine whether $Nil(R) = Z(R)$, by observing whether all of the vertices in $\Gamma^*(R)$ are looped. If so, by Theorem 2.2., we know we have a local ring.
**2)** For any $x \in \Gamma^*(R)$, determine the greatest value of $\zeta(x)$ in $\Gamma^*(R)$. This value will be equal to $n-1$, where $n$ is the length of the ideal chain.
**3)** Now count the $\delta(x)$ of the vertices with the greatest $\zeta(x)$. From Proposition 4.4., we know that $\delta(x) = p^m - 1$.
**4)** Check to see if $|M| = p^{m(n-1)}$. If this is the case, we may have an ideal lattice that consists of a principal maximal ideal, and so we proceed to Step 5, and if not we have a more complicated ideal lattice.
**5)** We now observe the vertices with the greatest $\zeta(x)$. They must be adjacent to only the vertices in the minimal ideal, all of which must be adjacent to every vertex in $\Gamma^*(R)$. If this is not the case, then we do not have a principal maximal ideal. If it is, proceed to step 6.

**6)** We can determine that $|R| = p^{mn}$, and we can determine that the ideal lattice is a single chain.

From this algorithm, we can determine the order of $R$, as well as its ideal lattice, so we can link the zero-divisor graph to a narrowed set of possible rings. We will exclude complete graphs from this algorithm, because as we will see in Section 7, complete graphs can correspond to ideal lattices that have a principal maximal ideal, as well as ideal lattices that have a maximal ideal that is not principal.

In an ideal lattice that forms a chain of length 2, $\Gamma^*(R)$ will be complete since all the elements in the minimal ideal must be adjacent to all vertices in $\Gamma^*(R)$, and since the length of the chain is 2 (and includes (0)), the minimal ideal is also the maximal ideal. We will prove these statements in Theorem 4.3. and Corollary 4.1.. We can also see that a ring with a longer chain will maintain this core complete graph structure, corresponding to the minimal ideal, and there will be numerous extensions off this core structure. This is because the vertices in the other ideals in the ideal chain won't necessarily be adjacent to one another. However, every vertex in $\Gamma^*(R)$ must be adjacent to the minimal ideal.

We can also find the degree of the loop which corresponds to where the ideal is in the ideal chain. So in the ideal, $M^j$, we expect all vertices $x \in M^j/M^{j+1}$ to have all the same degree of loops, where $M^{j+1} \subset M^j$. This is because if $M^j = (p^r)$, then $(M^j)^s = (p^r)^s = (0)$ where $1 \leq r \leq m$, where $m$ is the index of $M$ (the maximal ideal). Therefore, if $a \in (p^r)$, then $a^s = (cp^r)^s = c^s p^{rs}$ for any $c \in R$. This implies that $a^s = 0$ for any $s \in \mathbb{Z}$ such that $s \geq m/r$, and so the nilpotency of $a$ is the least $s \in \mathbb{Z}$ such that $s \geq m/r$.

The next theorems prove properties about the zero-divisor graphs of local Artinian rings with a principal maximal ideal.

**Theorem 4.3** *Suppose that $R$ is a local Artinian ring. If $M$ is a principal maximal ideal, then the minimal ideal will be adjacent to every vertex $x \in \Gamma^*(R)$.*
*Proof.* Since $M$ is principal then $M = (m)$ for some $m \in R$. Since $R$ is local, then $M = (m) = Z(R)$ and $M$ is nilpotent, therefore $M^n = (m)^n = (0)$ for some $n \in \mathbb{Z}$ with $n \geq 2$. Since $M$ is a principal maximal ideal, by Proposition 4.2, every nonzero ideal of $R$ is a power of $M$. So the minimal ideal will be $M^{n-1} = (m^{n-1})$. If $x \in (m^{n-1})$, then $x = am^{n-1}$ and $y \in (m)$, then $y = bm$ for $a, b \in R$. Therefore, $xy = abm^{n-1}m = abm^n = 0$. Therefore, every $x \in (m^{n-1})$ is adjacent to every $y \in Z(R)$. Therefore, every vertex in the minimal ideal is adjacent to every vertex $x \in \Gamma^*(R)$. □

**Theorem 4.4.** *Suppose that $R$ is a local Artinian ring. If $M$ is a principal maximal ideal, then every $x \in \Gamma^*(R)$ with $\zeta(x) = 1$ is adjacent to one another in $\Gamma^*(R)$.*
*Proof.* Since $M$ is principal then $M = (m)$ for some $m \in R$. Since $R$ is local, then $M$ is nilpotent, therefore $M^n = (m)^n = (0)$ for some $n \in \mathbb{Z}$ with $n \geq 2$. Since $M$ is a principal maximal ideal, by Proposition 4.2, every nonzero ideal of $R$ is a power of $M$. Therefore, if $M^2 \neq (0)$, then $M^n = (0)$ for some $n > 2$. We will have an ideal $M^{n-1}$, and $(M^{n-1})^2 = M^{2n-2}$. We can see that $\forall n \geq 2$, we get $2n - 2 \geq n$. Therefore $(M^{n-1})^2 = (0)$, so we have at least one ideal that is squared nilpotent. Any element $x$ in a squared nilpotent principal ideal must be squared nilpotent, so $\zeta(x) = 1$. Now we want to show that all elements in an ideal that

is squared nilpotent will be adjacent. Suppose that $(M^j)^2 = (0)$ for some $1 \leq j \leq n$. If $(M^j)^2 = (0)$ then $m^{2j} = 0$, so $2j \geq n$. Therefore, if $x, y \in (m^j)$, then $x = am^j$ and $y = bm^j$ for some $a, b \in R$. Therefore $xy = abm^j m^j = abm^2 j = 0$. So $x$ and $y$ are adjacent to one another in $\Gamma^*(R)$. $\square$

**Corollary 4.1.** *Suppose that $R$ is a local Artinian ring and $M$ is a principal maximal ideal. If $M$ is also the minimal ideal, then $\Gamma^*(R)$ is complete.*

*Proof.* If $M$ is both minimal and maximal, then it is the only nonzero ideal in the ideal lattice of $R$. Since $R$ is local, $M = Z(R)$ is nilpotent. Therefore, $M^n = (0)$ for some $n \in \mathbb{Z}$. However, if $n > 2$, then there would be a longer chain of ideals as all powers of $M$ will be in the ideal chain. Therefore, $M^2 = (0)$. So $M = (x) = \{ax | a \in R\}$. Therefore, there is an element $r = ax \in M$ and $s = bx \in M$. Clearly $rs = axbx = abx^2 = 0$. This implies that all elements in $M$ are adjacent to one another, so $\Gamma^*(R)$ is complete. $\square$

We have now established algebraic properties of local Artinian rings and the behaviors of the zero-divisor graphs that reflect the algebraic properties. We also developed an algorithm that can be applied to a zero-divisor graph to determine whether we have a local Artinian ring with a principal maximal ideal. Therefore, now if we are given a zero-divisor graph of any Artinian ring, we can determine whether we have a local ring with a principal maximal ideal, and if that is the case, we can narrow down to a set of possible associated rings simply by observing the structure of the graph. In the next sections, we will demonstrate how it works by applying it to $\mathbb{Z}_n$.

# 5 The Algebraic Structure of $\mathbb{Z}_n$

Rings of the form $\mathbb{Z}_n$ are examples of Artinian rings. Therefore, to provide a better understanding of the correspondence between the zero-divisor graph structure and the algebraic structure of Artinian rings, we will provide specific algorithms for $\mathbb{Z}_n$, and we will show how this is a specific example of a more general Artinian ring structure. We are looking to create an algorithm in which we can determine the ring simply by the structure of the zero-divisor graph. In the case of $\mathbb{Z}_n$ we will create algorithms assuming that we know that we have a ring of the form $\mathbb{Z}_n$.

We will first observe the Fundamental Theorem of Arithmetic, which allows us to realize that $n$ can be written as a product of one or more primes.

**Theorem 5.1. The Fundamental Theorem of Arithmetic** [9] *Each integer greater than 1 can be written as a product of primes, and except for the order in which these primes are written, this can be done in only one way.*

We also want to introduce the Chinese Remainder Theorem for Commutative Rings to understand how and when commutative rings can be decomposed into a direct product of rings.

**Theorem 5.2. The Chinese Remainder Theorem for Commutative Rings.** ([10]) *If $R$ is a commutative ring and $I_1, ..., I_k$ are ideals of $R$ that are pairwise coprime (meaning $I_i + I_j = R$ where $\forall i \neq j$), then the product of $I$ of these ideals is equal to their intersection, and the quotient ring $R/I$ is isomorphic to the product ring $R/I_1 \times ... \times R/I_k$ via the isomorphism*

$$f\colon R/I \to R/I_1 \times ... \times R/I_k$$
$$f(x + I) = (x + I_1, ..., x + I_k)$$

The following corollary allows us to understand the Chinese Remainder Theorem's application to commutative rings of the form $\mathbb{Z}_n$ specifically.

**Corollary 5.1.** *If $m, n$ are relatively prime, then $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ as rings. In particular, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of $n$, then $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{\alpha_k}\mathbb{Z}$ as rings.*

From this corollary, we see that $\mathbb{Z}_n$ can be decomposed into rings of the form $\mathbb{Z}_{p_k^{\alpha_k}}$, which are local rings. Therefore, we see that this corollary is a specific case of the Structure Theorem of Artinian Rings.

These theorems and corollary allow us to understand when the ring $\mathbb{Z}_n$ is decomposable and how it can be decomposed. They also highlight that the prime factorization of $n$ plays a key role in the ring decomposition. Therefore, we will structure our exploration of the zero-divisor graphs of $\mathbb{Z}_n$ into different cases depending on the factorization of $n$.

# 6 Zero-Divisor Graphs of $\mathbb{Z}_{p^m}$

We will start by observing zero-divisor graphs of $\mathbb{Z}_n$ where only one prime is involved in the prime factorization of $n$. These rings are the "building blocks" of $\mathbb{Z}_n$. In fact, we will see that they are the fields and local rings in $\mathbb{Z}_n$. Let us first observe the zero-divisor graph of $\mathbb{Z}_{p^m}$ where $m = 1$, so $\mathbb{Z}_p$.

**Theorem 6.1.** *$\mathbb{Z}_n$ is a field if and only if $n$ is a prime number. In this case, $\Gamma^*(\mathbb{Z}_n)$ will be empty.*

*Proof.* The first part of this theorem is known ([9]). Furthermore, a field $F$, by definition, is a ring in which $1 \neq 0$ and every non-zero element is a unit. Therefore, since every non-zero element is a unit, there can be no zero-divisors. Therefore, $\Gamma^*(\mathbb{Z}_n)$ will be empty. $\square$

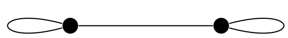
Therefore, we know that if we have $\mathbb{Z}_p$, we have a field and we will not have a zero-divisor graph. This corresponds with our proof in Section 3 that a field does not have an associated zero-divisor graph.

Let us now observe when we have $\mathbb{Z}_{p^m}$ where $m \in \mathbb{Z}$ and $m > 1$. The first thing that we should note about zero-divisor graphs of $\mathbb{Z}_{p^m}$, where $m > 1$, is that $\forall x \in \Gamma^*(\mathbb{Z}_n)$ will have $1 \leq \zeta(x) < \infty$. This means that the $Z(\mathbb{Z}_{p^m}) = Nil(\mathbb{Z}_{p^m})$, implying that $\Gamma^*(\mathbb{Z}_{p^m}) \cong \Gamma^*(Nil(\mathbb{Z}_{p^m}))$.

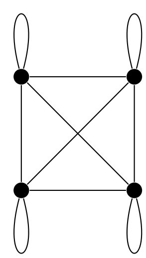
**Theorem 6.2.** $Z(\mathbb{Z}_{p^m}) = Nil(\mathbb{Z}_{p^m})$.
*Proof.* By the definition of zero divisors, if $\mathbb{Z}_{p^m}$ contains zero-divisors then we must have two non-zero elements, $x, y \in \mathbb{Z}_{p^m}$ such that $xy = 0$. Since we have the ring $\mathbb{Z}_{p^m}$, this implies that $p^m = 0$. Therefore, the ideal $(p^m) = (0)$, and it follows that $xy \in (p^m)$. So, we find $xy = rp^m$ $\forall r \in \mathbb{Z}_{p^m}$. Now suppose, for a contradiction, that $x \notin (p) = \{rp|r \in R\}$. This implies that $y \in (p^m)$, which implies that $y = 0$, which is a contradiction because $y \in Z(R)^*$. Therefore, $x \in (p)$. Similarly, we must have $y \in (p)$. Therefore, $Z(\mathbb{Z}_{p^m}) = (p)$. Since $p^m = 0$, then $(rp)^m = r^m p^m = 0$, $\forall r \in \mathbb{Z}_{p^m}$. Therefore, the ideal generated by $p$ is nilpotent, specifically $(p)^m = (0)$. Since $x, y \in (p)$, then $x^m = 0$ and $y^m = 0$. Therefore, all zero-divisors must be nilpotent, so $Z(\mathbb{Z}_{p^m}) = Nil(\mathbb{Z}_{p^m})$. □

Now let us observe some examples of $\Gamma^*(\mathbb{Z}_{p^2})$, enabling us to visualize a pattern that will form as the value of $m$ gets larger.

**Example 6.1.** Structure of $\Gamma^*(\mathbb{Z}_{p^2})$
If we have $\Gamma^*(\mathbb{Z}_{p^2})$, then we will have a complete graph with $p - 1$ vertices, where $(p)$ is the maximal ideal, and therefore every vertex is in the set $(p)$. Furthermore, from Theorem 2.4., we know that any vertex $x$ must be looped with $\zeta(x) = 1$.



$\Gamma^*(\mathbb{Z}_9) = \Gamma^*(\mathbb{Z}_{3^2})$



$\Gamma^*(\mathbb{Z}_{25}) = \Gamma^*(\mathbb{Z}_{5^2})$

If we observe the ideal lattice of the ring, we see that it's two ideals are $(p)$ and $(0)$, where $(p)$ is the principal maximal ideal. Therefore, this is an example of Corollary 4.1., as we see that the associated zero-divisor graph is a complete graph. Algorithm 4.1. does not work on these zero-divisor graphs, as they are complete. We will generate a specific algorithm for zero-divisor graphs of $\mathbb{Z}_n$, which we can apply to determine the associated rings with these complete zero-divisor graphs.

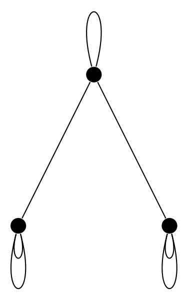
Although the structure of $\Gamma^*(\mathbb{Z}_{p^2})$ looks quite simple and discernible (as a complete graph), the structure of zero-divisor graphs of $\Gamma^*(\mathbb{Z}_{p^m})$ will get more complicated as $m$ gets greater. As the graphs get more complicated, not all vertices will be adjacent to one another. Instead, we expect to see a core complete graph structure corresponding to the minimal ideal, and we expect to see extensions off this complete graph structure as every vertex must be adjacent to the minimal ideal, but are not necessarily adjacent to every other vertex.

**Example 6.2.** Structure of $\Gamma^*(\mathbb{Z}_{p^3})$
We will observe that $\Gamma^*(\mathbb{Z}_{p^3})$ will look like a complete graph connected to an almost complete bipartite graph. This is an "almost" complete bipartite graph since we have a

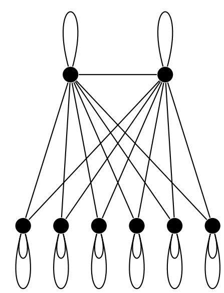
clear separation of the vertices into two sets, $X$ and $Y$, where all the vertices of $Y$ are adjacent to the vertices in $X$. However, it is not complete bipartite because the vertices in $X$ are adjacent to one another. We also note that all of the vertices are looped, which reflects the fact that $Z(\mathbb{Z}_{p^m})^* = Nil(\mathbb{Z}_{p^m})$. We want to classify the vertices of the graph structure, to reflect how they are generated and the degree of loops each have. In $\mathbb{Z}_{p^3}$, we will classify the vertices as follows:

Let $X = \{a \in (p^2)\}$. Any vertex $x \in X$ will have $\zeta(x) = 1$ and $|X| = p - 1$.
Let $Y = \{b \in (p) - (p^2)\}$. Any vertex $y \in Y$ will have $\zeta(y) = 2$, $\delta(y) = p - 1$, and $|Y| = p^2 - p$.

We can also see that the total number of vertices will be $|\Gamma^*(\mathbb{Z}_{p^3})| = p^2 - 1$.



$\Gamma^*(\mathbb{Z}_8) = \Gamma^*(\mathbb{Z}_{2^3})$



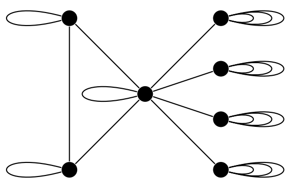
$\Gamma^*(\mathbb{Z}_{27}) = \Gamma^*(\mathbb{Z}_{3^3})$

**Example 6.3.** Structure of $\Gamma^*(\mathbb{Z}_{p^4})$

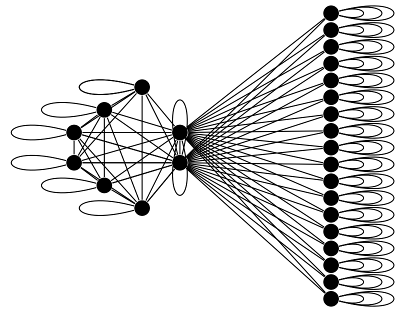
In $\Gamma^*(\mathbb{Z}_{p^4})$, we see that zero-divisor graph contains loops, where the vertex $x$ has $\zeta(x) = 3$. We will see a similar structure to the $\Gamma^*(\mathbb{Z}_{p^3})$ graphs, with a complete graph attached to an almost complete bipartite graph, however the number of elements in each will differ.

Let $X = \{a \in (p^3))\}$. Any vertex $x \in X$ will have $\zeta(x) = 1$ and $|X| = p - 1$.
Let $Y = \{b \in (p^2) - (p^3)\}$. Any vertex $y \in Y$ will have $\zeta(y) = 1$ and $|Y| = p^2 - p$.
Let $Z = \{c \in (p) - ((p^2) \cup (p^3))\}$. Any vertex $z \in Z$ will have $\zeta(z) = 3$, $\delta(z) = p - 1$, and $|Z| = p^3 - p^2$.

The vertices in the set $X$ and $Y$ will form the complete graph since $\zeta(x) = \zeta(y) = 1$, and the vertices in the set $Z$ will form the extending complete bipartite graph. We can see that in total, we have $|\Gamma^*(\mathbb{Z}_{p^4})| = p^3 - 1$.



$\Gamma^*(\mathbb{Z}_{16}) = \Gamma^*(\mathbb{Z}_{2^4})$



$\Gamma^*(\mathbb{Z}_{81}) = \Gamma^*(\mathbb{Z}_{3^4})$

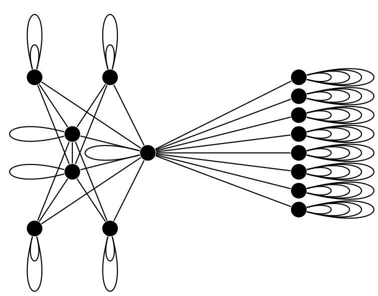
We are starting to get a clear structure for $\Gamma^*(\mathbb{Z}_{p^m})$. We expect to see a complete graph formed between vertices with loops where $\zeta(x) = 1$, and then an almost complete bipartite graph being formed out of those vertices that are looped to the greatest degree connected to the central loops generated by $(p^{m-1})$. We can also see that we can use the degree of loops to determine the degree of prime involved in our prime factorization of $n$.

**Example 6.4.** Structure of $\Gamma^*(\mathbb{Z}_{p^5})$

$\Gamma^*(\mathbb{Z}_{p^5})$ is slightly more complicated because we will have more loops of varying degree, since $((p^2)^3) = 0$, we will have loops with $\zeta(x) = 1, \zeta(y) = 2$, and $\zeta(z) = 4$. We again will classify our vertices as follows:

Let $X = \{a \in (p^4)\}$. Any vertex $x \in X$ will have $\zeta(x) = 1$ and $|X| = p - 1$.
Let $Y = \{b \in ((p^3) - (p^4))\}$. Any vertex $y \in Y$ will have $\zeta(y) = 1$ and $|Y| = p^2 - p$.
Let $Z = \{c \in (p^2) - (((p^4) \cup (p^3))\}$. Any vertex $z \in Z$ will have $\zeta(z) = 2$ and $|Z| = p^3 - p^2$.
Let $W = \{d \in (p) - ((p^2) \cup (p^3) \cup (p^3))\}$. Any vertex $w \in W$ will have $\zeta(w) = 4$, $\delta(w) = p - 1$, and $|W| = p^4 - p^3$.

We expect to see a complete bipartite graph being formed by the vertices in the sets $X$ and $Y$. Then there will be almost bipartite structures being formed with vertices in $Z$ and vertices in $W$.



$\Gamma^*(\mathbb{Z}_{32}) = \Gamma^*(\mathbb{Z}_{2^5})$

In general, we can see that any vertex $x, y \in \Gamma^*(M^{m-1})$, where $M^{m-1}$ is the minimal ideal, will have $\zeta(x) = 1$, $\zeta(y) = 1$, and $xy = 0$. This will be the core structure in $\Gamma^*(R)$.

The zero-divisor graph will get more complicated as the value of $m$ gets greater because this implies that there are more ideals in the ideal chain. This corresponds to our result in Theorem 4.4. We also must note the importance of the vertices of greatest degree. This is because we can use both the values of $\zeta(x)$ and $\delta(x)$ to determine the values of both $m$ and $p$.

Now that we have seen a couple of examples, we see a pattern forming. This we want to generalize for all $\Gamma^*(\mathbb{Z}_{p^m})$, thereby enabling us to distinguish different rings of the form $\mathbb{Z}_n$ correlating to different zero-divisor graph structures. The following theorem allows us to establish the amount of zero-divisors we expect to find in a zero-divisor graph of $\mathbb{Z}_{p^m}$.

**Theorem 6.3.** $|Z(\mathbb{Z}_{p^m})^*| = p^{m-1} - 1$.
*Proof.* In our proof of Theorem 6.2., we showed that $Z(\mathbb{Z}_{p^m}) = (p) = \{rp | r \in \mathbb{Z}_{p^m}\}$. We can determine $|(p)|$ by examining the total number of $r$ to have $rp < p^m$, therefore $r < p^{m-1}$. Since $r \in \mathbb{Z}_{p^m}$, we have $p^{m-1}$ choices for $r$, so $|(p)| = p^{m-1}$. Since $(p)$ includes 0, we have $p^{m-1} - 1$ non-zero choices for $r$. Therefore, $|Z(\mathbb{Z}_{p^m})^*| = p^{m-1} - 1$. □

**Theorem 6.4.** *If a vertex $x$ has the greatest value of $\zeta(x)$, then $\delta(x) = p - 1$.*
*Proof.* The vertices with the highest degree of loop must be generated by $(p) - ((p^2) \cup \ldots \cup (p^m))$. Therefore, the only vertices they are adjacent to in the zero-divisor graph must be those generated by $(p^{m-1})$. We know that $kp^{m-1} = 0$ if $k \geq p$. Therefore, we have $p$ choices for $k$. So $|(p^{m-1})| = p$. Since 0 does not appear in the zero-divisor graph, the vertices with the highest degree of loops will be adjacent to $p - 1$ vertices. Therefore, if vertex $x$ has the greatest value of $\zeta(x)$, then $\delta(x) = p - 1$. □

We can form an algorithm to distinguish the zero-divisor graphs of the ring $\mathbb{Z}_{p^m}$. It will be the following:

**Algorithm 6.1** *Suppose $R \cong \mathbb{Z}_{p^m}$. Determine $R$ knowing the structure of* $\Gamma^*(R)$

> **1)** Determine whether $Nil(R) = Z(R)$, by observing whether all of the vertices are looped.
> **2)** For any $x \in \Gamma^*(R)$, determine the greatest value of $\zeta(x)$. This value will be equal to $m - 1$.
> **3)** Now count the $\delta(x)$. From Theorem 6.4., $\delta(x) = p - 1$
> **4)** We now have the value of $p$ and the value of $m$, so we can determine the corresponding ring $R = \mathbb{Z}_{p^m}$ to the zero-divisor graph.

There are also certain properties that we can realize about the structure of $\Gamma^*(\mathbb{Z}_{p^m})$. The following theorem states that if we have $R \cong \mathbb{Z}_{p^m}$, then $\Gamma^*(R)$ must be star-shaped reducible.

**Theorem 6.5.** *When $R \cong \mathbb{Z}_{p^m}$, then $\Gamma^*(R)$ must be star-shaped reducible.*
*Proof.* If we have $R \cong \mathbb{Z}_{p^m}$, then we must have that $Z(R) = Nil(R)$. Therefore, every $x \in Z(R)$ is nilpotent. Thus, by Theorem 2.2., we know that if every $x \in Z(R)$ is nilpotent,

then there exists $b \in Z(R)$ such that $bZ(R) = 0$, and hence $\Gamma^*(R)$ is star-shaped reducible. $\square$

By Theorem 2.2., if $\Gamma^*(\mathbb{Z}_{p^m})$ is star-shaped reducible, then $\mathbb{Z}_{p^m}$ is local. In fact, we can note that all of rings of the form $\mathbb{Z}_{p^m}$ have one maximal ideal, $(p)$ and it is principal.

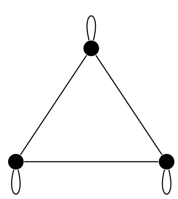
# 7 Zero-Divisor Graphs of Local Artinian Rings with a Maximal Ideal that is not Principal

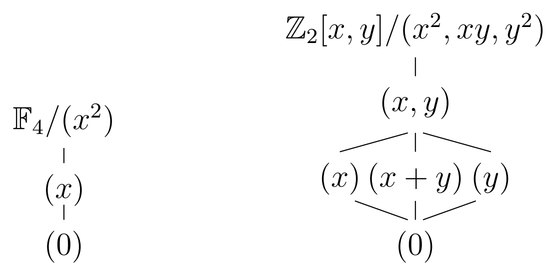
In Sections 5 and 6 we examined the zero-divisor graph properties of local Artinian rings with a principal maximal ideal. However, not all local Artinian rings have a principal maximal ideal. In fact, a local ring $R$ can have a unique maximal ideal $M$ that is generated by many elements, and therefore breaks down into numerous ideals. This ideal lattice will get very complicated, and in this section we will attempt to correlate the ideal lattice to zero-divisor graphs.

We saw in Corollary 4.1. that if we have a local Artinian ring with a principal maximal ideal, $M$ that is also a minimal ideal, then $\Gamma^*(R)$ is a complete graph. However, we can show that the converse of this corollary does not hold. The following example provides a counterexample.

**Example 7.1.** *Ideal Lattice of Complete Zero-Divisor Graphs.*

The following two rings have the same zero-divisor graph, which is a complete graph. However, the ideal lattice of the rings vary. $\mathbb{F}_4[x]/(x^2)$ has a principal maximal ideal of order 4 that is generated by one element. In comparison, $\mathbb{Z}_2[x,y]/(x^2,xy,y^2)$ has a unique maximal ideal that is principal and is generated by two elements. In the ideal lattice, this maximal ideal breaks into three smaller ideals, each of order 2.






$\Gamma^*(\mathbb{F}_4[x]/(x^2)) \cong \Gamma^*(\mathbb{Z}_2[x,y]/(x^2,xy,y^2))$

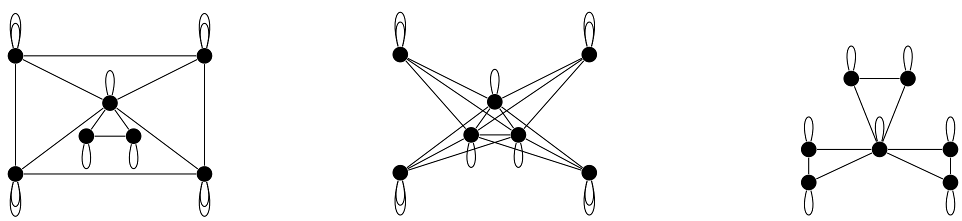
This example is significant because it shows us that we do not have a unique zero-divisor graph for a unique ideal lattice. Our zero-divisor graph does not distinguish between looped vertices of $\zeta(x) = 1$. Clearly all of the looped vertices annihilate themselves, but if we take any vertex in the graph of $\mathbb{F}_4[x]/(x^2)$ and set it as the generator of the principal ideal, we find that it generates all the elements in $\Gamma^*(\mathbb{F}_4[x]/(x^2))$. On the other hand, the vertices of $\mathbb{Z}_2[x,y]/(x^2,xy,y^2)$ are such that the each of them forms a principal ideal that consists of only 0 and the element itself.

This last example also shows us that local Artinian rings ideal lattices are not always simple, and the maximal ideal does not need to be principal. If this is the case, then the ideal chain will break up into three or more chains, which could break up into more if those subsequent ideals are also not principal. The next example shows zero-divisor graphs that correspond to more complicated ideal lattices:

**Example 7.2.** *Ideal Lattices of Local Rings with a Maximal Ideal that is not Principal*
All of the following rings, $R_1, R_2$, and $R_3$ are local Artinian rings each with a maximal ideal that is not principal.

$R_1 = \mathbb{Z}_4[x]/(x^2+2x)$, $\mathbb{Z}_8[x]/(2x, x^2+4)$, $\mathbb{Z}_2[x, y]/(x^2, y^2-xy)$, and $\mathbb{Z}_4[x, y]/(x^2, y^2-xy, xy-2, 2x, 2y)$.
$R_2 = \mathbb{Z}_2[x, y]/(x^3, xy, y^2)$, $\mathbb{Z}_8[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$.
$R_3 = \mathbb{Z}_4[x, y]/(x^2, y^2, xy-2, 2x, 2y), \mathbb{Z}_2[x, y]/(x^2, y^2), \mathbb{Z}_4[x]/(x^2)$

All of these rings are local and have $|R| = 16$. They are also the only local Artinian rings with 7 vertices in $\Gamma^*(R)$ that do not have a principal maximal ideal. ([11])



$\Gamma^*(R_1)$ $\Gamma^*(R_2)$ $\Gamma^*(R_3)$

We next want to form an algorithm that will allow us to determine the ideal lattice correlating to a more complicated local Artinian ring from the zero-divisor graph. In order to show how we can determine the ideal lattice from the zero-divisor graph, we first need to prove that annihilators form ideals.

**Theorem 7.1.** *If $R$ is a commutative unital ring with $a \in R$, then $Ann(a) = \{xa = 0 | x \in R\}$ forms an ideal.*
*Proof.* Let $x_1, x_2 \in Ann(a) = \{xa = 0 | x \in R\}$. This implies that $x_1 a = 0$ and $x_2 a = 0$. Therefore, it follows that $(x_1 \pm x_2)a = x_1 a \pm x_2 a = 0 \pm 0 = 0$. Therefore, we have $x_1 \pm x_2 \in Ann(a)$. Also, $\forall x \in Ann(a)$ and $r \in R$, we find that $(rx)a = r(xa) = r0 = 0$. Therefore, $rx \in Ann(a)$. So, $Ann(a)$ forms an ideal of $R$. $\square$

The zero-divisor graph enables us to visibly spot the annihilators of the ring. This is because, if we take any vertex in the zero-divisor graph, then we by our definition of a zero-divisor graph, we know that all the vertices adjacent to it annihilate it, as does 0. Therefore, all of these vertices form the annihilator ideal. This allows us to visually identify some of the ideals of a ring. The following algorithm describes an approach to identifying the ideal lattice of local Artinian rings from zero-divisor graphs.

**Algorithm 7.1.** *Determining a Local Ring with a Maximal Ideal that is not Principal from* $\Gamma^*(R)$

> **1)** Determine whether $Nil(R) = Z(R)$, by observing whether all of the vertices in $\Gamma^*(R)$ are looped. If so, by Theorem 2.2., we know we have a local ring.
> **2)** Determine $|M|$. Then determine the possible ideal lattices of this order.
> **3)** Follow the Algorithm 4.1. If the value the algorithm gives for $|M|$ does not equal $|M|$ in $\Gamma^*(R)$, then we know we do not have a principal maximal ideal.
> **4)** Since it is assumed that $R$ is a local, commutative, unital ring, we can also rule out ideal lattices corresponding to non-commutative rings and non-local rings.
> **5)** Label the vertices and observe the annihilators of each of the vertices. We know from Theorem 3.1.5. that these annihilators form ideals.
> Compare the number of annihilator ideals to the possible lattice structures. If the number of ideals and the structure of the ideals matches one of the ideal lattices, then, taking note of the index of each of the ideals, we can match the ideal lattice to the zero-divisor graph. If this is not the case, more ideals exist.
> In this case, if we know $|R|$, we can use the First Isomorphism Theorem to determine the size of the other ideals. Again, we want to take note of the index of the ideals by observing the $\zeta(x)$ of the vertices contained in each ideal.
> **6)** We can now match the zero-divisor graph with a certain ideal lattice, narrowing down the possibility of Artinian rings that it could be associated with.

This algorithm is more convoluted as the ideal lattice gets more complicated. We can notice that if we cannot visibly identify all of the ideals via annihilators, this algorithm becomes harder to follow and we need more information, namely $|R|$. To get a more concrete understanding of the algorithm, we will show it applied to Example 7.2.

**Example 7.3.** *Using Algorithm 7.1. on Example 7.2. to determine the ideal lattice of* $R$ *from* $\Gamma^*(R)$

> **1)** Clearly, in all the zero-divisor graphs, all of the elements are looped, therefore we are dealing with a local ring.
> **2)** $|M| = 8$. Therefore, we can look at the ideal lattices of rings of order 8 to determine whether they fit with our zero-divisor graph. Order 8 rings have the following ideal lattices ([12]):

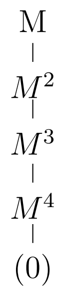



Figure 1

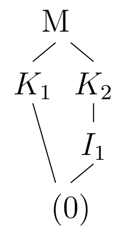



Figure 2

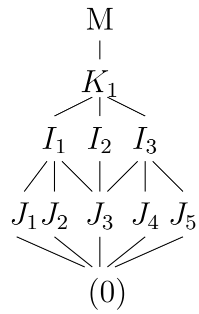



Figure 3

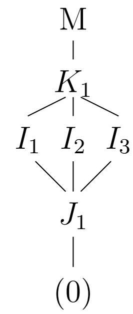



Figure 4

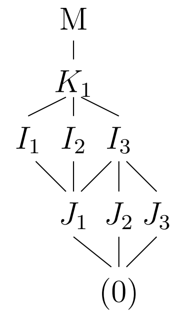



Figure 5

**3)** If we follow Algorithm 4.1., in $\Gamma^*(R_1)$ and $\Gamma^*(R_2)$ we find that $n = 3$ and $\delta(x) = p^m = 4$, and we find that $|M| = 16$, which is a contradiction.
In $\Gamma^*(R_3)$, we find that $|M| = 8$, however we see that the vertices with the greatest $\zeta(x)$ are not adjacent only to the center vertex.
Therefore, we do not have a principal maximal ideal in any of these rings, and so we cannot have Figure 1 as our ideal lattice corresponding to $R_1, R_2$, or $R_3$.
**4)** We can also rule out the possibilities of Figure 2 and Figure 3 as potential ideal lattices.
Figure 2 corresponds to a ring that is mixed (field and local ring direct product). This implies that the maximal ideal is not nilpotent, contradicting the fact that $R$ is local.
Figure 3 corresponds to a Dihedral group of order 8. Since Dihedral groups are non-commutative, Figure 3 cannot be the corresponding ideal lattice for $R$, a commutative ring.
Therefore, we either have the ideal lattices of Figure 4 and Figure 5 corresponding to $\Gamma^*(R_1), \Gamma^*(R_2)$, and $\Gamma^*(R_3)$.
**5)** Let us start with $\Gamma^*(R_1)$. Label the outside vertices $a, b, c, d$ and the inside triangle $x, y, z$, with $x$ being the center vertex adjacent to all vertices in the $\Gamma^*(R_1)$. We want to determine the annihilators in $\Gamma^*(R_1)$, since by Theorem 7.5, these form ideals.
Let $ann(y) = \{0, x, y, z\} = I_1$. Therefore, the inner triangle forms an ideal of order 4, and since they are all looped vertices with $\zeta(x) = \zeta(y) = \zeta(z) = 1$, we know that the index of $I_1$ is 2.
Now we can look at the annihilators of the outside vertices: $ann(a) = \{0, x, b, d\} = I_2$, and $ann(b) = \{0, x, a, c\} = I_3$. The index of both $I_2$ and $I_3$ is 3, since the highest $\zeta(b) = \zeta(a) = 2$. We see that all the possible ideal lattices of order 8 have at most 3 ideals of order 4. Therefore, we have found all possible ideals of order 4.
If $\Gamma^*(R_1)$ corresponds to Figure 5, then it would break down into 3 ideals of order 2. However, this is not the case.
Suppose, for a contradiction, that $(y)$ is an ideal of order 2. We know that $y^2 = 0$, so we clearly have 0 and $y$ in our ideal. However, $ay \in Z(R)^*$ since both $a, y \in Z(R)^*$, and $ay \neq 0$ since they are not adjacent. Therefore, $(y)$ cannot be an

ideal of order 2. Similarly, $(z)$ cannot be an ideal of order 2. On the other hand, $(x)$ is an ideal of order 2. It is adjacent to every zero-divisor, so $(x) = \{0, x\}$.
We can also prove that this ideal lattice, given the index of the ideals, uniquely corresponds to the zero-divisor graph $\Gamma^*(R_1)$.
To show this, first look at the ideal $J_1$. It is squared nilpotent, and therefore clearly annihilates itself.
Next, if we look at $I_1$, we know it has order 4, is squared nilpotent, and is principal by Theorem 3.5. This implies that the vertex that generates $I_1$, let us call it $y$ is such that $y^2 = 0$. Therefore, all vertices in $I_1$ are of the form $ry$, $\forall r \in R$. This implies that they are all adjacent, so we can form the interior triangle in the zero-divisor graph $\Gamma^*(R_1)$.
The index of $I_1$ is 2 , which implies that $ann(I_1) = I_1$. Therefore, $I_1$ does not annihilate either $I_2$ or $I_3$. However, $I_2$ and $I_3$ must have annihilators. Furthermore, these annihilators must have order 4. We know the index of $I_2$ and $I_3$ is degree 3, therefore the ideals do not annihilate themselves.
Since we do not have any other ideals of order 4, this implies that $ann(I_2) = I_3$, which means that $I_2 I_3 = (0)$. Therefore, we can form the remainder of our zero-divisor graph of $\Gamma^*(R_1)$.
**6)** Now let us observe $\Gamma^*(R_2)$. Let us similarly label the outside elements $a, b, c, d$ and the interior triangle $x, y, z$.
First we can determine $ann(a) = \{0, x, y, z\} = I_3$.
All the rings in Example 7.2. are such that $|R_1| = |R_2| = |R_3| = 16$.
From the First Isomorphism Theorem, for some $\phi : R \to R_a$ where $\phi(r) = ra$. We find that $R/Ker(\phi) \cong Im(\phi)$. In $R_a$, $|Im(\phi)| = |(a)|$.
Therefore, $|ann(a)| = |R_2|/|(a)|$. So, we find an ideal, $|(a)| = 4 = I_1$. Similarly, we can find the $|(b)| = 4 = I_2$.
Since we can have at most 3 ideals of order 4 in our ideal lattices of rings with order 8, we know we have found all possible ideals of order 4. Both the index of $I_1$ and $I_3$ will be 3, since $\zeta(a) = \zeta(b) = 2$.
Up to this point, $\Gamma^*(R_1)$ has the same ideal lattice as $\Gamma^*(R_2)$ and the index of each of the corresponding ideals in the lattices are equivalent. However, $\Gamma^*(R_2)$ cannot correspond to Figure 4, since in step 5 we showed that Figure 4, given the index of ideals $I_1, I_2$, and $I_3$ corresponds uniquely to $\Gamma^*(R_1)$. Also, we can see that in $\Gamma^*(R_2)$, the vertices $x, y, z$ all annihilate themselves and 0, and so are able to generate three ideals of order 2.
Therefore, we find that $\Gamma^*(R_2)$ corresponds to Figure 5.
**7)** Now let us observe $\Gamma^*(R_3)$. Let us label the outside vertices $a, b, c, d, e, f$ and the center vertex $x$.
Determine that $ann(a) = \{0, x, a, b\} = I_1$, $ann(c) = \{0, x, c, d\} = I_2$, and $ann(e) = \{0, x, e, f\} = I_3$. Furthermore, the index of $I_1, I_2$, and $I_3$ will be 2, since $\forall y \in \Gamma^*(R_3)$ have $\zeta(y) = 1$.
Also, $ann(x) = \{0, x\} = J_1$, therefore it forms an ideal of order 2. We cannot have any other ideals of order 2. This is because no outside vertex is adjacent to all other outside vertices. Therefore, $ae \in Z(R)^*$ and $ae \neq 0$.
Therefore, the ideal will have order greater than 2. So we can tell that $\Gamma^*(R_3)$

also corresponds to Figure 4, but the index of each of the ideals differs from the index of each of the ideals in the ideal lattice of $\Gamma^*(R_1)$.
Similarly to step 5, we can prove that this ideal lattice given that the index of $I_1, I_2$, and $I_3$ is 2, corresponds uniquely to $\Gamma^*(R_3)$. We can see that the ideal $|J_1| = 2$, and the index of $J_1$ is 2, and so corresponds to the center of $\Gamma^*(R_1)$.
We can then look at the annihilators of $I_1, I_2$, and $I_3$. We know that $|I_1| = |I_2| = |I_3| = 4$, and we find that $ann(I_1) = I_1$, $ann(I_2) = I_2$, and $ann(I_3) = I_3$. Therefore, they each annihilate themselves and are not adjacent to one another. This implies the structure $\Gamma^*(R_3)$.

The zero-divisor graphs in Example 7.2. are actually some of the simplest examples of zero-divisor graphs of local Artinian rings with maximals ideals that are not principal. The zero-divisor graphs and correlating rings' ideal lattices do not have long ideal chains and don't break into that many separate chains of ideals. However, we could find a plethora of examples where not only does the maximal ideal break into numerous different chains, but the ideals following also do. Therefore, our algorithm to determine these rings will get more complicated as the rings get larger. Algorithm 7.1 works by process of elimination, however this would be extremely hard in larger cases. Perhaps this method will motivate a more efficient algorithm for determining local rings with non-principal maximal ideals.

# 8 Zero-Divisor Graphs of a Direct Product of Fields

We have now examined the ideal lattices and zero-divisor graphs of fields and local rings. We now want to observe what happens when we take the direct products of different types of Artinian rings to attain more complicated Artinian rings. First, we will observe the zero-divisor graphs of a direct product of fields. To examine the properties of the zero-divisor graphs of these rings, we will use the Chinese Remainder Theorem previously introduced in Section 5, and we will review some algebraic properties of rings. The first is that any finite integral domain is a field.

**Theorem 8.1**. *Any finite integral domain is a field*
*Proof.* Suppose that $R$ is a finite integral domain with elements $\{0, x_1, ..., x_n\}$. If $x_i \neq 0$ $\forall x_i$ such that $1 \leq i \leq n$, then consider the set $x_iR = \{x_ix_1, ..., x_ix_n\} = \{x_ir | r \in R\}$. All $n$ elements of this set are distinct elements of $R$ because if $x_iy = x_iz$, then $y = z$. Therefore, $x_iR = R$. In particular, $1 \in x_iR$, so for some $r \in R$, $x_ir = rx_i = 1$. Thus $x_i$ has a multiplicative inverse, and $R$ is a field. □

**Theorem 8.2**. *Let $R$ be a finite commutative ring. Then for every prime ideal $P$, $R/P$ is a field and $P$ is maximal.*
*Proof.* Suppose that $R$ is a finite commutative ring, and let $P$ be a prime ideal. Since $P$ is prime, $R/P$ is an integral domain, and since $R$ is finite, then $R/P$ is finite. Therefore, $R/P$ is a finite integral domain, and so by Theorem 8.1., $R/P$ is a field. Since $R/P$ is a field, $P$ is maximal. □

Now we will show that any reduced ring is a finite product of rings.

**Theorem 8.3.** *Let $R$ be a finite reduced ring. Then $R \cong F_1 \times ... \times F_n$ where $F_1, ..., F_n$ are fields.*
*Proof.* Suppose that $R$ is a finite reduced ring. By Theorem 8.1., we know that any finite integral domain is a field. Next we want to look at the prime ideals of $R$. We know from Theorem 8.2., that the prime ideals are maximal. So in $R$, the nilradical, which is the intersection of all prime ideals, is equal to the Jacobson radical, which is the intersection of all maximal prime ideals, $M_i$, since every prime ideal is maximal. Then by the Chinese Remainder Theorem for Commutative rings and since the nilradical of $R$ is zero because $R$ is reduced and therefore has no nilpotent elements, we find that $R \cong R/M_1 \times ... \times R/M_n$. Furthermore, $R/M_1, ...R/M_n$ are all fields, since in Theorem 8.1. we showed that $R/P$ is a field. So let, $R/M_1 \cong F_1, ..., R/M_n \cong F_n$. Therefore, if $R$ is a reduced ring, then $R \cong F_1 \times ... \times F_n$ where $F_1, ..., F_n$ are fields. □

This theorem allows us to identify the zero-divisor graphs of rings that are a direct product of fields, as they will contain no nilpotent elements.

In Section 2, we proved that complete bipartite graphs with $m, n > 1$ had no nilpotent elements. Therefore, by Theorem 8.3., we know that this must be the zero-divisor graph of a direct product of fields. We will next observe specifically how we form complete bipartite graphs. We will start with star graphs before exploring complete bipartite graphs with $m, n > 2$.

We saw in Example 2.1. that in star graphs, such as $\Gamma^*(\mathbb{Z}_8)$, it is possible for there to exist nilpotent elements. Therefore, we want to distinguish when we have nilpotent elements, and when we do not have nilpotent elements in star graphs.

When approaching this problem, it is important to establish the algebraic properties that differ with zero divisor graphs of different diameters. We obtain the following result by ([13], Theorem 2.6):

**Theorem 8.4.** *Let $R$ be a ring.*
*(1) diam$(\Gamma^*(R)) = 0$ if and only if $R$ is (nonreduced and) isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$.*
*(2) diam$(\Gamma^*(R)) = 1$ if and only if $xy = 0$ for each distinct pair of zero divisors and $R$ has at least two nonzero divisors.*
*(3) diam$(\Gamma^*(R)) = 2$ if and only if either (i) $R$ is reduced with exactly two minimal primes and at least three nonzero zero divisors, or (ii) $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator.*
*(4) diam$(\Gamma^*(R)) = 3$ if and only if there are zero divisors $a \neq b$ such that $(0 : (a, b)) = (0)$ and either (i) $R$ is a reduced ring with more than two minimal primes, or (ii) $R$ is nonreduced.*

We know that when we have diam$(\Gamma^*(R)) = 1$, we have a complete graph. We saw in Theorem 2.4., that complete graphs contain looped vertices unless it is $\Gamma^*(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Therefore, in the case of diam$(\Gamma(R)^*)$=1, the only $\Gamma(R)^*$ that contains no nilpotent elements is a direct product of two fields, specifically $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We will now observe zero-divisor graphs with diam$(\Gamma^*(R)) = 2$. We will start with star graphs, observing when these types of graphs can be formed without nilpotent elements. Anderson and Livingston proved when a zero-divisor graph is a star-shaped graph ([2], Corollary 2.7).

**Theorem 8.5.** *Let $R$ be a finite commutative ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times F$, where $F$ is a finite field, or $R$ is local. Moreover, for some prime $p$ and integer $n \geq 1$, $|\Gamma^*(R)| = |F| = p^n$ if $R \cong \mathbb{Z}_2 \times F$, and $|\Gamma^*(R)| = p^n - 1$ if $R$ is local.*

From this theorem, we know the possible rings that have star-shaped zero-divisor graphs. In the next theorem ([3], Theorem 2.8), we will distinguish between the star graphs with nilpotent elements and star graphs without them.

**Theorem 8.6.** *Let $R$ be a finite commutative ring with identity such that*

$$\Gamma^*(R) = K_{1,n}$$

*for some $n \in \mathbb{Z}$ with a vertex $x$, called the center, such that $x$ is looped with $\zeta(x) = 1$ and adjacent to all vertices in $\Gamma^*(R)$. Then the following are equivalent:*
*(1) $Z(R)$ is an ideal;*
*(2) $x^2 = 0$;*
*(3) $R \cong \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_3[x]/(x^2)$, or $\mathbb{Z}_4[x]/(2x, x^2 - 2)$.*

From this last theorem, we now know exactly what rings form zero-divisor graphs that are star-shaped with a looped center, and therefore are not reduced rings. We can also see that the rings with nilpotent elements whose zero-divisor graph is a star-graph are indecomposable.

In the next theorem we will show that decomposable rings of the form $\mathbb{Z}_2 \times F$ will have star-shaped zero-divisor graphs with no nilpotent elements.

**Theorem 8.7.** *Let $R$ be a finite commutative ring and let $R \cong \mathbb{Z}_2 \times F$, where $F$ is a finite field. Then the zero-divisor graph $\Gamma^*(R)$ will be star-shaped and will not contain any loops.*
*Proof.* Suppose that $R \cong \mathbb{Z}_2 \times F$. We know from Theorem 8.5. that $\Gamma^*(R)$ is star-shaped. Furthermore, we know from Theorem 2.5. that none of the end-vertices of this graph are looped. Therefore, the only potential looped vertex is the center $a$. Since $R \cong \mathbb{Z}_2 \times F$, the center of the graph must be $(1, 0)$. Therefore, $a = (1, 0)$, and we can see that $a^2 \neq 0$, since $(1, 0)^2 = (1, 0)$. Therefore, $a$ is not looped. □

**Corollary 8.1.** *If $\Gamma^*(R)$ is star-shaped and contains no loops, then $\Gamma^*(R) = K_{1,p^m-1}$ for some prime $p$ where $m > 0$.*
*Proof.* From Theorem 8.7., if $\Gamma^*(R)$ is star-shaped and contains no loops then $R \cong \mathbb{Z}_2 \times F$. Since $(1, 0)$ is the center, it must be adjacent to all vertices of the form $(0, x)$, where $x \in F$. We know that the order of a finite field is $p^m$, therefore we have $p^m$ choices for $x$. However, our zero-divisor graph only shows non-zero vertices, so we have $p^m - 1$ non-zero choices of $x$. Therefore, $(1, 0)$ will be adjacent to $p^m - 1$ vertices, so our zero-divisor graph must be of

the form $K_{1,p^m-1}$. $\square$

Therefore, we now know that direct products of two fields, one of which is $\mathbb{Z}_2$, forms a star-graph that contains no nilpotent elements.

**Example 8.1.** Using the same rings as we did in Example 2.1, we can now clearly see just by looking at the zero-divisor graphs of $\Gamma^*(\mathbb{Z}_8)$ and $\Gamma^*(\mathbb{Z}_6)$, that $\mathbb{Z}_6$ is a decomposable ring and $\mathbb{Z}_8$ is indecomposable, so we can see that $\Gamma^*(\mathbb{Z}_6)$ forms a complete bipartite graph without loops, but $\Gamma^*(\mathbb{Z}_8)$ does not. $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$, and $\mathbb{Z}_8$ is indecomposable because it is isomorphic to a ring of the form $\mathbb{Z}_{p^n}$ where $p$ is a prime and $n$ is some positive integer. In this particular example, $\mathbb{Z}_8 = \mathbb{Z}_{2^3}$.

Therefore, we have shown the following statement about the simplest case of a bipartite graph:

**Theorem 8.8.** *The only zero-divisor graphs that contain a star graph and contain no loops is a direct product of fields.*

Now that we have observed all star-shaped graphs, we will now look at how zero-divisor graphs of the form $K_{m,n}$ where $m, n > 1$ are formed.

In order to observe this, we will first distinguish the type of zero-divisor graph with diameter 2 that can form complete bipartite graphs. We know from Theorem 8.4., that $\text{diam}(\Gamma(R)^*) = 2$ if and only if either (i) $R$ is reduced with exactly two minimal primes and at least three nonzero zero divisors, or (ii) $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator. Clearly, the first cases (i) is a direct product of two rings. We will next prove that the case (ii) cannot be a ring that is a direct product of fields.

In the case (ii), $Z(R)$ forms an ideal. We can use Theorem 2.2. to conclude the following result:

**Corollary 8.2.** *Let $R$ be a ring such that diam($\Gamma^*(R)) = 2$. If $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator, then $\Gamma^*(R)$ cannot be a complete bipartite graph.*

*Proof.* If $Z(R)$ forms an ideal, then, by Theorem 2.2., $\Gamma^*(R)$ is star-shaped reducible, and therefore has a looped center, $a$. Therefore, it contains at least one loop, and $\Gamma^*(R)$ cannot be a complete bipartite graph, since Theorem 2.6. proves that a complete bipartite graph cannot contain loops. $\square$

Therefore, in the case where $\text{diam}(\Gamma^*(R)) = 2$, the only possible zero-divisor graphs that could be a complete bipartite graph without loops are zero-divisor graphs of direct products of fields. In the next theorem, we will prove that complete bipartite graphs can only be formed by taking the direct product of two fields.

**Theorem 8.9**. *Let $R$ be a finite commutative ring. Then $\Gamma^*(R)$ will be a complete bipartite graph, $K_{p^n-1,q^m-1}$ with primes $p, q$ and $n, m > 1$, if and only if $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields.*

*Proof.* $\Leftarrow$ Suppose that $F_1$ and $F_2$ are fields. Therefore, $|F_1| = p^n$ and $|F_2| = q^m$ for some primes $p, q$. We know that every field is an integral domain. Therefore, by the definition of integral domains, $F_1$ and $F_2$ have no zero-divisors. If we take the direct product $F_1 \times F_2$, we see that $Z(F_1 \times F_2)^* = \{(x_1, 0), ..., (x_{p^n-1}, 0), (0, y_1), ..., (0, y_{q^m-1})\}$.

We can see that $\forall i, j$ such that $1 \leq i \leq p^n$ and $1 \leq j \leq q^m$, we will have $(x_i, 0)(0, y_j) = (0, 0)$, and clearly $(x_i, 0)^2 \neq (0, 0)$ since $x_i \in F_1$ and $F_1$ is an integral domain. Similarly, $(0, y_j)^2 \neq (0, 0)$.

Therefore, we have two distinct sets of vertices $X$ and $Y$, where $X = \{(x_1, 0), ..., (x_{p^n-1}, 0)\}$ and $Y = \{(0, y_1), ..., (0, y_{q^m-1})\}$, and every graph pair of graph vertices in the two sets, $X$ and $Y$ are adjacent, and no two graph vertices within the same set, $X$ or $Y$, is adjacent. Therefore, we have a complete bipartite graph, and by Theorem 2.6., no graph vertex in $X$ or $Y$ is looped. Therefore, if $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields then $\Gamma^*(R)$ is a complete bipartite graph, $K_{p^n-1,q^m-1}$.

$\Rightarrow$ Suppose that the zero-divisor graph of $\Gamma^*(R)$ is a complete bipartite graph, $K_{p^n-1,q^m-1}$ with $n, m > 1$. By Theorem 2.6., no graph vertex in $\Gamma^*(R)$ is looped. Therefore, $R$ contains no nilpotent elements, so by definition is a reduced ring. By Theorem 8.3., we know that a reduced ring is a finite product of fields. Since the graph of $\Gamma^*(R)$ is complete bipartite, then diam($\Gamma^*(R)$)$= 2$ and it contains no looped vertices. Thus, by Theorem 8.4., we know that $R$ is reduced with exactly two minimal primes and at least three nonzero zero divisors, so it is a direct product of two rings. Therefore, if the zero divisor graph $\Gamma^*(R)$ is a complete bipartite graph, $K_{p^n-1,q^m-1}$ where $p, q$ are primes and $m, n > 1$, then $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields. $\square$

We have now established that a complete bipartite graph will be formed when $R$ is a direct product of two fields. By Theorem 8.3., we know that a direct product of fields will be a reduced ring, and therefore its nilradical will be empty.

We now want to create an algorithm for determining a direct product of fields. However, before doing this, we first want to classify the vertices of $\mathbb{F}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{F}_{p_n^{\alpha_n}}$ as follows:

$X_1 = ((1, 0, \ldots, 0))$. $\forall x_1 \in X_1$, $\zeta(x_1) = 0$ and $\delta(x_1) = p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_n^{\alpha_n} - 1$.
$X_2 = ((0, 1, 0, \ldots, 0))$. $\forall x_2 \in X_2$, $\zeta(x_2) = 0$ and $\delta(x_2) = p_1^{\alpha_1} p_3^{\alpha_3} \cdots p_n^{\alpha_n} - 1$.
$\vdots$
$X_n = ((0, \ldots, 0, 1))$. $\forall x_n \in X_n$, $\zeta(x_n) = 0$ and $\delta(x_n) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}} - 1$.
$\vdots$
$Y_1 = ((1, \ldots, 1, 0) - ((1, 0, \ldots, 0) \cup \ldots \cup (0, \ldots, 0, 1, 0)))$. $\forall y_1 \in Y_1$, $\zeta(y_1) = 0$ and $\delta(y_1) = p_n^{\alpha_n} - 1$ vertices.
$\vdots$
$Y_n = ((0, 1, \ldots, 1) - ((0, 1, 0, \ldots, 0) \cup \ldots \cup ((0, \ldots, 0, 1)))$. $\forall y_n \in Y_n$, $\zeta(y_n) = 0$ and $\delta(y_n) = p_1^{\alpha_1} - 1$ vertices.

We can see that for all vertices in $\Gamma^*(R)$, the prime factorization of $\delta(x)$ will consist of a product of the order of the fields minus 1. We should note that we can have $p_i = p_j$. In this case, we will find a $\delta(x) = p_i^{\alpha_i} - 1$ and a $\delta(x) = p_i^{\alpha_i} p_i^{\alpha_j} - 1$. Therefore, we can determine that we have two distinct fields with order $p_i^{\alpha_i}$.

**Example 8.2.** The following two rings are direct product of fields. Let $R_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. We will show a method of determining that we have two fields with the same order $q = 2$ in $R_1$. Let $R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We will show a method of determining that there are three fields of the same order $q = 2$ in $R_2$. The zero-divisor graphs of $\Gamma^*(R_1)$ and $\Gamma^*(R_2)$ are the following:

$\Gamma^*(R_1)$ $\Gamma^*(R_2)$

For all vertices in $\Gamma^*(R_1)$, we can count the different values of $\delta(x)$. We can find 4 distinct values for $\delta(x) = \{1, 2, 3, 5\}$. Therefore, we have a $\delta(x) = 2 - 1$ and a $\delta(y) = 4 - 1$. Therefore,we must have two fields of the same order, specifically 2. And clearly, since we also have a $\delta(x) = 3 - 1$, we must also have a field of order 3. Therefore, we find that this graph is a direct product of $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

For all vertices in $\Gamma^*(R_2)$, we can count the different values of $\delta(x)$. We can find 2 distinct values of $\delta(y) = \{1, 3\}$. Therefore, we have a $\delta(y) = 2 - 1$ and a $\delta(y) = 4 - 1$. This implies that there must be at least two fields of the same order, specifically 2, in the decomposition of $R_2$. Since there are no other values of $\delta(y)$, we cannot have any fields of different orders in our direct product of fields. $R_2$ is not a direct product of 2 fields, since $\Gamma^*(R_2)$ is not complete bipartite. Therefore, we must have more than 2 fields involved of the same order. Since we only have $\delta(y) = 4 - 1$ and $\delta(y) = 2 - 1$, we know that we have 3 fields of order 2. Therefore we have $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 8.2. highlights the complications we have when we have rings that are a direct product of fields, in which two or more distinct fields have the same order. Now we can create an algorithm for $\Gamma^*(R)$, where $R$ is a direct product of fields:

**Algorithm 8.1.** *Determining a Direct Product of Fields*

**1)** Given any $\Gamma^*(R)$, determine whether $Nil(R) = \emptyset$, by observing if none of the vertices in $\Gamma^*(R)$ are looped. If so, by Theorem 8.3., we know we have a direct product of fields.
**2)** Beginning at the end vertices and moving in the zero-divisor graph, look at the vertices with $\delta(x) = p^m - 1$ for some prime $p$.
**3)** Determine the total number of distinct values of $\delta(x)$ whose value is $p^m - 1$. Therefore, we will have a number of distinct values $p_1^{\alpha_1} - 1$, $p_2^{\alpha_2} - 1$, $\ldots$, $p_s^{\alpha_s} - 1$.
**4)** In this step, we will examine if we have distinct fields of the same order.

If we have a complete bipartite graph and one value of $\delta(x) = p_i^{\alpha^i} - 1$, we can determine that we have a direct product of two fields of the same order. If our ring consists of a direct product of fields only of the same order and there are more than 2 distinct fields, then we expect to find $\delta(x_1) = p_i^{\alpha_i} - 1$, $\delta(x_2) = p_i^{2\alpha_i} - 1$, $\delta(x_3) = p_i^{3\alpha_i} - 1, \ldots, \delta(x_k) = p_i^{k\alpha_i} - 1$. In this case, we have $k+1$ distinct fields of the same order.
In the case where we have a direct product of fields, at least two of which have the same order and at least one field that has a different order. If we have $\delta(x_1) = p_i^{\alpha_i} - 1, \delta(x_2) = p_i^{2\alpha^i} - 1, \ldots, \delta(x_k) = p_i^{k\alpha_i} - 1$, then we have $k$ distinct fields of the same order.
This can be seen in Example 8.2.
**5)** Therefore, we can determine that $R \cong \mathbb{F}_{p_1^{\alpha_1}} \times \mathbb{F}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{F}_{p_s^{\alpha_s}}$

Clearly, when we are dealing with a direct product of more than two rings our graph will get extremely complicated. We next will look at direct products of fields in rings of the form $\mathbb{Z}_n$.

# 9 Zero-Divisor Graphs of $\mathbb{Z}_{p_1 \cdots p_m}$

In rings of the form $\mathbb{Z}_n$, we will find that our direct product of fields will be $\mathbb{Z}_{p_1 \cdots p_m}$. Therefore, we can see that we have different restrictions on the order of the fields in the decomposition of $\mathbb{Z}_n$ in comparison to the order of the fields in the decomposition of general Artinian rings. In $\mathbb{Z}_n$, the fields will have order $p$ as $\mathbb{Z}_n$ is a field if and only if $n$ is a prime. In general Artinian rings, in comparison, we may have fields of order $p^m$. We should also note that by Corollary 5.1., we can write the ring $\mathbb{Z}_{p_1 \ldots p_m}$ as $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_m}$ for $p_i \neq p_j$.

Before creating our algorithm for a direct product of fields in $\mathbb{Z}_n$, we will first establish the properties about the nilradical of $\mathbb{Z}_n$ in general.:

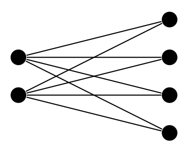
**Theorem 9.1.** *All nilpotent elements of $\mathbb{Z}_n$ must contain the same distinct primes in their prime factorization as the distinct primes in the prime factorization of $n$.*
*Proof.* By definition of a nilpotent element, $x \in \mathbb{Z}_n$ is nilpotent if $x^m = 0$. Since we are looking specifically at the case of $\mathbb{Z}_n$, this means that $x^m \equiv 0 (mod\ n)$. If the prime factorization of $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, then, by the Fundamental Theorem of Arithmetic, the only way to get $x^m = (\prod_{i=1}^{s} p_i^{\alpha_i})$ is if $x \in (\prod_{i=1}^{s} p_i)$. □

Now we can determine the size of the nilradical of $\mathbb{Z}_n$:

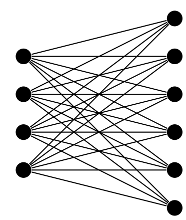
**Theorem 9.2.** *If $R \cong \mathbb{Z}_n$, where the prime factorization of $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, then $|Nil(R)| = n/(\prod_{i=1}^{k} p_i)$.*
*Proof.* Suppose that $m = \prod_{i=1}^{s} p_i$. By Theorem 9.1, we know that if $x \in \mathbb{Z}_n$ is nilpotent, then $x \in (m)$. We can determine $|(m)|$ by observing the total possible number of $k$ to make $km < 0$. In $\mathbb{Z}_n$, $n = 0$. Therefore, we need $km < n$. This means that $k < n/m$. So, we find that we have $n/m$ choices for $k$. Thus, $|Nil(R)| = n/m = n/(\prod_{i=1}^{k} p_i)$. □

Now we will explore the zero-divisor graph structures of $\mathbb{Z}_{p_1 \cdots p_n}$.

**Example 9.1.** Structure of $\Gamma^*(\mathbb{Z}_{pq})$
Let us start with the simplest case, $\Gamma^*(\mathbb{Z}_{pq})$. Using the Chinese Remainder Theorem for Rings and Corollary 5.1 we can decompose the ring. So we have $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$, implying that $\Gamma^*(\mathbb{Z}_{pq}) \cong \Gamma^*(\mathbb{Z}_p \times \mathbb{Z}_q)$. We determined previously that $\mathbb{Z}_p$ will be a field for any prime p. Therefore, both $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are fields, and so $\mathbb{Z}_{pq}$ is the direct product of two fields. We see that $\Gamma^*(\mathbb{Z}_{pq})$ is a complete bipartite graph.



$\Gamma^*(\mathbb{Z}_{15}) \cong \Gamma^*(\mathbb{Z}_3 \times \mathbb{Z}_5)$



$\Gamma^*(\mathbb{Z}_{35}) \cong \Gamma^*(\mathbb{Z}_5 \times \mathbb{Z}_7)$

We also can classify the vertices of these zero-divisor graphs, similarly to how we classified them in general Artinian rings that are a direct product of fields. By the definition of complete bipartite graphs, we will have two distinct sets of vertices:

> $X = \{a \in (p)\}$ and $|X| = q - 1$. For every $x \in X$, we have that $\zeta(x) = 0$ and $\delta(x) = p - 1$.
> $Y = \{b \in (q)\}$ and $|Y| = p - 1$. For every $y \in Y$, we have that every $\zeta(y) = 0$ and $\delta(y) = q - 1$.

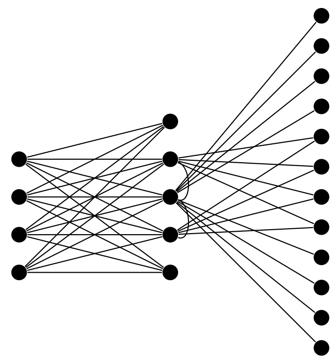
With this classification, we can see now that just by looking at the number of elements in each set, we can easily determine the ring corresponding to the zero-divisor graph. We can follow Algorithm 8.1. to determine the fields involved in this direct product of fields.

We can also note that this structure will have no nilpotent elements, since it is complete bipartite, and as we proved in Theorem 2.6., complete bipartite graphs do not contain any loops. This also is shown by Theorem 9.2., since $|Nil(\mathbb{Z}_{pq})| = 1 - 1 = 0$.

**Example 9.2.** Structure of $\Gamma^*(\mathbb{Z}_{pqr})$
Using the Chinese Remainder Theorem and Corollary 5.1., again we can see that we have $\mathbb{Z}_{pqr} \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$. Therefore, we can see that this ring is a direct product of fields. The more distinct primes that we have involved in the prime factorization of $n$, the more complicated the graph will get because this correlates to more fields involved in the decomposition of $\mathbb{Z}_n$.

We can classify the vertices of $\Gamma^*(\mathbb{Z}_{pqr})$ with the following classification:

> $X = \{a \in (pq)\}$. Every vertex $x \in X$ will have $\zeta(x) = 0$, $\delta(x) = pq - 1$, and $|X| = r - 1$.
> $Y = \{b \in (qr)\}$. Every vertex $y \in Y$ will have $\zeta(y) = 0$, $\delta(y) = qr - 1$, and



$\Gamma^*(\mathbb{Z}_{30}) \cong \Gamma^*(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$

$|Y| = p - 1$.
$Z = \{c \in (pr)\}$. Every vertex $z \in Z$ will have $\zeta(z) = 0$, $\delta(z) = pr - 1$ and $|Z| = q - 1$.
$W = \{d \in (r) - ((pr) \cup (qr))\}$. Every vertex $w \in W$ will have $\zeta(w) = 0$, $\delta(w) = r - 1$.
$R = \{e \in (p) - ((pq) \cup (pr))\}$. Every vertex $r \in R$ will have $\zeta(r) = 0$, $\delta(r) = p - 1$.
$S = \{f \in (q) - ((qr) \cup (qp))\}$. Every vertex $s \in S$ will have $\zeta(s) = 0$, $\delta(s) = q - 1$.

The zero-divisor graph will act similarly for a finite amount of primes in the prime factorization of $n$. We expect to have no nilpotent elements, and we will use $\delta(x)$ of a vertex $x$, to determine the primes involved in our factorization of $n$. So now we can now generate an algorithm for determining if the ring of the form $\mathbb{Z}_n$ in the zero-divisor graph is a direct product of fields:

**Algorithm 9.1** *Let $R \cong \mathbb{Z}_{p_1 p_2 \cdots p_s}$. Determine $R$ knowing the structure of* $\Gamma^*(R)$

**1)** Determine whether $Nil(R) = \emptyset$. If so, by Theorem 8.3., we know that the corresponding ring is of the form $\mathbb{Z}_{p_1 p_2 \cdots p_s}$, and we now have to determine the primes.
**2)** Beginning at the end vertices and moving in the graph, look at the vertices that have $\delta(x) = p - 1$ for some prime $p$.
**3)** Determine the total number of distinct values of $\delta(x)$, whose value is a prime minus one, which will be $p_1 - 1$, $p_2 - 1$,. . ., $p_s - 1$.
**4)** Now we can determine $R = \mathbb{Z}_{p_1 p_2 \cdots p_s}$.

Clearly, this algorithm is very similar to Algorithm 8.1., however there are two significant diffferences. The first difference is that the order of a field of the form $\mathbb{Z}_n$ is $p$, in comparison to a general Artinian field, which has order $p^m$. Therefore, the values of $\delta(x)$, we observe will be $p - 1$ in comparison to $p^m - 1$ for some prime $p$. The second difference is that we do not have step (4) of Algorithm 8.1.. This is because we will not have $p_i = p_j$ since $\mathbb{Z}_{p_i^2 p_1 \cdots p_n} \neq \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_n}$, so we will not have two distinct fields of the same order.

# 10 The Nilradicals of Direct Products of Local Rings and Fields

At this point, we have established properties about local rings, fields, and direct products of fields. Before we explore direct products of local rings and direct products of local rings with fields, we want to establish the following property about the nilradical of an Artinian ring. We are especially interested in the nilradical because we can easily identify it in a zero-divisor graph, and it is simpler and therefore easier to visualize. It also still enables us to extract information about the ring from the zero-divisor graph.

**Proposition 10.1.** *If $R \cong L_1 \times \cdots \times L_n$, where $L_1, \ldots, L_n$ are all local rings, then $Nil(R) \cong (Z(L_1) \times \cdots \times Z(L_n))$.*
*Proof.* By Fact 2.2., we know that the maximal ideal for a local ring is $Z(L)$, and so by Theorem 3.3., we know that the maximal ideals of $R \cong L_1 \times \cdots \times L_n$ will be $(Z(L_1) \times L_2 \times \cdots \times L_n)$, ..., $(L_1 \times \cdots \times L_{n-1} \times Z(L_n))$. Since the Jacobson Radical equals the Nilradical in Artinian rings, we find that $Nil(R) \cong (Z(L_1) \times \cdots \times Z(L_n))$. □

Therefore, since the maximal ideal of a field is (0), we can see that the maximal ideal of $F_1 \times \ldots \times F_n$ is $((0) \times \cdots \times (0))$. This result for a direct product of fields is expected since the product of fields is a reduced ring, and by definition, reduced rings have no nilpotent elements. In addition, we can see that $\Gamma^*(Nil(L \times F_1 \times \cdots \times F_n)) \cong \Gamma^*(Nil(L)) \cong \Gamma^*(L))$. It also allows us to visually identify certain properties about rings immediately from our zero-divisor graph. For example, if we see a zero-divisor graph with no looped vertices, then clearly it is corresponding to a ring that is a direct product of fields. We will find similar results for zero-divisor graphs corresponding to a direct product of local rings, as well as those corresponding to a direct product of local rings and fields.

# 11 Zero-Divisor Graphs of a Direct Product of Fields and Local Rings

We have examined the zero-divisor graphs of fields, local rings (that are not fields), and the direct product of fields. We now want to examine the zero-divisor graphs of direct products of local rings and fields. Our direct products will get more complicated, therefore we will use Proposition 10.1. to examine the zero-divisor graph of nilradical of the ring, in addition to the zero-divisor graph, to extract information.

We first want to create an algorithm for the direct product of local rings (that are all not fields):

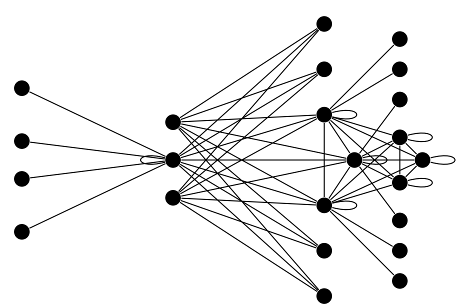
**Algorithm 11.1.** *Determining a Direct Product of Local Rings*

**1)** Determine if $|Nil(R)| \neq |Z(R)|$ and if $Nil(R) \neq \emptyset$. If so, look at $\Gamma^*(Nil(R))$ by observing the zero-divisor graph formed by only the looped vertices.
**2)** Determine the $\delta(x)$ of the vertices that are not looped or adjacent to any

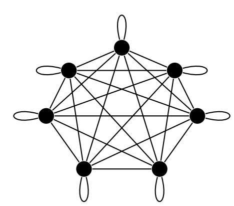
looped vertices. If there are no such vertices, then we have a direct product of local rings.
**3)** Observe $|\Gamma^*(Nil(R))|$. By Proposition 10.1., we know that $Nil(R) \cong (Z(L_1) \times \cdots \times Z(L_n))$. So, $|Nil(R)| = |Z(L_1)||Z(L_2)| \cdots |Z(L_m)|$.
We can use Lagrange's Theorem and observe $|\Gamma^*(R)|$ to determine $|Z(L_1)|, \cdots, |Z(L_m)|$. This can be seen in Example 11.1.
**4)** Once we have done this, we can look at the $\delta(x)$ of unlooped vertices to determine $|L_1|, \ldots, |L_m|$.
Therefore, we know the size of the zero-divisor graphs of the local rings involved in the decomposition of $R$, so $|R| = |L_1 \times \cdots \times L_m|$.

This algorithm is powerful because it enables us to identify zero-divisor graphs of rings that are a direct product of local rings (that are not fields) by observing that all of the vertices in $\Gamma^*(R)$ are looped or adjacent to looped vertices. However, step 3 of Algorithm 11.1 relies on the use of algebraic properties to eliminate possible $|Z(L_i)|$, which can get relatively convoluted. It also requires us to determine $|\Gamma^*(R)|$, which can be difficult when $R$ is large. We will next provide an example of the application of the algorithm:

**Example 11.1.** *Structure of* $R = L_1 \times L_2$
Suppose that $R \cong \mathbb{Z}_2[x,y]/(x^2, xy, y^2) \times \mathbb{Z}_4$, we can apply our Algorithm 11.1 to determine that we have a direct product of two local rings, one of order 8, for example $\mathbb{Z}_2[x,y]/(x^2, xy, y^2)$, and the other of order 4, for example $\mathbb{Z}_4$.



$\Gamma^*(R)$



$\Gamma^*(Nil(R))$

**1)** $|Nil(R)| \neq |Z(R)|$.
**2)** All of the vertices are looped or adjacent to looped vertices. Therefore, we have a direct product of local rings (not fields).
**3)** We can determine that $|\Gamma^*(Nil(R))| = 7$, implying that $|Nil(R))| = 8$. The size of each of the maximal ideals of the local rings involved in the direct product must multiply by one another to get 8. So, we either have three local rings with $|Z(L_1)| = |Z(L_2)| = |Z(L_3)| = 2$, or we have $|Z(L_1)| = 4$ and $|Z(L_2)| = 2$. However, we cannot have three local rings each of whose maximal ideal has size 2, because by Lagrange's Theorem this implies that each $|L_1|, |L_2|, |L_3| \geq 4$. Therefore, there would be at least 36 zero-divisors in $R$, since any $(x, y, z)$ having $x = 0, y = 0$, or $z = 0$, will be a zero-divisor. This contradicts $|\Gamma^*(R)| = 23$.

Therefore, we must have two rings where $|Z(L_1)| = 4$ and $|Z(L_2)| = 2$. This would imply that $|L_1| \geq 8$, $|L_2| \geq 4$. Therefore, we must have at least 10 vertices in $R$, which we do.
**4)** We can then look at the zero-divisor graph to determine $|R| = |L_1 \times L_2|$. If we observe $\delta(x)$ of any unlooped vertex, $x$ we find $\delta(x) = 1, 3$, or 7. This implies that our local rings must have $|L_1| = 8$ and $|L_2| = 4$, since we cannot have a local commutative, unital ring of order 2. Therefore, $|R| = 32$.

In this example, we can see that we can determine up to the size of $|L_1|$ and $|L_2|$, however we are unable to specifically identify what ring the zero-divisor graph represents. The power of this algorithm is that it enables us to narrow down the possibilities of associated rings.

We next want to examine direct products involving local rings (that are not a fields) with fields. We will use the following algorithm to identify these types of rings:

**Algorithm 11.2.** *Determining a Direct Product of Local Rings with Fields*

**1)** Determine if $|Nil(R)| \neq |Z(R)|$ and if $Nil(R) \neq \emptyset$. If so, we will be primarily interested $\Gamma^*(Nil(R))$.
**2)** Determine the $\delta(x)$ of the vertices that are not looped or adjacent to any looped vertices, these will correspond to the orders of the fields involved in the direct product. If there are no such values, then we have a direct product of local rings, so refer to Algorithm 11.1.
If there are such values, determine the total number of distinct values of $\delta(x)$ whose value is $p^m - 1$. We will have distinct values $p_1^{\alpha_1} - 1$, $p_2^{\alpha_2} - 1$, ..., $p_m^{\alpha_m} - 1$.
If we have $\delta(x_1) = p_i^{\alpha_i} - 1, \delta(x_2) = p_i^{2\alpha^i} - 1, \ldots, \delta(x_k) = p_i^{k\alpha_i} - 1$, then we have $k$ distinct fields of the same order.
**3)** Observe $\Gamma^*(Nil(R))$. Determine the possible methods of constructing $\Gamma^*(Nil(R))$ to narrow down the possible local rings (not fields) involved in the decomposition of $R$.
**4)** For any ring $R = L_1 \times \cdots \times L_n \times \mathbb{F}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{F}_{p_m^{\alpha_m}}$, we will have an element $a = (0, \ldots, 0, f)$ where $f \in \mathbb{F}_{p_m^{\alpha_m}}$. Furthermore, $\delta(a) = x_1 \cdots x_n p_1^{\alpha_1} \cdots p_{m-1}^{\alpha_{m-1}}$ where $x_1, \ldots, x_n$ are the respective order of $L_1, \ldots, L_n$ and $p_1^{\alpha_1}, \ldots, p_{m-1}^{\alpha_{m-1}}$ are the respective orders of $\mathbb{F}_{p_1^{\alpha_1}}, \ldots, \mathbb{F}_{p_{m-1}^{\alpha_{m-1}}}$.
In step 2, we determined the values of $p_1^{\alpha_1}, \ldots, p_{m-1}^{\alpha_{m-1}}$, and so we have $(\delta(a) + 1)/(p_1^{\alpha_1} \cdots p_{m-1}^{\alpha_{m-1}}) = x_1 \cdots x_n$.
Therefore, we can look at our findings in step 3 and determine the order of the local rings involved, by also observing the complexity of the structure of $\Gamma^*(R)$.
**5)** Therefore, we can determine that $R = L_1 \times \cdots \times L_n \times \mathbb{F}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{F}_{p_m^{\alpha_m}}$

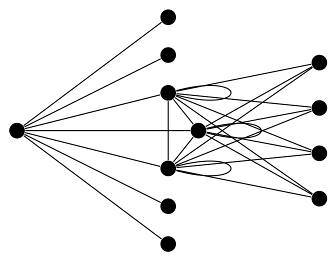
This algorithm will be harder to apply to larger rings. Example 11.2. below demonstrates an application of this algorithm on three different rings that are direct products of local rings with fields. We will also see examples applying a variation of this algorithm to $\Gamma^*(\mathbb{Z}_n)$, where local rings have a principal maximal ideal. This algorithm also holds for all types of local Artinian rings, whether the maximal ideal is principal or not.

**Example 11.2.** *Structure of $R_1 \cong L_1 \times F_1$, $R_2 \cong L_1 \times F_1$, and $R_3 \cong L_1 \times L_2 \times F_1$*

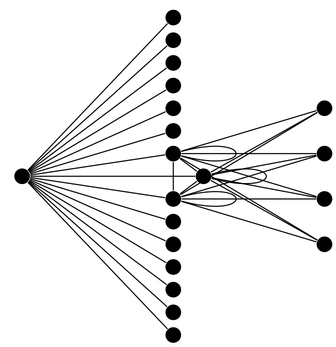
Below we have three direct products of local rings with fields. We will show how our Algorithm 11.2. can differentiate between each of the zero-divisor graphs. Suppose that:

$R_1 = \mathbb{Z}_2[x, y]/(x^2, xy, y^2) \times \mathbb{Z}_2$
$R_2 = \mathbb{F}_4[x]/(x^2) \times \mathbb{Z}_2$
$R_3 = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$.

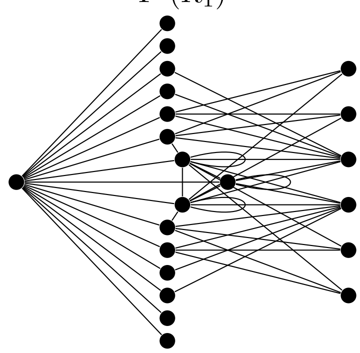
Therefore, generally our rings are $R_1 \cong L_1 \times F_1$, $R_2 \cong L_1 \times F_1$, and $R_3 \cong L_1 \times L_2 \times F_1$.



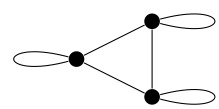
$\Gamma^*(R_1)$



$\Gamma^*(R_2)$



$\Gamma^*(R_3)$



$\Gamma^*(Nil(R_1)) \cong \Gamma^*(Nil(R_2)) \cong \Gamma^*(Nil(R_3))$

**1)** In all of these zero-divisor graphs, $|Nil(R)| \neq |Z(R)|$.
**2)** Without knowing the associated ring to the zero-divisor graph, we can then look at the $\delta(x)$ of the vertices that are not looped or adjacent to any looped vertices. In $\Gamma^*(R_1)$, $\Gamma^*(R_2)$, and $\Gamma^*(R_3)$, we find one distinct value $\delta(x) = 1$. Therefore, we have a field of order 2, $\mathbb{F}_2 \cong \mathbb{Z}_2$ involved in our direct product in all of these rings.
**3)** The $\Gamma^*(Nil(R_1)) \cong \Gamma^*(Nil(R_2)) \cong \Gamma^*(Nil(R_3))$ is a $K_3$ graph. The only way of attain this graph structure is if $|R| = 8$ or $|R| = 16$ [11]. Therefore, we have three options: 1) $\Gamma^*(Nil(R)) \cong \Gamma^*(L)$ where $|L| = 8$, 2) $\Gamma^*(Nil(R)) \cong \Gamma^*(L)$ where $|L| = 16$, or 3) $\Gamma^*(Nil(R)) \cong \Gamma^*(L_1 \times L_2))$ where $|L_1| = |L_2| = 4$.
**4)** In step 2, we found that all of these graphs contain one field, $\mathbb{Z}_2$. Therefore, there will be one vertex $y$ of the form $(0, 1)$ or $(0, 0, 1)$ depending on how many

local rings are involved in the direct product. In the case of $R \cong L_1 \times F_1$, this $y$ will be adjacent to all the vertices of the form $(a, 0)$ where $\forall a \in L_1$. Therefore, $\delta(y) = |L_1| - 1$. In the case of $R \cong L_1 \times L_2 \times F_2$, $y$ will be adjacent to $(a, b, 0)$ where $\forall a \in L_1$ and $\forall b \in L_2$. Therefore $\delta(y) = |L_1||L_2| - 1$.

**5)** We can see that in $\Gamma^*(R_1)$, this vertex $y$ has $\delta(y) = 7$. This implies that we have one local ring $L_1$ where $|L_1| = 8$. Therefore, we have determined that $\Gamma^*(R_1)$ has $R_1 \cong L_1 \times F_1$ where $|L_1| = 8$ and $|F_1| = 2$.

In $\Gamma^*(R_2)$, we see that this vertex $y$ has $\delta(y) = 15$. Therefore, we could have two local rings of order 4 each, or we could have just one local ring of order 16. When we look at $\Gamma^*(R_2)$, we see that the extensions from the complete bipartite graph that is formed by $F_1$ and $L$, are only adjacent to the looped vertices. Therefore, this simpler structure implies that we only have one local ring involved in our direct product. Therefore, we can conclude that $R_2 \cong L_1 \times F_1$ where $|L_1| = 16$ and $|F_1| = 2$.

In $\Gamma^*(R_3)$, we find that $\delta(y) = 15$. However, since the zero-divisor graph structure is more complicated, we can determine that we have the product of two local rings, which we previously determined must be order 4. Therefore, we can conclude that $R_3 \cong L_1 \times L_2 \times F_1$.

Algorithm 11.2. is powerful as it enables us to conclude that the corresponding ring $R$ to $\Gamma^*(R)$ is of the form $R = L_1 \times \cdots \times L_n \times F_1 \times \cdots \times F_m$ if $\Gamma^*(R)$ contains both looped and not looped vertices, and there exists at least one vertex that is not looped and not adjacent to any looped vertex. However, similarly to Algorithm 11.1., this algorithm becomes difficult to apply when $R$ is large.

# 12 Zero-Divisor Graphs of $\mathbb{Z}_{p_1^{\alpha_1} \cdots p_m^{\alpha_m}}$

We now want to observe the direct product of local rings and the direct product of local rings with fields in rings of the form $\mathbb{Z}_n$. We will begin by looking at the simplest example, $\Gamma^*(\mathbb{Z}_{p^m q})$. This is a direct product of a local ring with a field.

**Example 12.1.** Structure of $\Gamma^*(\mathbb{Z}_{p^m q})$

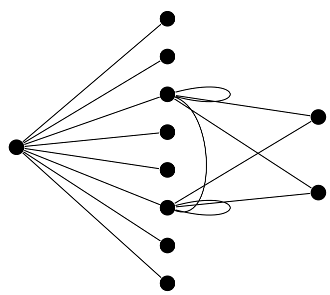
By Theorem 6.2., we know that $|Nil(\mathbb{Z}_{p^m q})| = p^{m-1}$. Immediately, we see a correspondence between this and $\Gamma^*(\mathbb{Z}_{p^m})$ since $|Nil(\mathbb{Z}_{p^m q})| = |Nil(\mathbb{Z}_{p^m})|$. In fact, we can extend this argument, and say that not only is the size of the nilpotent set the same, but $Nil(\mathbb{Z}_{p^m q}) \cong Nil(\mathbb{Z}_{p^m})$, and therefore $\Gamma^*(Nil(\mathbb{Z}_{p^m q}) \cong \Gamma^*(Nil(\mathbb{Z}_{p^m})) \cong \Gamma^*(\mathbb{Z}_{p^m})$.

The difference is that $\mathbb{Z}_{p^m q}$ is decomposable into $\mathbb{Z}_{p^m} \times \mathbb{Z}_q$ by Corollary 5.1, since $p^m$ and $q$ are relatively prime. We can observe the interaction of the vertices in the following sets, $(pq)$, $(q)$, $(p)$, $(p^2) - (p), \ldots, (p^m) - (p^{m-1})$. All the vertices in $(q)$ will be adjacent to all of the vertices in $(p^m) - (p^{m-1})$, since $qp^m = 0$. Therefore, they will form a complete bipartite structure. However, $\Gamma^*(\mathbb{Z}_{p^m})$ is not a complete bipartite graph because there are vertices in $(pq)$ are nilpotent elements. Since $pq^m = 0$, these elements in $(pq)$ will also be adjacent to vertices in $(p)$,$(p^2) - (p), \ldots, (p^{m-1}) - (p^{m-2})$. Therefore, we will form extensions off the complete bipartite graph.

We will see that the interior complete bipartite graph will have two distinct sets of vertices, $X$ and $Y$. The classification of vertices is as follows:

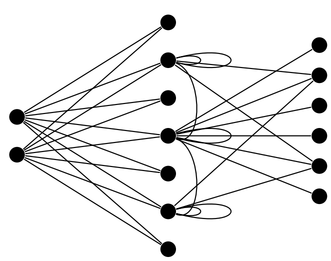
$X = \{b \in (q)\}$ and $|X| = p^m - 1$.
$Y_m = \{a \in (p^m)\}$ and $|Y_m| = q - 1$.
$Y_{m-1} = \{c \in (p^{m-1}) - (p^m)\}$ and $|Y_{m-1}| = q^{s-1} - q^{s-2}$.
$\vdots$
$Y_1 = \{d \in (p) - (\bigcup_{i=2}^{n} p^i)\}$ and $|Y_1| = q^2 - q$. Every vertex $y_1 \in Y_1$ will have $\delta(y_1) = p - 1$.

Similar to in $\Gamma^*(\mathbb{Z}_{p^m})$, this last set of vertices generated by $(p) - (\bigcup_{i=2}^{n} p^i)$ will enable us to understand the prime involved, because $\forall x \in (p) - (\bigcup_{i=2}^{n} p^i)$ we find that $\delta(x) = p - 1$.

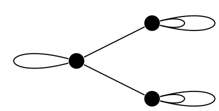


$\Gamma^*(\mathbb{Z}_{18}) \cong \Gamma^*(\mathbb{Z}_2 \times \mathbb{Z}_{3^2})$

$\Gamma^*(Nil(\mathbb{Z}_{18})) \cong \Gamma^*(\mathbb{Z}_{3^2})$



$\Gamma^*(\mathbb{Z}_{24}) \cong \Gamma^*(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})$



$\Gamma^*(Nil(\mathbb{Z}_{24})) \cong \Gamma^*(\mathbb{Z}_{2^3})$

**Algorithm 12.1.** *Let $R \cong \mathbb{Z}_{p^m p_1 p_2 \cdots p_s}$. Determine $R$ knowing the structure of $\Gamma^*(R)$*

**1)** Determine whether $|Nil(R)| \neq |Z(R)|$ and that $Nil(R) \neq \emptyset$. If so, look at $\Gamma^*(Nil(R))$.
**2)** If $\Gamma^*(Nil(R)) \cong \Gamma^*(\mathbb{Z}_{p^m})$, then we know that we have one prime of degree greater than 1 involved and some undetermined amount of other distinct primes. Furthermore, we know that $p^m$ will be the prime with the degree $m$ where $m > 1$.
**3)** Beginning at the end vertices and moving in the graph, look at the vertices that has $\delta(x) = q - 1$ for some prime $q$.
**4)** Determine the total number of distinct values of $\delta(x)$, whose value is a prime minus one. Therefore, we will find values $p_1 - 1$, $p_2 - 1$,. . ., $p_s - 1$

where $p_i \neq p_j$.

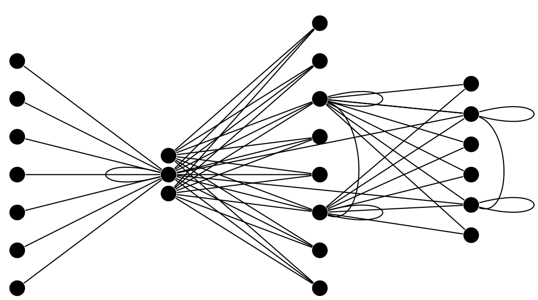
**5)** So we now can determine $R \cong \mathbb{Z}_{p^m p_1 p_2 \cdots p_s}$.

**Example 12.2.** Structure of $\Gamma^*(\mathbb{Z}_{p^m q^s})$

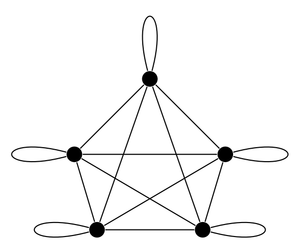
The structure of $\Gamma^*(\mathbb{Z}_{p^m q^s})$ will get complicated very fast, as we will have larger zero-divisor graphs with numerous nilpotent elements. By Corollary 5.1., we know that $\mathbb{Z}_{p^m q^s}$ is decomposable into $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^s}$. We expect to see an interior complete bipartite graph being formed from sets $X_m$ and $Y_s$. However as opposed to $\Gamma^*(\mathbb{Z}_{p^m q})$ where there was extensions off of one side, in $\Gamma^*(\mathbb{Z}_{p^m q^s})$ there will be extensions and nilpotent elements off of both sides of this interior complete bipartite graph.

Let us first classify the vertices, so we can better understand how this graph will be formed.

$X_m = \{a \in (p^m)\}$ and $|X_m| = q^s - 1$.
$X_{m-1} = \{c \in (p^{m-1}) - (p^m)\}$ and $|X_{m-1}| = q^{s-1} - q^{s-2}$.
$\vdots$
$X_1 = \{d \in (p) - (\bigcup_{i=2}^{n} p^i)\}$ and $|X_1| = q^2 - q$.
$Y_s = \{a \in (q^s)\}$ and $|Y_s| = p^m - 1$.
$Y_{s-1} = \{e \in (q^{s-1}) - ((q^s) \cup (\bigcup_{i=2}^{n} p^i))\}$ and $|Y_{s-1}| = p^{m-1} - p^{m-2}$.
$\vdots$
$Y_1 = \{f \in (q) - ((\bigcup_{i=2}^{n} q^i) \cup (\bigcup_{i=2}^{n} p^i))\}$ and $|Y_1| = p^2 - p$.



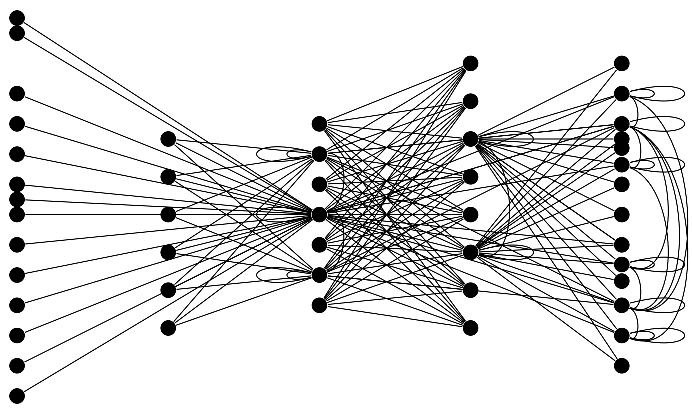
$\Gamma^*(\mathbb{Z}_{36}) \cong \Gamma^*(\mathbb{Z}_{3^2} \times \mathbb{Z}_{2^2})$



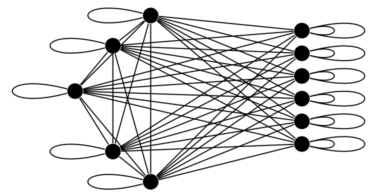
$\Gamma^*(Nil(\mathbb{Z}_{36}))$

As you can see, the zero-divisor graphs will get extremely complicated the bigger our zero-divisor graph gets. Therefore, we can look at the behavior of the nilradical zero-divisor graph instead. We noted before that $\Gamma^*(Nil(\mathbb{Z}_{p^m q})) \cong \Gamma^*(Nil(\mathbb{Z}_{p^m}))$. Therefore, by looking at the nilradical graph as well as the zero-divisor graph, we can determine that we are dealing with a zero-divisor of $\mathbb{Z}_n$ where $n$ has the prime factorization consisting of one prime to the degree greater than 1, and some other primes.

In $\Gamma^*(Nil(\mathbb{Z}_{p^m q^s}))$, we can determine the prime to the greatest degree based on the structure of the nilradical graph. For example, in $\Gamma^*(Nil(\mathbb{Z}_{36}))$, we have a graph structure similar to the $\Gamma^*(\mathbb{Z}_{p^2})$, so we expect the greatest degree of prime to be 2. However, we also know that we do not simply have one prime of degree greater than 1 involved for two reasons. The first is the fact that our graph of $\Gamma^*(\mathbb{Z}_{36})$ is complicated and has extensions on both sides of



$\Gamma^*(\mathbb{Z}_{72}) \cong \Gamma^*(\mathbb{Z}_{3^2} \times \mathbb{Z}_{2^3})$



$\Gamma^*(Nil(\mathbb{Z}_{72}))$

the interior complete bipartite graph. The second is that we can see that in the nilradical graph we do not have $|\Gamma^*(Nil(\mathbb{Z}_{36}))| = p - 1$. Instead, we have $|\Gamma^*(Nil(\mathbb{Z}_{36}))| = pq - 1$.

Similarly, for $\Gamma^*(Nil(\mathbb{Z}_{72}))$, we can tell that the greatest degree of prime we have involved is 3, since the graph of the nilradical looks similar to $\Gamma^*(p^3)$. We can also tell that we have more than one prime of degree greater than 1 involved in the prime factorization of $n$, since $|\Gamma^*(Nil(\mathbb{Z}_{72}))| = p^2q - 1$.

**Corollary 12.1** *If* $|Z(\mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}})^*| = m - 1$, *then* $m = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}$.

*Proof.* In Theorem 6.2., we know that if we have the prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, then $|Nil(R)| = n/(\prod_{i=1}^{k} p_i)$. Therefore, $Z(R)^* = n/(\prod_{i=1}^{k} p_i) - 1$. So, $m - 1 = n/(\prod_{i=1}^{k} p_i) - 1$. So $m = n/(\prod_{i=1}^{k} p_i)$. Therefore, $m = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}$. □

**Algorithm 12.2.** *Let* $R \cong \mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$. *Determine* $R$ *knowing the structure of* $\Gamma^*(R)$

**1)** Determine whether $|Nil(R)| \neq |Z(R)|$, and that $Nil(R) \neq \emptyset$. If we have neither then we can proceed to the next step.
**2)** If $\Gamma^*(Nil(R)) \cong \Gamma^*(\mathbb{Z}_{p^m})$, then we know that we have one prime of degree greater than 1 involved and some undetermined amount of other distinct primes, so go back to Algorithm 12.1.. If not, we look at $|\Gamma^*(Nil(R))| = m - 1$.
**3)** Since $m$ is not a prime, we take the prime factorization of $m$. This will be $m = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}$.
**4)** Beginning at the end vertices and moving in the graph, look at the vertices that has $\delta(x) = q - 1$ for some prime $q$.
**5)** Determine the total number of distinct values of $\delta(x)$ whose value is a prime minus 1, so we will get $p_1 - 1$, $p_2 - 1$,. . ., $p_s - 1$ for each $p_i \neq p_j$.
**6)** If there all of these primes are in the prime factorization of $m$, then we have $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. Therefore, $R \cong \mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$

Now our last type of $\Gamma^*(\mathbb{Z}_n)$ will be when $R \cong \mathbb{Z}_{p_1^{\alpha_1}\cdots p_s^{\alpha_s}p_r\cdots p_t}$. This is equivalent to a direct product of local rings (that are not fields) and fields, since by Corollary 5.1.,

$\mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}} \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}} \times \mathbb{Z}_{p_r} \times \cdots \times \mathbb{Z}_{p_t}$. This gets very complicated very fast, in fact the simplest example of this type of ring is $\Gamma^*(\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5) \cong \Gamma^*(\mathbb{Z}_{180})$.

We can however determine an algorithm for this type of ring. It will be the following:

**Algorithm 12.3.** *Let $R \cong \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} p_r \cdots p_t}$. Determine $R$ knowing the structure of $\Gamma^*(R)$*

**1)** Determine whether $|Nil(R)| \neq |Z(R)|$, and that $Nil(R) \neq \emptyset$. If we have neither then we can proceed to the next step.
**2)** If $\Gamma^*(Nil(R)) \cong \Gamma^*(\mathbb{Z}_{p^m})$, then we know that we have one prime of degree greater than 1 involved and some underdetermined amount of other distinct primes, so go back to Algorithm 12.1.. If not, we look at $|\Gamma^*(Nil(R))| = m - 1$.
**3)** Since $m$ is not a prime, we take the prime factorization of $m$. This will be $m = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_k^{\alpha_k - 1}$
**4)** Beginning at the end vertices and moving in the graph, look at the vertices that has $\delta(x) = q - 1$ for some prime $q$.
**5)** Determine the total number of distinct values of $\delta(x)$ whose value is a prime minus 1, which will be $p_1 - 1$, $p_2 - 1$,..., $p_s - 1$.
**6)** If there are additional distinct primes that are not in the prime factorization of $m$, then we have $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} p_r \cdots p_t$, where $p_r, \ldots, p_t$ are the distinct primes not in the prime factorization of $m$.

We have now examined all of the possible types of rings of the form $\mathbb{Z}_n$ and the behaviors of their corresponding zero-divisor graphs. We next want to generally apply an algorithm for $\Gamma^*(\mathbb{Z}_n)$ that will allow us to determine the associated ring to a zero-divisor graph.

# 13 Algorithms for determining the Algebraic structure of Artinian Rings from the Zero-Divisor Graphs

We have now examined all the cases of Artinian rings. We started by identifying the zero-divisor graph structures of fields and local rings, and built up to more complicated Artinian rings using direct products. Now, in order to tie all of this together, we will create a general algorithm that we can use when approaching a zero-divisor graph of an Artinian ring.

**Algorithm 13.1.** *Determine an Artinian ring $R$ knowing the structure of* $\Gamma^*(R)$

**1)** Observe $\Gamma^*(Nil(R))$
**2)**. If $\Gamma^*(Nil(R)) \cong \Gamma^*(R)$, then follow Algorithm 4.1.
**3)** If Algorithm 4.1. fails, but $\Gamma^*(Nil(R)) \cong \Gamma^*(R)$, then follow Algorithm 7.1.
**4)** If $\Gamma^*(Nil(R)) = \emptyset$, then follow Algorithm 8.1.
**5)** If $\forall x \in \Gamma^*(R)$ are adjacent to a looped vertex or is looped itself, then follow Algorithm 11.1.
**6)** If none of the above steps hold, then follow Algorithm 11.2.

This algorithm provides an approach for determining the associated rings to a zero-divisor graph. We saw throughout the paper that our specific Algorithms are unable to always get a specific ring, however we can find out a lot of information such as order, type of ring, and the number of rings in the decomposition.

Throughout this paper, we also observed algorithms for specifically identifying rings of the form $\mathbb{Z}_n$ given the zero-divisor graph. The following algorithm provides us with a method of approaching zero-divisor graphs with the knowledge that we have a ring of the form $\mathbb{Z}_n$:

**Algorithm 13.1.** *Determine* $\mathbb{Z}_n$ *knowing the structure of* $\Gamma^*(\mathbb{Z}_n)$

**1)** Observe $\Gamma^*(Nil(\mathbb{Z}_n))$
**2)**. If $\Gamma^*(Nil(\mathbb{Z}_n)) \cong \Gamma^*(\mathbb{Z}_n)$, then follow Algorithm 6.1.
**3)** If $\Gamma^*(Nil(\mathbb{Z}_n))$ is empty, then follow Algorithm 9.1.
**4)** If $\Gamma^*(Nil(\mathbb{Z}_n)) \cong \Gamma^*(\mathbb{Z}_{p^m})$, then follow Algorithm 12.1.
**5)** If $\Gamma^*(Nil(\mathbb{Z}_n))$ is neither of the above and $|\Gamma^*(Nil(\mathbb{Z}_n))| = m$, then follow Algorithm 12.2. or 12.3.

Both of these algorithms leave us questioning the uniqueness. The next section will explore whether we can uniquely correspond a ring, or an algebraic structure of a ring, to a zero-divisor graph.

# 14 Uniqueness of Zero-Divisor Graphs

We have now determined algorithms that allow us to identify the possible associated rings to zero-divisor graphs. Ideally we would like to narrow down the set of associated rings to a zero-divisor graph to as small a set as possible. Therefore, we want to explore when a zero-divisor graph corresponds to a unique ring. We also want to determine when a zero-divisor graph does not correspond uniquely to a ring, what other rings have the same zero-divisor graph.

We will first examine the uniqueness of the zero-divisor graphs corresponding to rings of the form $\mathbb{Z}_n$. The following theorem establishes that for each distinct $n$, we will have a distinct zero-divisor graph, $\Gamma^*(\mathbb{Z}_n)$.

**Theorem 14.1.** $\Gamma^*(\mathbb{Z}_n) \cong \Gamma^*(\mathbb{Z}_m)$ *if and only if* $m = n$.

*Proof.* $\Rightarrow$ Suppose that the prime factorizations of $n$ and $m$ are $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, and $m = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$. If $\Gamma^*(\mathbb{Z}_n) \cong \Gamma^*(\mathbb{Z}_m)$, then $|Nil(\mathbb{Z}_n)| = |Nil(\mathbb{Z}_m)|$. Now, suppose for a contradiction, that $m \neq n$. We know from Theorem 9.2. that $|Z(\mathbb{Z}_n)| = n/(\prod_{i=1}^k p_i)$ and $|Z(\mathbb{Z}_m)| = m/(\prod_{j=1}^l q_i)$. Therefore, we must have $n/(\prod_{i=1}^k p_i) = m/(\prod_{j=1}^l q_i)$. If $\prod_{i=1}^k p_i = \prod_{j=1}^l q_i$, then $n = m$, which is a contradiction. Therefore, $\prod_{i=1}^k p_i \neq \prod_{j=1}^l q_i$, so there is some $p_i \neq q_j$. Since $n/(\prod_{i=1}^k p_i) = m/(\prod_{j=1}^l q_i)$, we must have $p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} = q_1^{\alpha_1 - 1} \cdots q_s^{\alpha_s - 1}$. However, since $p_i \neq q_i$, this equality is a contradiction by the Fundamental Theorem of Arithmetic. Therefore, we must have $m = n$.
$\Leftarrow$ Clearly if $m = n$, then $\Gamma^*(\mathbb{Z}_n) \cong \Gamma^*(\mathbb{Z}_m)$. $\square$

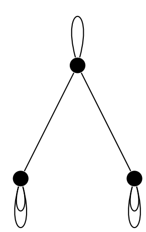
Theorem 14.1. allows us to understand that the corresponding zero-divisor graphs $\Gamma^*(R)$ to a ring $R \cong \mathbb{Z}_n$ is unique. Therefore, if we are given that $\Gamma^*(R)$ is a zero-divisor of a ring of the form $\mathbb{Z}_n$, we can determine the exact ring corresponding to the zero-divisor graph using simply the graph structure.

However, this uniqueness does not hold when examining zero-divisor graphs of Artinian rings. In Example 7.1 and Example 7.2., we saw numerous rings corresponding to each of the zero-divisor graphs. Even when looking at $\mathbb{Z}_n$, we cannot associate a ring uniquely to a zero-divisor graph unless we are given that $R$ is of the form $\mathbb{Z}_n$. This is shown in our next example where we find a multiplicative isomorphism between $Z(\mathbb{Z}_p[x]/(x^m))$ and $Z(\mathbb{Z}_{p^m})$. If we define a zero-divisor graph that enables us to visualize addition between the elements as well, then we will be able to distinguish the zero-divisor graphs.

**Example 14.1.** *Existence of a Multiplicative Isomorphism*
There exists a multiplicative isomorphism between $Z(\mathbb{Z}_p[x]/(x^m))$ and $Z(\mathbb{Z}_{p^m})$, defined by $\rho : Z(\mathbb{Z}_p[x]/(x^m)) \rightarrow Z(\mathbb{Z}_{p^m})$ where $\rho(x) = p$ and if $a \in \mathbb{Z}_p$, then $\rho(a) = a$.

Therefore, since the zero-divisor graphs only convey the multiplicative relationship between elements, a multiplicative isomorphism implies that the zero-divisor graphs will be isomorphic, so $\Gamma^*(\mathbb{Z}_p[x]/(x^m)) \cong \Gamma^*(\mathbb{Z}_{p^m})$.

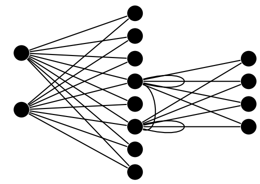
**Example 14.2.** $\Gamma^*(\mathbb{Z}_2[x]/(x^3)) \cong \Gamma^*(\mathbb{Z}_8)$.
There exists a multiplicative isomorphism from $\mathbb{Z}_2[x]/(x^3) \rightarrow \mathbb{Z}_8$ which is defined by the following: $\rho(1) = 1$, $\rho(x) = 2$, $\rho(x+1) = 3$, $\rho(x^2) = 4$, $\rho(x^2+1) = 5$, $\rho(x^2+x) = 6$, $\rho(x^2+x+1) = 7$.



$\Gamma^*(\mathbb{Z}_2[x]/(x^3)) \cong \Gamma^*(\mathbb{Z}_8)$

We can also see how decomposable rings are also not unique.

**Example 14.3.** (Example 3, [14]) $\Gamma^*(\mathbb{Z}_3 \times \mathbb{Z}_9) \cong \Gamma^*(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2))$.
There exists a multiplicative isomorphism, $\rho : Z(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)) \rightarrow Z(\mathbb{Z}_3 \times \mathbb{Z}_9)$ defined by the following $\rho((a,b)) = (a, \beta(b))$ where $\beta(1) = 1$, $\beta(2) = 8$, $\beta(x) = 3$, $\beta(x+1) = 4$, $\beta(x+2) = 2$, $\beta(2x) = 6$, $\beta(2x+1) = 7$, and $\beta(2x+2) = 5$.



$\Gamma^*(\mathbb{Z}_3 \times \mathbb{Z}_9) \cong \Gamma^*(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2))$

**Conjecture** $\Gamma^*(R_1) \cong \Gamma^*(R_2)$ *if and only if there exists a multiplicative isomorphism between* $Z(R_1)$ *and* $Z(R_2)$.

In Example 7.1., we saw that the zero-divisor graph was a complete graph and correlated to two different ideal lattices. However, in Example 7.3., we saw that the zero-divisor graphs corresponded to a unique algebraic ideal lattice, where the index of each of the ideals was known. This leads to the following conjecture:

**Conjecture** $\Gamma^*(R_1) \cong \Gamma^*(R_2)$ *if and only if* $R_1$ *and* $R_2$ *have the same structure of ideal lattice, and for each ideal* $I \subset R_1$, $J \subset R_2$ *where* $I$ *and* $J$ *are in the same place in each ideal lattice, the index of* $I$ *is equal to the index of* $J$.

# 15 Conclusion

The introduction of a looped zero-divisor graph allowed us to visualize the degree of nilpotence of vertices in the zero-divisor graph. This, in conjunction with the algorithms that we developed in this paper, allowed us to visually identify types of Artinian rings from the zero-divisor graph.

If all of the vertices are looped, then our corresponding Artinian ring is a local ring (that is not a field); if all of the vertices are not looped, then our corresponding ring is a direct product of fields; if there exists looped and unlooped vertices, but all of the vertices are either looped or adjacent to a looped vertex, then the corresponding ring is a direct product of local rings (that are not fields); and if there exists looped and unlooped vertices and at least one unlooped vertex is not adjacent to any looped vertex, then the corresponding ring is a direct product of local rings and fields.

Then, by counting the degrees of the vertices and by processes of elimination, we were able to extract more information from the zero-divisor graphs, which allowed us to narrow down a set of possible associated rings.

These algorithms are manageable for smaller Artinian rings, but when we are dealing with rings whose order is greater than 100, our algorithms become very inefficient. If we were to continue with this project, we may be motivated to create more efficient algorithms that could be applied to larger rings.

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