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Interval Orders with Restrictions on the Interval Lengths

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Abstract

This thesis examines several classes of interval orders arising from restrictions on the permissible interval lengths. We first provide an accessible proof of the characterization theorem for the class of interval orders representable with lengths between 1 and k for each $k \in \mathbb{Z}_{\geq 1}$. We then consider the interval orders representable with lengths exactly 1 and k for $k \in \mathbb{Z}_{\geq 0}$. We characterize the class of interval orders representable with lengths 0 and 1, both structurally and algorithmically. To study the other classes in this family, we consider a related problem, in which each interval has a prescribed length. We derive a necessary and sufficient condition for an interval order to have a representation with a given set of prescribed lengths. Using this result, we provide a necessary condition for an interval order to have a representation with lengths 1 and 2.

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Preface

This thesis examines several classes of interval orders arising from restrictions on the interval lengths. The two main families of interval orders we study are the interval orders representable with intervals of length *between* 1 and k for $k \in \mathbb{Z}_{\geq 1}$ and the interval orders representable with intervals of length *exactly* 1 and k for $k \in \mathbb{Z}_{\geq 0}$.

Chapter 1 introduces our main object of study, interval orders, and provides the necessary background for understanding the rest of the thesis, including general definitions, notation, and elementary results. It also includes an overview of the known results in the study of interval orders with restrictions on the interval lengths.

The goal of Chapter 2 is to provide an accessible proof of the characterization theorem for the class of interval orders representable with lengths between 1 and k for each $k \in \mathbb{Z}_{\geq 1}$. This result was previously shown by Fishburn [6] with a proof that is quite technical. Instead, we use a digraph model, based on the work of Isaak [13].

For the rest of the thesis, we shift our attention to the interval orders representable with lengths exactly 1 and k for $k \in \mathbb{Z}_{\geq 0}$. In Chapter 3, inspired by related work in graph theory [17], we examine the special case where $k = 0$. We characterize this class, both structurally and algorithmically. In Chapter 4, we study a related problem, in which each interval in the representation has a prescribed length. Using an adaptation of the tool of forcing cycles, developed by Gimbel and Trenk [9], we give a necessary and sufficient condition for an interval order to have an interval representation with a given set of positive prescribed lengths. We focus on the case where each interval's prescribed length is either 1 or 2 and derive a partial list of forbidden posets. Using this related problem, we study the class of interval orders representable with lengths 1 and 2 (with no prescribed lengths). We provide a necessary condition for an interval order to have a representation using only intervals of length 1 and 2 but no sufficient condition that guarantees such a representation.

Contents

Chapter 1

Introduction

In elementary school, we learn that given any two non-equal integers, we can always determine which of the two is smaller. This is not always possible in the case of other objects. For example, consider the Spring 2016 courses at Wellesley; we can say that one course is "less than" another if the former is a prerequisite for the latter, but if neither is a prerequisite for the other, then the two courses cannot be compared in this way. Examples like this give rise to a very important mathematical structure, called a partial order, which generalizes the familiar notion of a total order (e.g. the integers $\mathbb{Z}$ or the rationals $\mathbb{Q}$).

Let us now consider another partial order. Consider the schedule of events (e.g. lectures, performances, meetings) at Wellesley on a given day. We can say that one event is "less than" another if the former ends before the latter begins, and that the two events are "incomparable" if they overlap in time. The events with this ordering form a special type of partial order, called an interval order. A partial order can model the schedule and help answer questions such as, what is the minimum number of rooms needed to accommodate a particular schedule? In addition to planning and scheduling, interval orders and the closely-related interval graphs find applications in numerous other fields including archaeology, psychology, and genetics.

Partial orders that arise from scheduled events in the way described above have interesting mathematical properties. Mathematicians have determined which partial orders arise from schedules, both structurally and algorithmically. In some applications, it is desirable to impose additional constraints, for example on the duration of the events, and ask what partial orders arise from specific types of schedules. In this thesis, we will address questions such as the following: Which partial orders come from schedules in which all events have the same length? Which come from schedules in which each event is either one or two hours long? Which come from schedules in which the longest event is at most twice as long as the shortest event?

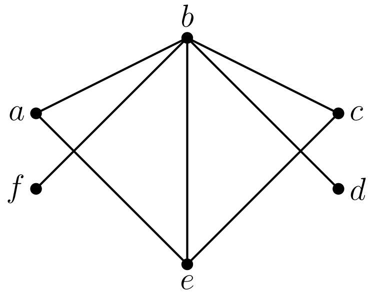


Figure 1.1: An example of a graph.

1.1 Graphs

Before we introduce partial orders, we start with a short introduction to graphs. Graphs and partial orders are closely related, and graphs provide us with additional machinery to study partial orders. All definitions not found here can be found in [18] or [24].

Definition 1.1. A *graph*, or *undirected graph*, is a pair $G = (V, E)$, where V is a non-empty finite set of *vertices*, and E is a set of unordered pairs $\{x, y\}$ with $x, y \in V$, called *edges*. We denote an edge between x and y by $\{x, y\}$ or xy for short.

Graphs are visualized using diagrams consisting of points, representing the vertices, and lines connecting the points, representing the edges. Figure 1.1 shows an example of a graph. In this case, the vertices are a, b, c, d, e, f; from the diagram we can see that ae is an edge while af is not. Graphs are a well-studied mathematical structure, for they are used in modelling a variety of real-world phenomena such as road networks or the world wide web.

1.2 Interval Graphs

We now shift our attention to a particular class of graphs, called interval graphs. Even though interval graphs have a number of interesting properties, here we only introduce the basic idea of an interval graph to give us another perspective on interval orders, which are introduced in the subsequent sections and are our main object of study. For further discussion of interval graphs, see [11].

Definition 1.2. Let $G = (V, E)$ be a graph. A collection of closed real intervals $\{I_v\}_{v \in V}$ is called an *interval representation* of G if, for all $x, y \in X$, we have $xy \in E$ if and only if $I_x \cap I_y \neq \emptyset$. A graph G is an *interval graph* if it has an interval representation.

Figures 1.4b and 1.4c show an interval representation and the resulting interval graph, respectively.

We conclude this section with an example to help us understand the importance of interval graphs. Let $G = (V, E)$ be a graph. In many applications, we want to find the minimum number of colors that can be used to color the vertices of a graph so that each vertex gets exactly one color and adjacent vertices get different colors. This number is called the *chromatic number* of G. There is no efficient algorithm for finding the chromatic number of a general graph; in fact, this is an NP-complete problem. For an interval graph, however, this problem can be solved in polynomial time by constructing an interval representation of G and finding the largest number of intervals containing the same point; see [10, Chapter 4, Chapter 8] for further discussion of this problem. Additionally, interval graphs arise naturally in many practical problems such as planning and scheduling, medical diagnosis, circuit design, physical mapping of DNA, etc. [16].

1.3 Partially Ordered Sets

We now introduce formally the fundamental object of study in this thesis.

Definition 1.3. Let X be a set. Let $\prec$ be a binary relation on X which is

1. irreflexive: for all $x \in X$, we have $x \not\prec x$;
2. antisymmetric: for all $x, y \in X$, if $x \prec y$, then $y \not\prec x$;
3. transitive: for all $x, y, z \in X$, if $x \prec y$ and $y \prec z$, then $x \prec z$.

The pair $(X, \prec)$ is called a *partially ordered set* (or *partial order* or *poset*). The set X is called the *ground set* and $\prec$ is referred to as "less than." If $x, y \in X$ with $x \prec y$ and there is no $z \in X$ such that $x \prec z \prec y$, then we say that y *covers* x. If $x, y \in X$ with $x \not\prec y$ and $y \not\prec x$, then we say that x and y are *incomparable* and write $x||y$. If $x, y \in X$ and x and y have precisely the same comparabilities, then x and y are said to be *twins*. If P contains no twins, then P is *twin-free*. If, for all distinct $x, y \in X$, we have either $x \prec y$ or $y \prec x$, then P is a *linear order* or a *total order*.

Partial orders can be infinite (e.g. $\mathbb{Z}$ with the usual "less than" relation) or finite (e.g. the power set of $\{x, y, z\}$ with strict set inclusion, illustrated in Figure 1.2).

Posets are visualized using *Hasse diagrams*. A Hasse diagram consists of points, representing the elements of the ground set, and line segments, representing the relation $\prec$. If $x, y \in X$ and $x \prec y$, then the point representing x is drawn below that representing y, and a line segment (edge) connects x and y if and only if y covers x, i.e., the edges implied by transitivity are omitted. Thus, if $x \prec y$, then there is an upward path from x to y, and if $x||y$, no such upward path exists. Figure 1.2 shows a Hasse diagram representing the partial order $(\mathscr{P}(\{x, y, z\}), \prec)$, where $\prec$ represents strict set inclusion. We can see from the diagram that in this partial order $\emptyset \prec \{x, y\}$ and $\{y\}||\{x, z\}$. Note that a Hasse diagram uniquely determines the corresponding partial order.

Hasse diagrams resemble the diagrams used to represent graphs. The most important difference between these two types of diagrams is that while the vertices in a graph diagram can

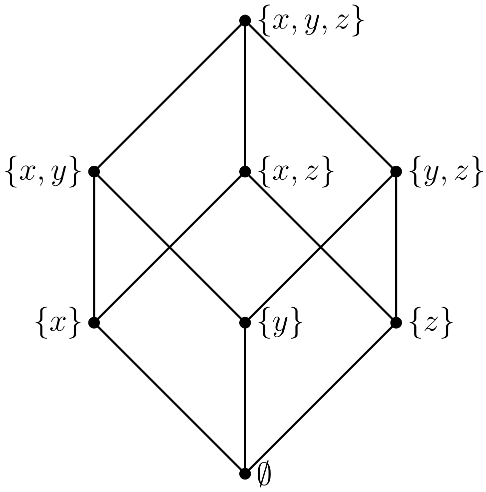


Figure 1.2: The poset $(\mathscr{P}(\{x, y, z\}), \prec)$, where $\prec$ represents strict set inclusion.

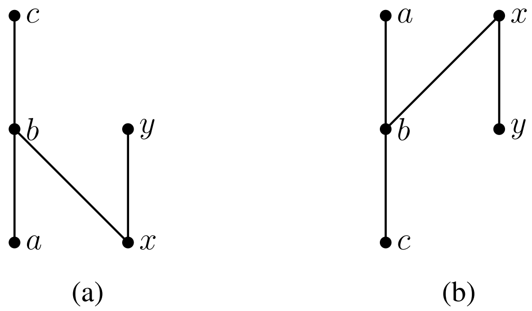


Figure 1.3: Different posets resulting from turning a Hasse diagram upside down.

be arranged in any way without changing the structure of the graph, flipping a Hasse diagram upside-down can result in a different poset. For example, the two posets shown in Figure 1.3 are different: in Figure 1.3a, we have $x \prec y$, while in Figure 1.3b, we have $y \prec x$.

Definition 1.4. Let $(X, \prec)$ be a poset. A subset $\{x_1, x_2, \ldots, x_n\}$ of X is called a *chain* if $x_1 \prec \cdots \prec x_n$.

For example, in the poset shown in Figure 1.2, the collection $\{\emptyset, \{x\}, \{x, y\}, \{x, y, z\}\}$ is a chain, as is $\{\{y\}, \{x, y, z\}\}$.

Definition 1.5. Let $(X, \prec)$ be a poset. A subset $\{x_1, x_2, \ldots, x_n\}$ of X is called an *antichain* if $x_i || x_j$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

For instance, in the poset in Figure 1.2, the collection $\{\{x\}, \{y\}, \{z\}\}$ is an antichain and so is $\{\{x\}, \{y, z\}\}$.

We now introduce an important family of partial orders.

Definition 1.6. Let $r, s \in \mathbb{Z}_{\geq 1}$. The poset $\mathbf{r} + \mathbf{s}$ is the poset consisting of two disjoint chains, one containing r elements and the other containing s elements.

Two examples of posets in this family are the poset $\mathbf{2} + \mathbf{2}$ shown in Figure 1.5 and the poset $\mathbf{3} + \mathbf{1}$ shown in Figure 1.7. We also have a shorthand way of describing posets in this family: we write $(x_1 \prec \cdots \prec x_r)||(y_1 \prec \cdots \prec y_s)$ to mean that the elements $x_1, \ldots, x_r$ form a chain of r elements, the elements $y_1, \ldots, y_s$ form a chain of s elements, and $x_i||y_j$ for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$. For instance, the poset $\mathbf{2} + \mathbf{2}$ in Figure 1.5 can be written as $(a \prec b)||(x \prec y)$.

Definition 1.7. Let $(X, \prec)$ be a poset. Let $x \in X$. Define the *down set* of x as $Down(x) = \{y \in X : y \prec x\}$, the *up set* of x as $Up(x) = \{y \in X : x \prec y\}$, and the *incomparability set* of x as $Inc(x) = \{y \in X : x||y\}$. We denote the collection of all down sets by $\mathcal{D}$ and the collection of all up sets by $\mathcal{U}$.

In the poset shown in Figure 1.2, we have $Down(\{x, z\}) = \{\emptyset, \{x\}, \{z\}\}$, $Up(\{x, z\}) = \{\{x, y, z\}\}$, and $Inc(\{x, z\}) = \{\{y\}, \{x, y\}, \{y, z\}, \{x, z\}\}$. Note that if P is a poset, then the elements u and v are twins in P if and only if $Down(u) = Down(v)$ and $Up(u) = Up(v)$.

Definition 1.8. Let $P = (X, \prec)$ be a poset and $Y \subseteq X$. Then the poset *induced* by Y, or the *restriction* of P to Y, is the poset $P' = (Y, \prec')$ where, for all $x, y \in Y$, we have $x \prec' y$ in P' if and only if $x \prec y$ in P. The restriction of P to Y is denoted by $P|_Y$. We say that P *contains an induced poset* R if there is $Y \subseteq X$ such that $P|_Y$ is isomorphic to R.

Informally, the poset $P|_Y$ induced by Y inherits precisely those comparabilities between elements of Y that exist in P. For instance, the poset $\mathbf{2} + \mathbf{2}$ shown in Figure 1.5 is induced in the poset in Figure 1.2 by the elements $\{x\}$, $\{x, y\}$, $\{z\}$, and $\{y, z\}$, but $\mathbf{2} + \mathbf{2}$ is not an induced poset in either Figure 1.3a or Figure 1.3b.

There is a close connection between graphs and posets, formalized by the following definition.

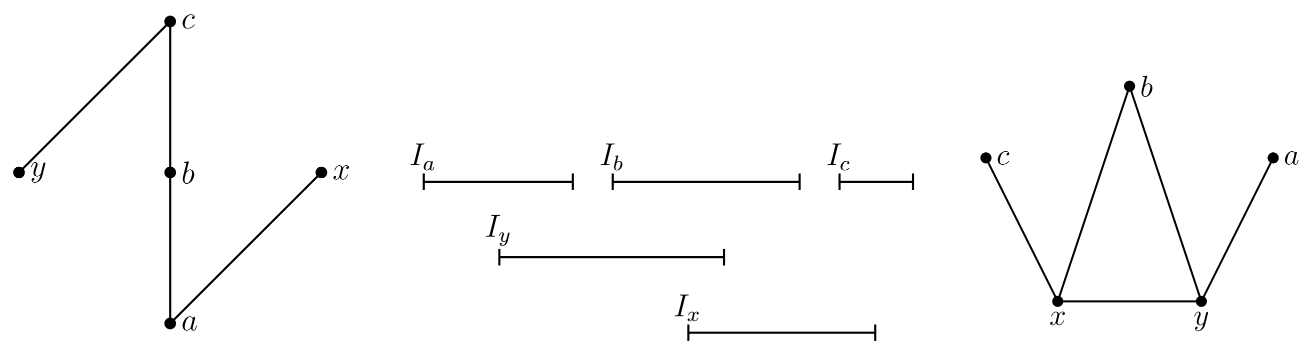
Definition 1.9. Let $P = (X, \prec)$ be a poset. The *incomparability graph* of P is the graph G with vertex set X and edge set E given by $xy \in E$ if and only if $x||y$ in P.

Figures 1.4a and 1.4c show a poset P and its incomparability graph G. It is not hard to see that each poset has a unique incomparability graph but this mapping is not one-to-one. In particular, the posets in Figures 1.3a and 1.3b have the same incomparability graph.

1.4 Interval Orders

1.4.1 Definition and Preliminaries

Definition 1.10. Let $P = (X, \prec)$ be a partial order. A collection of closed real intervals $\{I_x\}_{x \in X}$, where $I_x = [L(x), R(x)]$ for all $x \in X$, is called an *interval representation* of P if, for all $x, y \in X$, we have $x \prec y$ if and only if $R(x) < L(y)$. If P has an interval representation, then P is called an *interval order*.



(a) P (b) (c) G

Figure 1.4: An interval representation and the resulting interval order P and interval graph G.

It follows immediately from Definition 1.10 that if $\{I_x\}_{x\in X}$ is an interval representation of P, then $x||y$ if and only if $I_x \cap I_y \neq \emptyset$. Note also that if a poset is an interval order, then its incomparability graph is an interval graph. Figure 1.4 shows an example of an interval order P together with its incomparability graph G and a possible interval representation. Note that in this figure, the interval corresponding to each $v \in X$ is labelled I_v. From now on in figures we will label each interval with the name of the element to which it corresponds, i.e., we will write v instead of I_v.

As an aside, we note that the class of interval orders could also be defined as the class of partial orders that have representations that use only open intervals. It is well known that the open-interval definition is equivalent to the closed-interval definition for finite posets, i.e., they both define the same class of partial orders. This is a consequence of the fact that given an interval representation of a poset P, we can always modify this representation to make the endpoints of all intervals distinct (see for example [11, Lemma 1.5]). Some authors, for instance Shuchat, Shull and Trenk in [20], have considered representations that use both open and closed intervals. Here we will only consider closed-interval representations.

As mentioned earlier, interval orders arise naturally in practical problems such as planning and scheduling. Recall that we can define a relation $\prec$ on a collection of events as follows: one event is less than another if the former ends before the latter begins, and the two events are incomparable if they overlap in time. For example, if the intervals in Figure 1.4b represent the meetings that are scheduled in a company on a given day, then the poset in Figure 1.4a captures the comparabilities between them and allows us to analyze the schedule and the constraints it imposes (e.g. we need at least three different rooms to accommodate this schedule).

1.4.2 Helly Property

The next proposition is known as the *Helly property for intervals* (see for example [10, Section 4.5]). The Helly property will be very useful in some of the later chapters. We include a short proof here for completeness.

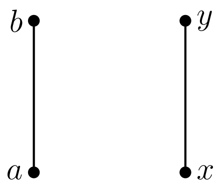


Figure 1.5: The poset $\mathbf{2}+\mathbf{2}$.

Proposition 1.1. *Let $\{I_i\}_{i=1}^n$ be a collection of intervals such that $I_i \cap I_j \neq \emptyset$ for all $i, j \in \{1, \ldots, n\}$. Then $\bigcap_{k\in\{1,\ldots,n\}} I_k \neq \emptyset$.*

Proof. Let $i \in \{1, \ldots, n\}$ be the index of the interval with the leftmost right endpoint, and let $j \in \{1, \ldots, n\}$ be the index of the interval with the rightmost left endpoint. We know that $I_i \cap I_j \neq \emptyset$ and $(I_i \cap I_j) \subseteq \bigcap_{k\in\{1,\ldots,n\}} I_k$, and thus $\bigcap_{k\in\{1,\ldots,n\}} I_k \neq \emptyset$. □

1.4.3 Characterization

Not every partial order is an interval order. Indeed, there is a necessary and sufficient condition to determine when a partial order has an interval representation, which is given in the following theorem.

Theorem 1.2 (Fishburn [4])**.** *A partial order is an interval order if and only if it contains no induced* $\mathbf{2}+\mathbf{2}$.

Figure 1.5 illustrates the partial order $\mathbf{2}+\mathbf{2}$. To see one direction of the proof of Theorem 1.2, suppose $\mathbf{2}+\mathbf{2}$ were an interval order and fix an interval representation of it with intervals I_a, I_b, I_x, I_y. Now $a \prec b$ implies that $R(a) < L(b)$, $b||x$ implies $L(b) \leq R(x)$, and $x \prec y$ implies that $R(x) < L(y)$. Combining these inequalities, we get $R(a) < L(y)$, so $a \prec y$, a contradiction. Thus, the poset $\mathbf{2}+\mathbf{2}$ has no interval representation, and if a poset contains an induced $\mathbf{2}+\mathbf{2}$, then we cannot construct an interval representation of it. The proof of the converse is omitted here but can be found in [4]. Other elegant proofs of this result can be found in [1] or [2]. Isaak [13] provides an alternative proof of this result using potentials in digraphs. From Theorem 1.2, it is immediate that the poset shown in Figure 1.2 is not an interval order because $\{x\}, \{x, y\}, \{z\}, \{y, z\}$ form an induced $\mathbf{2}+\mathbf{2}$.

1.4.4 Greenough Algorithm

The next two results yield a polynomial-time algorithm, quadratic in the size of the poset, for constructing an interval representation of a $(\mathbf{2}+\mathbf{2})$-free poset P. The first proposition is an elementary result in the study of partial orders and appears in [23]. We include the proof here for completeness.

Proposition 1.3. *Let $P = (X, \prec)$ be a partial order. The following are equivalent:*

(1) P contains no induced $\mathbf{2}+\mathbf{2}$.

(2) The down sets are ordered by inclusion, i.e., for all $x, y \in X$, we have $Down(x) \subseteq Down(y)$ or $Down(y) \subseteq Down(x)$.

(3) The up sets are ordered by inclusion, i.e., for all $x, y \in X$, we have $Up(x) \subseteq Up(y)$ or $Up(y) \subseteq Up(x)$.

Proof. We will prove $(1) \Rightarrow (2)$ and $(2) \Rightarrow (1)$ here. The proofs of $(1) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are similar.

$(1) \Rightarrow (2)$ Assume P contains no induced $\mathbf{2}+\mathbf{2}$. Suppose, for the sake of contradiction, that there exist $x, y \in X$ such that $Down(x) \nsubseteq Down(y)$ and $Down(y) \nsubseteq Down(x)$, i.e., assume there are $u, v \in X$ such that $u \in Down(x) - Down(y)$ and $v \in Down(y) - Down(x)$.

Note that $x||y$. Indeed, if $x \prec y$ or $y \prec x$, then $Down(x) \subseteq Down(y)$ or $Down(y) \subseteq Down(x)$ respectively. By the definition of the down set, we have $u \prec x$ and $v \prec y$. Now if $u \prec y$, then $u \in Down(y)$, contradicting the choice of u. If $y \prec u$, then by transitivity $y \prec u \prec x$, contradicting the fact that $x||y$. So $u||y$. Similarly $v||x$, thus implying that $u||v$. Hence x, y, u, and v form an induced $\mathbf{2}+\mathbf{2}$, a contradiction.

$(2) \Rightarrow (1)$ Assume now that the down sets can be ordered by inclusion. Suppose there is an induced $\mathbf{2}+\mathbf{2}$ $(u \prec x)||(v \prec y)$ for some $u, v, x, y \in X$. Then $u \prec x$ and $u \not\prec y$, i.e., $Down(x) \nsubseteq Down(y)$. Similarly $v \prec y$ with $v \not\prec y$, so $Down(y) \nsubseteq Down(x)$. Hence, the down sets are not ordered by inclusion. $\square$

As a result from Proposition 1.3, we can index the down sets and the up sets of an interval order P so that $D_1 \subseteq \cdots \subseteq D_{|\mathcal{D}|}$ and $U_1 \supseteq \cdots \supseteq U_{|\mathcal{U}|}$. The next proposition is based on the work of Greenough [12] and appears in [15] and [20].

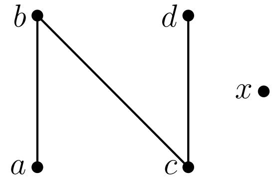
Theorem 1.4. *Let $P = (X, \prec)$ be an interval order. Let the down sets and up sets be indexed as above. For all $x \in X$, let $L(x) = i$ and $R(x) = j$, where $D_i = Down(x)$ and $U_j = Up(x)$. Then the following hold:*

(1) $|\mathcal{D}| = |\mathcal{U}|$.

(2) $L(x) \leq R(x)$ for all $x \in X$.

(3) If $I_x = [L(x), R(x)]$, then the collection $\{I_x\}_{x \in X}$ forms an interval representation of P.

The proof is omitted here but can be found in [20]. We refer to the representation given in Theorem 1.4 as the *Greenough representation* of a poset P. The Greenough representation has the interesting property that every value that appears as a left endpoint of some interval also appears as a right endpoint of some (possibly the same) interval. Due to this property, the Greenough representation is a useful initial representation in construction proofs. For further discussion of the Greenough representation, see [20]. The next example illustrates the Greenough representation of a particular poset.



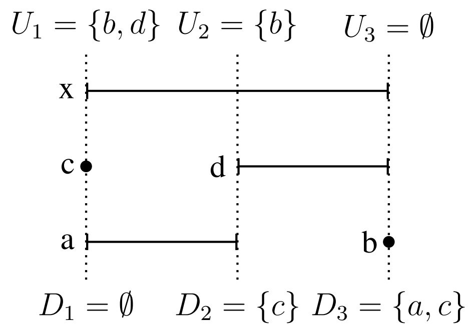


Figure 1.6: A poset with its Greenough representation.

Example 1.1. Consider the poset P in Figure 1.6a. We first compute the down set and the up set for each element and record these values in Table 1.1. The three down sets are $D_1 = \emptyset$, $D_2 = \{c\}$ and $D_3 = \{a, c\}$, and the three up sets are $U_1 = \{b, d\}$, $U_2 = \{b\}$ and $U_3 = \emptyset$. The fourth column of Table 1.1 shows the interval assigned to each element as described in Theorem 1.4 above, which is also illustrated in Figure 1.6b.

v	$Down(v)$	$Up(v)$	I_v
a	$\emptyset$	$\{b\}$	$[1, 2]$
b	$\{a, c\}$	$\emptyset$	$[3, 3]$
c	$\emptyset$	$\{b, d\}$	$[1, 1]$
d	$\{c\}$	$\emptyset$	$[2, 3]$
x	$\emptyset$	$\emptyset$	$[1, 3]$

Table 1.1: Down sets and up sets of the elements of the poset in Figure 1.6a.

1.5 Restrictions on the Interval Lengths

For some applications, it might be required not only that a poset be an interval order but also that it have a representation satisfying certain constraints (e.g. on the positions of the endpoints or on the lengths of the intervals). Restricting the set of permissible interval lengths gives rise to a number of interesting classes of interval orders.

The most well-studied class arising from restricting the set of permissible interval lengths is the *unit interval orders*. A unit interval order is an interval order that has a representation in which all intervals have the same length (usually set to one). Unit interval orders are characterized by the following theorem.

Theorem 1.5 (Scott-Suppes [19]). *A partial order is a unit interval order if and only if it contains neither an induced* $\mathbf{2} + \mathbf{2}$ *nor an induced* $\mathbf{3} + \mathbf{1}$.

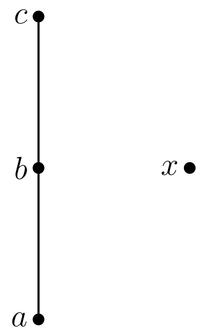


Figure 1.7: The poset $\mathbf{3}+\mathbf{1}$.

We have already shown that a poset containing an induced $\mathbf{2}+\mathbf{2}$ cannot be an interval order. We now show that the poset $\mathbf{3}+\mathbf{1}$, illustrated in Figure 1.7, is not a unit interval order. Suppose it were and fix a unit interval representation of it with intervals I_a, I_b, I_c, I_x. Since $a \prec b \prec c$, we have $R(a) < L(b) \leq R(b) < L(c)$. So $L(c) - R(a) > |I_b|$. Now I_x must intersect all of I_a, I_b, and I_c, so we must have $L(x) \leq R(a)$ and $L(c) \leq R(x)$. Hence $R(x) - L(x) \geq L(c) - R(a) > |I_b|$, i.e., I_x cannot have the same length as I_b. Therefore, the poset $\mathbf{3}+\mathbf{1}$ has no unit interval representation, and thus no poset containing an induced $\mathbf{3}+\mathbf{1}$ can have a unit interval representation. Again, we omit the proof of the converse here, but it can be found in [19]. An elegant inductive proof of this result can be found in [1]. Isaak [13] provides an alternative proof of this result using potentials in digraphs. The posets in Figures 1.3a and 1.3b are examples of interval orders that are not unit interval orders. Indeed, each of these posets contains an induced $\mathbf{3}+\mathbf{1}$, formed by a, b, c, and y in both cases, but no induced $\mathbf{2}+\mathbf{2}$.

Having considered the class of unit interval orders, we now explore two main ways to generalize this idea: we can allow variations in the *number* or in the *range* of permissible lengths. The goal in each case is to obtain a characterization of the respective class. We discuss each of these ideas in turn, but in this thesis, we focus on the second one.

Before we continue, we introduce some notation. We will say that an interval order P is *representable* with (at most) k distinct lengths if P has an interval representation that uses no more than k distinct interval lengths. Similarly, we will say that an interval order P is *representable* with a set of lengths S if P has an interval representation $\{I_x\}_{x\in X}$ such that $|I_x| \in S$ for all $x \in X$. For $n \in \mathbb{Z}_{\geq 1}$, we will write $\mathscr{P}_n$ to denote the class of interval orders representable with at most n distinct lengths. If $S \subseteq \mathbb{R}$, we will use $\mathscr{P}_S$ to denote the class of interval orders representable with interval lengths in S. For example, $\mathscr{P}_3$ is the class of interval orders representable with at most three distinct lengths, $\mathscr{P}_{[2,3]}$ is the class of interval orders representable with lengths between 2 and 3, and $\mathscr{P}_{\{5,7\}}$ is the class of interval orders representable with lengths exactly 5 and 7.

The classes $\mathscr{P}_n$:

We first consider varying the number of permissible lengths. In [5], Fishburn shows that even though the class $\mathscr{P}_1$ of the unit interval orders is axiomatizable by a universal sentence, the same is not true for the class $\mathscr{P}_n$ when $n \geq 2$. That is, when $n \geq 2$, there is no finite set $\mathscr{F}_n$ such that $P \in \mathscr{P}_n$ if and only if P does not contain an induced suborder isomorphic to an element of $\mathscr{F}_n$.

In a different paper [7], Fishburn considers the class $\mathscr{P}_2$ and some of the anomalies exhibited by the partial orders in this class. If $P \in \mathscr{P}_2$ and $L(P) = \{\alpha : P$ has an interval representation with lengths 1 and $\alpha\}$, the author shows that there are posets P for which $L(P)$ is not connected and contains arbitrarily many gaps. In particular, he shows that for each integer $m \geq 2$, there exist interval orders $P_1, P_2, P_3 \in \mathscr{P}_2$ satisfying the following:

- $L(P_1) = (1, m)$;
- $L(P_2) = (2 - \frac{1}{m}, 2) \cup (m, \infty)$;
- $L(P_3) = (m, 2m - 1) \cup (2m - 1, \infty)$.

In this paper, Fishburn also shows that for all $n \in \mathbb{Z}_{\geq 1}$, there is an interval order P such that $L(P)$ is the union of n disjoint open intervals.

In [14], Joos et al. study the class of interval graphs corresponding to $\mathscr{P}_2$. In their problem, the authors consider a connected graph $G = (V, E)$ with a partition of the vertex set into two sets A and B. They provide a polynomial-time algorithm that determines whether there is an interval representation of G such that all elements in A are assigned an interval of length L_A and all elements of B are assigned an interval of length L_B for some $L_A, L_B \in \mathbb{R}$ with $L_A < L_B$.

The study of the classes $\mathscr{P}_n$ is related to the *interval count problem* for interval orders: given an interval order P, determine the minimum number of distinct interval lengths required to represent P. Cerioli et al. [3] provide a survey of the known results about this problem.

The classes $\mathscr{P}_S$:

Now consider varying the range of permissible lengths. In [6], Fishburn studies the class $\mathscr{P}_{[m,n]}$ of interval orders that have representations with interval lengths in $[m, n]$, where $m, n \in \mathbb{Z}_{\geq 1}$ with $\gcd(m, n) = 1$. The paper provides a structural characterization of $\mathscr{P}_{[m,n]}$ for arbitrary values of m, n. In particular, in this paper Fishburn characterizes $\mathscr{P}_{[1,k]}$ for $k \in \mathbb{Z}_{\geq 1}$ as the class of partial orders with no induced $\mathbf{2} + \mathbf{2}$ or $(\mathbf{k} + \mathbf{2}) + \mathbf{1}$. In Chapter 2, we provide a more accessible proof of this result using potentials in digraphs. In [8], Fishburn and Graham study the classes $\mathscr{P}_{[1,\alpha]}$ for $\alpha \in \mathbb{R}_{\geq 1}$ in the context of interval graphs. They find that $\mathscr{P}_{[1,\alpha]} \subseteq \mathscr{P}_{[1,\beta]}$ if $\alpha < \beta$ and that the class $\mathscr{P}_{[1,\alpha]}$ expands at each rational value of α.

Authors have also considered the classes $\mathscr{P}_S$ when the set S is discrete. Several authors have studied the class of interval graphs representable with lengths 0 and 1, corresponding to the class $\mathscr{P}_{\{0,1\}}$ of interval orders. Skrien [21] provides a cubic-time recognition algorithm for this class, and Rautenbach and Szwarcfiter [17] derive a linear-time recognition algorithm and a characterization in terms of forbidden induced subgraphs. In Chapter 3, we discuss the class $\mathscr{P}_{\{0,1\}}$ of interval orders; we derive a characterization in terms of a forbidden substructure and a polynomial-time recognition algorithm. There is no known characterization for any $\mathscr{P}_{\{1,k\}}$ with $k \in \mathbb{Z}_{\geq 2}$. In Chapter 4, we discuss a related problem to help us better understand this family of classes: we consider the case where the length of each interval is pre-determined and we establish the conditions under which a poset has a representation with the given lengths.

Specified lengths for the intervals

In [16], Pe'er and Shamir study the problem of constructing an interval representation satisfying certain constraints for a given interval graph G. One of the problems they consider is the following: given a graph G and a fixed length for each interval, determine whether or not G has an interval representation with the specified interval lengths. The authors find that this problem in strongly NP-complete, i.e., there is no efficient algorithm to solve this problem. On the other hand, it is known that given an interval order P and a prescribed length for each interval, we can determine in polynomial time (using linear programming) whether or not such an interval representation of P exists. The approach used in this case is similar to the one discussed by Isaak in [13] and the one we employ in Chapter 2. In Chapter 4, we also provide a direct algorithm for constructing such a representation.

Chapter 2

Interval Orders Representable with Lengths between 1 and k

2.1 Introduction

The goal of the current chapter is to characterize the class of interval orders representable with intervals of length between 1 and k, where $k \in \mathbb{Z}_{\geq 1}$, as the class of partial orders that contain no induced $\mathbf{2}+\mathbf{2}$ and no induced $(\mathbf{k}+\mathbf{2})+\mathbf{1}$. Note that in the case $k=1$, this result reduces to Theorem 1.5 discussed in Chapter 1. This characterization theorem was proven by Fishburn, for example in [6], but the proof presented in that paper is very technical. We provide a more accessible proof of this result that uses potentials in digraphs, a model developed by Garth Isaak [13]. Before we present the proof, we review some preliminaries from graph theory. All definitions not found here can be found in [18] or [24].

2.2 Preliminaries from Graph Theory

In Chapter 1, we defined the notion of an undirected graph (or just graph). We now introduce a related structure, called a directed graph.

Definition 2.1. A *directed graph*, or *digraph*, is a pair $G=(V,E)$, where V is a finite set of *vertices*, and E is a set of ordered pairs (x,y) with $x,y \in V$, called *arcs*. If in addition there is a function $w : E \to \mathbb{R}$ with $(x,y) \mapsto w_{xy}$, then G is a *weighted digraph* and w_{xy} is called the *weight* of the arc (x,y). We denote an arc from x to y by (x,y) or $x \to y$, and in a weighted digraph, by $x \xrightarrow{w_{xy}} y$, where w_{xy} denotes the weight of the arc. The *underlying graph* of a digraph $G = (V, E)$ is the graph $G' = (V, E')$, where $xy \in E'$ if and only if $(x, y) \in E$ or $(y, x) \in E$.

The digraphs we will use in our model will have an additional property, described in the following definition.

Definition 2.2. A graph is $G = (V, E)$ is *bipartite* if there exists a partition of V into two sets A and B such that every edge contains one vertex in A and one in B. A digraph is *bipartite* if the underlying graph is bipartite.

We now present some basic definitions and elementary results about digraphs.

Definition 2.3. A *path* from x to y in a digraph $G = (V, E)$ is a sequence of (not necessarily distinct) vertices $W : x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that $(x_{i-1}, x_i) \in E$ for all $i \in \{1, \ldots, n\}$. In this case, we say that x is the *starting vertex* of W and y is the *end vertex* of W, or that the path *starts* at x and *ends* at y. A path W is called a *closed path* or *cycle* if $x = y$. The *length* of a path is the number of arcs it includes. In a weighted graph, the *weight* of a path is the sum of the weights of all of the arcs it includes.

We note that some authors use the term "path" only when all vertices in the sequence are distinct and the term "walk" to refer to a sequence of not necessarily distinct vertices.

The following theorem, which appears in [24, Section 1.2.], is an elementary result from graph theory that characterizes bipartite graphs.

Theorem 2.1. *A graph G is bipartite if and only if G contains no odd-length cycles.*

Definition 2.4. A vertex y is said to be *reachable* from vertex x in a digraph $G = (V, E)$ if there exists a path from x to y in G.

Definition 2.5. A *negative-weight cycle* in a digraph, or *negative cycle* for short, is a cycle whose weight is negative.

Definition 2.6. A *shortest path* from x to y in a weighted digraph $G = (V, E)$ is a path from x to y that has minimum weight. The *distance* from x to y, denoted by $d(x, y)$, is the weight of a shortest path from x to y.

The following result is well known (see for example [18, Chapter 8]). We include the proof here for completeness.

Lemma 2.2. *Let $G = (V, E)$ be a digraph. A shortest path from x to y in G is defined for all $x, y \in V$ such that y is reachable from x if and only if G contains no negative-weight cycles.*

Proof. ($\Rightarrow$) Suppose G contains a negative cycle. If $x, y \in V$ are part of a negative cycle, then y is reachable from x but there is no shortest path from x to y. So it is not possible to find a shortest path between each pair of vertices.

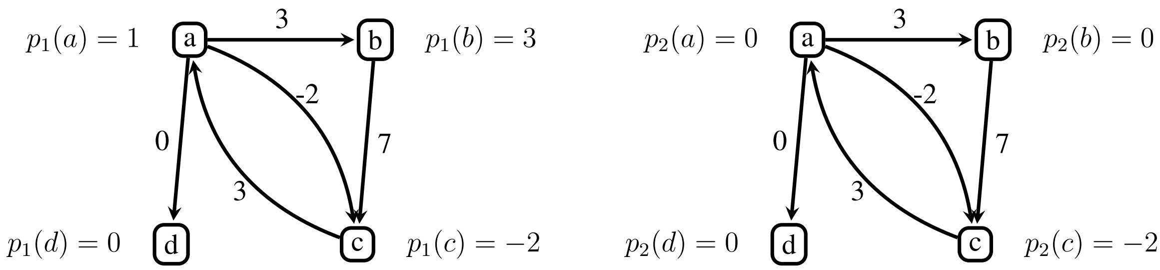


Figure 2.1: A digraph with two different potential functions.

($\Leftarrow$) Suppose G contains no negative cycles. Then all cycles in the digraph must have nonnegative weight. Let $x, y \in V$ so that y is reachable from x. Note that x and y need not be distinct. Suppose $W : x = x_0, x_1, \ldots, x_n = y$ is a path from x to y and $x_i = x_j$ for some $i, j \in \{0, \ldots, n\}$ with $i < j$. Replacing the segment from x_i to x_j by x_i gives a path, whose weight is at most that of the original path. This implies that, to find a shortest path between x and y, we only need to consider paths that do not repeat any vertices. In a finite digraph, there are only finitely many such paths between x and y; therefore, we can find a path with minimum weight. $\square$

Next we introduce a special kind of function on the vertex set of a graph, called a potential function. It turns out that the problem of computing an interval representation with lengths between 1 and k for a poset P reduces to the problem of finding a potential function on a particular digraph.

Definition 2.7. Let $G = (V, E)$ be a weighted digraph. A *potential function* $p : V \to \mathbb{R}$ is a function satisfying $p(v) - p(u) \leq w_{uv}$ for all $(u, v) \in E$.

Example 2.1. Consider the weighted digraph in Figure 2.1. It is not hard to check that p_1 and p_2 are both potential functions on this graph. From this example we can see that potential functions are not necessarily unique. We will also see shortly that not every graph has a potential function.

The following result, which appears in [18, Section 8.2.], characterizes those digraphs for which there exist potential functions. We include the proof here for completeness.

Theorem 2.3. *A weighted digraph has a potential function if and only if it contains no negative cycles.*

Proof. Let $G = (V, E)$ be a weighted digraph.

($\Leftarrow$) Suppose G contains no negative cycles. By Lemma 2.2, a shortest path from x to y is defined for all $x, y \in V$ such that y is reachable from x. Define the function $p : V \to \mathbb{R}$, where $p(x)$ is the minimum weight of a path ending at x. We show that this is a potential function. Note that the potential function p_2 shown in Figure 2.1 is exactly the function p on the given digraph. Let $x, y \in V$ with $(x, y) \in E$ and let $x_0, \ldots, x_n = x$ be a shortest path ending at x. Then $x_0, \ldots, x_n, y$ is a path ending at y. Since $p(y)$ is the minimum over all such paths, it follows that $p(y)$ must be no greater than the weight of this particular path, which is given by $p(x) + w_{xy}$, i.e., $p(y) \leq p(x) + w_{xy}$ and therefore $p(y) - p(x) \leq w_{xy}$ as desired.

($\Rightarrow$) Now assume that G contains a negative cycle $x_0, \ldots, x_n$ with total weight $c < 0$. Suppose, for the sake of contradiction, that G has a potential function p. Then we have the following inequalities:

$$\begin{aligned} p(x_1) - p(x_0) &\leq w_{x_0 x_1} \\ p(x_2) - p(x_1) &\leq w_{x_1 x_2} \\ &\vdots \\ p(x_n) - p(x_{n-1}) &\leq w_{x_{n-1} x_n}. \end{aligned}$$

Adding all the inequalities together, we get

$$[p(x_1) - p(x_0)] + [p(x_2) - p(x_1)] + \cdots + [p(x_n) - p(x_{n-1})] \leq w_{x_0 x_1} + w_{x_1 x_2} + \cdots + w_{x_{n-1} x_n}.$$

On the left-hand side, we get $p(x_n) - p(x_0)$, which is 0 since $x_0 = x_n$, and on the right-hand side, we get the weight of the cycle, namely c. So $0 \leq c$, but $c < 0$ by assumption, contradicting the existence of a potential function. $\square$

2.3 Main Result

In [13], Isaak develops a digraph model and uses it to prove Fishburn's Theorem (Theorem 1.2), characterizing finite interval orders, and Scott-Suppes' Theorem (Theorem 1.5), characterizing finite unit interval orders. In each case, given a partial order P, Isaak constructs an associated digraph so that P has an interval representation or a unit interval representation if and only if the associated digraph has a potential function. In this chapter, we will construct a digraph $G_{P,k}$ so that $G_{P,k}$ has a potential function if and only if P has an interval representation with lengths between 1 and k.

Let $P = (X, \prec)$ be a partial order. If P has an interval representation $\mathcal{I} = \{I_x\}_{x \in X} = \{[L(x), R(x)]\}_{x \in X}$, the endpoints must satisfy the following inequalities for some small positive constant γ:

(1) $R(x) \leq L(y) - \gamma$ for all $x, y \in X$ with $x \prec y$;

(2) $R(x) \geq L(y)$ for all $x, y \in X$ with $x || y$ or $x = y$.

Note that the second condition ensures that each I_x is indeed an interval, i.e., that $L(x) \leq R(x)$, for each $x \in X$. If, in addition, we want the intervals in this representation to have lengths between 1 and k, the endpoints need to satisfy two additional inequalities:

(3) $R(x) \geq 1 + L(x)$ for all $x \in X$;

(4) $R(x) \leq k + L(x)$ for all $x \in X$.

Note that once we add inequality (3), we no longer need to consider inequality (2) when $x = y$ because (3) imposes a stronger condition on the endpoints than (2). Rewriting the four inequalities above, we get the following conditions:

(1) $R(x) - L(y) \leq -\gamma$ for all $x, y \in X$ with $x \prec y$;

(2) $L(y) - R(x) \leq 0$ for all $x, y \in X$ with $x||y$;

(3) $L(x) - R(x) \leq -1$ for all $x \in X$;

(4) $R(x) - L(x) \leq k$ for all $x \in X$.

We now construct a weighted digraph $G_{P,k}$ that has two vertices for each element in a given poset P.

Definition 2.8. Let $P = (X, \prec)$ be a partial order. Define $G_{P,k}$ to be the weighted digraph with vertices $\{l_x, r_x\}_{x \in X}$ and the following arcs:

- (l_y, r_x) with weight $-\gamma$ for all $x, y \in X$ with $x \prec y$;
- (r_x, l_y) with weight 0 for all $x, y \in X$ with $x||y$;
- (r_x, l_x) with weight -1 for all $x \in X$;
- (l_x, r_x) with weight k for all $x \in X$.

We will soon show that inequalities (1)-(4) above are exactly the conditions which need to be satisfied by a potential function on the graph $G_{P,k}$. We are now ready to state the main result of this chapter.

Theorem 2.4. *Let $P = (X, \prec)$ be a partial order and let $k \in \mathbb{Z}_{\geq 1}$. The following are equivalent:*

(1) P has an interval representation with lengths between 1 and k.

(2) P contains no induced $\mathbf{2} + \mathbf{2}$ or $(\mathbf{k} + \mathbf{2}) + \mathbf{1}$.

(3) The weighted digraph $G_{P,k}$ contains no negative cycles.

Proof. (1) $\Rightarrow$ (3) Suppose P has an interval representation $\mathcal{I} = \{I_x\}_{x \in X}$, where $I_x = [L(x), R(x)]$, with lengths between 1 and k. Then, as discussed above, the endpoints satisfy the following inequalities:

(1) $R(x) - L(y) \leq -\gamma$ for all $x, y \in X$ with $x \prec y$;

(2) $L(y) - R(x) \leq 0$ for all $x, y \in X$ with $x||y$;

(3) $L(x) - R(x) \leq -1$ for all $x \in X$;

(4) $R(x) - L(x) \leq k$ for all $x \in X$.

Define $p : \{r_x, l_x\}_{x \in X} \to \mathbb{R}$ by

$$p(y) = \begin{cases} L(x) \text{ if } y = l_x \text{ for some } x \in X \\ R(x) \text{ if } y = r_x \text{ for some } x \in X \end{cases}.$$

Then p satisfies

1. $p(r_x) - p(l_y) \leq -\gamma$ for all $x, y \in X$ with $x \prec y$;
2. $p(l_y) - p(r_x) \leq 0$ for all $x, y \in X$ with $x||y$;
3. $p(l_x) - p(r_x) \leq -1$ for all $x \in X$;
4. $p(r_x) - p(l_x) \leq k$ for all $x \in X$.

Thus, for all $u, v \in \{l_x, r_x\}_{x \in X}$, we have $p(v) - p(u) \leq w_{uv}$, so by Definition 2.7, this p is a potential function on $G_{P,k}$. Then by Theorem 2.3, we know that $G_{P,k}$ contains no negative cycles.

$(3) \Rightarrow (1)$ If $G_{P,k}$ contains no negative cycles, then by Theorem 2.3, there exists a potential function p on $G_{P,k}$. For each $x \in X$, let $L(x) = p(l_x)$, $R(x) = p(r_x)$, and $I_x = [L(x), R(x)]$. Using a similar argument as above, we can show that the inequalities p needs to satisfy as a potential function on $G_{P,k}$ can be rewritten in terms of $L(x)$ and $R(x)$, which then guarantees that $\{I_x\}_{x \in X}$ forms a valid interval representation of P with lengths between 1 and k.

For concreteness, we prove $(3) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ for $k = 2$, i.e., we will show that $G_{P,2}$ contains a negative cycle if and only if P has an induced $\mathbf{2} + \mathbf{2}$ or $\mathbf{4} + \mathbf{1}$; the proof for all other positive integer values of k is analogous.

$(3) \Rightarrow (2)$ If P contains an induced $\mathbf{2} + \mathbf{2}$, say $(a \prec x)||(b \prec y)$, then $l_x \xrightarrow{-\gamma} r_a \xrightarrow{0} l_y \xrightarrow{-\gamma} r_b \xrightarrow{0} l_x$ is a negative cycle in $G_{P,2}$ (and more generally, in $G_{P,k}$) with weight -2γ. Similarly, if P contains an induced $\mathbf{4} + \mathbf{1}$, say $(a \prec b \prec c \prec d)||x$, then $r_x \xrightarrow{0} l_d \xrightarrow{-\gamma} r_c \xrightarrow{-1} l_c \xrightarrow{-\gamma} r_b \xrightarrow{-1} l_b \xrightarrow{-\gamma} r_a \xrightarrow{0} l_x \xrightarrow{2} r_x$ is a negative cycle in $G_{P,2}$ with weight -3γ. More generally, if $k \in \mathbb{Z}_{\geq 1}$ and P contains an induced $(\mathbf{k} + \mathbf{2}) + \mathbf{1}$, then the associated digraph $G_{P,k}$ has a negative cycle of weight $-(k + 1)\gamma$.

$(2) \Rightarrow (3)$ Assume now that $G_{P,2}$ contains a negative cycle, and let C be a negative cycle in $G_{P,2}$ of minimal length.

We now make a few useful observations about C. First notice that every arc in $G_{P,2}$ has the form (l_x, r_y) or (r_x, l_y); hence $G_{P,2}$ is bipartite. Also observe that since $G_{P,2}$ is bipartite, a cycle must contain an even number of arcs, and they must alternate between arcs of the form (l_x, r_y) and arcs of the form (r_x, l_y). Thus, we can assume that C takes the form $l_{x_1} \to r_{x_2} \to l_{x_3} \to \cdots \to r_{x_n} \to l_{x_1}$ for some $x_1, x_2, \ldots, x_n \in X$. By Definition 2.8, it is impossible to construct a negative cycle containing only two arcs, so we can assume that C contains at least four arcs. If C contains at most one arc with weight $-\gamma$, then all remaining arcs from a left to a right endpoint have weight 2, and thus the cycle can never become negative. So C must contain at least two arcs with weight $-\gamma$.

We proceed by considering three different cases.

Case 1: Suppose that C contains no arcs with weight 2 or -1, i.e., C only arcs with weight 0 and $-\gamma$. Then C has the form $l_{x_1} \xrightarrow{-\gamma} r_{x_2} \xrightarrow{0} l_{x_3} \xrightarrow{-\gamma} \dots \xrightarrow{0} l_{x_1}$. We have $x_2 \prec x_1$, $x_2||x_3$, and $x_4 \prec x_3$. Consider now the relationship between x_1 and x_4. If $x_1 \prec x_4$, then $x_2 \prec x_1 \prec x_4 \prec x_3$, a contradiction. If $x_4 \prec x_1$, then (l_{x_1}, r_{x_4}) is an arc with weight $-\gamma$. Thus, we can replace the three arcs $l_{x_1} \xrightarrow{-\gamma} r_{x_2} \xrightarrow{0} l_{x_3} \xrightarrow{-\gamma} r_{x_4}$ by the single arc $l_{x_1} \xrightarrow{-\gamma} r_{x_4}$ to get a shorter negative cycle, which contradicts the minimality of C. Thus $x_1||x_4$ and $(x_2 \prec x_1)||(x_4 \prec x_3)$ forms an induced $\mathbf{2+2}$.

Case 2: Suppose now that C contains arcs with weight -1 but no arcs with weight 2. Note that the first arc must have weight $-\gamma$. Rewrite C if necessary so that the second arc (r_{x_2}, l_{x_3}) has weight -1. Then $l_{x_1} \xrightarrow{-\gamma} r_{x_2}$, i.e. $x_2 \prec x_1$, and $r_{x_2} \xrightarrow{-1} l_{x_3}$, i.e. $x_2 = x_3$. Now $l_{x_3} \xrightarrow{-\gamma} r_{x_4}$, i.e., $x_4 \prec x_3$. Hence $x_4 \prec x_2 = x_3 \prec x_1$. Thus (l_{x_1}, r_{x_4}) is an arc with weight $-\gamma$ in $G_{P,2}$. Observe that every cycle containing arcs with weight $0, -1$, and $-\gamma$ is nonpositive; thus, if C is negative, replacing the segment $l_{x_1} \xrightarrow{-\gamma} r_{x_2} \xrightarrow{-1} l_{x_3} \xrightarrow{-\gamma} r_{x_4}$ by $l_{x_1} \xrightarrow{-\gamma} r_{x_4}$ results in a shorter negative cycle, contradicting the minimality of C.

Case 3: Assume now that C contains an arc with weight 2. By the above observations about C, we know that C must contain at least two arcs with weight $-\gamma$. Since C contains both arcs of weight 2 and arcs of weight $-\gamma$, it must contain a segment of the form $l_x \xrightarrow{2} r_y \to l_z \xrightarrow{-\gamma} r_w$. Now, if the arc (r_y, l_z) has weight -1, then we can get a shorter negative cycle by replacing $l_x \xrightarrow{2} r_y \xrightarrow{-1} l_z$ by l_x, a contradiction to the minimality of C. Thus, the arc (r_y, l_z) must have weight 0, i.e., C must contain a segment of the form $l_x \xrightarrow{2} r_y \xrightarrow{0} l_z \xrightarrow{-\gamma} r_w$.

We begin by choosing a starting point for C so that $l_{x_1} \xrightarrow{2} r_{x_2} \xrightarrow{0} l_{x_3} \xrightarrow{-\gamma} r_{x_4}$. Then we know that $x_1 = x_2$, $x_1 = x_2||x_3$, and $x_4 \prec x_3$. If $x_4 \prec x_1$, then we can replace the segment $l_{x_1} \xrightarrow{2} r_{x_2} \xrightarrow{0} l_{x_3} \xrightarrow{-\gamma} r_{x_4}$ by $l_{x_1} \xrightarrow{-\gamma} r_{x_4}$ to get a shorter negative cycle, a contradiction. If $x_1 \prec x_4$, then $x_1 \prec x_4 \prec x_3$, contradicting the fact that $x_1||x_3$. Hence $x_1||x_4$.

Suppose $r_{x_4} \xrightarrow{0} l_{x_5}$, i.e., $x_4||x_5$. If $x_1 = x_5$, then replacing $l_{x_1} \to \cdots \to l_{x_5}$ by l_{x_1} results in a shorter negative cycle, so $x_1 \neq x_5$. (If C contains only a single vertex after the replacement, then C could not have been negative originally.) We now consider the next arc in C. If $l_{x_5} \xrightarrow{-\gamma} r_{x_6}$, then $l_{x_3} \xrightarrow{-\gamma} r_{x_4} \xrightarrow{0} l_{x_5} \xrightarrow{-\gamma} r_{x_6}$, and by a similar argument as in Case 1, we know that P contains an induced $\mathbf{2+2}$. Otherwise $l_{x_5} \xrightarrow{2} r_{x_6}$ and $x_5 = x_6$. If $x_5 = x_6 \prec x_3$, then we can replace $l_{x_3} \xrightarrow{-\gamma} r_{x_4} \xrightarrow{0} l_{x_5} \xrightarrow{2} r_{x_6}$ by the single arc $l_{x_3} \xrightarrow{-\gamma} r_{x_6}$ to get a shorter negative cycle. If $x_3 \prec x_5 = x_6$, then $x_4 \prec x_3 \prec x_5$, a contradiction. Then $x_3||x_5 = x_6$. Consider now the relationship between x_1 and x_5. If $x_1 \prec x_5$ or $x_5 \prec x_1$, then x_1, x_3, x_4, and x_5 form a $\mathbf{2+2}$. Otherwise, we have $x_1||x_5$. Since C contains at least two arcs with weight $-\gamma$, we can replace the segment $r_{x_2} \xrightarrow{0} l_{x_3} \xrightarrow{-\gamma} r_{x_4} \xrightarrow{0} l_{x_5}$ by $r_{x_2} \xrightarrow{0} l_{x_5}$ to get a shorter negative cycle. Again, we get a contradiction.

Hence $r_{x_4} \xrightarrow{-1} l_{x_5}$. As before, if $l_{x_5} \xrightarrow{2} r_{x_6}$, then we can replace $r_{x_4} \xrightarrow{-1} l_{x_5} \xrightarrow{2} r_{x_6}$ with l_{x_4} to get a shorter negative cycle, again contradicting the minimality of C. So $l_{x_5} \xrightarrow{-\gamma} r_{x_6}$. Now if $x_6 \prec x_1$, then $l_{x_1} \xrightarrow{-\gamma} r_{x_6}$, and we can replace $l_{x_1} \xrightarrow{2} r_{x_2} \xrightarrow{0} l_{x_3} \xrightarrow{-\gamma} r_{x_4} \xrightarrow{-1} l_{x_5} \xrightarrow{-\gamma} r_{x_6}$ by $l_{x_1} \xrightarrow{-\gamma} r_{x_6}$ to obtain a shorter negative cycle, a contradiction. If $x_1 \prec x_6$, then $x_1 \prec x_6 \prec x_5 = x_4$, which is again a contradiction. Hence $x_1||x_6$. By a similar argument, we have $r_{x_6} \xrightarrow{-1} l_{x_7} \xrightarrow{-\gamma} r_{x_8}$ and $x_1||x_8$. Therefore $x_1||(x_8 \prec x_7 = x_6 \prec x_5 = x_4 \prec x_3)$ forms an induced $\mathbf{4} + \mathbf{1}$. $\square$

Note that the last argument above fails for the following arc $r_{x_8} \to l_{x_9}$. Since the current weight of the cycle has already become negative, we do not need to have another arc following $r_{x_8} \to l_{x_9}$. We can simply close the cycle by having $r_{x_8} \xrightarrow{0} l_{x_9}$ with $x_1 = x_9$. If we do not have $r_{x_8} \xrightarrow{0} l_{x_1}$, we can replace the remaining portion $r_{x_8} \to l_{x_9} \to \dots r_{x_n} \to l_{x_1}$ of the cycle by the single arc $r_{x_8} \xrightarrow{0} l_{x_1}$ to get a shorter negative cycle. Note that the fact that $k = 2$ is used mainly in this last part. For a general value of k, the negative cycle will have the same form, except that the first arc will have weight k and thus the cycle will continue alternating between arcs of weight -1 and arcs of weight $-\gamma$ until there are enough arcs of weight -1 to cancel the k. We can use induction to show that the cycle will indeed take this form. Then we can close the cycle with an arc of weight 0 to produce a cycle containing one arc of weight k, two arcs of weight 0, k arcs of weight -1 and $k+1$ arcs of weight $-\gamma$. Hence, the total weight of the cycle is precisely $-(k+1)\gamma$.

We conclude with a proposition demonstrating how we can use the above theorem to algorithmically construct an interval representation of a partial order P with lengths between 1 and k or determine that no such representation exists.

Proposition 2.5. *Let $P = (X, \prec)$ be a partial order and let $k \in \mathbb{Z}_{\geq 1}$. In polynomial time, we can either construct an interval representation of P in which all interval lengths are between 1 and k or determine that no such representation exists.*

Proof. Given a partial order $P = (X, \prec)$ and an integer $k \in \mathbb{Z}_{\geq 1}$, construct the associated weighted digraph $G_{P,k}$ using Definition 2.8. Use a standard shortest-paths algorithm (e.g. the Floyd-Warshall algorithm) on $G_{P,k}$ to compute the weight of a shortest path between each pair of vertices or detect a negative cycle. If a negative cycle is detected, then by Theorem 2.4, there is no interval representation of P in which all interval lengths are between 1 and k. If the digraph contains no negative cycles, then the function $p : \{l_x, r_x\}_{x \in X} \to \mathbb{R}$, where $p(y)$ is the minimum weight of a path in $G_{P,k}$ ending at y, is a potential function on $G_{P,k}$. Then, as we showed in the proof of $(3) \Rightarrow (1)$ of Theorem 2.4, we can construct an interval for each element of the poset such that this collection of intervals forms a valid interval representation of P with lengths between 1 and k. Note that each step in this process takes at most polynomial time, so the entire construction can be carried out in polynomial time. For example, this procedure runs in $O(|X|^3)$ if the Floyd-Warshall shortest-paths algorithm is used. $\square$

Chapter 3

Interval Orders Representable with Lengths 0 and 1

3.1 Introduction

In this chapter, we examine the class $\mathscr{P}_{\{0,1\}}$ of interval orders representable with lengths exactly 0 and 1. This class belongs to the family $\{\mathscr{P}_{\{1,k\}}\}_{k\in\mathbb{Z}_{\geq 0}}$, but its behavior is very different from that of any one the other classes in this family.

By using an appropriate scaling, it is not hard to show that the class $\mathscr{P}_{[0,1]}$ of interval orders representable with lengths between 0 and 1 is the same as the class of interval orders. The class $\mathscr{P}_{\{0,1\}}$ of interval orders representable with lengths 0 and 1, however, differs from each of the other classes of interval orders we have discussed so far. We will show that $\mathscr{P}_{\{0,1\}} \subsetneq \mathscr{P}_2$ and that for each $k > 0$, we have $\mathscr{P}_{\{0,1\}} \not\subseteq \mathscr{P}_{\{1,k\}}$ and $\mathscr{P}_{\{0,1\}} \not\supseteq \mathscr{P}_{\{1,k\}}$ if $k \neq 1$. By definition, we know that $\mathscr{P}_{\{0,1\}} \subseteq \mathscr{P}_2$. The next example shows that this is a strict inclusion and that $\mathscr{P}_{\{0,1\}} \not\supseteq \mathscr{P}_{\{1,k\}}$ if $k \neq 1$.

Example 3.1. Consider the interval order shown in Figure 3.1. It is not hard to check that for any $k > 1$, this interval order has an interval representation in which a, b, c, d, e get intervals of length 1 and x gets an interval of length k. However, this poset has no representation with lengths 0 and 1. To see why, suppose it does. Then a, b, c, x form a $\mathbf{3}+\mathbf{1}$, and thus, we must have $L(x) \leq R(a) < L(b) \leq R(b) < L(c) \leq R(x)$, i.e., the interval I_b has to be a proper subinterval of I_x. So we must have $|I_b| = 0$ and $|I_x| = 1$. But now, since $b||(d \prec e)$, we need to have $L(b) \leq R(d) < L(e) \leq R(b)$, so $|I_b| > 0$, a contradiction. So this poset belongs to both $\mathscr{P}_2$ and $\mathscr{P}_{\{1,k\}}$ for all $k > 1$ but not to $\mathscr{P}_{\{0,1\}}$.

Let $k \geq 1$. To see why $\mathscr{P}_{\{0,1\}} \not\subseteq \mathscr{P}_{\{1,k\}}$, consider the poset $(\mathbf{k}+\mathbf{2})+\mathbf{1}$. This poset has a

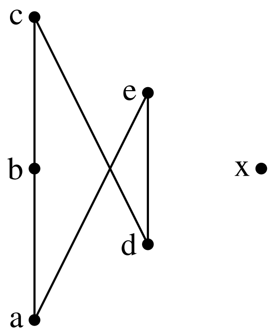


Figure 3.1: A poset in $\mathscr{P}_{\{1,k\}}$ for all $k > 1$ but not in $\mathscr{P}_{\{0,1\}}$.

representation with lengths 0 and 1, but no representation with lengths between 1 and k as shown in Chapter 2 and thus no representation with lengths exactly 1 and k.

There is no known characterization of $\mathscr{P}_{\{1,k\}}$ for $k \geq 2$ (see Chapter 4 for further discussion of these classes). However, the class $\mathscr{P}_{\{0,1\}}$ turns out to be easier to work with than the other classes in this family. In this chapter, we establish a necessary and sufficient condition for an interval order to have a representation with intervals of length 0 and 1. We characterize the class $\mathscr{P}_{\{0,1\}}$ in terms of a forbidden substructure and provide a polynomial-time recognition algorithm.

As noted in Chapter 1, Skrien [21] and Rautenbach and Szwarcfiter [17] have characterized the analogous class of interval graphs. Skrien provides a cubic-time recognition algorithm for this class, while Rautenbach and Szwarcfiter derive a linear-time recognition algorithm and a characterization in terms of forbidden induced subgraphs.

3.2 Characterization

Recall from Chapter 1 that for any poset $P = (X, \prec)$ and for all $x \in X$, we define the incomparability set $Inc(x)$, down set $Down(x)$ and up set $Up(x)$ as follows: $Inc(x) = \{y \in X : x||y\}$, $Down(x) = \{y \in X : y \prec x\}$, and $Up(x) = \{y \in X : x \prec y\}$. We now define a partition of the elements of the ground set.

Definition 3.1. Let $P = (X, \prec)$ be a poset. Define $U(P) = \{x \in X : Inc(x) \text{ is an antichain}\}$ and $V(P) = X - U(P) = \{x \in X : Inc(x) \text{ is not an antichain}\}$.

Before establishing the main result, we show a number of auxiliary results concerning this partition. First we discuss some properties of the set $U(P)$. Recall the Greenough representation of an interval order from Theorem 1.4.

Lemma 3.1. *Let $P = (X, \prec)$ be an interval order. The point $x \in X$ is in $U(P)$ if and only if $|I_x| = 0$ in the Greenough representation of P.*

Proof. Let $\mathcal{I} = \{I_x\}_{x \in X}$ be the Greenough representation of P.

($\Rightarrow$) Assume $x \in U(P)$ and suppose for the sake of contradiction that $|I_x| > 0$, i.e., $L(x) < R(x)$. Recall that in the Greenough representation, each endpoint is both a left and a right endpoint. Therefore, there exist $u, v \in X$ such that $L(x) = R(u)$ and $R(x) = L(v)$. Each of the intervals I_u and I_v intersects I_x, so $u, v \in Inc(x)$. Since $L(x) < R(x)$, we have $R(u) < L(v)$. So $u \prec v$, and thus $Inc(x)$ is not an antichain, a contradiction.

($\Leftarrow$) Assume that $|I_x| = 0$ in the Greenough representation. For $u, v \in Inc(x)$, the intervals I_u and I_v must intersect I_x. Since I_x is a point, both I_u and I_v must contain I_x and thus will intersect each other. So $u||v$. Hence $Inc(x)$ is an antichain, and $x \in U(P)$. □

In a twin-free interval order, no two points can get identical intervals. Hence the next corollary follows immediately from Lemma 3.1.

Corollary 3.2. *If P is a twin-free interval order, then the partial order $P|_{U(P)}$ is a chain.*

Lemma 3.3. *If a poset P has an interval representation with lengths 0 and 1, then P has a representation with lengths 0 and 1 in which $|I_x| = 0$ for all $x \in U(P)$.*

Proof. Fix an interval representation of P in which all intervals have length either 0 or 1. Let $x \in U(P)$. By the definition of $U(P)$, we know that $Inc(x)$ is an antichain. Thus, for all $u, v \in Inc(x)$, we have $I_u \cap I_v \neq \emptyset$. Thus, by the Helly property discussed in Chapter 1, we have $\bigcap_{v \in Inc(x)} I_v \neq \emptyset$. Hence I_x can be contracted to a single point in $\bigcap_{v \in Inc(x)} I_v$. □

Using Definition 3.1, it is clear that if P is representable with lengths 0 and 1, then each element $x \in V(P)$ must get an interval of length 1. By Lemma 3.3, we know that each element $x \in U(P)$ can be assigned an interval of length 0. The ability to pre-determine the length of the interval corresponding to each element in the ground set simplifies the subsequent construction. This is one of the key differences between this class and each of the other $\mathscr{P}_{\{1,k\}}$ with $k \geq 2$, where there is no known efficient way to pre-determine the lengths of *all* intervals (even though we can do it for some).

We now turn our attention to the set $V(P)$. If $P \in \mathscr{P}_{\{0,1\}}$, then, in any representation of P with lengths 0 and 1, all intervals corresponding to elements in $V(P)$ will have the same length; thus $P|_{V(P)}$ must be a unit interval order. In this case, we want to construct a unit interval representation of $P|_{V(P)}$ satisfying certain properties. In order to determine the order of the left endpoints in this representation, we define a sorting on the elements in $V(P)$. It turns out that this sorting also lets us determine whether or not $P \in \mathscr{P}_{\{0,1\}}$.

Definition 3.2. Let $P = (X, \prec)$ be an interval order. Define the order $\prec_s$ on X by $x \prec_s y$ if either $Down(x) \subsetneq Down(y)$, or $Down(x) = Down(y)$ and $Up(y) \subseteq Up(x)$.

Lemma 3.4. *Let $P = (X, \prec)$ be a twin-free interval order. Then $\prec_s$ is a linear order on any X.*

Proof. Let $x, y \in X$. Since P is twin-free, we know that either $Down(x) \neq Down(y)$ or $Up(x) \neq Up(y)$. We know from Proposition 1.3 in Chapter 1 that in an interval order, the down sets are ordered by inclusion and the up sets are also ordered by inclusion. If $Down(x) \subsetneq Down(y)$, then $x \prec_s y$. Similarly, if $Down(y) \subsetneq Down(x)$, then $y \prec_s x$. The only other possibility is $Down(x) = Down(y)$. Then we must have either $Up(x) \subsetneq Up(y)$ or $Up(y) \subsetneq Up(x)$, implying that $y \prec_s x$ or $x \prec_s y$ respectively. $\square$

The next example illustrates the order $\prec_s$ on a specific interval order, and we see that that $\prec_s$ is indeed a linear order.

Example 3.2. Consider again the twin-free poset P from Figure 3.1. We have $U(P) = \{a, c\}$ and $V(P) = \{b, d, e, x\}$. If we compute the down set and up set of each $v \in X$, we get Table 3.1 below.

v	$Down(v)$	$Up(v)$
a	$\emptyset$	$\{b, c, e\}$
b	$\{a\}$	$\{c\}$
c	$\{a, b, d\}$	$\emptyset$
d	$\emptyset$	$\{c, e\}$
e	$\{a, d\}$	$\emptyset$
x	$\emptyset$	$\emptyset$

Table 3.1: Down sets and up sets of the poset in Figure 3.1.

Thus, the relation $\prec_s$ on X gives us the linear order $a \prec_s d \prec_s x \prec_s b \prec_s e \prec_s c$. Thus here $\prec_s$ is indeed a linear order.

Note that for all $y \in X$, we have $y \notin Down(y)$. If $x \in X$ and $y \prec x$, then $y \in Down(x)$, so $Down(x) \not\subseteq Down(y)$ and $x \not\prec_s y$. Hence we make the following observation.

Observation 3.1. *For all $x, y \in X$, if $x \prec_s y$, then $x \prec y$ or $x||y$.*

Lemma 3.5. *Let $P = (X, \prec)$ be a twin-free interval order. Then P contains no induced $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$ and $u, v \in X$ if and only if, for all $x, y \in V(P)$ with $x \prec_s y$, we have $Up(y) \subseteq Up(x)$.*

Proof. ($\Leftarrow$) To prove the contrapositive, assume that $(u \prec y \prec v)||x$ induces in P a $\mathbf{3+1}$ for some $x, y \in V(P)$ and $u, v \in X$. Now $u \in Down(y)$ and $u \notin Down(x)$. Since P is an interval order, by Proposition 1.3, we know that the down sets are ordered by inclusion. Therefore, we must have $Down(x) \subsetneq Down(y)$. By Definition 3.2, we have $x \prec_s y$. Additionally $y \prec v$ and $x \not\prec v$, and so $v \in Up(y) - Up(x)$. So there exist $x, y \in V(P)$ with $x \prec_s y$ and $Up(y) \not\subseteq Up(x)$.
($\Rightarrow$) Again we show the contrapositive. Assume now that there exist $x, y \in V(P)$ such that $x \prec_s y$ with $Up(y) \not\subseteq Up(x)$. Then there is $v \in Up(y) - Up(x)$, i.e., $y \prec v$ but $x \not\prec v$. Now, if $Down(x) = Down(y)$, by the definition of $\prec_s$, we have $Up(y) \subseteq Up(x)$, a contradiction. So $Down(x) \subsetneq Down(y)$ and thus there exists $u \in Down(y) - Down(x)$. Hence $u \prec y$ but

$u \not\prec x$. Now we show that $u||x$ and $v||x$. Since $u \not\prec x$, we must have $u||x$ or $x \prec u$. If $x \prec u$, then by transitivity $x \prec u \prec y \prec v$. But $x \not\prec v$, so $u||x$. Similarly $v||x$.

Finally, we show that $x||y$. If $x \prec y$, then $x \prec y \prec v$, contradicting the fact that $x \not\prec v$. Similarly if $y \prec x$, then $u \prec y \prec x$, a contradiction. So $x||y$ and $(u \prec y \prec v)||x$ forms a $\mathbf{3}+\mathbf{1}$. □

The following corollary is an immediate consequence of Lemma 3.5.

Corollary 3.6. *Let $P = (X, \prec)$ be a twin-free interval order containing no induced $\mathbf{3}+\mathbf{1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$ and $u, v \in X$. Then, for all $x, y \in V(P)$, we have $x \prec_s y$ if and only if $Down(x) \subseteq Down(y)$ and $Up(y) \subseteq Up(x)$.*

Proof. ($\Rightarrow$) Let $x, y \in V(P)$ and assume that $x \prec_s y$. Then by Definition 3.2, we know that $Down(x) \subseteq Down(y)$. By Lemma 3.5, it follows that $Up(y) \subseteq Up(x)$.
($\Leftarrow$) Let $x, y \in V(P)$ and assume that $Down(x) \subseteq Down(y)$ and $Up(y) \subseteq Up(x)$. Suppose for a contradiction that $y \prec_s x$. Then by Definition 3.2, we have $Down(y) \subseteq Down(x)$. So $Down(x) = Down(y)$. Then, because $y \prec_s x$, by the definition of $\prec_s$ we must have $Up(x) \subseteq Up(y)$. So $Up(x) = Up(y)$, and thus x and y are twins, a contradiction. So $x \prec_s y$. □

The next example illustrates Lemma 3.5 on a particular poset.

Example 3.3. Consider again the poset from Figure 3.1. In Example 3.2, we found that $a \prec_s d \prec_s x \prec_s b \prec_s e \prec_s c$. We have $U(P) = \{a, c\}$ and $V(P) = \{b, d, e, x\}$. Now $x \prec_s b$ and $Up(b) = \{c\} \not\subseteq \emptyset = Up(x)$, and so P contains an induced $\mathbf{3}+\mathbf{1}$, namely $(a \prec b \prec c)||x$, with $b, x \in V(P)$.

Lemma 3.7. *Let $Q = (Y, \prec)$ be a unit interval order. Let $\{I_y\}_{y \in Y}$, where $I_y = [L(y), R(y)]$ for all $y \in Y$, be any unit interval representation of Q in which twins get identical intervals. Then, for all $x, y \in Y$, we have $x \prec_s y$ if and only if either x and y are twins or $L(x) < L(y)$ (and hence $R(x) < R(y)$).*

Proof. ($\Rightarrow$) Assume that $x \prec_s y$ and that x and y are not twins. Fix a unit interval representation $\{I_y\}_{y \in Y}$ of Q. If $Down(x) = Down(y)$, by Definition 3.2, we know that $Up(y) \subseteq Up(x)$; since x and y are not twins, we must have $Up(y) \subsetneq Up(x)$. So there is $z \in Up(x) - Up(y)$. Then $R(x) < R(y)$, and so $L(x) < L(y)$. If $Down(x) \subsetneq Down(y)$, then there is $w \in Down(y) - Down(x)$. So $L(x) < L(y)$ and $R(x) < R(y)$.
($\Leftarrow$) Assume now that $\{I_y\}_{y \in Y}$ is a unit interval representation of Q. Let $x, y \in Y$ and suppose $L(x) < L(y)$ and thus $R(x) < R(y)$. If $u \in X$ and $u \prec x$, then $R(u) < L(x) < L(y)$, so $u \prec y$. So $Down(x) \subseteq Down(y)$. Similarly $Up(y) \subseteq Up(x)$. So $x \prec_s y$. □

The next couple of lemmas will be used to validate two steps of the construction argument in the proof of the main theorem. One step of this argument is to create a unit interval representation of the interval order $P|_{V(P)}$, and the first lemma will be useful for verifying the correctness of the constructed representation. After some modification of this initial unit interval representation, we will add intervals corresponding to the elements of $U(P)$ to the representation, which is where the second lemma will be used.

Lemma 3.8. *Let $Q = (Y, \prec)$ be a twin-free unit interval order. If the elements of Y are indexed $s_1, \ldots, s_n$ so that $s_{i-1} \prec_s s_i$ for all $i \in \{2, \ldots, n\}$, then, for all $i, j \in \{1, \ldots, n\}$ with $i < j$, if $s_i || s_j$, then $s_k || s_l$ for all $k, l \in \{i, \ldots, j\}$.*

Proof. First we show that if $x, y \in Y$ with $x \prec_s y$, then $Down(x) \subseteq Down(y)$ and $Up(y) \subseteq Up(x)$. Fix a unit interval representation $\{I_y\}_{y \in Y}$ of Q. Let $x, y \in Y$ with $x \prec_s y$. Since x and y cannot be twins, by Lemma 3.7, we know that $L(x) < L(y)$ and $R(x) < R(y)$. So, for all $z \in Y$, if $z \prec x$, then $z \prec y$. So $Down(x) \subseteq Down(y)$. Similarly $Up(y) \subseteq Up(x)$.

Suppose, for the sake of contradiction, that there are $i, j \in \{1, \ldots, n\}$ with $i < j$ such that $s_i || s_j$ and that there exist $k, l \in \{i, \ldots, j\}$ such that $s_k \prec s_l$. Note that by Observation 3.1, we must have $s_k \prec_s s_l$. Thus, we have

$$Down(s_i) \subseteq Down(s_k) \subseteq Down(s_l) \subseteq Down(s_j)$$

and

$$Up(s_i) \supseteq Up(s_k) \supseteq Up(s_l) \supseteq Up(s_j).$$

Now since $s_k \in Down(s_l)$, we have $s_k \in Down(s_j)$, i.e., $s_k \prec s_j$. But then $s_j \in Up(s_k) \subseteq Up(s_i)$, so $s_i \prec s_j$, a contradiction.

□

The next lemma can be proven for unit interval orders using the unit interval representation. Here we show that it actually holds for a broader class of interval orders.

Lemma 3.9. *Let $P = (X, \prec)$ be a twin-free interval order with no induced $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$. Let $a, b \in V(P)$ and $c \in X$. If $a \prec c$ and $b||c$, then $a \prec_s b$. Similarly, if $c \prec a$ and $b||c$, then $b \prec_s a$.*

Proof. Since P contains no induced $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$, by Corollary 3.6, for all $x, y \in V(P)$, we have $x \prec_s y$ if and only if $Down(x) \subseteq Down(y)$ and $Up(y) \subseteq Up(x)$. Suppose that $a, b \in V(P)$ and $c \in X$ with $a \prec c$ and $b||c$. Then $c \in Up(a)$ but $c \notin Up(b)$. Since up sets are ordered by inclusion, we must have $Up(b) \subsetneq Up(a)$. Now Lemma 3.5 implies that $a \prec_s b$.

The other argument is analogous. □

Note that Lemma 3.9 is not true if P contains an induced $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$. For example, consider the poset in Figure 3.1. Then $b, x \in V(P)$, and we have $b \prec c$ and $x||c$. However, as we saw in Example 3.2, we have $x \prec_s b$ rather than $b \prec_s x$.

We are now ready to prove the main result, characterizing the class $\mathscr{P}_{\{0,1\}}$.

Theorem 3.10. *Let $P = (X, \prec)$ be a twin-free interval order. Then P has an interval representation with lengths 0 and 1 if and only if P contains no $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$ and $u, v \in X$.*

Proof. $(\Rightarrow)$ Assume P contains an induced $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$ and $u, v \in X$. To represent this $\mathbf{3+1}$, we must have $|I_y| < |I_x|$ because all of I_u, I_y, I_v need to intersect I_x. So $|I_x| = 1$ and $|I_y| = 0$. Since $y \in V(P)$, there exist $s, t \in Inc(y)$ such that $s \prec t$, and it is impossible to place the intervals I_s and I_t such that they both intersect I_y without intersecting each other. Hence P cannot be represented with lengths 0 and 1.
$(\Leftarrow)$ Our proof of the second direction is constructive. Assume P contains no $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $y \in V(P)$. By Corollary 3.6, we can assume that $x \prec_s y$ if and only if $Down(x) \subseteq Down(y)$ and $Up(y) \subseteq Up(x)$. The construction involves three main steps: constructing an initial representation of $P|_{V(P)}$, modifying the initial representation to make all endpoints distinct, and adding points corresponding to the elements in $U(P)$ to the representation. These steps are illustrated for a particular poset in Example 3.4.

Since P contains no $\mathbf{3+1}$ $(u \prec y \prec v)||x$ with $x, y \in V(P)$, we know that $P|_{V(P)}$ contains no induced $\mathbf{3+1}$. Therefore $P|_{V(P)}$ is a unit interval order. Hence we will construct a unit interval representation of it. Let $V(P) = \{s_1, \ldots, s_n\}$, where $s_{i-1} \prec_s s_i$ for all $i \in \{2, \ldots, n\}$. Note that even though P is twin-free, the order $P|_{V(P)}$ may not be twin-free. Our goal is to construct a representation in which twins get identical intervals. From the definition of $\prec_s$, it follows directly that if s_i and s_j are twins in $P|_{V(P)}$ for some $i, j \in \{1, \ldots, n\}$ with $i < j$, then s_i and s_k are twins in $P|_{V(P)}$ for all $i < k \leq j$. Then we proceed as follows.

First, we create a new list $s'_1, \ldots, s'_m$ from $\{s_1, \ldots, s_n\}$ by removing all but one representative from each twin class while maintaining the order of the elements. We assign s'_1 the interval $[0, 1]$, and set the current position p to 0. For each $i \in \{2, \ldots, m\}$, we find the largest index j such that $s'_j \prec s'_i$. If such a j exists, we move the position marker p to $\max\{p, R(s'_j)\} + \frac{1}{2^i}$, and set $I_{s'_i} = [p, p+1]$. Otherwise, we set $p := p + \frac{1}{2^i}$. Finally, for each $i \in \{1, \ldots, n\}$, if s_i has not yet been assigned an interval, and s_i is twins with s'_j for some $j \in \{1, \ldots, m\}$, then we assign $I_{s_i} = I_{s'_j}$.

We verify that the collection $\{I_v\}_{v \in V(P)}$ is indeed a valid unit interval representation of $P|_{V(P)}$. It is clear that if $x \prec y$, then $R(x) < L(y)$. Now suppose $x||y$, and without loss of generality assume that $x \prec_s y$. Then $L(x) \leq L(y)$. Assume that x and y are not twins and that x is twins with s'_i and y is twins with s'_j for some $i, j \in \{1, \ldots, m\}$ with $i < j$. Then, by Lemma 3.8, we know that $s'_k||s'_l$ for all $k, l \in \{i, \ldots, j\}$. Hence, the largest possible value of r such that $s'_r \prec s'_j$ is $i - 1$. By construction, we know that $L(s'_{i-1}) + \frac{1}{2^i} \leq L(s'_i)$ and hence $R(s'_{i-1}) + \frac{1}{2^i} \leq R(s'_i)$. So $R(s'_{i-1}) \leq R(s'_i) - \frac{1}{2^i}$. We also know that $L(s'_i) \leq R(s'_i) - \frac{1}{2^i}$. Then $L(s'_j)$ can be at most $\max\{L(s'_i), R(s'_{i-1})\} + \frac{1}{2^{i+1}} + \ldots \frac{1}{2^j} \leq R(s'_i) - \frac{1}{2^i} + \frac{1}{2^{i+1}} + \ldots \frac{1}{2^j} < R(s'_i)$, so the intervals I_{s_i} and I_{s_j} intersect.

Note that in this representation, twins get identical intervals and all other endpoints are distinct, i.e., two intervals I_x and I_y share an endpoint (and hence both endpoints) if and only if x and y are twins in $P|_{V(P)}$. Since $P|_{V(P)}$ is a unit interval order, by Lemma 3.7, it follows that if $x, y \in V(P)$ with $x \prec_s y$ and x and y are not twins in $P|_{V(P)}$, then $L(x) < L(y)$ and $R(x) < R(y)$ in this representation.

The goal now is to extend this representation of $P|_{V(P)}$ to a representation of P by adding intervals corresponding to the elements of $U(P)$. Let ϵ be the smallest positive distance between endpoints in our representation.

To construct a representation of P, we first need to make all endpoints in the unit interval representation of $P|_{V(P)}$ distinct. Note that "being twins" is an equivalence relation on $V(P)$. We proceed by traversing the elements of $V(P)$ in the order given by $\prec_s$. For each $i \in \{2, \ldots, n\}$, let j be the smallest index such that s_j and s_i are twins. Shift I_{s_i} to the right by $\frac{i-j}{2n}\epsilon$.

Note that this procedure does not change the comparabilities and incomparabilities between the elements of $V(P)$. Moreover, all endpoints in this representation are distinct, and the relative order of the endpoints is preserved, i.e., for all $x, y \in V(P)$, if $L(x) < L(y)$ before, then $L(x) < L(y)$ after the above procedure is executed. Observe also that the smallest distance between endpoints in the new representation is at least $\frac{\epsilon}{2n} > 0$.

It remains to place each element $x \in U(P)$ in the correct position. Let $x \in U(P)$. Then I_x must be placed inside of $\bigcap_{v \in Inc(x)} I_v$. Since $Inc(x) \subseteq V(P)$ and $Inc(x)$ forms an antichain in both P and $P|_{V(P)}$, we know that for all $v, w \in Inc(x)$, we have $I_v \cap I_w \neq \emptyset$. So by the Helly property, it follows that $\bigcap_{v \in Inc(x)} I_v \neq \emptyset$ in our representation.

We now claim that there is an interval $J_x \subseteq \bigcap_{v \in Inc(x)} I_v$ such that $J_x \cap I_t = \emptyset$ for all $t \in V(P) - Inc(x)$. Let $w \in Inc(x)$ be the element with the leftmost right endpoint, and $w' \in Inc(x)$ be the element with the rightmost left endpoint. Then $\bigcap_{v \in Inc(x)} I_v = I_w \cap I_{w'}$. Let $t_1, t_2 \in V(P) - Inc(x)$. If $t_1 \prec x$, then since $w||x$, by Lemma 3.9, we have $t_1 \prec_s w$, so $R(t_1) < R(w)$. If $x \prec t_2$, then since $w'||x$, again by Lemma 3.9, we have $w' \prec_s t_2$. So $L(w') < L(t_2)$. Thus, no single $t \in V(P) - Inc(x)$ can contain the interval $I_w \cap I_{w'}$. If $t_1 \prec x$ and $x \prec t_2$, by transitivity we have $t_1 \prec x \prec t_2$. So $t_1 \prec t_2$. Therefore $R(t_1) < L(t_2)$. So it is not possible for $I_{t_1} \cup I_{t_2}$ to contain the whole interval $I_w \cap I_{w'}$. Hence, there is an interval $J_x \subseteq I_w \cap I_{w'}$ such that $J_x \cap I_t = \emptyset$ for all $t \in V - Inc(x)$, and I_x can be placed inside of J_x. If there are $x_1, \ldots, x_p \in U(P)$ such that $Inc(x_1) = \cdots = Inc(x_p)$, then all of $I_{x_1}, \ldots, I_{x_p}$ have length 0 and can be placed inside of J_{x_1}.

This completes the construction, and so P has a representation with lengths 0 and 1. □

We conclude with an example illustrating the steps of the algorithm described above.

Example 3.4. Consider the poset P shown in Figure 3.2. We have $U(P) = \{b, c\}$ and $V(P) = \{a, d, x\}$. We then compute the down sets and up sets of the elements, as we did in Example 1.1, to get Table 3.2 below.

So the order $\prec_s$ gives $c \prec_s a \prec_s x \prec_s d \prec_s b$. Restricting $\prec_s$ to $V(P)$, we have $a \prec_s x \prec_s d$. Note that $\prec_s$ sorts the down sets by inclusion and the up sets by containment, i.e., $Down(a) \subseteq$

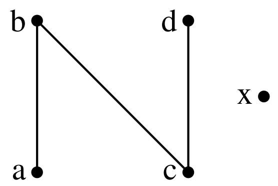


Figure 3.2: A poset which has an interval representation with lengths 0 and 1.

v	$Down(v)$	$Up(v)$
a	$\emptyset$	$\{b\}$
b	$\{a, c\}$	$\emptyset$
c	$\emptyset$	$\{b, d\}$
d	$\{c\}$	$\emptyset$
x	$\emptyset$	$\emptyset$

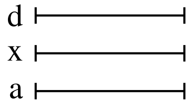
Table 3.2: Down sets and up sets of the poset in Figure 3.2.

$Down(x) \subseteq Down(d)$ and $Up(a) \supseteq Up(x) \supseteq Up(d)$. Hence, by Lemma 3.5, this P does not contain an induced $\mathbf{3} + \mathbf{1}$ $(u \prec y \prec v)||w$ with $w, y \in V(P)$, and, by Theorem 3.10, it follows that P has an interval representation with lengths 0 and 1.

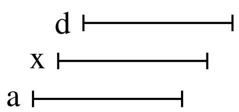
First, we construct a unit interval representation of $P|_{V(P)}$ as shown in Figure 3.3a. Note that the three elements $a, d,$ and x get identical intervals because they are all twins in $P|_{V(P)}$.

Then we shift the intervals so that their endpoints are distinct and their left endpoints are arranged in the order specified by $\prec_s$. Note that $n = |V(P)| = 3$, and let $s_1 = a, s_2 = x$, and $s_3 = d$. The minimum distance between endpoints is 1, so $\epsilon = 1$. The interval corresponding to $s_1 = a$ remains unchanged. Now consider $s_2 = x$. The smallest index j such that s_j and s_2 are twins in $P|_{V(P)}$ is 1. So I_x is shifted to the right by $\frac{2-1}{2n}\epsilon = \frac{1}{6}$. Similarly, we consider s_3 and we find that I_{s_3} gets shifted to the right by $\frac{3-1}{2n}\epsilon = \frac{1}{3}$. We obtain the representation shown in Figure 3.3b.

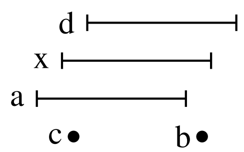
Finally, we add points corresponding to the elements b and c. First consider c. We have $c||a$, $c||x$, and $c \prec d$. So $w = a$ and $w' = x$, so c needs to be placed inside of $(I_a \cap I_x) - I_d$. Similarly, we place b inside of $(I_x \cap I_d) - I_a$ to obtain a representation of P with lengths 0 and 1 (see Figure 3.3c).



(a) Initial unit interval representation of $V(P)$.



(b) Unit interval representation of $V(P)$ with distinct endpoints.



(c) Interval representation of P with lengths 0 and 1.

Figure 3.3: Main steps in constructing an interval representation with lengths 0 and 1 for the poset in Figure 3.2.

Chapter 4

Interval Orders Representable with Lengths 1 and 2

4.1 Introduction

In this chapter, we examine the class of interval orders representable with lengths exactly 1 and 2. As we mentioned in Chapter 1, there is no known characterization for $\mathscr{P}_{\{1,k\}}$ when $k > 1$. Moreover, related results suggest that obtaining a characterization for a class in this family is difficult. One reason this might be a difficult task is that unlike in the case of lengths 0 and 1, here we know of no efficient way to pre-determine the length of each interval. This is why we consider the following related problem, in which each interval has a prescribed length, to help us gain insight into the behavior of classes in this family. In this thesis, we focus specifically on the class $\mathscr{P}_{\{1,2\}}$.

Problem 1. *Let $P = (X, \prec)$ be an interval order. Suppose each $x \in X$ is colored red or black. Determine if there is an interval representation of P such that if $x \in X$ is black, then $|I_x| = 1$, and if $x \in X$ is red, then $|I_x| = 2$.*

This problem has an efficient solution using linear programming. In particular, we can use the digraph model discussed in Chapter 2. Given a poset P, we define G_P to be the weighted digraph with vertices $\{l_x, r_x\}_{x \in X}$ and the following arcs:

- (l_y, r_x) with weight $-\gamma$ for all $x, y \in X$ with $x \prec y$;
- (r_x, l_y) with weight 0 for all $x, y \in X$ with $x||y$ or $x = y$;
- (r_x, l_x) with weight -1 for all $x \in X$ such that x is black;

- (l_x, r_x) with weight 1 for all $x \in X$ such that x is black;
- (r_x, l_x) with weight $-k$ for all $x \in X$ such that x is red;
- (l_x, r_x) with weight k for all $x \in X$ such that x is red.

By similar reasoning as in Chapter 2, we can show that G_P has a potential function if and only if P has an interval representation satisfying the given constraints.

In fact, this digraph model can be extended to help us solve the generalized version of this problem, stated in Problem 2. Before we can state the problem, we need the following definition.

Definition 4.1. Let $P = (X, \prec)$ be an interval order. A *length function* is a function $f : X \to \mathbb{R}_{>0}$ such that $f(x)$ is the prescribed length of I_x for all $x \in X$. For each $x \in X$, we will refer to the value of $f(x)$ as the *length* of x.

Note that for the purposes of this chapter, we will assume that all prescribed lengths are positive.

Problem 2. *Let $P = (X, \prec)$ be an interval order. Let $f : X \to \mathbb{R}_{>0}$ be a length function. Determine whether or not it is possible to construct an interval representation of P with the given prescribed lengths, i.e., a representation such that for all $x \in X$, the length of I_x is exactly $f(x)$.*

The digraph model is particularly suitable for the problem discussed in Chapter 2, where the constraints on the lengths of the intervals are inequalities. In that case, the digraph model provides us with an elegant way to characterize the classes $\mathscr{P}_{[1,k]}$ for $k \in \mathbb{Z}_{\geq 1}$, while the direct approach is messy and yields some very technical conditions (see [6]). When the constraints are equalities, however, the digraph model seems to add unnecessary overhead. In this case, the direct approach yields a more elegant solution and provides more structural insight.

In the first two sections of this chapter, we develop tools that can be used to solve Problem 2 discussed above. We then use these tools to derive results about the specific class $\mathscr{P}_{\{1,2\}}$.

4.2 Forcing Cycles

Here we extend the notion of a forcing cycle, defined by Gimbel and Trenk in [9].

Definition 4.2. Let $P = (X, \prec)$ be a partial order. A *forcing trail* C in P is a sequence of elements $C : x_0, x_1, \ldots, x_n$, where for all $k \in \{1, \ldots, n\}$, we have either $x_{i-1} \prec x_i$ or $x_{i-1} || x_i$. If in addition we have $x_0 = x_n$, then C is called a *forcing cycle*. We say that a forcing trail is *simple* if $x_0, \ldots, x_n$ are all distinct, except possibly x_0 and x_n.

For example, in the poset shown in Figure 4.1, the sequence $C_1 : a, b, c, e, d$ is a forcing trail, while the sequence $C_2 : a, b, d, e, x, a$ is a simple forcing cycle.

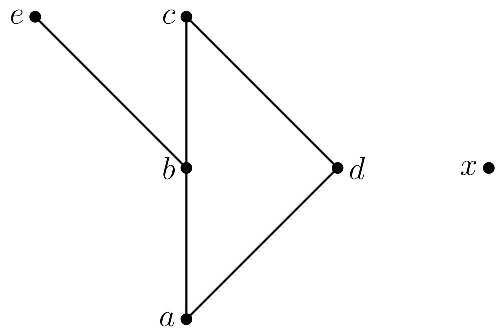


Figure 4.1: An example of a partial order.

Definition 4.3. Let $C : x_0, x_1, \ldots, x_n$ be a forcing trail in a partial order $P = (X, \prec)$. We define $up(C) = |\{i : x_{i-1} \prec x_i\}|$ and $side(C) = |\{i : x_{i-1}||x_i\}|$.

Considering the forcing cycle C_2 in the example above, we have $a \prec b||d||e||x||a$, so $up(C_2) = 1$ and $side(C_2) = 4$.

We now define the weight of each relation ($\prec$ or $||$) in a forcing cycle. The weight of the ith relation (the one relating x_{i-1} and x_i) depends on the length of either x_{i-1} or x_i. If the ith relation is $\prec$, then its weight depends on the length of x_{i-1}; otherwise it depends on the length of x_i.

Definition 4.4. Let $P = (X, \prec)$ be an interval order and $f : X \to \mathbb{R}$ be a length function. Let $C : x_0, x_1, \ldots, x_n$ be a forcing trail in P. For all $i \in \{1, \ldots, n\}$, we define the *weight* $w_i(C)$ to be

$$w_i(C) = \begin{cases} f(x_{i-1}) & x_{i-1} \prec x_i \\ f(x_i) & x_{i-1}||x_i \end{cases}.$$

Then we define the *up weight* to be

$$W_U(C) = \sum_{\{i : x_{i-1} \prec x_i\}} w_i(C),$$

and the *side weight* to be

$$W_S(C) = \sum_{\{i : x_{i-1}||x_i\}} w_i(C).$$

Suppose that we have the length function f given in Table 4.1 for the poset in Figure 4.1.

v	a	b	c	d	e	x
$f(v)$	1	2	2	1	2	3

Table 4.1: Length function f for the poset in Figure 4.1.

Then for the forcing cycle C_2 above, we have $w_1(C_2) = f(a) = 1, w_2(C_2) = f(d) = 1, w_3(C_2) = f(e) = 2, w_4(C_2) = f(x) = 3, w_5(C_2) = f(a) = 1$. Thus $W_U(C_2) = w_1(C_2) = 1$ and $W_S(C_2) = w_2(C_2) + w_3(C_2) + w_4(C_2) + w_5(C_2) = 1 + 2 + 3 + 1 = 7$.

4.3 Existence of a Representation and Construction Algorithm

We will use forcing cycles to establish a necessary and sufficient condition to determine whether a given interval order P has an interval representation satisfying $|I_x| = f(x)$ for a given length function f. We first begin with a lemma that will simplify the subsequent construction argument.

Lemma 4.1. *Let $P = (X, \prec)$ be an interval order and $f : X \to \mathbb{R}_{>0}$ be a length function. If P contains a forcing cycle C such that $W_U(C) \geq W_S(C)$, then P contains a simple forcing cycle with the same property.*

Proof. Suppose, for the sake of contradiction, that P contains a forcing cycle C with $W_U(C) \geq W_S(C)$ but no simple forcing cycle with the same property. Let $C : x_0, x_1, \ldots, x_n$ be a minimal forcing cycle in P with $W_U(C) \geq W_S(C)$. Without loss of generality, assume that $x_0 = x_i = x_n$ for some $i \in \{1, \ldots, n-1\}$. Consider the forcing cycles $C_1 : x_0, \ldots, x_i$ and $C_2 : x_i, \ldots, x_n$. If $W_U(C_1) < W_S(C_1)$ and $W_U(C_2) < W_S(C_2)$, then $W_U(C_1) + W_U(C_2) < W_S(C_1) + W_S(C_2)$. But $W_U(C_1) + W_U(C_2) = W_U(C)$ and $W_S(C_1) + W_S(C_2) = W_S(C)$, and so $W_U(C) < W_S(C)$, a contradiction. So $W_U(C_1) \geq W_S(C_1)$ or $W_U(C_2) \geq W_S(C_2)$, which contradicts the minimality of C.

□

Before we state the main result of the current chapter, we need one more definition.

Definition 4.5. A poset $P = (X, \prec)$ is *separable* if it is possible to partition the set X nontrivially into two sets X_1 and X_2 so that $u \prec v$ for all $u \in X_1$ and $v \in X_2$. If P is not separable, then it is *inseparable*.

Note that if $P = (X, \prec)$ is separable with $X = X_1 \cup X_2$ and $f : X \to \mathbb{R}_{>0}$ is a length function, then P has an interval representation with the lengths given by f if and only if $P|_{X_1}$ and $P|_{X_2}$ have interval representations with the lengths given by $f|_{X_1}$ and $f|_{X_2}$ respectively. Also observe that if P is separable and $C : x_0, \ldots, x_n$ is a forcing cycle in P, then either $x_i \in X_1$ for all $i \in \{0, \ldots, n\}$ or $x_i \in X_2$ for all $i \in \{0, \ldots, n\}$. Additionally, if $Q = (Y, \prec)$ is any poset, it is possible to decompose Q into inseparable posets $Q_1, \ldots, Q_r$ in $O(|Y|^2)$ time (for details, see [22]).

The next theorem establishes a necessary and sufficient condition for an interval order to have an interval representation with a set of prescribed lengths.

Theorem 4.2. *Let $P = (X, \prec)$ be an interval order and $f : X \to \mathbb{R}_{>0}$ be a length function. Then there exists an interval representation of P in which $|I_x| = f(x)$ for all $x \in X$ if and only if P contains no forcing cycle C such that $W_U(C) \geq W_S(C)$.*

Proof. ($\Rightarrow$) Suppose there exists a forcing cycle $C : x_0, x_1, \ldots, x_n$ such that $W_U(C) \geq W_S(C)$. We will show that it is impossible to construct an interval representation of the poset induced by $\{x_1, \ldots, x_n\}$ in which $|I_{x_i}| = f(x_i)$ for all $i \in \{1, \ldots, n\}$. It will then follow that it is impossible to construct such a representation of all of P.

Suppose, for the sake of contradiction, that there exists an interval representation $\mathcal{I} = \{I_{x_i}\}_{i=1}^n$, where $I_{x_i} = [L(x_i), R(x_i)]$ for all $i \in \{1, \ldots, n\}$, such that $R(x_i) - L(x_i) = |I_{x_i}| = f(x_i)$ for all $i \in \{1, \ldots, n\}$. By the definition of an interval representation, for each $i \in \{1, \ldots, n\}$ such that $x_{i-1} \prec x_i$, the inequality $L(x_i) > L(x_{i-1}) + f(x_{i-1})$ must be satisfied. Similarly, for each $i \in \{1, \ldots, n\}$ such that $x_{i-1}||x_i$, the inequality $L(x_i) \geq L(x_{i-1}) - f(x_i)$ must be satisfied. Thus, adding all of these inequalities together, we get

$$L(x_1) + \cdots + L(x_n) > L(x_0) + \cdots + L(x_{n-1}) + \sum_{\{i : x_{i-1} \prec x_i\}} f(x_{i-1}) - \sum_{\{i : x_{i-1}||x_i\}} f(x_i),$$

i.e.,

$$L(x_1) + \cdots + L(x_n) > L(x_0) + \cdots + L(x_{n-1}) + \sum_{\{i : x_{i-1} \prec x_i\}} w(i) - \sum_{\{i : x_{i-1}||x_i\}} w(i).$$

Thus

$$L(x_n) > L(x_0) + W_U(C) - W_S(C).$$

Since $L(x_0) = L(x_n)$, we get $0 > W_U(C) - W_S(C)$, and so $W_S(C) > W_U(C)$, contradicting the assumption that $W_U(C) \geq W_S(C)$.

($\Leftarrow$) Suppose there is no forcing cycle C with $W_U(C) \geq W_S(C)$. We describe an algorithm, inspired by the work of Gimbel and Trenk [9], to construct a desired representation of P.

Recall that if the poset P is separable with $X = X_1 \cup X_2$, then to construct an interval representation of P with the prescribed lengths, it suffices to construct an interval representation of each of $P|_{X_1}$ and $P|_{X_2}$ with the prescribed lengths. Also, if P is separable, we can decompose it into inseparable posets $P_1, \ldots, P_r$ in $O(|X|^2)$ time. We can then run the algorithm below on each P_i and combine the resulting interval representations of all P_i to get an interval representation of P; if any P_i fails to have a representation with the prescribed lengths, then do does P. Thus, following the approach of Gimbel and Trenk [9] and Trenk [22], we will assume that the interval order P is inseparable.

Our algorithm computes a lower and upper bound for the position of the left endpoint of each interval, and either determines a representation or finds that no such representation exists and returns a forcing cycle C with $W_U(C) \geq W_S(C)$. For each $x \in X$, we will denote the lower bound by $lb(x)$ and the upper bound by $ub(x)$.

Note that if $R = (Y, \prec)$ is an interval order, $\{I_y\}_{y \in Y}$ is an interval representation of R, and δ is a small positive constant, then R has an another interval representation in which the distance between the leftmost endpoint and the rightmost endpoint is at most $\sum\limits_{y \in Y} |I_y| + \delta$.

We begin with an arbitrary element $x_0 \in X$ and set $lb(x_0) = ub(x_0) = 0$. We then let M be a sufficiently large positive constant, say $2 \sum\limits_{x \in X} f(x)$, and initialize $lb(x) = -M$ and $ub(x) = M$ for all other $x \in X$. We then repeat the steps below until convergence, narrowing the range for each left endpoint. For each element $x \in X$, we keep track of the last element which has caused each of $lb(x)$ and $ub(x)$ to change, which will be used to recover a forcing cycle C with $W_U(C) \geq W_S(C)$ if necessary.

In each iteration, we consider each pair $x, y \in X$, arranged so that either $x \prec y$ or $x||y$. For some sufficiently small positive constant ϵ, we do the following:

- If $x \prec y$:
  - If $lb(y) \leq lb(x) + f(x)$, set $lb(y) := lb(x) + f(x) + \epsilon$.
  - If $ub(x) \geq ub(y) - f(x)$, set $ub(x) := ub(y) - f(x) - \epsilon$.
- If $x||y$:
  - If $lb(x) < lb(y) - f(x)$, set $lb(x) := lb(y) - f(x)$.
  - If $ub(x) > ub(y) + f(y)$, set $ub(x) := ub(y) + f(y)$.

Notice that in all cases, lower bounds only increase and upper bounds only decrease. If we have $ub(x) < lb(x)$ for some $x \in X$, we stop and return that no representation is possible. Otherwise, if no changes have occurred in the last iteration, we set $I_x = [lb(x), lb(x) + f(x)]$ for each $x \in X$ and return this representation. If at least one change has occurred, we continue to the next iteration.

We claim that this algorithm always halts, either producing a representation satisfying the given constraints, or determining that no representation exists and producing a forcing cycle with $W_U(C) \geq W_S(C)$.

First we show that this algorithm terminates after a finite number of steps. Suppose not. At every step, either a lower bound gets increased by at least ϵ or an upper bound gets decreased by at least ϵ. Since there are infinitely many such steps, by the pigeonhole principle, we know that at least one bound gets changed infinitely many times. Without loss of generality, assume it is the lower bound of some element $x \in X$. Then, there is N such that, after N steps, the value of $lb(x)$ will exceed M. Since $ub(x) \leq M$, we know that at some point, we will have $ub(x) < lb(x)$ and the procedure will terminate, a contradiction.

Now assume the algorithm halts after an iteration during which no changes occur. We claim that the collection of intervals $\{I_x\}_{x \in X}$ with $I_x = [lb(x), lb(x) + f(x)]$ for each $x \in X$

is a valid interval representation of P. Suppose, for the sake of contradiction, that there are elements $x, y \in X$ such that I_x and I_y do not form a valid interval representation of the poset induced by $\{x, y\}$. Without loss of generality, assume that $L(x) \leq L(y)$. Thus, if $x \prec y$, we have $L(y) \leq R(x)$, i.e., $lb(y) \leq lb(x) + f(x)$, which cannot be the case after the algorithm has converged. The case where $x||y$ is similar. Therefore, if the algorithm converges, setting $I_x = [lb(x), lb(x) + f(x)]$ for all $x \in X$ gives us a valid interval representation of P with the lengths given by f.

Suppose now that the algorithm halts with $ub(x_0) < lb(x_0)$ for some $x_0 \in X$. We backtrack and recover the sequence of changes that has caused the bounds to cross. Let C be a forcing cycle, initially containing only x_0.

Let us first consider the lower bounds. Let $x_1 \in X$ be the last element that has caused $lb(x_0)$ to change. Add x_1 to the left of x_0 in C. Note that we must have either $x_1 \prec x_0$ or $x_1||x_0$, i.e., x_1 and x_0 can be ordered this way in a forcing cycle. Then add the element $x_2 \in X$, which has caused the last change of $lb(x_1)$, to the left of x_1. Continue adding elements in this fashion until the starting element is reached.

We proceed in a similar way with the upper bounds, except here we attach elements to the right of x_0 until we reach the starting element. Thus C becomes a cycle.

Let $C : x_n, \ldots, x_1, x_0 = y_0, y_1, \ldots, y_m$. Now we claim that $W_U(C) \geq W_S(C)$.

We know that for all $i \in \{1, \ldots, n\}$, we have $lb(x_{i-1}) = lb(x_i) + f(x_i) + \epsilon$ if $x_i \prec x_{i-1}$ and $lb(x_{i-1}) = lb(x_i) - f(x_i)$ if $x_i||x_{i-1}$. Adding all of these equations together, we get

$$lb(x_0) = lb(x_n) + \sum_{\{i:x_{i+1} \prec x_i\}} w(i) - \sum_{\{i:x_{i+1}||x_i\}} w(i) + |\{i : x_{i+1} \prec x_i\}|\epsilon.$$

Similarly, we get

$$ub(x_0) = ub(y_m) - \sum_{\{j:y_{j-1} \prec y_j\}} w(j) + \sum_{\{j:x_{j-1}||x_j\}} w(j) - |\{j : x_{j-1} \prec x_j\}|\epsilon.$$

Now $lb(x_n) = ub(y_m) = 0$ and $ub(x_0) < lb(x_0)$, so

$$- \sum_{\{j:y_{j-1} \prec y_j\}} w(j) + \sum_{\{j:x_{j-1}||x_j\}} w(j) - |\{j : x_{j-1} \prec x_j\}|\epsilon = ub(x_0) <$$
$$< lb(x_0) = \sum_{\{i:x_{i+1} \prec x_i\}} w(i) - \sum_{\{i:x_{i+1}||x_i\}} w(i) + |\{i : x_{i+1} \prec x_i\}|\epsilon.$$

Rearranging this inequality so that the summations taken over incomparable terms are combined, we get

$$\sum_{\{j: x_{j-1} || x_j\}} w(j) + \sum_{\{i: x_{i+1} || x_i\}} w(i) <$$
$$< \sum_{\{i: x_{i+1} \prec x_i\}} w(i) + \sum_{\{j: y_{j-1} \prec y_j\}} w(j) + |\{i : x_{i+1} \prec x_i\}|\epsilon + |\{i : x_{j-1} \prec x_j\}|\epsilon,$$

i.e.,

$$W_S(C) < W_U(C) + up(C)\epsilon.$$

Since ϵ can be arbitrarily small, we conclude that $W_S(C) \leq W_U(C)$.

□

The following example illustrates the algorithm described above for a particular poset.

Example 4.1. Consider the poset $\mathbf{3} + \mathbf{1}$, say $(a \prec b \prec c)||x$, with $f(a) = f(b) = f(c) = 1$ and $f(x) = 2$. It is not difficult to check that this poset contains no forcing cycle C with $W_U(C) \geq W_S(C)$. Hence the algorithm described above will find a valid interval representation with the prescribed lengths. Table 4.2 illustrates how the algorithm computes the lower bounds (lb) and upper bounds (ub) for the left endpoints of the intervals. We note that the order in which the comparisons are made depends on the specific implementation; here we illustrate one possibility.

We start with the element a. We choose $M = 2(f(a) + f(b) + f(c) + f(x)) = 10$ and initialize all other lower bounds to $-M$ and all other upper bounds to M, as illustrated in the Initial Bounds column of Table 4.2. The subsequent columns show how the bounds change during each iteration. Each cell in the subsequent columns shows the new value of each bound together with the element that has caused the bound to change. An empty cell indicates that the corresponding bound has not changed during the given iteration.

	Initial bounds		Iteration 1		Iteration 2		Iteration 3	
v	lb	ub	lb	ub	lb	ub	lb	ub
a	0	0						
b	-10	10	$1+\epsilon, a$	$9-\epsilon, c$		$3, x$		$2-\epsilon, c$
c	-10	10	$1+\epsilon, a$		$2+2\epsilon, b$	$3, x$		
x	-10	10	$-2, a$	$1, a$			$2\epsilon, c$	

Table 4.2: Execution of the algorithm described in the proof of Theorem 4.2 for the poset $(a \prec b \prec c)||x$ with $f(a) = f(b) = f(c) = 1$ and $f(x) = 2$.

During the fourth iteration, the algorithm finds that no other bounds need to be changed and halts. Setting $I_y = [lb(y), lb(y) + f(y)]$ for each element y of this poset, we get $I_a = [0, 1], I_b =$

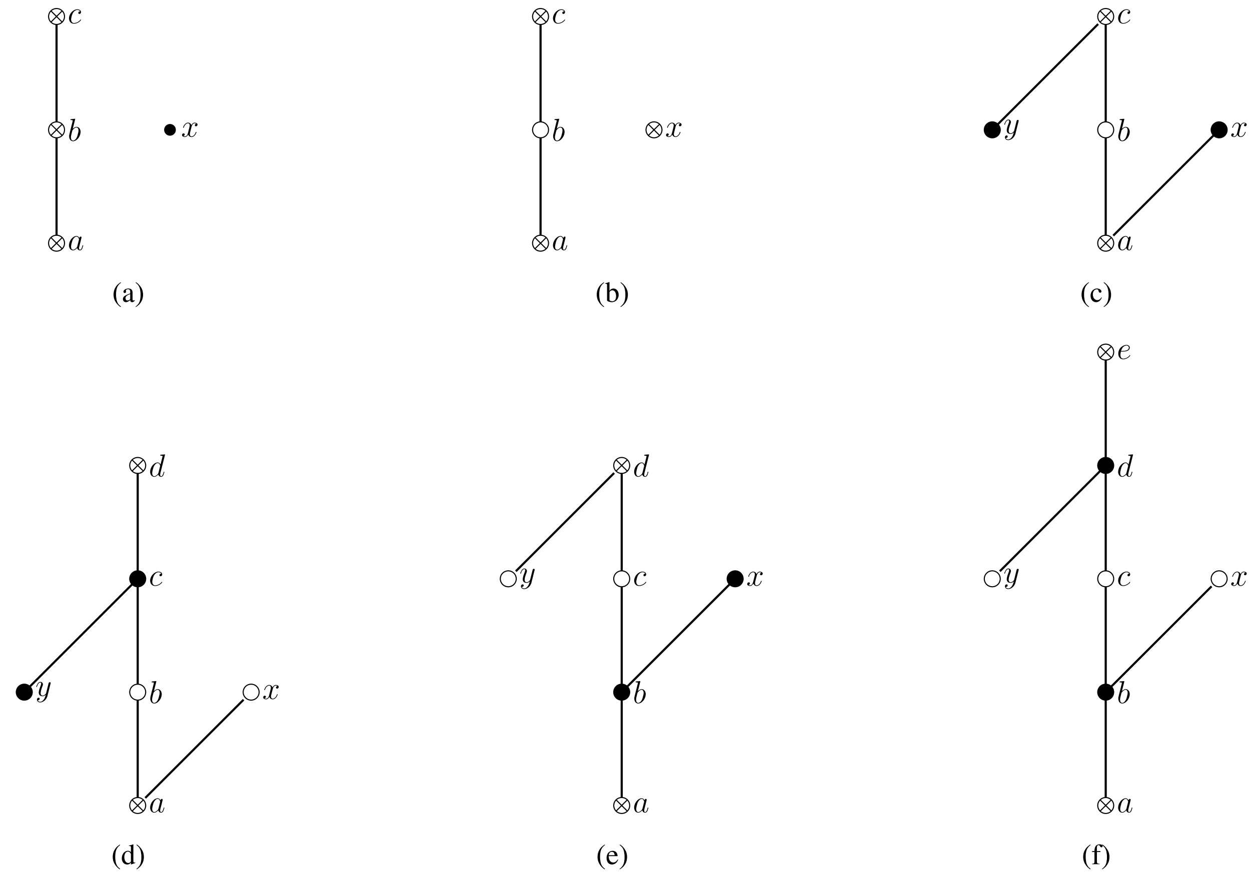


Figure 4.2: Forbidden colored posets, where ○ means red, ● means black, and $\otimes$ means that the element can be either color.

$[1+\epsilon, 2+\epsilon]$, $I_c = [2+2\epsilon, 3+2\epsilon]$, and $I_x = [2\epsilon, 2+2\epsilon]$ to get a valid interval representation of the given poset with the prescribed lengths.

We note here that the algorithm described in the proof of Theorem 4.2 is not guaranteed to run in polynomial time when no representation exists. We are currently developing an improved version of this algorithm, which first performs a sequence of comparisons to determine whether or not it is possible to find a representation and then uses the current algorithm to compute a representation if one exists.

4.4 Forbidden Substructures

We now turn our attention to the class $\mathscr{P}_{\{1,2\}}$. First we discuss Problem 1, and establish a necessary (but not sufficient) condition for a colored interval order to have an interval representation satisfying the constraints given by the coloring.

Proposition 4.3. *Let $P = (X, \prec)$ be an interval order, and suppose each element of $x \in X$ is colored either black or red. Suppose P has an interval representation such that for all $x \in X$, if*

x is black, then $|I_x| = 1$, and if x is red, then $|I_x| = 2$. Then P does not contain the poset $\mathbf{4}+\mathbf{1}$ or any of the posets in Figure 4.2 with the specified colors.

Proof. We will show the contrapositive, i.e., that if P contains an induced $\mathbf{4}+\mathbf{1}$ or one of the posets in Figure 4.2, then P has no representation satisfying the constraints specified by the coloring. Clearly, by Theorem 2.4 in Chapter 2, if P has a representation with lengths 1 and 2, then P contains no induced $\mathbf{4}+\mathbf{1}$. By Theorem 4.2, it suffices to check that each of the posets in Figure 4.2 has a forcing cycle C with $W_U(C) \geq W_S(C)$. We will use $f(x)$ to denote the prescribed length of I_x for all $x \in X$.

(a) Consider the poset in Figure 4.2a and the forcing cycle $C : a, b, c, x, a$. Then $W_U(C) = f(a) + f(b)$ and $W_S(C) = f(x) + f(a) = 1 + f(a)$. Since $f(b) \geq 1$, we have $W_U(C) \geq W_S(C)$.

(b) Consider poset in Figure 4.2b and the same forcing cycle C as above. Then $W_U(C) = f(a) + f(b) = f(a) + 2$ and $W_S(C) = f(x) + f(a)$. Since $f(x) \leq 2$, we have $W_U(C) \geq W_S(C)$.

(c) Consider the poset in Figure 4.2c and the forcing cycle $C : a, b, c, x, y, a$. Then $W_U(C) = f(a) + f(b) = f(a) + 2$ and $W_S(C) = f(x) + f(y) + f(a) = 2 + f(a)$. So $W_U(C) = W_S(C)$.

(d) Consider the poset in Figure 4.2d and the forcing cycle $C : a, b, c, d, x, y, a$. Then $W_U(C) = f(a) + f(b) + f(c) = f(a) + 3$ and $W_S(C) = f(x) + f(y) + f(a) = 3 + f(a)$. So $W_U(C) = W_S(C)$.

(e) Considering the poset in Figure 4.2e and the same forcing cycle C as in the previous part, we get $W_U(C) = W_S(C)$.

(f) Consider the poset in Figure 4.2f and the forcing cycle $C : a, b, c, d, e, x, y, a$. Then $W_U(C) = f(a) + f(b) + f(c) + f(d) = f(a) + 4$ and $W_S(C) = f(x) + f(y) + f(a) = 4 + f(a)$. So $W_U(C) = W_S(C)$.

In all of the above cases P does not have an interval representation with the prescribed interval lengths. □

We now conclude by briefly returning to the class $\mathscr{P}_{\{1,2\}}$ (with no prescribed lengths). The following corollary is an immediate consequence of Proposition 4.3.

Corollary 4.4. *Let $P = (X, \prec)$ be an interval order. If P contains the poset $\mathbf{4}+\mathbf{1}$ or if every coloring of the elements using red and black leads to one of the posets in Figure 4.2, then P has no representation with lengths 1 and 2.*

As we can see, characterizing the class $\mathscr{P}_{\{1,2\}}$ remains an open problem. Additionally, it is not clear whether the results we have derived about $\mathscr{P}_{\{1,2\}}$ can be easily extended to any of the other classes $\mathscr{P}_{\{1,k\}}$ for $k \geq 3$. We do hope, however, that the tools developed in this chapter will be useful in identifying forbidden substructures and deriving stronger results about the classes in this family. In particular, establishing an upper bound on the length of a forbidden simple forcing cycle for each $k \in \mathbb{Z}_{\geq 2}$ or showing that no such upper bound exists can answer the question of whether the list of forbidden posets is finite or infinite.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

The aim of this thesis was to investigate the behavior of two families of interval orders. First we studied the classes of interval orders representable with lengths between 1 and k for $k \in \mathbb{Z}_{\geq 1}$. For each $k \in \mathbb{Z}_{\geq 1}$, the class $\mathscr{P}_{[1,k]}$ of interval orders representable with lengths between 1 and k is characterized in terms of two forbidden partial orders. We provided an alternative and more accessible proof of this characterization theorem using potentials in weighted directed graphs. Next we studied the classes of interval orders representable with lengths exactly 1 and k for $k \in \mathbb{Z}_{\geq 0}$. We found both an algorithmic and a structural characterization of the class $\mathscr{P}_{\{0,1\}}$ of interval orders representable with lengths 0 and 1. We were unable to characterize any $\mathscr{P}_{\{1,k\}}$ for $k \geq 2$. We considered a related problem, in which each interval has a prescribed length. We established a necessary and sufficient condition for an interval order to have a representation with a given set of positive prescribed lengths. We focused on the case where each prescribed length is either 1 or 2 and derived a partial list of forbidden colored substructures. We used this result to establish a necessary condition for membership in $\mathscr{P}_{\{1,2\}}$.

5.2 Open Problems and Future Directions

As we discussed at the end of Chapter 4, it is still unclear whether any of the classes $\mathscr{P}_{\{1,k\}}$ for $k \geq 2$ can be characterized in terms of a finite set of forbidden substructures. It is also interesting to explore whether there is an efficient algorithm to determine whether or not a poset has an interval representation with lengths 1 and k. We would also like to investigate whether the classes in this family exhibit common patterns in their behavior, that is, whether results about, say, the class $\mathscr{P}_{\{1,2\}}$ can be generalized to $\mathscr{P}_{\{1,k\}}$ for arbitrary values of k.

We hope to expand the list of forbidden colored substructures in the case where each prescribed length is either 1 or 2 so that we get a forbidden colored substructure characterization of $\mathscr{P}_{\{1,2\}}$. We hope to derive similar lists for $\mathscr{P}_{\{1,k\}}$ with $k > 2$. Finding a characterization in terms of forbidden colored substructures may give insight into finding a forbidden poset characterization of $\mathscr{P}_{\{1,2\}}$, and more generally, of $\mathscr{P}_{\{1,k\}}$. We are also interested in finding out whether, given $k \in \mathbb{Z}_{\geq 2}$, we can find an upper bound on the size of a minimal forbidden colored poset or prove that no such upper bound can exist.

The algorithm we developed in Chapter 4 does not run in polynomial time currently. Since the problem this algorithm is designed to solve has an efficient solution using linear programming, we believe that our algorithm can be improved so that it runs in polynomial time as well. We are currently developing an improved version of this algorithm, which first determines whether or not an interval order has a representation with a given set of prescribed lengths and then uses the current version of the algorithm to compute a representation if one exists.

Bibliography

[1] B. Balof and K. Bogart. Simple inductive proofs of the Fishburn and Mirkin theorem and the Scott–Suppes theorem. *Order*, 20(1):49–51, 2003.

[2] K. P. Bogart. An obvious proof of Fishburn's interval order theorem. *Discrete Mathematics*, 118(1):239–242, 1993.

[3] M. R. Cerioli, F. de S. Oliveira, and J. L. Szwarcfiter. On representing an interval graph using the minimum number of interval lengths. *Matemática Contemporânea*, 39:59–67, 2010.

[4] P. C. Fishburn. Intransitive indifference with unequal indifference intervals. *Journal of Mathematical Psychology*, 7(1):144–149, 1970.

[5] P. C. Fishburn. Restricted thresholds for interval orders: a case of nonaxiomatizability by a universal sentence. *Journal of Mathematical Psychology*, 24(3):276–283, 1981.

[6] P. C. Fishburn. Threshold-bounded interval orders and a theory of picycles. *SIAM Journal on Algebraic Discrete Methods*, 4(3):290–305, 1983.

[7] P. C. Fishburn. Paradoxes of two-length interval orders. *Discrete Mathematics*, 52(2):165–175, 1984.

[8] P. C. Fishburn and R. L. Graham. Classes of interval graphs under expanding length restrictions. *Journal of Graph Theory*, 9(4):459–472, 1985.

[9] J. G. Gimbel and A. N. Trenk. On the weakness of an ordered set. *SIAM Journal on Discrete Mathematics*, 11(4):655–663, 1998.

[10] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York, 1980.

[11] M. C. Golumbic and A. N. Trenk. *Tolerance Graphs*, volume 89. Cambridge University Press, Cambridge, 2004.

[12] T. L. Greenough. *Representation and Enumeration of Interval Orders and Semiorders*. PhD thesis, Dartmouth College, Hanover, 1976.

[13] G. Isaak. Interval order representation via shortest paths. In S. Brams and Roberts F. S. Gehrlein, W. V., editors, *The Mathematics of Preference, Choice and Order*, pages 303–311. Springer, Berlin, 2009.

[14] F. Joos, C. Löwenstein, F. de S. Oliveira, D. Rautenbach, and J. L. Szwarcfiter. Graphs of interval count two with a given partition. *Information Processing Letters*, 114(10):542–546, 2014.

[15] M. Keller and T. Trotter. *Applied Combinatorics*. Preliminary edition, 2015.

[16] I. Pe'er and R. Shamir. Realizing interval graphs with size and distance constraints. *SIAM Journal on Discrete Mathematics*, 10(4):662–687, 1997.

[17] D. Rautenbach and J. L. Szwarcfiter. Unit and single point interval graphs. *Discrete Applied Mathematics*, 160(10):1601–1609, 2012.

[18] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*, volume 24. Springer, Berlin, 2003.

[19] D. Scott and P. Suppes. Foundational aspects of theories of measurement. *Journal of Symbolic Logic*, pages 113–128, 1958.

[20] A. Shuchat, R. Shull, and A. N. Trenk. Unit interval orders of open and closed intervals. *Order*, pages 1–15, 2015.

[21] D. Skrien. Chronological orderings of interval graphs. *Discrete Applied Mathematics*, 8(1):69–83, 1984.

[22] A. N. Trenk. On k-weak orders: Recognition and a tolerance result. *Discrete Mathematics*, 181(1):223–237, 1998.

[23] W. T. Trotter. *Combinatorics and Partially Ordered Sets: Dimension Theory*. Johns Hopkins University Press, Baltimore, 1992.

[24] D. B. West. *Introduction to Graph Theory*. Prentice Hall, Upper Saddle River, 2 edition, 2001.