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Fractional Weak Discrepancy and Interval Orders

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ABSTRACT

The *fractional weak discrepancy* $wd_F(P)$ of a poset $P = (V, \prec)$ was introduced in [9] as the minimum nonnegative k for which there exists a function $f : V \to \mathbf{R}$ satisfying (i) if $a \prec b$ then $f(a)+1 \leq f(b)$ and (ii) if $a \parallel b$ then $|f(a) - f(b)| \leq k$. In this paper we generalize results in [10, 11] on the range of the wd_F function for semiorders (interval orders with no induced $\mathbf{3}+\mathbf{1}$) to interval orders with no $\mathbf{n}+\mathbf{1}$, where $n \geq 3$. In particular, we prove that the range for such posets P is the set of rationals that can be written as r/s, where $0 \leq s-1 \leq r < (n-2)s$. If $wd_F(P) = r/s$ and P has an optimal forcing cycle C with $\text{up}(C) = r$ and $\text{side}(C) = s$, then $r \leq (n-2)(s-1)$. Moreover when $s \geq 2$, for each r satisfying $s-1 \leq r \leq (n-2)(s-1)$ there is an interval order having such an optimal forcing cycle and containing no $\mathbf{n}+\mathbf{1}$.

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1 Introduction

In this paper we will consider irreflexive posets $P = (V, \prec)$ and write $x \parallel y$ when elements x and y in V are incomparable. Of particular importance to us will be posets with no induced $\mathbf{r} + \mathbf{s}$, where $\mathbf{r} + \mathbf{s}$ is the poset consisting of two disjoint chains, one with r elements and one with s elements. An *interval order* is a poset with no induced $\mathbf{2} + \mathbf{2}$; equivalently, P is an interval order if each element $x \in V$ can be assigned an interval I_x on the real line so that $x \prec y$ precisely when I_x is completely to the left of I_y [2]. A *semiorder* (*unit interval order*) is an interval order with a representation in which each interval has the same length. Equivalently, a semiorder is a poset with no induced $\mathbf{2} + \mathbf{2}$ and no induced $\mathbf{3} + \mathbf{1}$ [8]. As Fishburn and Monjardet [4] show, these ideas have their roots in early work of Wiener.

A *weak order* is a poset with no induced $\mathbf{2} + \mathbf{1}$. Alternatively, a weak order can be defined as a poset P to which we can assign a real-valued function $f : V \to \mathbf{R}$ so that $a \prec b$ if and only if $f(a) < f(b)$ [1]. We can think of such a function as ranking the elements of P in a way that respects the ordering $\prec$ and gives incomparable elements equal rank.

We will consider the fractional weak discrepancy of a poset, introduced in [9] as a generalization of Trenk's concept of integer-valued weak discrepancy (originally called "weakness" in [14]). The weak discrepancy is a measure of how far a poset is from being a weak order.

Definition 1 The *fractional weak discrepancy* $wd_F(P)$ of a poset $P = (V, \prec)$ is the minimum nonnegative real number k for which there exists a function $f : V \to \mathbf{R}$ satisfying

(i) if $a \prec b$ then $f(a) + 1 \leq f(b)$ ("up" constraints)

(ii) if $a \parallel b$ then $|f(a) - f(b)| \leq k$. ("side" constraints)

To define the (*integer*) *weak discrepancy* $wd(P)$, we take the minimum $k \in \mathbf{Z}, k \geq 0$, for which there is a $\mathbf{Z}$-valued function f satisfying (i) and (ii).

In [9] we express $wd(P)$ as the optimal solution to an integer program and $wd_F(P)$ as the optimal solution to its linear relaxation, and prove that $wd(P) = \lceil wd_F(P) \rceil$. We can interpret $wd(P)$ and $wd_F(P)$ as bounding the discrepancy in ranking between incomparable elements of V, where ranks must be integers (as in salary or grade levels) or not (as in actual salaries). See [13] for other interpretations.

For example, the poset $P = \mathbf{3} + \mathbf{2}$ is illustrated in Figure 1 with a labeling that is in fact optimal. The presence of an $\mathbf{r} + \mathbf{s}$ in a poset P gives a substructure whose elements we can traverse by traveling up one chain, then to the bottom of the second chain, then up the second chain, and then back to the bottom of the first chain, all the while respecting the ordering in P. This is generalized in the following key definition.

Definition 2 A *forcing cycle* C of poset $P = (V, \prec)$ is a sequence $C : x_0, x_1, \ldots, x_m = x_0$ of $m \geq 2$ elements of V for which $x_i \prec x_{i+1}$ or $x_i \parallel x_{i+1}$ for each $i : 0 \leq i < m$.

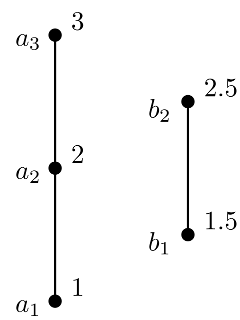

Figure 1: $P = \mathbf{3}+\mathbf{2}$ and an optimal labeling showing $wd_F(P) = 1.5, wd(P) = 2$.

If C is a forcing cycle, we write $\text{up}(C) = |\{i : x_i \prec x_{i+1}\}|$ and $\text{side}(C) = |\{i : x_i \parallel x_{i+1}\}|$.

For example, one forcing cycle of $P = \mathbf{3}+\mathbf{2}$ in Figure 1 is $C_1 : a_1 \prec a_2 \prec a_3 \parallel b_1 \prec b_2 \parallel a_1$, which has $\text{up}(C_1) = 3$ and $\text{side}(C_1) = 2$. Another forcing cycle is $C_2 : a_1 \prec a_2 \parallel b_1 \prec b_2 \parallel a_1$, which has $\text{up}(C_2) = 2$ and $\text{side}(C_2) = 2$. Note that in any forcing cycle C, we have $\text{up}(C) \geq 0$ and $\text{side}(C) \geq 2$. Forcing cycles provide the main tool for proving results about (fractional) weak discrepancy, as shown in the following theorem.

Theorem 3 ([6, 9]) Let $P = (V, \prec)$ be a poset with at least one incomparable pair. Then $wd(P) = \left\lceil \max_C \frac{\text{up}(C)}{\text{side}(C)} \right\rceil$ and $wd_F(P) = \max_C \frac{\text{up}(C)}{\text{side}(C)}$, where the maximum is taken over all forcing cycles C in P.

The maximum ratio for the poset $P = \mathbf{3}+\mathbf{2}$ in Figure 1 is achieved by the forcing cycle C_1. So $wd(P) = \lceil 3/2 \rceil = 2$ and $wd_F(P) = 3/2$.

One consequence of Theorem 3 is that the fractional weak discrepancy of a poset is always a nonnegative rational number. This raises two important questions:

Question 1: Which nonnegative rational numbers can be achieved as $wd_F(P)$ for which posets P?
Question 2: If $wd_F(P) = r/s$ for integers $r \geq 0$ and $s \geq 2$, does there exist a forcing cycle C in P with $\text{up}(C) = r$ and $\text{side}(C) = s$?

The authors' earlier papers [10] and [11] gave partial answers to these questions. Theorem 4 answers Question 1 for three classes of posets.

Theorem 4 ([10, 11]) The range of $wd_F(P)$ can be described as follows:

(i) $\{wd_F(P) : P \text{ a semiorder}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$.

				$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
				5/6	6/6	7/6	8/6	9/6	10/6	$\cdots$
			4/5	5/5	6/5	7/5	8/5	9/5	10/5	$\cdots$
		3/4	4/4	5/4	6/4	7/4	8/4	9/4	10/4	$\cdots$
	2/3	3/3	4/3	5/3	6/3	7/3	8/3	9/3	10/3	$\cdots$
1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	9/2	10/2	$\cdots$

Figure 2: The range of positive values for $wd_F(P)$. The solid boxes show the range for semiorders. The dashed boxes illustrate Theorem 12 when $n = 4$. They show the r-s pairs ($r \geq 1, s \geq 2$) for which there is an interval order P containing a $\mathbf{3+1}$ but no $\mathbf{4+1}$, where $wd_F(P) = r/s$ and P has an optimal forcing cycle C with $r = \text{up}(C),\ s = \text{side}(C)$.

(ii) $\{wd_F(P) : P \text{ not a semiorder}\} = \{q \geq 1 : q \in \mathbf{Q}\}$.

(iii) $\{wd_F(P) : P \text{ an interval order but not a semiorder}\} = \{q \geq 1 : q \in \mathbf{Q}\}$.

In [11], we answered Question 2 in the negative by exhibiting a poset P with $wd_F(P) = 3/2$ but having no forcing cycle C with $\text{up}(C) = 3$ and $\text{side}(C) = 2$.

The current paper extends Theorem 4 in the following way. We generalize the notion of a semiorder (interval order with no $\mathbf{3+1}$) to the larger class of interval orders with no induced $\mathbf{n+1}$, when $n \geq 3$. In Theorem 5 we give a sufficient condition in terms of forcing cycles for an interval order to contain an $\mathbf{n+1}$. In Theorem 12 we give conditions that guarantee the existence of an interval order that contains no $\mathbf{n+1}$. These two results lead to Corollary 18, where we answer Question 1 for this class for each n: the range of $wd_F(P)$ is the set of rationals that can be written as r/s, where $0 \leq (s-1) \leq r < (n-2)s$. When $s \geq 2$ and $r \leq (n-2)(s-1)$ these orders have an optimal forcing cycle with $r = \text{up}(C),\ s = \text{side}(C)$. Figure 2 illustrates the case $n = 4$, where $r \geq 1, s \geq 2$. The solid boxes show the values where $r = s-1$ and correspond to the semiorders. The dashed boxes show the r-s pairs where $s-1 < r \leq 2(s-1)$, and correspond to interval orders that contain a $\mathbf{3+1}$ but no $\mathbf{4+1}$ and have an optimal forcing cycle with $r = \text{up}(C),\ s = \text{side}(C)$. As the figure shows, such an order can exist for some but not all pairs that represent the same rational number, e.g., 6/4 is in a dashed box but 3/2 is not. An example of this is discussed immediately after Corollary 18.

We also consider the following variant of Question 2:

Question 2′: Let $r \geq 1$ and $s \geq 2$ be integers for which r/s can be achieved as the fractional weak discrepancy of some poset. Does there exist a poset P with an optimal forcing cycle C such that $wd_F(P) = r/s, \text{up}(C) = r$, and $\text{side}(C) = s$?

We answer Question 2′ in the affirmative and show in Corollary 17 that when n is sufficiently large ($n \geq 2 + \frac{r}{s-1}$) we can take P to be an interval order with no $\mathbf{n+1}$.

2 Interval orders containing an induced $\mathbf{n+1}$

The main result of this section, Theorem 5, gives a sufficient condition for an interval order to contain an induced $\mathbf{n+1}$, generalizing the case of $n = 3$. If C is a forcing cycle of an interval order P with $\text{side}(C) = r + 1$ and $\text{up}(C) > r$, Theorem 3 implies $wd_F(P) \geq 1$. Thus Theorem 4(i) implies that P is not a semiorder. Since semiorders are interval orders with no induced $\mathbf{3+1}$, P must contain a $\mathbf{3+1}$. The bound $\text{up}(C) > r$ is tight since we have shown ([10], Proposition 16) how to construct, for each $r > 0$, an interval order P possessing an optimal forcing cycle C with $\text{side}(C) = r + 1$ and $\text{up}(C) = r$ but no induced $\mathbf{3+1}$. In the case $n = 3$, we can express this result as saying that if $\text{up}(C) > (n-2)r$ and $\text{side}(C) = r + 1$, then P must contain an $\mathbf{n+1}$. This generalization is contained in Theorem 5, whose proof appears in Section 2.3.

Theorem 5 *Let n be an integer with $n \geq 3$. If an interval order P contains a forcing cycle C with $\text{up}(C) > (n-2)(\text{side}(C) - 1)$, then P contains an induced $\mathbf{n+1}$.*

In Theorem 12 we will construct, for each integer $r > 0$, an interval order P possessing an optimal forcing cycle with $\text{side}(C) = r + 1$ and $\text{up}(C) = (n-2)r$ but with no induced $\mathbf{n+1}$. Thus the bound given in Theorem 5 is the best possible for interval orders. The inequality in Theorem 5 cannot be generalized to include all noninterval orders since $P = (\mathbf{n-1}) + (\mathbf{n-1})$ has no induced $\mathbf{n+1}$ but has an optimal forcing cycle C with $\text{up}(C) = 2(n-2), \text{side}(C) = 2$, and $wd_F(P) = n - 2$. When $n \geq 3$, this poset P contains a $\mathbf{2+2}$ and so is not an interval order. However, we have shown that a slightly weaker bound holds for all posets: if P is any poset for which $wd_F(P) > n - 2$, then P contains an $\mathbf{n+1}$ ([14, 11]).

We will prove Theorem 5 by contradiction. Let C be a forcing cycle with $\text{up}(C) > (n-2)(\text{side}(C) - 1)$ and suppose P contains no $\mathbf{n+1}$. We will apply an algorithm to C that will lead to a contradiction. First we will outline the algorithm and illustrate it with an example, and then we will prove the theorem. The algorithm is based on moving along C through successive sequences of u_j up steps and s_j side steps. It builds a stack K of elements taken from C and then derives a contradiction from it.

j	u_j	s_j	λ_j	$\Sigma_{l=1}^{j}(u_l - \lambda_l)$
1	2	2	3	–1
2	2	1	1	0
3	5	2	3	2

Table 1: Example 6 before preprocessing, with $n = 4$.

2.1 The algorithm and an example

The algorithm consists of the following three stages:

1. Preprocessing: If necessary, relabel C to start the cycle at the beginning of a sequence of up steps and so that the partial sums of $\Sigma(u_j - \sigma_j)$ are nonnegative, where σ_j is a certain function of s_j and n.
2. Initialization: (step 0) Place the first element of C on the stack.
3. Iteration: Let p be the number of alternating sequences of up and side steps in C. For each $j = 1, 2, \ldots, p$,

   (step j_u) Add the next u_j elements of C, corresponding to the next sequence of up steps, to the top of K.

   (step j_s) Remove the top σ_j elements from K.

   Iterate until we return to the beginning of C. We will prove that the stack never empties.

Suppose that P does not contain an $\mathbf{n+1}$. We will show that in any interval representation of P, after each step of the algorithm the order of elements on the stack K respects the partial order of P. We will then use the structure of K to show that C is not a forcing cycle, a contradiction. It will therefore follow that P must contain an $\mathbf{n+1}$.

In order to define σ_j, we first introduce the parameter λ_j. For $j = 1, 2, \ldots, p$ let

$$\lambda_j = (n-2)(s_j - 1) + 1. \tag{1}$$

Example 6 *Consider an interval order P that contains the following forcing cycle C, shown in Figure 3.*

$$C : x_0 \prec x_1 \prec x_2 \parallel x_3 \parallel x_4 \prec x_5 \prec x_6$$
$$\parallel x_7 \prec x_8 \prec x_9 \prec x_{10} \prec x_{11} \prec x_{12} \parallel x_{13} \parallel x_{14} = x_0.$$

We will let $n = 4$ in this example and assume as above that P does not contain an $\mathbf{n+1}$. Here $p = 3, \mathrm{up}(C) = 9$, and $\mathrm{side}(C) = 5$, so we have $\lambda_j = 2s_j - 1$ and $(n-2)(\mathrm{side}(C) - 1) = 8$. The values of the various parameters and of the partial sums of $\Sigma(u_j - \lambda_j)$ are shown in Table 1.

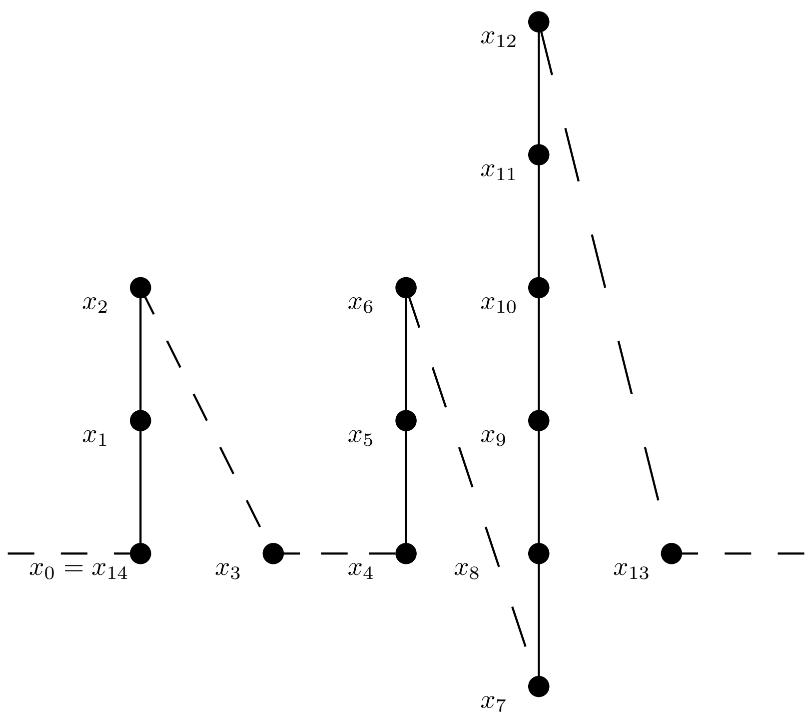

Figure 3: A forcing cycle in an interval order P (there may be other elements, comparabilities, and incomparabilities that are not shown).

j	u_j	s_j	σ_j	$\Sigma_{l=1}^{j}(u_l - \sigma_l)$
1	2	1	1	1
2	5	2	3	3
3	2	2	3	2

Table 2: Example 6 after preprocessing, with $n = 4$.

step	u_j	σ_j	K
0			y_0
1_u	2		y_0, y_1, y_2
1_s		1	y_0, y_1
2_u	5		$y_0, y_1, y_4, y_5, y_6, y_7, y_8$
2_s		3	y_0, y_1, y_4, y_5
3_u	2		$y_0, y_1, y_4, y_5, y_{11}, y_{12}$
3_s		3	y_0, y_1, y_4

Table 3: Evolution of the stack K in Example 6.

The details of the preprocessing step will be explained during the proof of Proposition 9. In Example 6, preprocessing will start the forcing cycle at x_4:

$$x_4 \prec x_5 \prec x_6 \parallel x_7 \prec x_8 \prec x_9 \prec x_{10} \prec x_{11} \prec x_{12} \\ \parallel x_{13} \parallel x_{14} = x_0 \prec x_1 \prec x_2 \parallel x_3 \parallel x_4.$$

To simplify the notation, we relabel the elements and denote the new forcing cycle again by C,

$$C : y_0 \prec y_1 \prec y_2 \parallel y_3 \prec y_4 \prec y_5 \prec y_6 \prec y_7 \prec y_8 \\ \parallel y_9 \parallel y_{10} \prec y_{11} \prec y_{12} \parallel y_{13} \parallel y_{14} = y_0.$$

We again denote the number of alternating up and side steps in the cycle by u_j, s_j, redefine λ_j accordingly, and then define

$$\sigma_j = \begin{cases} 0, & \text{if } j = p \text{ and } s_p = 1 \\ \lambda_j, & \text{otherwise.} \end{cases} \tag{2}$$

That is, we remove the top $\sigma_j = \lambda_j$ elements from the stack unless there is only one side step at the end of the forcing cycle. In that case $\lambda_p = 1$ and $\sigma_p = 0$, and we remove no elements in step p_s.

Table 2 gives the characteristics of the cycle after preprocessing. In this example, $\sigma_p = \lambda_p$, i.e., the exceptional case in (2) does not arise. Table 3 shows how the stack K evolves when we apply the algorithm to C.

Continuing Example 6, we will give samples of the reasoning we will use in the proof of Theorem 5 to show how the evolution of the stack determines the form of any interval representation of P.

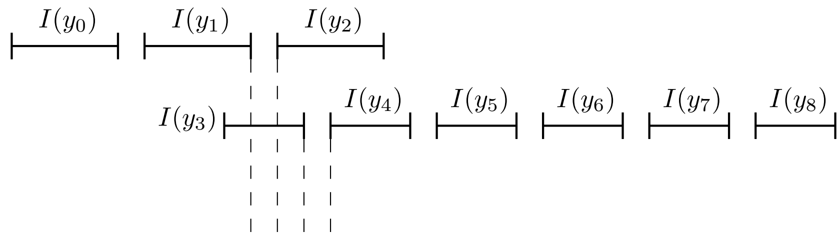

Figure 4: A possible interval representation of P after step 2_u in Example 6, where one element was removed from the stack in the preceding step 1_s.

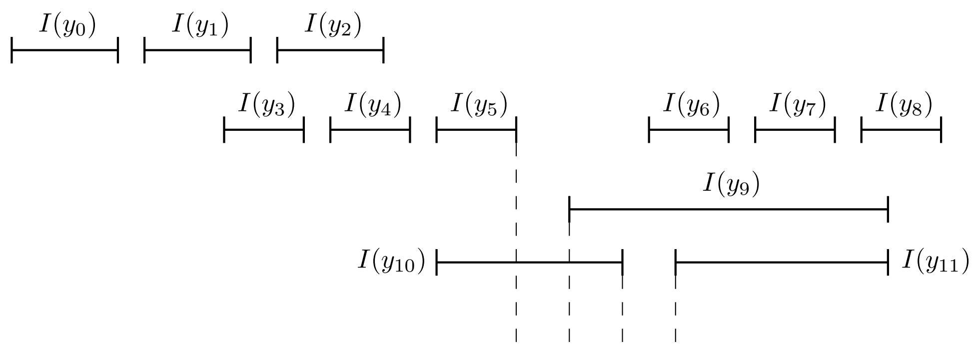

Figure 5: A possible interval representation of P after step 3_u in Example 6, where three elements were removed from the stack in the preceding step 2_s.

Figure 4 shows a possible configuration of the intervals assigned to elements $y_0, y_1, \ldots, y_8$ in P after step 2_u. Note that y_1 precedes y_4 in the stack. We assign to each y_i the interval $I(y_i) = [L(y_i), R(y_i)]$. We show that in any interval representation of P, the vertical dashed lines must appear in the order shown. That is $R(y_1) < L(y_2) \leq R(y_3) < L(y_4)$ because in P we have $y_1 \prec y_2$, $y_2 \parallel y_3$, and $y_3 \prec y_4$. Thus we may conclude that $y_1 \prec y_4$ in P.

Figure 5 shows a possible configuration of the intervals assigned to $y_0, y_1, \ldots, y_{11}$ after step 3_u. We will show that since P does not contain a $\mathbf{4}+\mathbf{1}$ the vertical dashed lines must appear in the order shown, that is $R(y_5) < L(y_9) \leq R(y_{10}) < L(y_{11})$. First, we have $y_8 \parallel y_9$ hence $L(y_8) \leq R(y_9)$. Since $y_5 \prec y_6 \prec y_7 \prec y_8$, if we had $L(y_9) \leq R(y_5)$ then $(y_5 \prec y_6 \prec y_7 \prec y_8) \parallel y_9$ would be an induced $\mathbf{4}+\mathbf{1}$. Hence $R(y_5) < L(y_9)$, the first of our three inequalities. The second and third inequalities follow directly from $y_9 \parallel y_{10}$ and $y_{10} \prec y_{11}$, respectively. In particular, we have shown that $y_5 \prec y_{11}$.

2.2 Preprocessing to obtain a good starting point

We started with a forcing cycle C with $\mathrm{up}(C) > (n-2)(\mathrm{side}(C)-1)$. Let $r = \mathrm{up}(C), s = \mathrm{side}(C)$. We may choose to start the cycle at an element x_0 that is the beginning of a sequence of up steps, i.e., if C contains m elements then $x_{m-1} \parallel x_m = x_0 \prec x_1$. We call x_0 an *upward starting point* for C. Then C consists of p alternating sequences of u_j up steps and s_j side steps.

Our goal in preprocessing is to find an upward starting point for C so that the stack K never empties during the algorithm. We will accomplish this by finding an upward starting point for which the partial sums of $\sum_{j=1}^{p}(u_j - \sigma_j)$ are all nonnegative. This is done in Proposition 9, which we prove after the following lemmas.

Lemma 7 *Let $n \geq 4$, let the forcing cycle C begin at an upward starting point, and suppose $r > (n-2)(s-1)$. Then $\sum_{j=1}^{p}(u_j - \lambda_j) \geq p-1$.*

Proof.

$$\begin{aligned}
\sum_{j=1}^{p}(u_j - \lambda_j) &= \sum_{j=1}^{p} u_j - \sum_{j=1}^{p} \lambda_j \\
&= r - \sum_{j=1}^{p}((n-2)(s_j-1)+1) \qquad \text{(by (1))} \\
&= r - (n-2)s + (n-2)p - p \\
&> (n-2)(s-1) - (n-2)s + (n-2)p - p \qquad (\text{since } r > (n-2)(s-1)) \\
&= (n-2)(p-1) - p \\
&\geq 2(p-1) - p \qquad (\text{since } n \geq 4) \\
&= p-2.
\end{aligned}$$

So $\sum_{j=1}^{p}(u_j - \lambda_j) > p-2$, and since both sides are integers the result follows. □

We will also need the following technical lemma stating that whenever the sum of a finite number of reals is positive, there is a cyclic permutation of the terms that makes all the partial sums positive.

Lemma 8 *Let $\tau_p : t_1, t_2, \ldots, t_p$ be a finite sequence of real numbers with $\sum_{j=1}^{p} t_j > 0$. There exists an index q with $1 \leq q \leq p$ for which the partial sums of the sequence $\tau_q : t_{q+1}, t_{q+2}, \ldots, t_p, t_1, t_2, \ldots, t_q$ are all positive.*

Proof. If all partial sums of τ_p are positive, we are done. Otherwise, choose q so that $\sum_{j=1}^{q} t_j = k \leq 0$ is the minimum of the partial sums, and q is the *largest* index to achieve this minimum value. Since the sum of the entire sequence is positive, $q \leq p-1$.

We will show all the partial sums of τ_q are positive. For $q+1 \leq i \leq p$, we have $\sum_{j=q+1}^{i} t_j > 0$ because of the way q was chosen, i.e., it is nonnegative because k is minimum, and nonzero because q is the largest index to achieve the value k. For similar reasons, when $1 \le i \le q$ we have

$$\sum_{j=q+1}^{p} t_j + \sum_{j=1}^{i} t_j \ge \sum_{j=q+1}^{p} t_j + \sum_{j=1}^{q} t_j = \sum_{j=1}^{p} t_j > 0.$$

□

Proposition 9 *There is an upward starting point for C for which the partial sums of $\sum_{j=1}^{p}(u_j - \sigma_j)$ are all nonnegative.*

Proof. Recall that $\sigma_j = \lambda_j$ except that if $\lambda_p = 1$ then $\sigma_p = 0$. Thus $\sum_{j=1}^{p}(u_j - \sigma_j) \ge \sum_{j=1}^{p}(u_j - \lambda_j)$. So it suffices to find an upward starting point for which all partial sums of $\sum_{j=1}^{p}(u_j - \lambda_j)$ are nonnegative.

If all partial sums of $\sum_{j=1}^{p}(u_j - \lambda_j)$ are nonnegative then we simply start C at the current upward starting point x_0. In particular, Lemma 7 implies this is the case when $p = 1$ and there is only one term in the sum, $u_1 - \sigma_1 \ge p - 1 = 0$.

Now suppose some partial sum is negative, so that $p \ge 2$. Since $\sum_{j=1}^{p}(u_j - \lambda_j) \ge p - 1 > 0$, by Lemma 8 there is an index q for which the partial sums of

$$\sum_{j=q+1}^{p} (u_j - \lambda_j) + \sum_{j=1}^{q} (u_j - \lambda_j)$$

are all positive.

This corresponds to letting the starting point of C be

$$y_0 = x_{u_1+s_1+\cdots+u_q+s_q},$$

the element that completes the first q alternating sequences of up and side steps. We then relabel the elements of C as $y_0, y_1, y_2, \ldots, y_{m-1}, y_m = y_0$ and relabel the u_j, λ_j, σ_j accordingly, so that u_1 is now the number of up steps beginning at y_0, etc. Then y_0 is an upward starting point for C for which the partial sums of $\sum_{j=1}^{p}(u_j - \lambda_j)$ are all positive. □

In Example 6, it is now easy to check that the procedure in Proposition 9 for choosing a new starting point for C gives $q = 1$ and $y_0 = x_4$.

2.3 Initialization and iteration

Let y_0 denote an upward starting point of C for which the partial sums of $\sum_{j=1}^{p}(u_j - \sigma_j)$ are all nonnegative. We initialize the stack K with y_0 and then iteratively add the next sequence of u_j elements and subtract σ_j elements from the stack, for $j = 1, 2, \ldots, p$. The following remark is easy to verify and will be useful in the proof of Proposition 11.

Remark 10 *Let $P = (V, \prec)$ be a poset and n be an integer, $n \ge 3$. If P contains a chain $c_1 \prec \cdots \prec c_n$ and an element d incomparable to both c_1 and c_n, then $c_1 \prec \cdots \prec c_n$ and d form an induced $\mathbf{n+1}$ in P.*

Proposition 11 (a) *The stack K never empties during the algorithm.*

(b) *Suppose P does not contain an induced $\mathbf{n+1}$. Then after each step j_u and j_s, $j = 1, 2, \ldots, p$, the order of elements on the stack respects the partial order in P.*

Proof. We will use the following notation to help describe the evolution of the stack K during the algorithm. Let β_j be the first element added to the stack during step j_u and let α_j be the top element of the stack after step j_s. For example, using Table 3 we see that $\beta_2 = y_4$ and $\alpha_2 = y_5$. Denote the elements on the stack after step j_u, from the top of the stack down, by $b_1, b_2, \ldots$. Then $b_{u_j} = \beta_j$ and the top u_j elements of K correspond to the j^{th} sequence of up steps in C, namely $U_j : \beta_j = b_{u_j} \prec \cdots \prec b_2 \prec b_1$.

In the forcing cycle C, U_j is followed by s_j elements corresponding to the next sequence of side steps, $S_j : d_1 \parallel d_2 \parallel \cdots \parallel d_{s_j}$. At the ends of this sequence we have

$$\begin{aligned} b_1 \parallel d_1 \qquad & \text{for } 1 \le j \le p \\ d_{s_j} \prec \beta_{j+1} \qquad & \text{for } 1 \le j \le p-1. \end{aligned} \tag{3}$$

Proof of (a). The number of elements on the stack after step j_u of the algorithm is $1 + \sum_{l=1}^{j-1}(u_l - \sigma_l) + u_j$. The number after the succeeding step j_s is $1 + \sum_{l=1}^{j}(u_l - \sigma_l)$. Since the partial sums of $\sum_{j=1}^{p}(u_j - \sigma_j)$ are all nonnegative, there are always at least two elements on the stack after j_u and at least one after j_s. Thus the stack never empties during the algorithm.

Proof of (b). Suppose P does not contain an $\mathbf{n+1}$. Since no elements are added to the stack during j_s, it suffices to prove (b) only for j_u. We will do this by induction on j.

For step 1_u, (b) is true since y_0 is an upward starting point for C. Now suppose (b) is true for $1, 2, \ldots, j$, where $1 \le j \le p-1$, and prove it is true for $j+1$. We seek to show $\alpha_j \prec \beta_{j+1}$, i.e., the top element α_j of K after step j_s precedes the first element β_{j+1} to be added to K in step $(j+1)_u$.

In the preceding step j_s we removed the top $\sigma_j = (n-2)(s_j-1)+1$ elements from the stack K, so that $\alpha_j = b_{(n-2)(s_j-1)+2}$. We consider the cases $s_j = 1$ and $s_j \ge 2$ separately.

If $s_j = 1$ then we removed only the top element, b_1. So we have $\alpha_j = b_2$ and $b_1 \parallel d_1 \prec \beta_{j+1}$. Also, $b_2 \prec b_1$ by the induction assumption for j. Thus in any interval representation of P we have $R(\alpha_j) = R(b_2) < L(b_1) \le R(d_1) < L(\beta_{j+1})$, which implies $\alpha_j \prec \beta_{j+1}$ as desired. Figure 4 illustrates this reasoning for Example 6 where $j = 1$ and $\alpha_j = y_1$, $\beta_j = y_1$, $d_1 = y_3$ and $\beta_{j+1} = y_4$. Since step $(j+1)_u$ consists of adding U_{j+1} to the stack on top of α_j, (b) is true for $j+1$.

Now suppose $s_j \ge 2$. We can think of step j_s as having removed first b_1 and then groups of $n-2$ elements, one group at a time, until we have removed σ_j elements. If we have removed i groups then define e_i to be the element at the top of the stack, $e_i = b_{(n-2)i+2}$. We will prove by induction on i that

$$e_i \prec d_i \quad \text{for} \quad 1 \leq i \leq s_j - 1. \tag{4}$$

For $i = 1$ we need to show $e_1 = b_n \prec d_1$. By the induction assumption for j we have $b_n \prec b_{n-1} \prec \ldots \prec b_1$. If $b_n \parallel d_1$, then since $b_1 \parallel d_1$, Remark 10 implies that this chain and d_1 form an $\mathbf{n+1}$, a contradiction. If $d_1 \prec b_n$ then we have $d_1 \prec b_n \prec b_1$, which contradicts $b_1 \parallel d_1$. Thus $e_1 \prec d_1$.

Suppose (4) is true for $1, 2, \ldots, i-1$. We will prove it is true for i. By the induction assumptions for j and for $i-1$, we have

$$\begin{aligned} e_i \quad &= b_{(n-2)i+2} = b_{(n-2)(i-1)+n} \prec b_{(n-2)(i-1)+n-1} \prec \ldots \\ &\prec b_{(n-2)(i-1)+2} = e_{i-1} \prec d_{i-1}. \end{aligned} \tag{5}$$

If $e_i \parallel d_i$ then by Remark 10, d_i and the chain in (5) form an $\mathbf{n+1}$, a contradiction. If $d_i \prec e_i$ then $d_i \prec e_i \prec e_{i-1} \prec d_{i-1}$, contradicting $d_i \parallel d_{i-1}$. Thus $e_i \prec d_i$, which completes the induction on i.

By setting $i = s_j - 1$ we conclude that $\alpha_j = e_{s_j-1} \prec d_{s_j-1}$. To complete the induction on j it suffices to show $\alpha_j \prec \beta_{j+1}$. In any interval representation of P we must have $R(\alpha_j) < L(d_{s_j-1})$. We also know $d_{s_j-1} \parallel d_{s_j}$, so $L(d_{s_j-1}) \leq R(d_{s_j})$. By (3), $d_{s_j} \prec \beta_{j+1}$, and so we have $R(d_{s_j}) < L(\beta_{j+1})$. Combining these inequalities, we get $R(\alpha_j) < L(\beta_{j+1})$, hence $\alpha_j \prec \beta_{j+1}$ as desired. Figure 5 illustrates this reasoning for Example 6 where $j = 2$ and $\alpha_j = y_5$, $d_{s_j-1} = y_9$, $d_{s_j} = y_{10}$ and $\beta_{j+1} = y_{11}$. This completes the induction on j and the proof of (b). □

We can now prove Theorem 5.

Proof of Theorem 5. When $n = 3$ the result follows from Theorems 3 and 4(i). Now let $n \geq 4$ and let P be an interval order with a forcing cycle C with $r = \text{up}(C)$, and $s = \text{side}(C)$ and for which $r > (n-2)(s-1)$. Suppose P does not contain an induced $\mathbf{n+1}$. Apply the algorithm to $C : y_0, y_1, \ldots, y_{m-1}, y_m = y_0$.

Now consider the possible forms of the stack K after the final step p_s. By the initialization step and Proposition 11(a), the bottom element of K is y_0. Suppose $s_p = 1$. Since y_0 is an upward starting point for C we then have $y_{m-2} \prec y_{m-1} \parallel y_m = y_0$. By equation (2) we have $\sigma_p = 0$, so we remove no elements from the stack during p_s and the top element of K is $\alpha_p = y_{m-1}$. By Proposition 11(b) with $j = p$, it follows that $y_0 \prec y_{m-1}$, a contradiction.

Now suppose $s_p \geq 2$. The top element of K after p_s is now $\alpha_p = e_{s_p-1}$. By (4) with $j = p$, we have $\alpha_p = e_{s_p-1} \prec d_{s_p-1}$. Since y_0 is on the bottom of K, either $y_0 \prec \alpha_p$ or $y_0 = \alpha_p$. Each contradicts $d_{s_p-1} \parallel d_{s_p} = y_m = y_0$.

Since all possible forms of K after step p_s lead to a contradiction, the poset P must contain an induced $\mathbf{n+1}$ and the proof of Theorem 5 is complete. □

3 The range of wd_F for interval orders with no $\mathbf{n+1}$

Theorem 5, together with Theorem 3, implies that if P is an interval order with no $\mathbf{n+1}$ (for $n \geq 3$) and if $wd_F(P) = r/s$ for integers $r \geq 0$ and $s \geq 2$, with an optimal forcing cycle C where $r = \mathrm{up}(C), s = \mathrm{side}(C)$, then $r \leq (n-2)(s-1)$.

In this section we show that this bound is achieved and determines the range of $wd_F(P)$ for such interval orders. Indeed, we can construct interval orders P with no $\mathbf{n+1}$ and having $wd_F(P) = r/s$ for each value of r between $s-1$ and $(n-2)(s-1)$. After stating this formally in Theorem 12, we outline the construction and give some preliminary lemmas before proving the theorem. The range of $wd_F(P)$ for interval orders with no $\mathbf{n+1}$ is given in Corollary 18.

Theorem 12 *Let n, s be positive integers with $n \geq 2$. There exists an interval order P with no induced $\mathbf{n+1}$ and with $wd_F(P) = \frac{r}{s}$, for all integers r such that $s-1 \leq r \leq (n-2)(s-1)$. If $s \geq 2$ there is an optimal forcing cycle C with $\mathrm{up}(C) = r,\ \mathrm{side}(C) = s$.*

When $n = 2$ or $s = 1$, the only value of r satisfying the inequalities is $r = 0$. We may let P be any weak order (no $\mathbf{2+1}$), since $wd(P) = \lceil wd_F(P) \rceil$ and P is a weak order if and only if $wd(P) = 0$ (e.g., [6]). When either $n = 3$ or when $n \geq 4$ and $r = s-1$, the theorem asserts there exists an interval order P with $wd_F(P) = \frac{r}{r+1}$. By [10], this is true for all integers $r \geq 0$ and in fact P is a semiorder.

We now assume $n \geq 4, r \geq s \geq 2$, and $r \leq (n-2)(s-1)$. We will construct a partial order $P = (V, \prec)$ by giving an interval representation for it. The ground set V will consist of $r+s$ elements of two kinds, x_i corresponding to singleton sets of real numbers and y_j corresponding to intervals of positive length. The construction will guarantee that P is an interval order with no induced $\mathbf{n+1}$. Then we will prove that $wd_F(P) = \frac{r}{s}$ and give an optimal forcing cycle with r up arcs and s side arcs.

Let $x_0 < x_1 < \cdots < x_r$ be any increasing sequence of $r+1$ real numbers and let $I(x_i) = \{x_i\}$. We will choose some of these x_i to serve as endpoints for the intervals assigned to the y_j. In particular, we select a subsequence $x_0 = x_{h(0)} < x_{h(1)} < \cdots < x_{h(s-2)} < x_r$, which we write more simply as $x_0 = z_0 < z_1 < \cdots < z_{s-1} = x_r$, as follows.

Definition 13 *For $j = 0, 1, \ldots, s-1$, let*

$$h(0) = 0$$
$$h(j) = \min\left\{h(j-1) + n - 2, \left\lfloor (j+1)\frac{r}{s} \right\rfloor\right\}, \quad j = 1, \ldots, s-1.$$

In the definition of $h(j)$, the two expressions over which we take the minimum will serve to guarantee that P contains no $\mathbf{n+1}$ and that the labels on

j	$h(j-1)+n-2$	$\lfloor (j+1)r/s \rfloor$	$h(j)$	z_j
0			0	x_0
1	2	2	2	x_2
2	4	4	4	x_4
3	6	5	5	x_5
4	7	7	7	x_7
5	9	8	8	x_8
6	10	10	10	x_{10}

Table 4: The subsequence $z_j = x_{h(j)}$ for $n = 4, s = 7, r = 10$.

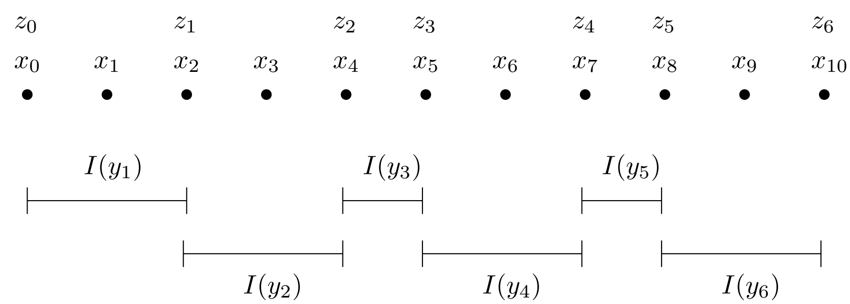

Figure 6: An interval representation for a poset P with no induced $\mathbf{4}+\mathbf{1}$. The forcing cycle $C : x_0 \prec \cdots \prec x_{10} \parallel y_6 \parallel \cdots \parallel y_1 \parallel x_0$ and the labeling determined from Table 5 show that $wd_F(P) = 10/7$.

incomparable elements are no more than r/s apart. We will show below that the indices $h(j)$ are strictly increasing for $j \geq 0$ and that $h(s-1) = r$. Taking this for granted for now, we let $z_j = x_{h(j)}, j = 0, 1, 2, \ldots, s-1$. We then define $I(y_j) = [z_{j-1}, z_j], j = 1, \ldots, s-1$.

That is, the union of the intervals $I(y_j)$ is $[x_0, x_r]$, their left and right endpoints are strictly increasing, adjacent intervals intersect in exactly one of the x_i, and non-adjacent intervals are disjoint. (We could have taken any intervals with these properties instead of letting the z_j be the endpoints.)

Now let $V = \{x_0, x_1, \ldots, x_r, y_1, y_2, \ldots, y_{s-1}\}$. We define a partial order $P = (V, \prec)$ with $u \prec v$ if and only if $I(u)$ is completely to the left of $I(v)$. The intervals $I(x_i), I(y_j)$ then give a representation of P as an interval order. We illustrate this construction in Table 4 for $n = 4, wd_F(P) = r/s = 10/7$. The corresponding interval representation is shown in Figure 6.

We must show this representation is well-defined, i.e., the intervals $I(y_j)$ are nonempty. We will need these two properties that are easy to verify and the

i, j	0	1	2	3	4	5	6	7	8	9	10
$g(x_i) = is$	0	7	14	21	28	35	42	49	56	63	70
$g(y_j) = jr$		10	20	30	40	50	60				

Table 5: The values of g for $n = 4, s = 7, r = 10$. These give rise to the labeling $f(u) = g(u)/s$.

lemma that follows.

$$\lfloor a + b \rfloor \leq \lfloor a \rfloor + \lceil b \rceil \tag{6}$$

$$\text{If } 2 \leq a \leq b, \text{ then } \frac{a-1}{b-1} \leq \frac{a}{b}. \tag{7}$$

Lemma 14 *For each $j = 0, \ldots, s-1$, we have $h(j) \geq \lfloor j \frac{r}{s-1} \rfloor$.*

Proof. We prove the lemma by induction on j. The inequality is immediate for $j = 0$. Now take some $j = 1, \ldots, s-1$ and suppose the inequality is true for $j-1$. We prove it is true for j. The minimization in Definition 13 leads to two cases.

Case 1. Suppose $h(j) = h(j-1) + n - 2$. Since $r \leq (n-2)(s-1)$ and $n-2$ is an integer, we have $\frac{r}{s-1} \leq \lceil \frac{r}{s-1} \rceil \leq n-2$. Thus

$$\begin{aligned}
h(j) &\geq h(j-1) + \left\lceil \frac{r}{s-1} \right\rceil \\
&\geq \left\lfloor (j-1)\frac{r}{s-1} \right\rfloor + \left\lceil \frac{r}{s-1} \right\rceil && \text{(by the induction assumption)} \\
&\geq \left\lfloor (j-1)\frac{r}{s-1} + \frac{r}{s-1} \right\rfloor && \text{(by (6))} \\
&= \left\lfloor j\frac{r}{s-1} \right\rfloor.
\end{aligned}$$

Case 2. Suppose $h(j) = \lfloor (j+1)\frac{r}{s} \rfloor$. Since $2 \leq j+1 \leq s$, it follows from (7) that

$$\frac{j+1}{s} \geq \frac{j}{s-1},$$

$$h(j) = \left\lfloor (j+1)\frac{r}{s} \right\rfloor \geq \left\lfloor j\frac{r}{s-1} \right\rfloor.$$

This completes the proof of Lemma 14. □

Corollary 15 *The function $h(j)$ is strictly increasing for $j = 0, 1, \ldots, s-1$ and $h(s-1) = r$. Thus the partial order $\prec$ is well-defined on V.*

Proof. Let j be any index with $1 \leq j \leq s-1$ and consider the two cases in the proof of Lemma 14. In Case 1 it is immediate that $h(j-1) < h(j)$, since we have assumed $n \geq 4$. In Case 2, $h(j-1) \leq \lfloor j\frac{r}{s} \rfloor \leq \lfloor (j+1)\frac{r}{s} \rfloor = h(j)$. Since $r \geq s$ we have $\lfloor j\frac{r}{s} \rfloor < \lfloor (j+1)\frac{r}{s} \rfloor$, and thus $h(j-1) < h(j)$. So in both cases $h(j)$ is strictly increasing.

Setting $j = s-1$ in Lemma 14 implies $h(s-1) \geq r$, while doing this in Definition 13 implies $h(s-1) \leq r$. So $h(s-1) = r$. □

Lemma 14 gives a lower bound for $h(j)$. It will also be useful to have the following bound, which is tighter in some but not all cases.

Lemma 16 *For each $j = 0, \ldots, s-1$, we have $h(j) \geq j\frac{r}{s}$.*

Proof. We prove the lemma by induction on j. The inequality is immediate for $j = 0$. Now take some $j = 1, \ldots, s-1$ and suppose the inequality is true for $j-1$. Then $h(j)$ is defined by one of the two cases in the proof of Lemma 14.

Case 1. Suppose $h(j) = h(j-1) + n - 2$. In this case,

$$\begin{aligned} h(j) &= h(j-1) + n - 2 \\ &\geq (j-1)\frac{r}{s} + n - 2 && \text{(by the induction assumption)} \\ &\geq j\frac{r}{s} - \frac{r}{s} + \frac{r}{s-1} && \text{(since } n-2 \geq \frac{r}{s-1}\text{)} \\ &> j\frac{r}{s}. \end{aligned}$$

Case 2. Suppose $h(j) = \lfloor (j+1)\frac{r}{s} \rfloor$. In this case, since $r \geq s$ we have

$$h(j) = \left\lfloor (j+1)\frac{r}{s} \right\rfloor \geq (j+1)\frac{r}{s} - 1 = j\frac{r}{s} + \frac{r}{s} - 1 \geq j\frac{r}{s}.$$

This completes the proof of Lemma 16. □

We can describe the up and side relations for the poset P in four up and two side cases:

Case ($\prec xx$): $x_i \prec x_k$ if and only if $i < k$
Case ($\prec yy$): $y_j \prec y_l$ if and only if $j \leq l-2$
Case ($\prec xy$): $x_i \prec y_j$ if and only if $i < h(j-1)$
Case ($\prec yx$): $y_j \prec x_i$ if and only if $h(j) < i$
Case ($\parallel yy$): $y_j \parallel y_{j+1}$
Case ($\parallel xy$): $x_i \parallel y_j$ if and only if $h(j-1) \leq i \leq h(j)$, i.e., $z_{j-1} \leq x_i \leq z_j$.

We now proceed to the proof of the main result of this section.

Proof of Theorem 12. We have reduced the proof to the case where $n \geq 4$ and $2 \leq s \leq r \leq (n-2)(s-1)$, and constructed an interval order P. First, we show that P contains no induced $\mathbf{n+1}$. Since each x_i is incomparable to at most two elements, an $\mathbf{n+1}$ would consist of some y_j and a chain of n elements, each of which is incomparable to y_j. See Figure 6 for examples. The chain is then a subset of $\{y_{j-1}, x_{h(j-1)} = z_{j-1}, x_{h(j-1)+1}, \ldots, x_{h(j)} = z_j, y_{j+1}\}$. (If $j = 1$ there is no element y_0 and if $j = s-1$ there is no y_s.) Since $y_{j-1} \parallel z_{j-1}$ and $y_{j+1} \parallel z_j$, the chain contains at most one element from each of these pairs. By Definition 13, the chain thus contains at most $h(j) - h(j-1) + 1 \leq n - 2 + 1 = n - 1$ elements, so P contains no induced $\mathbf{n+1}$.

It remains to show $wd_F(P) = r/s$. The cycle $C = x_0 \prec x_1 \prec \cdots \prec x_r \parallel y_{s-1} \parallel y_{s-2} \parallel \cdots \parallel y_1 \parallel x_0$ is a forcing cycle in P. Since $\text{up}(C) = r$, $\text{side}(C) = s$, Theorem 3 implies $wd_F(P) \geq \frac{r}{s}$. We now prove the reverse inequality.

We need to find a labeling of the elements of P that satisfies Definition 1 with $k = r/s$. Let

$$g(x_i) = is \text{ for } i = 0, 1, \ldots, r$$
$$g(y_j) = jr \text{ for } j = 1, 2, \ldots, s-1.$$

Then define the labeling $f : V \to \mathbf{Q}$ by $f(u) = g(u)/s$, i.e., $f(x_i) = i, f(y_j) = j\frac{r}{s}$. For example, Table 5 shows the values of $g(u)$ when $n = 4,\ s = 7,\ r = 10$.

To prove f satisfies Definition 1 it suffices to prove

(i) if $a \prec b$ then $g(a) + s \leq g(b)$ ("up" constraints)

(ii) if $a \parallel b$ then $|g(a) - g(b)| \leq r$. ("side" constraints)

To prove (i), we consider the four up cases stated earlier.

Case ($\prec xx$): Let $x_i \prec x_k$, i.e., $i < k$. Then $g(x_i) + s = (i+1)s \leq ks = g(x_k)$.

Case ($\prec yy$): Let $y_j \prec y_l$, i.e., $j \leq l - 2$. Since $s \leq r$,

$$g(y_j) + s = jr + s \leq (l-2)r + r < lr = g(y_l).$$

Case ($\prec xy$): Let $x_i \prec y_j$, i.e., $i < h(j-1)$. Then $i + 1 \leq h(j-1)$, so

$$g(x_i) + s = (i+1)s \leq h(j-1) \cdot s \leq \left\lfloor j\frac{r}{s} \right\rfloor s \leq jr = g(y_j).$$

Case ($\prec yx$): Let $y_j \prec x_i$, i.e., $h(j) < i$. Since $h(j)$ is an integer, we can say $i \geq h(j) + 1$. By Lemma 16,

$$g(x_i) = is \geq (h(j)+1)s \geq \left(j\frac{r}{s}\right) s + s = jr + s = g(y_j) + s.$$

To prove (ii), we consider the two side cases.

Case ($\parallel yy$): Let $y_j \parallel y_{j+1}$. It follows immediately that $|g(y_{j+1}) - g(y_j)| = r$.

Case ($\parallel xy$): Let $x_i \parallel y_j$, i.e., $h(j-1) \leq i \leq h(j)$. We will prove that $|g(x_i) - g(y_j)| \leq r$. Since $g(x_k)$ increases with k, it suffices to consider $i = h(j-1)$, where $x_i = z_{j-1}$, and $i = h(j)$, where $x_i = z_j$. When $j = 1, \ldots, s-1$ we must prove

$$-r \leq h(j-1)s - jr \leq r$$
$$-r \leq h(j)s - jr \leq r.$$

Since $h(j-1) < h(j)$, this is equivalent to proving

$$(j-1)\frac{r}{s} \leq h(j-1) < h(j) \leq (j+1)\frac{r}{s}.$$

The right-hand inequality follows from Definition 13 and the left hand-inequality from Lemma 16.

The existence of such a labeling shows that $wd_F(P) \leq \frac{r}{s}$, and so we conclude that $wd_F(P) = \frac{r}{s}$. This completes the proof of Theorem 12. □

We next state a corollary giving the following interpretation of Theorem 12. We wish to construct a poset having any desired fractional weak discrepancy among the positive rationals r/s that can be achieved. When n is sufficiently large, we can accomplish this with an interval order containing no $\mathbf{n+1}$. (If $s = 1$, we can apply the corollary by taking the equivalent rational $\frac{2r}{2s}$.)

Corollary 17 *Let r, s be integers with $s \geq 2$ and $s - 1 \leq r$. For all integers $n \geq 2 + \frac{r}{s-1}$ there exists an interval order P with no induced $\mathbf{n+1}$ and with $wd_F(P) = r/s$.*

Finally, we can now describe the range of the fractional weak discrepancy function over the set of posets containing no $\mathbf{n+1}$, for $n \geq 3$.

Corollary 18 *Let $n \geq 3$. The range of wd_F for interval orders containing no induced $\mathbf{n+1}$ is the set W of rationals that can be expressed as r/s, where $0 \leq s - 1 \leq r < (n-2)s$.*

Proof. Let $q \in W$, i.e., $q = r/s$ where $0 \leq s - 1 \leq r < (n-2)s$. Suppose $r \leq (n-2)(s-1)$. By Theorem 12 there is an interval order P with no induced $\mathbf{n+1}$ for which $wd_F(P) = r/s$. Otherwise, $(n-2)(s-1) < r < (n-2)s$. In particular, $r \geq s$. We will show that the equivalent representation $q = \frac{(n-2)r}{(n-2)s}$ satisfies the hypotheses of Theorem 12, i.e. that

$$0 \leq (n-2)s - 1 \leq (n-2)r \leq (n-2)[(n-2)s - 1].$$

First, since $s \leq r$ we have $0 \leq (n-2)s - 1 \leq (n-2)r - 1 < (n-2)r$. Next, since r is an integer and $r < (n-2)s$, we have $(n-2)r \leq (n-2)[(n-2)s - 1]$. We can thus apply Theorem 12 and conclude that in this case too there is an interval order P with no induced $\mathbf{n+1}$ for which $wd_F(P) = q$.

Conversely, let P be an interval order with no $\mathbf{n+1}$. If $wd_F(P) = 0$ we let $r = 0$ and $s = 1$, so $wd_F(P) = r/s \in W$. If $wd_F(P) > 0$, then P has an incomparable pair and thus an optimal forcing cycle C. Let $r = \mathrm{up}(C), s = \mathrm{side}(C)$, so $s \geq 2$ and $wd_F(P) = r/s$. By Theorem 5, $r \leq (n-2)(s-1)$. By Theorem 4, either $r = s - 1$ or $r \geq s$. Thus $wd_F(P) \in W$. □

Figure 2 illustrates Corollary 18. The pairs in the solid boxes, where $r = s-1$ and P is a semiorder, clearly satisfy the inequalities in the corollary. Consider one of the remaining r-s pairs in the figure, where $r \geq s$, and let $q = r/s$. If $r \leq (n-2)(s-1)$ then q is in the range W and this pair is in the dashed box in row s, where we number the rows by the denominators of their entries.

If $(n-2)(s-1) < r < (n-2)s$, then we can still conclude that $q \in W$ because $q = \frac{(n-2)r}{(n-2)s}$ and this representation appears in the dashed box in row $(n-2)s$. However, if $r \geq (n-2)s$ then no positive integer c satisfies $cr \leq (n-2)(cs-1)$.

So no representation $q = \frac{cr}{cs}$ satisfies the inequalities in the corollary and $q \notin W$ for the given value of n.

When $n = 4$, the case shown in Figure 2, the only pair with $2(s-1) < r < 2s$ has $r = 2s - 1$ and we consider $\frac{2r}{2s}$ instead of r/s. For example, this shows that $3/2 \in W$ and there is an interval order P with $wd_F(P) = 3/2$ that contains no $\mathbf{4} + \mathbf{1}$ and has an optimal forcing cycle C with $\mathrm{up}(C) = 6,\ \mathrm{side}(C) = 4$.

4 Future directions

In this article we generalized results in [10, 11] about semiorders to interval orders with no induced $\mathbf{n} + \mathbf{1}$. We can generalize semiorders in a different way by describing them as posets with no induced $\mathbf{2} + \mathbf{2}$ or $\mathbf{3} + \mathbf{1}$, i.e., no induced $\mathbf{m} + \mathbf{n}$ with $m + n = 4$. Posets having no induced $\mathbf{m} + \mathbf{n}$ with $m + n = M$ are called $(M, 2)$-free [14], e.g., semiorders are $(4, 2)$-free. Orders that are $(5, 2)$-free, i.e., contain no $\mathbf{4} + \mathbf{1}$ or $\mathbf{3} + \mathbf{2}$, are called *subsemiorders* [3]. A question for future consideration is the extent to which the results in [10, 11] extend to subsemiorders.

This problem is complicated by there being no known characterization of subsemiorders in terms of representations. However, we have obtained partial results in this direction by considering *split semiorders* [3, 5]. These are $(5, 2)$-free orders $P = (V, \prec)$ for which each element $v \in V$ is represented by a unit interval $I(v) = [L(v), R(v)]$ and a point $C(v) \in I(v)$, such that $x \prec y$ if and only $C(x) < L(y)$ and $R(x) < C(y)$. In particular, we prove in [12] that for any rational number $q > 0$, there exists a split semiorder P with $wd_F(P) = q$ if and only if $q = r/s$ for some integers r, s with $0 \le s - 1 \le r < 2s$.

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