

2005

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### Recommended Citation

Magid, M. A. (2005). Lorentzian isothermic surfaces in  $R_j(n)$ . *Rocky Mountain Journal of Mathematics*, 35(2), 627–640.  
doi:10.1216/rmjm/1181069750

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## LORENTZIAN ISOTHERMIC SURFACES IN $\mathbf{R}_j^n$

M.A. MAGID

ABSTRACT. If a time-like surface in  $\mathbf{R}_j^n$  is isothermic then there is another, uniquely determined surface which is either anti-conformal or conformal and sign-reversing with the same Gauss map. The choice of mapping depends on whether the shape operator is diagonalizable over  $\mathbf{R}$  or  $\mathbf{C}$ .

**1. Introduction.** There has been much recent work on isothermic surfaces in Euclidean spaces, see, for example, [1, 2, 4] and this paper begins a companion study for the timelike case. We will prove, for timelike surfaces in  $\mathbf{R}_j^n$ , the analogue of Palmer's result in [9]. In the positive definite setting, of course, all shape operators are diagonalizable over  $\mathbf{R}$ , and the existence of shape operators which are not diagonalizable over  $\mathbf{R}$  makes the timelike case different. Here we have two cases to consider and the possibility of sign-reversing mappings arise for the first time.

### 2. Preliminaries.

2.1 *Definitions and notations for timelike surfaces in  $\mathbf{R}_j^n$ .* The metric in  $\mathbf{R}_j^n$ , for  $n \geq 3$  and  $n - 1 \geq j \geq 1$  is denoted by

$$\langle \vec{v}, \vec{w} \rangle = -v_1w_1 - v_2w_2 + \cdots - v_jw_j + v_{j+1}w_{j+1} + \cdots + v_nw_n$$

for  $\vec{v} = (v_1, v_2, \dots, v_n)$  and  $\vec{w} = (w_1, w_2, \dots, w_n)$ . We begin with isothermal coordinates  $(t, s)$  on  $X$ , by which we mean that, for the induced metric  $h$  on  $X$  and for the associated coordinate vectors  $\partial t$  and  $\partial s$ , there is a nonzero function  $\lambda$  so that

$$(2.1) \quad \begin{aligned} h(\partial t, \partial t) &= -\lambda^2 \\ h(\partial t, \partial s) &= 0 \\ h(\partial s, \partial s) &= \lambda^2. \end{aligned}$$

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AMS *Mathematics subject classification.* 53A15.  
Received by the editors on August 7, 2001.

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Let  $N_i$ ,  $i = 1, 2, \dots, n-2$  denote a basis of normal vectors to the image of  $X$  which are orthogonal and have length  $\pm 1$ . The fundamental equations are:

$$\begin{aligned}
 X_{tt} &= \frac{\lambda_t}{\lambda} X_t + \frac{\lambda_s}{\lambda} X_s + \sum_{i=1}^{n-2} e_i N_i \\
 X_{ts} &= \frac{\lambda_s}{\lambda} X_t + \frac{\lambda_t}{\lambda} X_s + \sum_{i=1}^{n-2} f_i N_i \\
 X_{ss} &= \frac{\lambda_t}{\lambda} X_t + \frac{\lambda_s}{\lambda} X_s + \sum_{i=1}^{n-2} g_i N_i \\
 (N_{it})^T &= \langle N_i, N_i \rangle \left( \frac{e_i}{\lambda^2} X_t - \frac{f_i}{\lambda^2} X_s \right) \\
 (N_{is})^T &= \langle N_i, N_i \rangle \left( \frac{f_i}{\lambda^2} X_t - \frac{g_i}{\lambda^2} X_s \right),
 \end{aligned}
 \tag{2.2}$$

where, for example,  $(N_{it})^T$  is the tangential component of  $N_{it}$ . Thus, the shape operator corresponding to  $N_i$  is

$$A_i = \langle N_i, N_i \rangle \begin{bmatrix} -e_i/\lambda^2 & -f_i/\lambda^2 \\ f_i/\lambda^2 & g_i/\lambda^2 \end{bmatrix}.
 \tag{2.3}$$

The mean curvature vector,  $\eta$ , is

$$\eta = \sum_{i=1}^{n-2} \frac{(g_i - e_i)}{2\lambda^2} N_i.
 \tag{2.4}$$

Recall that, at each point on a timelike surface, there are three possibilities for a shape operator. The shape operator is either a) diagonalizable over  $\mathbf{R}$ , b) diagonalizable over  $\mathbf{C}$  but not  $\mathbf{R}$  or c) not diagonalizable over  $\mathbf{C}$ , and has a single real eigenvalue [5, 8]. Case a) occurs if and only if

$$\left( \frac{e_i + g_i}{2} \right)^2 - f_i^2 > 0,$$

while case b) occurs if and only if the same quantity is negative.

A Riemannian surface is called isothermic if there is an isothermal coordinate system for which every shape operator is diagonalized. For a timelike surface we need to adjust the definition. Essentially we will call a surface isothermic if there is an isothermal coordinate system for which each shape operator has one of the three canonical forms.

*Definition.* A timelike immersion is called *isothermic* if there is some isothermal coordinate system  $(t, s)$  such that each  $A_i$  with respect to the basis  $\{\partial t, \partial s\}$  has one of the following forms:

$$\text{a) } \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{b) } \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{c) } \begin{bmatrix} \nu \pm a/2 & -a/2 \\ a/2 & \nu \mp a/2 \end{bmatrix}.$$

We call the first case real isothermic and the second complex isothermic. In this paper we are working locally and assume that, in a neighborhood of a point, the immersion is either real isothermic or complex isothermic. Note that, in the case of complex isothermic, each shape operator satisfies  $e_i + g_i = 0$ . Finally note that we allow umbilic points and a totally umbilic surface satisfies all three conditions.

**2.3 Isothermal coordinate systems.** One of the key elements of the proof is changing from one isothermal coordinate system to another. We will need the following

**Proposition 2.1.** *Let  $(t, s)$  be an isothermal coordinate system on  $X$ . Any other isothermal coordinate system  $(u, v)$  on  $X$  with  $J(u, v)/J(t, s) > 0$  satisfies*

$$(2.5) \quad \begin{aligned} u_t &= \sigma \rho \cosh(\theta) & v_t &= \rho \sinh(\theta) \\ u_s &= \rho \sinh(\theta) & v_s &= \sigma \rho \cosh(\theta) \end{aligned}$$

with

$$(2.6) \quad \rho_t = \sigma \rho \theta_s \quad \text{and} \quad \rho_s = \sigma \rho \theta_t.$$

*Conversely, given any  $\rho$  and  $\theta$  satisfying (2.6) then there exist a new coordinate system  $(u, v)$  satisfying (2.5).*

Here  $g(X_v, X_v) = \mu^2$  and  $g(X_s, X_s) = \lambda^2$ ,  $\rho^2 = \lambda^2/\mu^2$ ,  $\rho > 0$  and  $\sigma = \pm 1$ .

The proof of this proposition is a straightforward computation.

Note that (2.6) implies  $\theta_{tt} = \theta_{ss}$ . In addition  $\theta_{tt} = \theta_{ss}$  implies the existence of the coordinate system as well. We can find a function  $\beta$  so that  $\beta_t = \theta_s$  and  $\beta_s = \theta_t$ . Setting  $\rho = e^\beta$  gives the desired  $\theta$  and  $\rho$ .

**Proposition 2.2.** *With the new variables  $\{u, v\}$  given in Proposition 2.1, the normal components of the fundamental equations are:*

$$(2.7) \quad \begin{aligned} e_{iuv} &= \frac{ch^2(\theta)}{\rho^2} e_i - \sigma \frac{sh(2\theta)}{\rho^2} f_i + \frac{sh^2(\theta)}{\rho^2} g_i \\ f_{iuv} &= -\sigma \frac{sh(2\theta)}{2\rho^2} e_i + \frac{ch(2\theta)}{\rho^2} f_i - \sigma \frac{sh(2\theta)}{2\rho^2} g_i \\ g_{iuv} &= \frac{sh^2(\theta)}{\rho^2} e_i - \sigma \frac{sh(2\theta)}{\rho^2} f_i + \frac{ch^2(\theta)}{\rho^2} g_i \end{aligned}$$

(Here, for example,  $e_{iuv}$  is the coefficient of  $N_i$  in the expansion of  $X_{uv}$ .)

### 3. Main theorem.

**Theorem.** *Let  $X : (M, h) \rightarrow (\mathbf{R}_j^n, \langle \cdot, \cdot \rangle)$  be a conformal immersion and let  $\gamma : X(M) \rightarrow G(\mathbf{R}_1^2 \subset \mathbf{R}_j^n)$  be the Gauss map into the Grassmannian of Lorentzian two-planes in  $\mathbf{R}_j^n$ .*

*There exists an anti-conformal immersion  $\tilde{X} : (M, h) \rightarrow (\mathbf{R}_j^n, \langle \cdot, \cdot \rangle)$  such that  $\tilde{\gamma} \circ \tilde{X} = \gamma$  where  $\tilde{\gamma}$  is the Gauss map on the image of  $\tilde{X}$  if and only if  $X(M)$  is real isothermic.*

*There exists a sign-reversing conformal immersion  $\hat{X} : (M, h) \rightarrow (\mathbf{R}_j^n, \langle \cdot, \cdot \rangle)$  such that  $\hat{\gamma} \circ \hat{X} = \gamma$  where  $\hat{\gamma}$  is the Gauss map on the image of  $\hat{X}$  if and only if  $X(M)$  is complex isothermic.*

*$\tilde{X}$  and  $\hat{X}$  is unique, up to similarity, if  $M$  has no umbilic points.*

Note that  $\hat{X}$  is sign-reversing and conformal if and only if, for some

$\mu \neq 0$ ,

$$(3.1) \quad \begin{aligned} \langle \hat{X}_t, \hat{X}_t \rangle &= \mu^2 \\ \langle \hat{X}_t, \hat{X}_s \rangle &= 0 \\ \langle \hat{X}_s, \hat{X}_s \rangle &= -\mu^2. \end{aligned}$$

*Proof.* First consider the case where the immersion  $X$  is real isothermic, so that we may assume  $f_i = 0$  for  $i = 1, 2, \dots, n-2$ . We define a new immersion, for  $\sigma = \pm 1$

$$\tilde{X}_t = -\frac{\sigma}{\lambda^2} X_t, \quad \tilde{X}_s = \frac{\sigma}{\lambda^2} X_s.$$

It is easy to check that  $\tilde{X}_{ts} = \tilde{X}_{st}$ , so that  $\tilde{X}$  does define an immersion.

Conversely, suppose  $X$  and  $\tilde{X}$  exist as in the first part of the theorem. Since the Gauss maps agree we must have:

$$\tilde{X}_t = aX_t + bX_s, \quad \tilde{X}_s = cX_t + dX_s,$$

with  $ad - bc < 0$ . By the definition of anti-conformality we have:

$$\begin{aligned} \langle \tilde{X}_t, \tilde{X}_t \rangle &= -\mu^2 = -a^2\lambda^2 + b^2\lambda^2 \\ \langle \tilde{X}_s, \tilde{X}_s \rangle &= \mu^2 = -c^2\lambda^2 + d^2\lambda^2 \\ \langle \tilde{X}_t, \tilde{X}_s \rangle &= 0 = -ac + bd. \end{aligned}$$

Thus, if we set  $\nu^2 = \mu^2/\lambda^2$  we see that

$$(3.2) \quad a = \nu\tau ch(\theta), \quad b = \nu sh(\theta), \quad c = -\nu sh(\theta), \quad \text{and} \quad d = -\nu\tau ch(\theta),$$

for some function  $\theta$  and  $\tau = \pm 1$ .

From  $\tilde{X}_{ts} = \tilde{X}_{st}$  we arrive at three equations defined in terms of the  $\lambda$ ,  $e_i$ ,  $f_i$ , and  $g_i$  associated to  $X(t, s)$ :

$$(I1) \quad a_s + b_t + 2a\frac{\lambda_s}{\lambda} + 2b\frac{\lambda_t}{\lambda} = 0,$$

$$(I2) \quad b_s + a_t + 2a\frac{\lambda_t}{\lambda} + 2b\frac{\lambda_s}{\lambda} = 0,$$

$$(I3) \quad 2a\left(\sum f_i N_i\right) + b\left(\sum e_i N_i\right) + b\left(\sum g_i N_i\right) = 0.$$

Note the last equation gives  $2af_i + b(e_i + g_i) = 0$  for  $i = 1, \dots, n - 2$ .

We want to find an angle  $\psi$  so that, for each  $i$ ,

$$f_{iuv} = 0 = -\sigma \frac{sh(2\psi)}{2\rho^2} e_i + \frac{ch(2\psi)}{\rho^2} f_i - \sigma \frac{sh(2\psi)}{2\rho^2} g_i.$$

In other words, we want  $\psi$  so that

$$\tanh(2\sigma\psi) = \frac{2f_i}{e_i + g_i} = -\frac{b}{a}.$$

We need only set  $\psi = -\tau\sigma\theta/2$ . We have to check that  $\psi_{tt} - \psi_{ss} = 0$ , or, equivalently, that  $\theta_{tt} - \theta_{ss} = 0$ . We note that  $\theta$  is proportional to the arctanh  $(b/a)$  and that

$$\begin{aligned} \theta_t &= \frac{ab_t - ba_t}{a^2 - b^2} \\ \theta_s &= \frac{ab_s - ba_s}{a^2 - b^2}, \end{aligned}$$

so that the numerator of  $\theta_{tt} - \theta_{ss}$  is

$$(3.3) \quad (a^2 - b^2)(ab_{tt} - ba_{tt} - ab_{ss} + ba_{ss}) - 2(ab_t - ba_t)(aa_t - bb_t) \\ + 2(ab_s - ba_s)(aa_s - bb_s).$$

By differentiating (I1) and (I2) with respect to  $s$  and  $t$  and substituting for  $a_{tt}$ ,  $a_{ss}$ ,  $b_{tt}$  and  $b_{ss}$  and then, again, using (I1) and (I2) to substitute for  $a_t$  and  $b_t$  the equation (3.3) can be seen to be zero. Thus  $\tilde{X}$  can be seen to be real isothermic. We postpone the proof of the uniqueness of  $\tilde{X}$  to the end of this proof.

Next suppose that  $X$  is complex isothermic, so that  $e_i + g_i = 0$ . We define  $\hat{X}$  by

$$\hat{X}_t = -\frac{\sigma}{\lambda^2} X_s, \quad \hat{X}_s = \frac{\sigma}{\lambda^2} X_t.$$

Again,  $\hat{X}_{ts} = \hat{X}_{st}$ . Next, assume that  $X$  and  $\hat{X}$  exist, so that

$$\hat{X}_t = aX_t + bX_s, \quad \hat{X}_s = cX_t + dX_s,$$

with  $ad - bc > 0$ . By the definition of a sign-reversing conformal immersion we have:

$$\begin{aligned}\langle \hat{X}_t, \hat{X}_t \rangle &= \mu^2 = -a^2\lambda^2 + b^2\lambda^2 \\ \langle \hat{X}_s, \hat{X}_s \rangle &= -\mu^2 = -c^2\lambda^2 + d^2\lambda^2 \\ \langle \hat{X}_t, \hat{X}_s \rangle &= 0 = -ac + bd.\end{aligned}$$

Thus, if we set  $\nu^2 = \mu^2/\lambda^2$  we see that

$$a = \nu sh(\theta), \quad b = \tau \nu ch(\theta), \quad c = -\tau \nu ch(\theta), \quad \text{and} \quad d = -\nu sh(\theta),$$

for some function  $\theta$  and  $\tau = \pm 1$ .

From  $\hat{X}_{ts} = \hat{X}_{st}$  we arrive at the same three equations as before: (I1), (I2) and (I3).

We want to find an angle  $\psi$  so that  $e_{i\{uv\}} + g_{i\{uv\}} = 0$ , or, for each  $i$ ,

$$\frac{ch(2\psi)}{\rho^2}e_i - 2\sigma \frac{sh(2\psi)}{\rho^2}f_i + \frac{ch(2\psi)}{\rho^2}g_i = 0.$$

Thus we set

$$\tanh(2\sigma\psi) = \frac{e_i + g_i}{2f_i} = -\frac{a}{b} = -\tau \tanh(\theta),$$

or  $\psi = -\tau\sigma\theta/2$ . The only difference here is that we have  $a/b$  instead of  $b/a$ . But the equations (I1) and (I2) are interchanged if  $a$  and  $b$  are interchanged. Thus we again see that  $\psi_{tt} - \psi_{ss} = 0$ .

To see uniqueness, suppose we have two  $\tilde{X}$  which are anti-conformal to the original  $X$ . Then, for some choice of isothermal coordinates, each  $f_i = 0$ . By (I3) we get  $b(e_i + g_i) = 0$ . Since  $M$  is not umbilic at any point, for some  $i$ ,  $(e_i + g_i) \neq 0$  at every point. Then  $b = 0$  and, so,  $\theta = 0$ . Finally, using, (I1) and (I2) we find that  $a = -k/\lambda^2$  for some constant  $k$ . Thus any two  $\tilde{X}$  are similar. The same argument will work for  $\hat{X}$ .  $\square$

In the next proposition we record the basic properties of  $\tilde{X}$  and  $\hat{X}$ . For simplicity we consider surfaces in Lorentz 3-space



**Proposition.** *Let  $X : (M, h) \rightarrow (\mathbf{R}_1^3, \langle \cdot, \cdot \rangle)$  be a conformal immersion.*

1. *If  $X$  is real isothermic then  $\tilde{X}$  is real isothermic and, if  $X$  is neither umbilic nor minimal, then  $\tilde{\tilde{X}} = X$ .*

2. *If  $X$  is complex isothermic and if  $X$  is neither umbilic nor minimal then  $\hat{X}$  is complex isothermic and  $\hat{\hat{X}} = X$ .*

*Proof.* The statement follows from looking at the shape operators of  $\tilde{X}$  and  $\hat{X}$ . Indeed, if the shape operator of  $X$  is

$$A = \begin{bmatrix} -e/\lambda^2 & 0 \\ 0 & g/\lambda^2 \end{bmatrix},$$

then the shape operator for  $\tilde{X}$  is

$$\tilde{A} = \begin{bmatrix} \sigma e & 0 \\ 0 & \sigma g \end{bmatrix}.$$

In the same way, if the shape operator of  $X$  is

$$A = \begin{bmatrix} -e/\lambda^2 & -f/\lambda^2 \\ f/\lambda^2 & -e/\lambda^2 \end{bmatrix},$$

then

$$\hat{A} = \begin{bmatrix} -\sigma f & \sigma e \\ -\sigma e & -\sigma f \end{bmatrix}.$$

Thus, if the immersion is neither umbilic nor minimal then  $\tilde{X}$  or  $\hat{X}$  can be called the dual of  $X$ .  $\square$

#### 4. Examples.

4.1. Surfaces in Lorentz space with constant mean curvature, including minimal surfaces, are isothermic. (We assume that the shape operator is either diagonalizable over  $\mathbf{R}$  or  $\mathbf{C}$ .) We give a proof of this fact, since it does not seem to appear in the literature.

Suppose that  $X : M \rightarrow \mathbf{R}_1^3$  is an isometric immersion with constant mean curvature. The Codazzi equations in this case become, from the fundamental equations with  $e_1 = e$ ,  $f_1 = f$  and  $g_1 = g$ ,

$$(4.1) \quad (e + g)_t = 2f_s \quad \text{and} \quad (e + g)_s = 2f_t.$$

This implies that  $f_{tt} = f_{ss}$ . Assume first that the surface has a shape operator which is diagonalizable over  $\mathbf{R}$ . As before, we seek a function  $\theta(t, s)$  so that the normal component of  $X_{uv}$  is zero. Equivalently we want

$$\theta = \frac{\sigma}{2} \operatorname{arctanh} \left( \frac{2f}{e + g} \right).$$

As in the proof of the theorem, if we use (4.1) and  $f_{tt} = f_{ss}$ , we can see that  $\theta_{tt} = \theta_{ss}$  and the surface is isothermic.

If the shape operator is diagonalizable over  $\mathbf{C}$  then we want

$$\theta = \frac{\sigma}{2} \operatorname{arctanh} \left( \frac{e + g}{2f} \right).$$

We see again that  $\theta_{tt} = \theta_{ss}$ .

For an explicit complex isothermic example, consider a Lorentzian helicoid:

$$X(t, s) = (\cosh(s) \sinh(t), \sinh(t) \sinh(s), s),$$

with normal vector

$$N(t, s) = \frac{1}{\cosh(t)} (-\sinh(s), -\cosh(s), \sinh(t)).$$

The surface looks like Figure 1.

It is easy to see that the dual surface to this surface is:

$$\hat{X}(t, s) = \frac{1}{\cosh(t)} (\sinh(s), \cosh(s), -\sinh(t)),$$

which, as we know, is a parameterization of part of the standard immersion of the Lorentzian sphere.

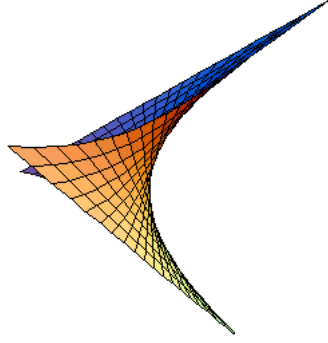


FIGURE 1.

4.2. *Lorentzian rotation surfaces in  $\mathbf{R}_1^3$  are real isothermic.* There are several different kinds of rotation surfaces in  $\mathbf{R}_1^3$ , depending on length of the axis of rotation. Details are given in [7].

a)  $X(t, s) = (x(s) \sinh(t), x(s) \cosh(t), y(s))$ , with  $x(s) \neq 0$ . We can see that

$$N(t, s) = \frac{1}{\sqrt{x'^2 + y'^2}}(y' \sinh(t), y' \cosh(t), -x').$$

b)  $X(t, s) = (x(t) \cosh(s), x(t) \sinh(s), y(t))$ , with  $(x(t), y(t))$  a time-like curve. We can see that

$$N(t, s) = \frac{1}{\sqrt{x'^2 - y'^2}}(y' \cosh(s), y' \sinh(s), x').$$

c)  $X(t, s) = (x(t), y(t) \cos(s), y(t) \sin(s))$ , with  $y(t) \neq 0$  and  $(x(t), y(t))$  a timelike curve.

$$N(t, s) = \frac{1}{\sqrt{x'^2 - y'^2}}(y', \cos(s)x', \sin(s)x').$$

d)

$$X(t, s) = \left( \frac{a(t) - b(t)}{\sqrt{2}} + \frac{a(t)s^2}{2\sqrt{2}}, \frac{a(t) + b(t)}{\sqrt{2}} - \frac{a(t)s^2}{2\sqrt{2}}, sa(t) \right)$$

with  $a'(t)b'(t) < 0$  and

$$N(t, s) = \left( \frac{1}{\sqrt{-2a'(t)b'(t)}} \right) \left( \frac{a'(t)+b'(t)}{\sqrt{2}} + \frac{a'(t)s^2}{2\sqrt{2}}, \frac{a'(t)-b'(t)}{\sqrt{2}} - \frac{a'(t)s^2}{2\sqrt{2}}, sa(t)' \right).$$

These surfaces are all real isothermic. The parameterizations above all have diagonalized shape operators, but the coordinates are isothermal only if the profile curves are parameterized properly.

4.3. Every member of a Bonnet family is isothermic [3, 6].

## APPENDIX

**1. Null Coordinates.** If we use null coordinates  $\{x, y\}$  on our surface, then many of the proofs are simplified but lose some of the geometric flavor. In this appendix some of the proofs are given with respect to null coordinates, defined by

$$\begin{aligned} x &= \frac{t-s}{\sqrt{2}}, & y &= \frac{t+s}{\sqrt{2}} \\ t &= \frac{x+y}{\sqrt{2}}, & s &= \frac{y-x}{\sqrt{2}}. \end{aligned}$$

Thus  $\langle X_x, X_y \rangle = -\lambda^2$  and the fundamental equations become:

$$\begin{aligned} X_{xx} &= 2\frac{\lambda_x}{\lambda}X_x + \sum_{i=1}^{n-2} \left( \frac{e_i + g_i}{2} - f_i \right) N_i \\ X_{xy} &= \sum_{i=1}^{n-2} \frac{e_i - g_i}{2} N_i \\ X_{yy} &= 2\frac{\lambda_y}{\lambda}X_y + \sum_{i=1}^{n-2} \left( \frac{e_i + g_i}{2} + f_i \right) N_i. \end{aligned} \tag{5.1}$$

We write the coefficients of the normal components as

$$\begin{aligned} \bar{e}_i &= \frac{e_i + g_i}{2} - f_i \\ \bar{f}_i &= \frac{e_i - g_i}{2} \\ \bar{g}_i &= \frac{e_i + g_i}{2} + f_i \end{aligned} \tag{5.2}$$

**Proposition 5.1.** *If new null variables  $\{\tilde{x}, \tilde{y}\}$  are given so that  $x = \alpha(\tilde{x})$  and  $y = \beta(\tilde{y})$ , then the normal components of fundamental equations become:*

$$\begin{aligned}\bar{e}_{i\tilde{x}\tilde{y}} &= \alpha'^2 \bar{e}_i \\ \bar{f}_{i\tilde{x}\tilde{y}} &= \alpha' \beta \bar{f}_i \\ \bar{g}_{i\tilde{x}\tilde{y}} &= \beta^2 \bar{g}_i\end{aligned}$$

The expressions for  $\tilde{X}$  and  $\hat{X}$  are

$$\begin{aligned}\tilde{X}_x &= -\frac{\sigma}{\lambda^2} X_y, & \tilde{X}_y &= -\frac{\sigma}{\lambda^2} X_x \\ \hat{X}_x &= -\frac{\sigma}{\lambda^2} X_y, & \hat{X}_y &= \frac{\sigma}{\lambda^2} X_x.\end{aligned}$$

In the proof of the main theorem we can, of course, use null coordinates. For example, in the case of  $\hat{X}$ , suppose that we are given

$$\begin{aligned}\hat{X}_x &= aX_x + bX_y \\ \hat{X}_y &= cX_x + dX_y\end{aligned}$$

with  $ad - bc > 0$  and  $\langle \hat{X}_x, \hat{X}_y \rangle > 0$ . This yields that  $a = 0 = d$  and  $bc < 0$ .  $\hat{X}_{xy} = \hat{X}_{yx}$  gives

$$(5.3) \quad \begin{aligned}b_y + \frac{2b\lambda_y}{\lambda} &= 0 \\ c_x + \frac{2c\lambda_x}{\lambda} &= 0 \\ b\bar{g}_i &= c\bar{e}_i.\end{aligned}$$

We are looking for a change of null coordinates  $\tilde{x} = \alpha(x)$ ,  $\tilde{y} = \beta(y)$  so that  $\bar{e}_i + \bar{g}_i = 0$ . From (5.3) we find

$$b = \frac{\gamma(x)}{\lambda^2}, \quad c = \frac{\delta(y)}{\lambda^2}.$$

We note that  $b$  and  $c$  have opposite signs; assume, without loss of generality, that  $b < 0$ . We need to find  $\alpha(x)$  and  $\beta(y)$  so that

$$\alpha'(x)^2 \bar{e}_i + \beta'(y)^2 \bar{g}_i = 0.$$

We know that

$$\gamma(x)\bar{g}_i = \delta(y)\bar{e}_i$$

with  $\gamma(x) < 0$  and  $\delta(y) > 0$ , or

$$\frac{\bar{g}_i}{\delta(y)} - \frac{\bar{e}_i}{\gamma(x)} = 0.$$

Just set

$$\alpha'(x)^2 = -\frac{1}{\gamma(x)}$$

$$\beta'(y)^2 = \frac{1}{\delta(y)}.$$

#### REFERENCES

1. F.E. Burstall, *Isothermic surfaces: Conformal geometry, Clifford algebras and integrable systems*, preprint arXiv:math.DG/003096 v2.
2. F. Burstall, U. Hertrich-Jeromin, F. Pedit and U. Pinkall, *Curved flats and isothermic surfaces*, Math. Z. **225** (1997), 199–209.
3. A. Fujioka and J.-I. Inoguchi, *Timelike Bonnet surfaces in space forms*, preprint.
4. G. Kamberov, F. Pedit and U. Pinkall, *Bonnet pairs and isothermic surfaces*, Duke Math. J. **92** (1998), 637–644.
5. Martin Magid, *Isometric immersions of Lorentz space with parallel second fundamental forms*, Tsukuba J. Math. **8** (1984), 31–54.
6. ———, *A theorem of Cartan for timelike Bonnet surfaces forms*, preprint.
7. Louise McNertny, *One-parameter families of surfaces with constant curvature in Lorentz 3-space*, Doctoral thesis, Brown University.
8. Barrett O’Neill, *Semi-Riemannian geometry*, Academic Press, New York, 1983.
9. Bennet Palmer, *Isothermic surfaces and the Gauss map*, Proc. Amer. Math. Soc. **104** (1988), 876–884.

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