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Björling problem for timelike surfaces in the Lorentz-Minkowski space

R. M. B. Chaves *, M. P. Dussan *, M. Magid †

Abstract

We introduce a new approach to the study of timelike minimal surfaces in the Lorentz-Minkowski space through a split-complex representation formula for this kind of surface. As applications, we solve the Björling problem for timelike surfaces and obtain interesting examples and related results. Using the Björling representation, we also obtain characterizations of minimal timelike surfaces of revolution as well as of minimal ruled timelike surfaces in the Lorentz-Minkowsi space.

MSC 53A10
Key words: Timelike surface, Björling problem, Lorentz-Minkowski space.

1 Introduction

Minimal surfaces play an important role in Differential Geometry and also in Physics, especially in problems related to General Relativity. In Euclidean space, the classical Björling problem (see [14], [9]) was proposed by Björling [7] in 1844 and consists of the construction of a minimal surface in $\mathbb{R}^3$ containing a prescribed analytic strip. The solution to this problem was obtained by Schwarz in [24]. These results have inspired many authors to obtain analogous results in other ambient spaces. For instance, in [2], J. Aledo, R. Chaves and A. Gálvez studied the Björling problem in the context of affine geometry, and more recently Aledo, Martinez and Milan in [1] generalized the results in [2] when solving the Björling problem for affine maximal surfaces. In [5], Asperti and Vilhena studied the problem for spacelike surfaces in $\mathbb{L}^4$. Moreover, F. Mercuri and I. Onnis, in [22], studied the problem when the ambient space is a Lie group, while Aledo, Galvez and Mira, in [3], solved a Bjorling-type problem for flat surfaces in the 3-sphere. For other references see [13] and [18].

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The local geometry of surfaces in the Lorentz-Minkowski space $\mathbb{L}^3$ is more complicated than that of surfaces in $\mathbb{R}^3$, since in $\mathbb{L}^3$ the vectors have different causal characters which yield more cases to be analysed. Hence we could consider spacelike, timelike or lightlike surfaces in $\mathbb{L}^3$. In this paper we consider timelike minimal surfaces in $\mathbb{L}^3$. Although timelike minimal surfaces neither maximize nor minimize surface area, they have nice geometric properties similar to minimal surfaces in Euclidean space $\mathbb{R}^3$. For instance, they also admit an Enneper-Weierstrass representation, introduced by M. Magid in [21]. But it is well known there are many differences in their behaviour.

The Björling problem for spacelike surfaces in $\mathbb{L}^3$ was considered by L. Alías, R. Chaves and P. Mira in [4] via a complex representation formula, while in [23] Mira and Pastor proposed the problem of establishing a Björling type formula for timelike minimal surfaces. As it is well known and as was pointed in [23], any general method for studying timelike minimal surfaces seems to be of interest, since this theory is not much developed. So, we were motivated to investigate the Björling problem for the timelike surfaces in $\mathbb{L}^3$. Our approach considers the ring of split-complex numbers, denoted henceforth by $\mathbb{C}'$, which plays a role similar to that played by the ordinary complex-numbers in the spacelike case. We note that the split-complex analysis and split-complex geometry depending on the split-complex numbers have been appropriately used to study timelike surfaces, as well as their applications in physics; see for instance [10], [12] and [15].

In this paper we assume that all the real-valued functions $\gamma(t)$ are real analytic. This condition allows us, after extending the function γ to a subset $\mathcal{O} \subset \mathbb{C}'$, to construct appropiate split-holomorphic functions defined in $\mathcal{O}$, whose real part will represent an analytic solution of the Björling problem in the timelike setting. In others words we construct split-complex representation formulas for analytic timelike minimal surfaces that are solutions of the Björling problem. Since in the timelike setting we need to consider the causal character of the analytic curves, there are two Björling problems which we have to study: Assuming that the analytic strip contains either an analytic timelike curve or an analytic spacelike curve $\gamma : I \to \mathbb{L}^3$, and that an orthogonal unit analytic spacelike vector field W is defined along γ. We call these respectively the timelike or spacelike Björling problem. Then using our split-complex representation formulas we prove the existence and uniqueness of analytic solutions to the two Björling problems as well as obtaining important results and many interesting examples of minimal surfaces with prescribed geometric properties. We note that since there is non-uniqueness of the solution of the Björling problem when considering the initial data $\gamma(t)$ as a null curve (as we can see from Example 3.2 below), we have just to consider the two cases above.

We prove the following split-complex representation formula in order to solve the timelike Björling problem. A similar formula for the spacelike Björling problem is

proven in Theorem 3.3.

Theorem 1.1. *(Timelike Björling representation) Let $X : \mathfrak{O} \subset \mathbb{C}' \to \mathbb{L}^3$ be a timelike minimal surface in $\mathbb{L}^3$ and set $\gamma(t) = X(t,0)$, $W(t) = N(t,0)$ on a real interval I in $\mathfrak{O}$. Choose any simply connected region $\mathfrak{R} \subset \mathfrak{O}$ containing I over which we can define split-holomorphic extensions $\gamma(z), W(z)$ for all $z \in \mathfrak{R}$. Then for all $z \in \mathfrak{R}$ we have:*

$$X(z) = Re\left(\gamma(z) + k' \int_{t_o}^{z} W(w) \times \gamma'(w) dw\right), \tag{1}$$

where t_o is an arbitrary fixed point in I and the path integral is taken over any path in $\mathfrak{R}$ from t_o to z.

Using the Björling approach we also give alternative proofs of the well known characterizations of minimal timelike surfaces of revolution and minimal timelike ruled surfaces in $\mathbb{L}^3$ (Woestijne [26]). More specifically, we prove the following results.

Theorem 1.2. *Every analytic minimal timelike surface of revolution in $\mathbb{L}^3$ is congruent to a part of one of the following surfaces:*

i) a Lorentzian elliptic catenoid.
ii) a Lorentzian hyperbolic catenoid
iii) a Lorentzian surface with spacelike profile curve.
iv) a Lorentzian parabolic catenoid.

Theorem 1.3. *Every analytic minimal timelike ruled surface of $\mathbb{L}^3$ is congruent to a part of one of the following surfaces:*

i) A Lorentzian plane
ii) The helicoid of the 1st kind.
iii) The helicoid of the 2nd kind.
iv) The helicoid of the 3rd kind.
v) The conjugate surface of Enneper of the 2nd kind.
vi) A flat B-scroll over a null curve.

We have organized the paper as follows. In Section 2 we fix the notation and give some preliminaries involving the split-complex numbers. Section 3 contains the key results regarding the analysis of extending real analytic functions to split-holomorphic functions. In Section 3 we also state and solve the Björling problems determining minimal timelike surfaces in $\mathbb{L}^3$, constructed in terms of its Björling data. Sections 4 and 5 contain, respectively, the characterization of analytic minimal timelike surfaces of revolution and analytic minimal timelike ruled surfaces, Theorems 1.2 and 1.3, together with some examples.

2 Preliminaries

Following the notation in [15], we begin with a definition:

Definition 2.1. *The split-complex numbers* $\mathbb{C}' = \{t + k's | t, s \in \mathbb{R}, k'^2 = 1, 1k' = k'1\}$ *is a commutative algebra over* $\mathbb{R}$. *If* $z = t + k's$ *then* $Re(z) = t$, $Im(z) = s$, $\bar{z} = t - k's$. *The indefinite metric on* $\mathbb{C}'$ *is given by* $-z\bar{z} = -t^2 + s^2$.

Definition 2.2. *A function* $f : \mathbb{C}' \to \mathbb{C}'$, $f(z) = f(t + k's) = u(t, s) + k'v(t, s)$ *is split-holomorphic if and only if* $u_t = v_s$ *and* $u_s = v_t$. *(See [10], [15]).*

Note that in this case

$$\frac{df}{dz} = f'(z) = \frac{1}{2}(\partial_t + k'\partial_s)(u + k'v) = u_t + k'v_t = v_s + k'u_s.$$

Proposition 2.1. *If* C *is a curve in the* $\mathbb{C}'$ *plane and* $f(z)$ *is a split-holomorphic function on* C *with a continuous derivative* $f'(z)$, *then*

$$\int_C f'(z)dz = f(z)|_C \tag{2}$$

and the integral is clearly path independent.

Proof. We use the standard definition of a line integral.

$$\int_C f'(z)dz = \int_C f'(t + k's)(dt + k'ds) =$$
$$\int_C u_t dt + u_s ds + k'(v_t dt + v_s ds) = \int_C d(u + k'v).$$

□

Proposition 2.2. *If* $f = u + k'v$ *is split holomorphic, then there is a split-holomorphic function* g *so that* $g' = f$.

Proof. We know that $u_t = v_s$ and $u_s = v_t$. Take $\beta, \alpha : \mathbb{R}^2 \to \mathbb{R}$ so that $\beta_s = u, \beta_t = v$ and $\alpha_t = u, \alpha_s = v$. Let $g = \alpha + k'\beta$. Then $g' = \alpha_t + k'\beta_t = u + k'v = f$. □

Thus every line integral of a split-holomorphic function is path independent.

Proposition 2.3. *If* $f = u + k'v$ *is a split-holomorphic function with* $u, v \in C^2$ *then* f' *is again split-holomorphic.*

Proof. $f'(t, s) = u_t + k'v_t = v_s + k'u_s$. We must show that $u_{tt} = v_{ts}, u_{ts} = v_{tt}$. We know that $u_t = v_s$ and $u_s = v_t$, so we are done, as long as the mixed partials are equal. □

We also note that using the following theorem and the Real Analytic Inverse Function Theorem ([19]), we can see that a split-holomorphic analytic function has a split-holomorphic analytic inverse if the determinant of the Jacobian is non-zero.

Theorem 2.1. *Let $U \subset \mathbb{R}^2$ be an open subset and let $f_1 : U \to \mathbb{R}$ and $f_2 : U \to \mathbb{R}$ have continuous partial derivatives. Consider the equations:*

$$f_1(t, s) = u \tag{3}$$
$$f_2(t, s) = v \tag{4}$$

near a given solution (t_o, s_o), (u_o, v_o). If the Jacobian matrix $J_f(t_o, s_o)$ is non-zero then the equation (3) can be solved uniquely as $(t, s) = g(u, v)$ for (t, s) near (t_o, s_o) and (u, v) near (u_o, v_o). Moreover, the function g has continuous partial derivatives.

Definition 2.3. *$\mathbb{L}^3$ is $\mathbb{R}^3$ with the indefinite inner product*

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Let $X : M_1^2 \to \mathbb{L}^3$ be a timelike surface, i.e., a surface which inherits a non-degenerate metric h of signature $(1, 1)$ on every tangent space. Following [25] we call the pair $(M_1^2, [h])$ a Lorentz surface defined by h where $[h]$ denotes the class of metrics conformal to h by a positive factor. This is the analog of a Riemann surface in the timelike setting.

Let $z = t + k's$, where t and s are conformal coordinates and

$$\phi_j = \frac{\partial X_j}{\partial z} = \frac{1}{2}\left(\frac{\partial X_j}{\partial t} + k'\frac{\partial X_j}{\partial s}\right),$$

where X_j represents a component of timelike immersion.

Observe that

$$-\phi_1^2 + \phi_2^2 + \phi_3^2 = \langle X_s, X_s \rangle + \langle X_t, X_t \rangle + 2\langle X_t, X_s \rangle = 0.$$

If we set $|a + k'b|^2 = b^2 - a^2$ then

$$-|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = \langle X_s, X_s \rangle - \langle X_t, X_t \rangle > 0.$$

Consider the split-complex 1-forms defined by $\Phi_j = \phi_j dz$. By looking at a conformal change of coordinates and using Proposition 2.1 in [21], we can see this defines a global form on M_1^2.

In her unpublished thesis [6] Berard shows that X is minimal if and only if $X_{jss} = X_{jtt}$ with respect to isothermal coordinates $\{t, s\}$. Thus the ϕ_j are split-holomorphic if and only if X is a minimal immersion in $\mathbb{L}^3$.

Note that if Φ_k are globally defined, then, following [11], pp. 77–78, we have, in a local coordinate patch:

$$\begin{aligned} 2Re\int_\gamma \phi_j dz &= Re\int_\gamma \frac{1}{2}\left(\frac{\partial X_j}{\partial t} + k'\frac{\partial X_j}{\partial s}\right)(dt + k'ds) \\ &= \int_\gamma \left(\frac{\partial X_j}{\partial t}dt + \frac{\partial X_j}{\partial s}ds\right) = \int_\gamma dX_j = X_j\,|_\gamma\ . \end{aligned}$$

Thus the integral over any closed curve has real part zero. The converse is also true.

Theorem 2.2. *Let Σ be a Lorentzian surface and choose three split-holomorphic one-forms Φ_1, Φ_2, Φ_3 globally defined on Σ satisfying:*

1. $-\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$.
2. $-|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 > 0$.
3. *Each Φ_j has no real periods.*

Then the map $X : \Sigma \to \mathbb{L}^3$ given by

$$X(z) = 2Re\int_{\gamma_z} (\Phi_1, \Phi_2, \Phi_3)\, dz,$$

where γ_z is a path from the fixed basepoint z is a minimal immersion in $\mathbb{L}^3$.

Remark 2.1. *We could also use the split-complex variable $w = k'z = s + k't$ in the above formulas, setting*

$$\psi_j = \frac{\partial X_j}{\partial w} = \frac{1}{2}\left(\frac{\partial X_j}{\partial s} + k'\frac{\partial X_j}{\partial t}\right).$$

After replacing Φ_j by Ψ_j the formulas are the same, except that

$$-|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 = -\langle X_s, X_s\rangle + \langle X_t, X_t\rangle < 0.$$

We will use this alternative choice of variable in Subsection 3.1.

3 Proofs of the main results

Throughout these last sections we assume that all the real-valued functions $\gamma(t)$ are C^ω, i.e., they are real analytic (have power series representations).

We begin by proving results about extending $\gamma(t)$ as a split-holomorphic function and proving that a split-complex analytic function $f(z)$ on a domain $\mathcal{D} \subset \mathbb{C}'$ is uniquely determined in $\mathcal{D}$ by knowledge of the derivatives $f^{(k)}(\alpha)$ at a point $\alpha \in \mathcal{D}$.

Proposition 3.1. *Let $\gamma(x)$ be a real analytic function given by $\gamma(x) = \sum_{k=0}^{\infty} \gamma_k x^k$ which converges in $|x| < R$. This function can be extended split-holomorphically in a neighborhood of $0 \in \mathbb{C}'$.*

Proof. First consider the new series

$$\sum_{k=0}^{\infty} 2^{k-1} \gamma_k y^k. \tag{5}$$

Using Hadamard's formula for the radius of convergence of a power series, we can see that the radius of convergence is $R/2$ ([19]). We are considering

$$\begin{aligned} \gamma(t + k's) &= u(t,s) + k'v(t,s) = \sum_{k=0}^{\infty} \gamma_k (t + k's)^k = \gamma_0 + \gamma_1 t + \gamma_2(t^2 + s^2) \\ &+ \gamma_3(t^3 + 3ts^2) + \cdots + k'(\gamma_1 s + \gamma_2(2ts) + \gamma_3(3t^2 s + s^3) + \ldots. \end{aligned}$$

It is clear that both the real and imaginary part converges on $|t| + |s| < R/2$, using the series defined in (5).

Now we prove that $u_t = v_s$ and $u_s = v_t$. In fact, we just prove first equality on the even terms of $u(t,s)$ since the other the equalities follow in a similar way.

For k even, the k^{th} term of $u(t,s)$ is $\gamma_k(t^k + \binom{k}{2} t^{k-2}s^2 + \binom{k}{4} t^{k-4}s^4 + \cdots + s^k)$. Then in u_t this yields:

$$\gamma_k(kt^{k-1} + (k-2)\binom{k}{2} t^{k-3}s^2 + (k-4)\binom{k}{4} t^{k-5}s^4 + \cdots + 2\binom{k}{k-2} ts^{k-2}).$$

The k^{th} term of $v(t,s)$ is: $\gamma_k(kt^{k-1}s + \binom{k}{3} t^{k-3}s^3 + \binom{k}{5} t^{k-5}s^5 + \cdots + \binom{k}{k-1} ts^{k-1})$.

We can then see that v_s has the term:

$$\gamma_k(kt^{k-1} + 3\binom{k}{3} t^{k-3}s^2 + 5\binom{k}{5} t^{k-5}s^4 + \cdots + (k-1)\binom{k}{k-1} ts^{k-2}).$$

The proof finishes when we note that

$$(k-2j)\binom{k}{2j} = (2j+1)\binom{k}{2j+1}.$$

□

Definition 3.1. *Let $U \subset \mathbb{R}^2$ be an open subset. The function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is called real analytic on U if, for each $p \in U$, f may be represented by a convergent power series in some neighborhood of p. We write "f is C^ω".*

Proposition 3.2. *If $\gamma(t)$ is a real-valued analytic function with two split-holomorphic extensions $u + k'v$ and $a + k'b$ satisfying $a(t,0) = u(t,0) = \gamma(t)$ in an open set, then they agree everywhere they are defined.*

Proof. We can see, using analytic continuation of real analytic functions of one variable, (p. 14 ([19])), that $a(t,0) = u(t,0) = \gamma(t)$ for all t. We know the extension of $\gamma(t)$, $f(t,s) = u(t,s) + k'v(t,s)$ is split-complex holomorphic. Thus, following [10] if we let

$$t = \frac{x+y}{2} \quad \text{and} \quad s = \frac{x-y}{2}$$

then

$$f(x,y) = F(x) + G(y) + k'(F(x) - G(y)).$$

If $x = y$ then $f(x,0) = \gamma(x) = F(x) + G(x)$ and $F(x) = G(x)$. We see that

$$f(x,y) = (1/2)(\gamma(x) + \gamma(y)) + (1/2)k'(\gamma(x) - \gamma(y)).$$

Since we did not assume anything about how $f(x,y)$ was constructed, only that it was split-complex holomorphic and real-valued on a part of the real axis we can see that there is only one extension. □

Example 3.1. *If $\gamma(t) = sinh(t)$, then using the power series expansion of $sinh(t)$, $f(t + k's) = sinh(t)cosh(s) + k'cosh(t)sinh(s)$. Look at*

$$\begin{aligned} f(x,y) &= sinh\left(\frac{x+y}{2}\right) cosh\left(\frac{x-y}{2}\right) + k'cosh\left(\frac{x+y}{2}\right) sinh\left(\frac{x-y}{2}\right) \\ &= (1/2)(sinh(x) + sinh(y)) + k'/2(sinh(x) - sinh(y)). \end{aligned}$$

Theorem 3.1. *Let f and g be split-complex analytic functions on a domain D (open, connected subset) of $\mathbb{C}'$. If $f(z) = g(z)$ in a neighborhood of some $\alpha \in D$ then $f = g$ in D.*

We need two lemmas before we begin its proof.

Lemma 3.1. *If $f(z) = \sum_{k=0}^{\infty} \gamma_k z^k$ then $\gamma_k = \frac{f^{(k)}(0)}{k!}$, $\quad k = 0, 1, 2....$*

This follows from [10] Lemma 1.5, which states that $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$. It is easy to see, using this result that, $f'(z) = \sum_{k=1}^{\infty} \gamma_k k z^{k-1}$ and the formula follows by induction.

Lemma 3.2. *If $F(z)$ is split-complex analytic in a neighborhood of $\alpha \in D$ and $F(z) \equiv 0$ in that neighborhood then $F^k(\alpha) = 0$ for all k.*

Again we use the difference quotient definition to see that

$$F'(\alpha) = \lim_{h \to 0} \frac{F(\alpha + h) - F(\alpha)}{h} = 0$$

in the neighborhood. The higher derivatives are zero in the same way.

Proof of Theorem 3.1: This just follows the one-variable proof given in Levinson and Redheffer's book [20], page 147-8. In fact, set $F(z) = f(z) - g(z)$. Suppose there is a $\beta \in D$ with $F(\beta) \neq 0$. Join α to β by a piecewise linear, connected curve $z = \zeta(t)$ in D with $a \leq t \leq b$, $\zeta(a) = \alpha$ and $\zeta(b) = \beta$. We define

$$S = \{t \in [a, b] \,|\, F^{(k)}[\zeta(t)] = 0 \,|\, k = 0, 1, 2, ...\}.$$

We see that $a \in S$ by Lemma 3.2 hence S is not empty, so that S has a least upper bound t_o. By definition we can take a sequence $\{t_j\} \subset S$ which converges to $t_o \in S$. By continuity $F^{(k)}(\zeta(t_0)) = \lim_{j \to \infty} F^{(k)}(\zeta(t_j)) = 0$, so that $t_0 \in S$. Using the series representation for F in a neighborhood of $\zeta(t_0)$ we see that t_0 cannot be the upper bound of S and so no β exists.

3.1 Timelike and spacelike Björling Problem for Lorentzian surfaces

The classical Björling problem asks for the existence and uniqueness of a minimal surface in $\mathbb{R}^3$ that passes through a real analytic curve with a prescribed analytic unit normal along this curve. Now, in this paper, since we are working with a Lorentzian metric we can study two forms of the Björling problem for minimal surfaces in $\mathbb{L}^3$, namely, when the initial data $\gamma(t)$ is timelike or spacelike curve. In fact, even though, we can state a Björling problem when the initial data $\gamma(t)$ is a null curve, there is not uniqueness to the solution for this problem, as the following example shows.

Example 3.2. *Take the null cubic curve*

$$x(u) = (\frac{4}{3}u^3 + u, \frac{4}{3}u^3 - u, 2u^2),$$

with a unit normal field $N(u) = (2u, 2u, 1)$ *along it. Let*

$$\begin{aligned} y(v) &= \left(\tfrac{1}{2}sinh(2v), v, \tfrac{1}{2}(cosh(2v) - 1)\right) \\ z(v) &= (v, v, 0) \end{aligned}$$

be two null curves. Then the two surfaces $f(u, v) = x(u) + y(v)$ *and* $g(u, v) = x(u) + z(v)$ *are two distinct minimal surfaces containing the curve* $x(u)$ *with* $N(u)$ *a normal field along it.*

Hence it remains only two problems to be studied.

Assume that $\gamma : I \to \mathbb{L}^3$ is a regular analytic timelike (respectively spacelike) curve in $\mathbb{L}^3$ and $W : I \to \mathbb{L}^3$ is a unit analytic spacelike vector field along γ such that $\langle \gamma', W \rangle = 0$. The Björling problem is to determine a minimal Lorentzian surface $X : \mathfrak{O} \subset \mathbb{C}' \to \mathbb{L}^3$ such that $X(t,0) = \gamma(t)$ and $N(t,0) = W(t)$ (respectively $X(0,s) = \gamma(s)$ and $N(0,s) = W(s)$ for all $s \in I$). In our case, $\mathfrak{O}$ is a split-complex domain with $I \subset \mathfrak{O}$ and $N : \mathfrak{O} \to \mathbb{L}^3$ is the Gauss map of the surface.

When γ is timelike this problem is called the *timelike Björling problem* and if γ is spacelike, we call it the *spacelike Björling problem.*

The following theorem describes the split-complex representation formula in the timelike Björling problem. We follow the notation established in Section 2 for $z = t + k's$ where t and s are conformal coordinates and note that the Lorentzian cross-product used in the theorem is defined by

$$\langle (u \times v), w \rangle = det[u, v, w].$$

For its proof one follows the Proof of Theorem 3.1 in [4].

Theorem 3.2. *(Timelike Björling representation) Let $X : \mathfrak{O} \subset \mathbb{C}' \to \mathbb{L}^3$ be a timelike minimal surface in $\mathbb{L}^3$ and set $\gamma(t) = X(t,0)$, $W(t) = N(t,0)$ on a real interval I in $\mathfrak{O}$. Choose any simply connected region $\mathfrak{R} \subset \mathfrak{O}$ containing I over which we can define split-holomorphic extensions $\gamma(z), W(z)$ for all $z \in \mathfrak{R}$. Then for all $z \in \mathfrak{R}$ we have:*

$$X(z) = Re\left(\gamma(z) + k' \int_{t_o}^{z} W(w) \times \gamma'(w) dw\right), \tag{6}$$

where t_o is an arbitrary fixed point in I and the path integral is taken over any path in $\mathfrak{R}$ from t_o to z.

Proof. Since X is a minimal immersion we can look at:

$$\phi(z) = \frac{\partial X}{\partial z}(z),$$

which is split-holomorphic over $\mathfrak{O}$. We have

$$X(z) = 2Re \int_{\gamma_z} \phi(w) dw$$

with the constant of integration being the one that makes the expression $X(t,0) = \gamma(t)$ holds for all $t \in I$. We know that $N \times X_s = X_t$ so that:

$$\phi(z) = \frac{1}{2}(X_t + k' X_s) = \frac{1}{2}(X_t + k' N \times X_t).$$

From the definition of γ, W we have $\phi(t,0) = \frac{1}{2}(\gamma'(t) + k'W(t) \times \gamma'(t))$. This is a mapping from I to $\mathbb{C}'^3$. Then the argument in Proposition 3.2 shows that this has a unique extension to:

$$\phi(z) = \frac{1}{2}(\gamma'(z) + k'W(z) \times \gamma'(z)).$$

As in [4] we end up with

$$X(z) = Re\left(\gamma(z) + k' \int_{s_o}^{z} W(w) \times \gamma'(w)dw\right).$$

□

For the spacelike Björling problem the alternative choice of variable $w = k'z = s + k't$, described in the end of Section 2, is more convenient. It will allow us to get a spacelike Björling representation as follows.

Theorem 3.3. *(Spacelike Björling representation) Let $X : \mathfrak{O} \subset \mathbb{C}' \to \mathbb{L}^3$ be a timelike minimal surface in $\mathbb{L}^3$ and set $\gamma(s) = X(0,s)$, $W(s) = N(0,s)$ on a real interval I in $\mathfrak{O}$. Choose any simply connected region $\mathfrak{R} \subset \mathfrak{O}$ containing I over which we can define split-holomorphic extensions $\gamma(w), W(w)$ for all $w \in \mathfrak{R}$. Then for all $w \in \mathfrak{R}$ we have:*

$$X(w) = Re\left(\gamma(w) + k' \int_{s_o}^{w} W(\zeta) \times \gamma'(\zeta)d\zeta\right), \tag{7}$$

where s_o is an arbitrary fixed point in I and the path integral is taken over any path in $\mathfrak{R}$ from s_o to w.

Proof. Since X is a minimal immersion we can look at:

$$\psi(w) = \frac{\partial X}{\partial w}(w),$$

is split-holomorphic over $\mathfrak{O}$. We have

$$X(w) = 2Re \int_{\gamma_w} \psi(\zeta)d\zeta$$

with the constant of integration being the one that makes the expression $X(0,s) = \gamma(s)$ holds, for all $s \in I$. We know that $X_t = N \times X_s$ so that:

$$\psi(w) = \frac{1}{2}(X_s + k'X_t) = \frac{1}{2}(X_s + k'N \times X_s).$$

From the definition of γ, W we have $\psi(s,0) = \frac{1}{2}(\gamma'(s) + k'W(s) \times \gamma'(s))$. This is a mapping from I to $\mathbb{C}'^3$. The argument in Proposition 3.2 shows that this has a unique extension to:

$$\psi(w) = \frac{1}{2}(\gamma'(w) + k'W(w) \times \gamma'(w)).$$

As in the previous case we end up with

$$X(w) = Re\left(\gamma(w) + k'\int_{s_o}^{w} W(\zeta)\times\gamma'(\zeta)d\zeta\right).$$

□

Example 3.3. *The Lorentzian helicoid of 3rd kind can be parametrized by*

$$X(t,s) = (sinh(t)cosh(s), sinh(t)sinh(s), s).$$

Note that $sinh(t+k's) = sinh(t)cosh(s) + k'cosh(t)sinh(s)$ *and* $cosh(t+k's) = cosh(t)cosh(s) + k'sinh(t)sinh(s)$. *Let*

$$\gamma(t) = (sinh(t), 0, 0), \qquad W(t) = (0, -1/cosh(t), sinh(t)/cosh(t).$$

We also know that $(a+k'b)^{-1} = \frac{a-k'b}{a^2-b^2}$. *Thus*

$$\gamma(z) = (sinh(t)cosh(s) + k'cosh(t)sinh(s), 0, 0),$$
$$\gamma'(z) = (cosh(t)cosh(s) + k'sinh(t)sinh(s), 0, 0),$$

and

$$W(z) = \left(0, \frac{2(-cosh(t)cosh(s) + k'sinh(t)sinh(s))}{cosh(2s)+cosh(2t)}, \frac{sinh(2t)+k'sinh(2s)}{cosh(2s)+cosh(2t)}\right).$$

Finally we see that $W(w)\times\gamma'(w) = (0, sinh(w), 1)$, *and*

$$Re\left(\gamma(z) + k'(0, cosh(z), z)\right) = (sinh(t)cosh(s), sinh(t)sinh(s), s).$$

Now let us show how to recover that Lorentzian helicoid through the spacelike Björling representation.

In fact, $\gamma(s) = X(0,s) = (0,0,s)$ and $W(s) = N(0,s) = (-sinh(s), -cosh(s), 0)$ are spacelike vectors. The extensions are $\gamma(w) = (0,0,w)$ and $W(w) = (-sinh(w), -cosh(w), 0)$. Then, we see that $W(\zeta)\times\gamma'(\zeta) = (cosh(\zeta), sinh(\zeta), 0)$. By using Theorem 3.3 we obtain:

$$\begin{aligned} X(s+k't) &= Re\left(\gamma(s+k't) + k'\int_{s_0}^{s+k't}(cosh(\zeta), sinh(\zeta), 0)\right) \\ &= (cosh(s)sinh(t), sinh(t)sinh(s), s). \end{aligned}$$

We observe that this Lorentzian helicoid is a ruled surface and it will be considered again in Example 5.1.

Since it is simple to move from the timelike solutions to the spacelike ones, in the following we will focus on the results corresponding to the timelike case.

The next result proves that the timelike Björling problem has a unique solution.

Theorem 3.4. *There exists a unique solution to the timelike Björling problem for minimal surfaces. In fact, if γ, W are defined as in the formulation of the timelike Björling problem, then:*

(1) there exists a simply connected open set $\mathfrak{O} \subset \mathbb{C}'$ containing I for which γ, W admit split-holomorphic extensions $\gamma(z), W(z)$ over $\mathfrak{O}$ and the mapping $X : \mathfrak{O} \to \mathbb{L}^3$ given by

$$X(z) = Re\left(\gamma(z) + k' \int_{t_o}^{z} W(w) \times \gamma'(w) dw\right), \tag{8}$$

is a solution to the timelike Björling problem. Here $t_o \in I$ is fixed but arbitrary.

(2) If $X_1 : \mathfrak{O}_1 \subset \mathbb{C}' \to \mathbb{L}^3$, $X_2 : \mathfrak{O}_2 \subset \mathbb{C}' \to \mathbb{L}^3$, are two different solutions to the timelike Björling problem, then X_1 and X_2 coincide over the non-empty open set $\mathfrak{O}_1 \cap \mathfrak{O}_2$.

Proof. We start by proving (2). The timelike Björling representation shows that every solution of the Björling problem is given by (6) on any simply connected open set for which $\gamma(z)$ and $W(z)$ exist. So we can construct the two split-holomorphic extensions, which are equal in a neighborhood of I in the plane $\mathbb{C}'$. It follows then from Theorem 3.1 that they agree in $\mathfrak{O}_1 \cap \mathfrak{O}_2$.

For (1), let $\mathfrak{O} \subset \mathbb{C}'$ be a open set such that $I \subset \mathfrak{O}$ and over which the split-holomorphic extensions $\gamma(z), W(z)$ exist. We define the split-holomorphic mapping $\phi : \mathfrak{O} \subset \mathbb{C}' \to \mathbb{C}'^3$:

$$\phi(z) = \frac{1}{2}(\gamma'(z) + k' W(z) \times \gamma'(z)).$$

So, if $\phi = (\phi_1, \phi_2, \phi_3)$, it follows that

$$-\phi_1(z)^2 + \phi_2(z)^2 + \phi_3(z)^2 = 0,$$

and

$$-|\phi_1(t,0)|^2 + |\phi_2(t,0)|^2 + |\phi_3(t,0)|^2 = \frac{1}{4}(1 + |W(t) \times \gamma'(t)|^2) > 0.$$

Now we assume that $\mathfrak{O}$ is simply connected and that for all $z \in \mathfrak{O}$, $-|\phi_1(z)|^2 + |\phi_2(z)|^2 + |\phi_3(z)|^2 > 0$. Since

$$2Re \int_{\gamma} \phi_k dz = \int_{\gamma} \left(\frac{\partial \psi_k}{\partial t} dt + \frac{\partial \psi_k}{\partial s} ds\right) = \int_{\gamma} d\psi_k = 0,$$

Theorem 2.2 assures us that

$$X(z) = 2Re \int_{t_o}^{z} (\phi_1(w), \phi_2(w), \phi_3(w)) dw$$

is a minimal immersion in $\mathbb{L}^3$, i.e, $X : \mathfrak{O} \subset \mathbb{C}' \to \mathbb{L}^3$ given by

$$X(z) = Re\left(\gamma(z) + k' \int_{t_o}^{z} W(w) \times \gamma'(w) dw\right)$$

is minimal surface. Finally, X satisfies the conditions of the Björling problem. In fact, since $\gamma(z)$ and $W(z)$ are real when restricted to I , we have $X(t,0) = \gamma(t)$. Moreover, one has

$$\frac{\partial X}{\partial t}(t,0) = \gamma'(t), \quad \frac{\partial X}{\partial s}(t,0) = W(t) \times \gamma'(t),$$

which implies that

$$W(t) \times \frac{\partial X}{\partial t}(t,0) = \frac{\partial X}{\partial s}(t,0) = N(t,0) \times \frac{\partial X}{\partial t}(t,0),$$

and so $N(t,0) = W(t)$. □

Corollary 3.1. *Let $\gamma : I \to \mathbb{L}^3$ be a regular analytic timelike curve in $\mathbb{L}^3$, and let $W : I \to \mathbb{L}^3$ be a spacelike analytic unit vector field along γ such that $\langle \gamma', W \rangle = 0$. There exists a unique analytic minimal immersion in $\mathbb{L}^3$ whose image contains $\gamma(I)$ and such that its Gauss map along γ is W.*

Proof. It remains to prove the uniqueness since the existence comes from Theorem 3.4. Assume that $X : M_1^2 \to \mathbb{L}^3$ is a minimal immersion with a local isothermal coordinates system (U, ψ), where U is an open set in M_1^2 and $\psi(U) = V$. Choose $J \subset I$ so that $\gamma(J) \subset X(U)$. Locally $X|_U$ can be written as a minimal surface $\chi : V \to \mathbb{L}^3$ defined by $X(\psi^{-1}(V))$. There is an $\alpha : J \to V$ such that $\chi(\alpha(t)) = \gamma(t)$ and $N(\alpha(t)) = W(t)$ for all $t \in J$. We can see that α is real analytic as follows. The Jacobian of χ has rank 2. At any point $\alpha(t_o) = p_o$ pick two coordinates such that (χ_1, χ_2) have invertible Jacobian at p_o. Then $\alpha(t) = (\chi_1, \chi_2)^{-1} \circ \gamma(t)$ is real analytic and so has a split-holomorphic extension $\alpha(z) : O \subset \mathbb{C}' \to \mathbb{C}'$, where O is open and $J \subset O$. Writing $\alpha(t) = \alpha_1(t) + k'\alpha_2(t)$ and using the fact that $\gamma(t)$ is a regular curve we obtain $\alpha_1'^2 - \alpha_2'^2 \neq 0$. Then one can apply the inverse function theorem in a neighborhood of a point $t_o \in J$ for which $\gamma(z)$ has non-null derivatives at t_o. In fact, since the split-holomorphic complexification of the real-analytic function $\alpha_j(t)$ is given by:

$$f_j(t,s) = \frac{1}{2}(\alpha_j(t+s) + \alpha_j(t-s)) + \frac{1}{2}k'((\alpha_j(t+s) - \alpha_j(t-s)),$$

the split-holomorphic extension is:

$$\begin{aligned} f_1(t,s) + k' f_2(t,s) &= \frac{1}{2}(\alpha_1(t+s) + \alpha_1(t-s) + \alpha_2(t+s) - \alpha_2(t-s)) \\ &+ \frac{k'}{2}((\alpha_1(t+s) - \alpha_1(t-s) + \alpha_2(t+s) + \alpha_2(t-s)), \end{aligned}$$

whose Jacobian determinant is $(\alpha_1'(t+s)+\alpha_2'(t+s))(\alpha_1'(t-s)-\alpha_2'(t-s))$. Hence, as the original curve $\gamma(t)$ is timelike, for $s=0$ the determinant is non-zero. Thus we obtain a split-biholomorphic mapping $\alpha(z): A\subset\mathbb{C}'\to B\subset\mathbb{C}'$, where A is open subset of V which contains a real interval $(t_o-\epsilon,t_o+\epsilon)$ and B is an open subset of V. Hence the minimal surface $X|_B: B\subset V\to\mathbb{L}^3$ can be expressed as $\varphi: A\subset\mathbb{C}'\to\mathbb{L}^3$ with $\varphi(z)=X(\alpha(z))$. Moreover, for all $t\in(t_o-\epsilon,t_o+\epsilon)$ we have

$$\varphi(t,0)=X(\alpha(t,0)=X(\alpha(t))=\gamma(t), \tag{9}$$

$$N_\varphi(t,0)=N(\alpha(t,0)=N(\alpha(t))=W(t). \tag{10}$$

Hence it follows from the uniquess of $\varphi(z)$ that $X: M_1^2\to\mathbb{L}^3$ is also unique. □

Now let us consider the restricted timelike Björling problem: Let $\gamma: I\to\mathbb{L}^3$ be a real analytic curve in $\mathbb{L}^3$ with $<\gamma',\gamma'>=-1$ and such that $\gamma''(t)$ is spacelike for all $t\in I$. Construct a minimal Lorentzian surface in $\mathbb{L}^3$ containing γ as a geodesic.

The next corollary, whose proof is similar to Corollary 3.5 in [4], gives the answer for the above problem.

Corollary 3.2. *Let $\gamma: I\to\mathbb{L}^3$ be a constant speed analytic timelike curve in $\mathbb{L}^3$ such that $\gamma''(t)$ is spacelike for all $t\in I$. There exists a unique minimal Lorentzian immersion in $\mathbb{L}^3$ which contains γ as a geodesic.*

Following [4], it is possible to construct examples of minimal immersions containg a given curve as geodesic. In the next example, we start with a pseudo-circle in $\mathbb{L}^3$, i.e, a planar timelike curve with non-zero constant curvature.

Example 3.4. *Any pseudo-circle contained in a timelike plane in $\mathbb{L}^3$ is congruent to a curve of the form $-x_1^2+x_3^2=R^2$, and may be parametrized by $\gamma(t)=R(sinh(t),0,cosh(t))$. It follows from Corollary 3.2 that there is a unique minimal immersion in $\mathbb{L}^3$ containing γ as a geodesic. So, taking $W=-\frac{\gamma''}{|\gamma''|}$ i.e., $W(t)=-(sinh(t),0,cosh(t))$ we get*

$$\gamma(t)+k'\int^t W(\tau)\times\gamma'(\tau)d\tau=R(sinh(t),-k't,cosh(t)).$$

Hence the minimal immersion contaning γ as geodesic, is given by

$$X(t,s)=R(sinh(t)cosh(s),-s,cosh(t)cosh(s))$$

for $(t,s)\in\mathbb{R}\times(-\frac{\pi}{2},\frac{\pi}{2})$.

We point out that this is a surface of revolution and it will be contained in formula (13).

Observe that if the minimal immersion in $\mathbb{L}^3$ contains a pseudo-circle as a geodesic, the plane in which the pseudo-cirle is contained is timelike. Hence we have a similar consequence to Proposition 3.6 of [4], namely:

Proposition 3.3. *Any minimal timelike immersion in $\mathbb{L}^3$ containing a pseudo-circle as a geodesic is congruent to a piece of a Lorentzian surface given by Example 3.4.*

For the spacelike Björling problem we obtain analogous results to Theorem 3.4, Corollaries 3.1, 3.2 and Proposition 3.3.

4 Minimal timelike surfaces of revolution

Here we will give an alternative proof for the classification of timelike minimal surfaces of revolution in $\mathbb{L}^3$ given by Woestijne in [26], where one can also find the graphics of those surfaces. In our proof we show that those surfaces can be characterized as solutions of certain timelike or spacelike Björling problems.

We start by considering the different kinds of surfaces of revolution in $\mathbb{L}^3$, depending on the causal caracter of the axis of revolution, as obtained in [6]. They can be parametrized by:

$$a) \qquad X(t,s) = (a(t), b(t)\ cos(s), b(t)\ sin(s)), \tag{11}$$

where $(a(t), b(t))$ is a timelike curve and $b(t) \neq 0$.

$$b) \qquad X(t,s) = (a(t)\ cosh(s), a(t)\ sinh(s), b(t)), \tag{12}$$

where $(a(t), b(t))$ is a timelike curve and $a(t) \neq 0$.

$$c) \qquad X(t,s) = (a(s)\ sinh(t), a(s)\ cosh(t), b(s)), \tag{13}$$

with $a(s) \neq 0$, $a'^2 + b'^2 \neq 0$.

$$d) \qquad X(t,s) = (\frac{a(t) - b(t)}{\sqrt{2}} + \frac{a(t)s^2}{2\sqrt{2}}, \frac{a(t) + b(t)}{\sqrt{2}} - \frac{a(t)s^2}{2\sqrt{2}}, sa(t)) \tag{14}$$

with $a'(t)b'(t) < 0,\ a(t) \neq 0$.

It is also known that all of these surfaces can be conformally parametrized if the profile curves are parametrized properly.

Next we see examples of surfaces in $\mathbb{L}^3$ which will be necessary for the classification of timelike minimal surfaces of revolution. We begin with the following lemma.

Lemma 4.1. *let $\gamma(t)$ be a timelike analytic curve in $\mathbb{L}^3$ contained in the timelike coordinate plane x_1, x_3 or x_1, x_2-plane. Then there exists a unique timelike minimal*

immersion in $\mathbb{L}^3$*, that intersects orthogonally that plane along of* γ*, and is parametrized respectively by:*

$$a)\ X(z) = (Re\ a(z), Im \int_{t_o}^{z} \sqrt{a'^2 - b'^2} d\tau, Re\ b(z)),\ \ if\ \gamma(t) = (a(t), 0, b(t)), \tag{15}$$

$$b)\ X(z) = (Re\ a(z), Re\ b(z), Im \int_{t_o}^{z} \sqrt{a'^2 - b'^2} d\tau),\ \ if\ \gamma(t) = (a(t), b(t), 0). \tag{16}$$

Proof. In order to prove a) the Gauss map along the curve γ is orthogonal to γ' and $e = (0, 1, 0)$. So, one has that $N(t) = \frac{\gamma'(t) \times e}{|\gamma'(t) \times e|}$ and from Corollary 3.1 we obtain the existence and uniqueness. Finally the explicit formula above comes directly from the timelike Björling representation. The proof of b) is similar. □

Example 4.1. *(Lorentzian elliptic catenoid) Let* $\gamma(t) = A(t,\ cos(t-\theta), 0)$ *with* $A > 0$*,* $\theta \in \mathbb{R}$ *and* $t \in (\theta - \pi/2, \theta + \pi/2)$*. Changing to a new parameter* $u = t - \theta$ *and using that*

$$sin(z) = sin(t + k's) = cos(s)\ sin(t) + k'\ cos(t)\ sin(s)$$

$$cos(z) = cos(t + k's) = cos(t)\ cos(s) - k'\ sin(t)\ sin(s),$$

we have from Lemma 4.1, a timelike minimal surface which may be parametrized by

$$X(u, v) = A(u + \theta, cos(u)cos(v), cos(u)sin(v)), \quad where\ \ (u, v) \in (-\pi/2, \pi/2) \times \mathbb{R}.$$

Example 4.2. *(Lorentzian hyperbolic catenoid) Let* $\gamma(t) = A(sinh(t + \theta), 0, t)$ *with* $A > 0$*,* $\theta \in \mathbb{R}$ *defined for all* $t > -\theta$*. Now, setting* $u = t + \theta$*, one obtains from Lemma 4.1 a timelike minimal surface which may be parametrized by*

$$X(u, v) = A(sinh(u)cosh(v), sinh(u)sinh(v), u - \theta), \quad where\ \ u > 0, \quad v \in \mathbb{R}.$$

Example 4.3. *(Lorentzian surface with spacelike profile curve) Let* $\gamma(s) = A(0, cosh(s + \theta), s)$ *where* $A > 0$*,* $\theta \in \mathbb{R}$ *and* $s > -\theta$*. Choose a new parameter* $v = s + \theta$ *and consider a simple variation of Lemma 4.1, for analytic spacelike curves parametrized by* $(0, a(s), b(s))$*. Then taking* $e = (1, 0, 0)$ *and* $N(s) = \frac{e \times \gamma'(s)}{|e \times \gamma'(s)|}$*, we have the existence and uniqueness of the timelike minimal immersion given explicit by:*

$$X(w) = X(s + k't) = (Im \int_{s_o}^{w} \sqrt{a'^2 + b'^2} d\tau, Re\ a(w), Re\ b(w)), \tag{17}$$

where $w = k'z = s + k't$*. By applying this result to the curve* $\gamma(s) = A(0, cosh(s+\theta), s)$ *we get a timelike minimal surface parametrized by*

$$X(u, v) = A(cosh(v)sinh(u), cosh(v)cosh(u), v - \theta), \quad where\ \ v > 0, \quad u \in \mathbb{R}.$$

The next example corresponds to the Lorentzian parabolic catenoid. To simplify the computations, we will use a null frame of $\mathbb{L}^3$ given by

$$L_1 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), \quad L_2 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), \quad L_3 = (0, 0, 1).$$

Example 4.4. *(Lorentzian parabolic catenoid) Applying Lemma 4.1 to the analytic timelike curve $(p(t), q(t), 0)$ written with respect to the null frame $\{L_1, L_2, L_3\}$, we obtain existence and uniqueness of the timelike minimal immersion given by :*

$$X(z) = (Re\ p(z), Re\ q(z), Im \int_{t_o}^{z} \sqrt{-2p'(\tau)q'(\tau)}d\tau). \tag{18}$$

Applying this result to the curve

$$\gamma(t) = A(\frac{1}{6}t^3 + \frac{B}{2}t^2 + \frac{B^2}{2}t, -(t+B), 0), \quad where \ \ A > 0,\ B \in \mathbb{R},\ t < -B,$$

we get a timelike minimal surface which may be parametrized by:

$$X(t, s) = A(\frac{1}{6}t^3 + \frac{B}{2}t^2 + \frac{B^2}{2}t + \frac{s^2}{2}(t+B), -(t+B), s(t+B)),$$

with respect to the null frame.

Proof of Theorem 1.2: Consider a timelike minimal surface of revolution parametrized by the conformal immersion (11): In this case, the x_1, x_2-plane intersects the surface orthogonally along the curve $\gamma(t) = X(t, 0) = (a(t), b(t), 0)$. Here $N(t, 0) \times \gamma'(t)$ is collinear with $e = (0, 0, 1)$, and from Björling representation one sees that the split-holomorphic extensions $a(z)$ and $b(z)$ should satisfy

$$Re\ a(z) = a(t), \qquad Re\ b(z) = b(t)\ cos(s).$$

Thus $\frac{\partial}{\partial t} Re\ a(z) = \frac{\partial}{\partial s} Im\ a(z)$ and $\frac{\partial}{\partial s} Re\ a(z) = \frac{\partial}{\partial t} Im\ a(z)$. Since $Re\ a(z) = a(t)$, one obtains $a(t) = At + B$ for A, B constants. Now applying the split-holomorphic conditions for $b(z)$, we find that $b(t) = C_1\ cos(t) + C_2\ sin(t)$, where C_1, C_2 are constants. Since the immersion is conformal, we have that $|\gamma'(t)|^2 = -b^2(t)$, which implies that $A^2 = C_1^2 + C_2^2$ i,e., there is $\theta \in \mathbb{R}$ such that $C_1 = A\ cos(\theta)$ and $C_2 = A\ sin(\theta)$. Substituting those values in $b(t)$ one has $b(t) = Acos(t - \theta)$. Using Lemma 4.1, the surface is a piece of the Lorentzian elliptic catenoid.

Now, let us consider a minimal surface of revolution parametrized by (12), in which the rotation of the timelike analytic curve $(a(t), 0, b(t))$ is around the x_3-axis. Following

a), we get $b(t) = At + B$ and $a(t) = Asinh(t + \theta)$, where $\theta \in \mathbb{R}$ and A, B are constants. From Lemma 4.1 the resulting surface is congruent to a piece of the Lorentzian hyperbolic catenoid.

Now we will consider parametrization (13). For that case, the $\gamma(s) = (0, a(s), b(s))$ is an analytic spacelike curve which is rotated around of the x_3-axis. Following the same idea above, one obtains

$$Re\ a(w) = a(s)cosh(t), \quad Re\ b(w) = b(s),$$

where $w = k'z = s + k't$. Hence the split-holomorphic conditions for both $a(w)$ and $b(w)$, imply that $b(s) = As + B$ and $a(s) = C_1 cosh(s) + C_2 sinh(s)$, where A, B, C_1, C_2 are constants. As the immersion is conformal, $|\gamma'(s)|^2 = a^2(s)$, $C_1^2 - C_2^2 = A^2$, and so $a(s) = A\ cosh(s + \theta)$, $\theta \in \mathbb{R}$. Now the surface obtained is congruent to a piece of a Lorentzian surface with spacelike profile curve.

Finally we consider a minimal surface of revolution parametrized by the conformal immersion (14) written as

$$X(t, s) = (b(t) - \frac{a(t)s^2}{2}, a(t),\ sa(t)),$$

with respect to the null frame $\{L_1, L_2, L_3\}$. Then the x_1, x_2-plane intersects the surface orthogonally along the curve $\gamma(t) = X(t, 0) = (b(t), a(t), 0)$. We also obtain $N(t, 0) \times \gamma'(t) = (0, 0, \pm\sqrt{-2a'b'})$ along γ. Hence using representation (18) one gets that

$$Re\ b(z) = b(t) - \frac{a(t)s^2}{2}, \quad Re\ a(z) = a(t).$$

Since $-2a'(t)b'(t) = a^2$, it follows that

$$a(t) = -A(t + B), \quad b(t) = A(\frac{1}{6}t^3 + \frac{B}{2}t^2 + \frac{B^2}{2}t).$$

Applying the same variation of Lemma 4.1 used in Example 4.4, the surface is congruent to a piece of the Lorentzian parabolic catenoid. □

5 Minimal timelike ruled surfaces

In this section we study the minimal timelike ruled surfaces in $\mathbb{L}^3$. Using our split-complex Björling representation, we will give an alternative proof to the classification obtained by Woestijne in [26], where the graphics of those surfaces can be found.

We begin identifying the timelike ruled surfaces in $\mathbb{L}^3$, following Kim and Yoon in [16] and [17].

A ruled surface in $\mathbb{L}^3$ is defined by:

$$X(t,s) = \alpha(t) + s\beta(t), \quad t \in J_1,\ s \in J_2, \tag{19}$$

with J_1 and J_2 open intervals in $\mathbb{R}$ and where $\alpha = \alpha(t)$ is a curve in $\mathbb{L}^3$ defined on J_1 and $\beta = \beta(t)$ is a transversal vector field along α. The curve $\alpha = \alpha(t)$ is called the *base curve* and $\beta = \beta(t)$ the *director vector field*. In particular if β is constant, the ruled surface is called cylindrical, and non-cylindrical otherwise.

First, we suppose the base curve α is spacelike or timelike. In this case, the director vector field β can be naturally chosen to be orthogonal to α. In addition, since the ruled surface is timelike, we get different cases, depending on the causal character of the base curve α and the director vector field β, as follows:

Case 1 The base curve α is spacelike and β is timelike. In this case β' must be spacelike since it lies in $[\beta]^{\perp}$. This surface will be denoted by X^3_+.

Case 2 α is timelike and β' is non-null. In this case the director vector field β is always spacelike and the surface will be denoted by X^1_-.

Case 3 α is timelike and β' is lightlike. In this case the director vector field β is always spacelike and the surface will be denoted by X^2_-.

But if the base curve α is a lightlike curve and the vector field β along α is a lightlike vector field, then the ruled surface is called a *null scroll*. In particular, a null scroll with Cartan frame is said to be a B-scroll ([16], [17]). It is also a timelike surface.

We first give some examples of minimal timelike ruled surfaces.

Example 5.1. *(Timelike helicoid of the 3rd kind) Let*

$$\begin{cases} \gamma(t) = (t,0,0), \\ W(t) = \frac{1}{\sqrt{1+c^2t^2}}(0,-ct,1). \end{cases} \tag{20}$$

Changing the parameter to $ct = sinh(u)$, one gets

$$\gamma(u) = \frac{1}{c}(sinh(u),0,0), \qquad W(u) = \frac{1}{cosh(u)}(0,-sinh(u),1).$$

Using the timelike Björling representation, we obtain the solution of the timelike Björling problem with respect to the given data (γ, W), parametrized by

$$X(z) = \frac{1}{c}(sinh(u)cosh(v), v, sinh(u)sinh(v)), \quad z = u + k'v.$$

Example 5.2. *(Timelike helicoid of the 1st kind) Let us consider the data*

$$\begin{cases} \gamma(s) = (0, s, 0), \\ W(s) = \frac{1}{\sqrt{1-c^2s^2}}(cs, 0, 1). \end{cases} \tag{21}$$

Changing the parameter to $cs = sin(v)$*, one gets*

$$\gamma(v) = \frac{1}{c}(0, sin(v), 0), \qquad W(v) = \frac{1}{cos(v)}(sin(v), 0, 1).$$

Applying the Björling representation to the spacelike curve $\gamma(s)$*, the solution of the Björling problem is parametrized by:*

$$X(w) = \frac{1}{c}(u, sin(v)cos(u), sin(v)sin(u)), \quad w = v + k'u.$$

Example 5.3. *(Timelike helicoid of the 2nd kind) Let*

$$\begin{cases} \gamma(s) = (0, s, 0), \\ W(s) = \frac{1}{\sqrt{c^2s^2-1}}(1, 0, cs). \end{cases} \tag{22}$$

Changing the parameter to $cs = cosh(v)$*, one gets*

$$\gamma(v) = \frac{1}{c}(0, cosh(v), 0), \qquad W(v) = \frac{1}{sinh(v)}(1, 0, cosh(v)).$$

Hence the solution of the spacelike Björling problem is parametrized by:

$$X(w) = \frac{1}{c}(cosh(v)sinh(u), cosh(v)cosh(u), u), \quad w = v + k'u.$$

Example 5.4. *(Conjugate of Enneper's timelike surface of the 2nd kind) Let*

$$\begin{cases} \gamma(s) = (0, s, 0), \\ W(s) = \frac{1}{\sqrt{1-2sc}}(cs, 0, 1 - cs) \end{cases} \tag{23}$$

with $c \neq 0$ *and* $1-2sc > 0$*. Changing the parameter to* $v^2 = 1-2cs$*,* $v > 0$*, the solution of the Björling problem is parametrized by:*

$$X(w) = -\frac{1}{6c}(3u + 3uv^2 + u^3,\ 3v^2 + 3u^2 - 3,\ 3u - 3uv^2 - u^3), \quad w = v + k'u.$$

Example 5.5. *(B-scroll) Let $\alpha = \alpha(t)$ be a lightlike curve in $\mathbb{L}^3$ with Cartan frame $\{A, B, C\}$ i.e., A, B, C are vector fields along α in $\mathbb{L}^3$ satisfying the following conditions:*

$$\begin{array}{l} \langle A, A\rangle = \langle B, B\rangle = 0,\ \langle A, B\rangle = 1, \\ \langle A, C\rangle = \langle B, C\rangle = 0,\ \langle C, C\rangle = 1, \\ \alpha' = A, \quad C' = -aA - c(t)B, \end{array} \tag{24}$$

where a is a constant and $c(t)$ a nowhere vanishing function.

The surface defined by $X(t, s) = \alpha(t) + sB(t)$ is a timelike surface in $\mathbb{L}^3$ called a B-scroll. Following [26] a B-scroll is minimal if and only if it is flat, i. e., $B'(t) \equiv 0$ and $C' = -c(t)B$.

It is possible to study this surface in the context of timelike (spacelike) Björling problem. In fact, let us reparametrize it by taking the curve $\gamma(t) = \alpha(t) + s(t)B(t)$ with $s'(t) < 0$ $(s'(t) > 0)$. Then $\langle\gamma'(t), \gamma'(t)\rangle = s'(t) < 0 (> 0)$ and $\gamma(t)$ is a timelike (spacelike) curve. In order to simplify the computations, take $s(t) = -t$ $(s(t) = t)$. Now $\gamma(t) = \alpha(t) - tB(t)$ $(\gamma(t) = \alpha(t) + tB(t))$ and $W(t) = C(t)$ are the timelike (spacelike) Björling data. Using (24) we have $\langle\gamma'(t), W(t)\rangle = 0$. Using formula (6) we obtain the parametrization of the timelike (spacelike) Björling problem.

For instance, taking the lightlike curve $\alpha(t) = (\frac{-t^3}{6\sqrt{2}} - \frac{t}{\sqrt{2}}, \frac{-t^2}{2}, \frac{-t^3}{6\sqrt{2}} + \frac{t}{\sqrt{2}})$ and the lightlike vector field $B(t) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ we obtain the B-scroll

$$X(s, t) = \alpha(t) + sB(t) = (\frac{-t^3}{6\sqrt{2}} - \frac{t}{\sqrt{2}} + \frac{s}{\sqrt{2}}, \frac{-t^2}{2}, \frac{-t^3}{6\sqrt{2}} + \frac{t}{\sqrt{2}} + \frac{s}{\sqrt{2}}).$$

Using the reparametrization given above, we obtain the Björling data:

$$\gamma(t) = \left(\frac{-t^3}{6\sqrt{2}} - \sqrt{2}\, t, \frac{-t^2}{2}, \frac{-t^3}{6\sqrt{2}}\right),$$

and

$$W(t) = C(t) = A(t) \times B(t) = (\frac{t}{\sqrt{2}}, 1, \frac{t}{\sqrt{2}}).$$

After once more using formula (6), we obtain the surface parametrized by $X(t, s) = (X_1(t, s), X_2(t, s), X_3(t, s))$, where

$$\begin{array}{l} X_1(t, s) = -\dfrac{s^3 + 3s^2t + 3st^2 + t^3 + 12t}{6\sqrt{2}}; \\ X_2(t, s) = -\dfrac{(s + t)^2}{2}; \\ X_3(t, s) = -\dfrac{s^3 + 3s^2t + 3s\left(t^2 - 4\right) + t^3}{6\sqrt{2}}. \end{array} \tag{25}$$

Observe that setting $s = 0$ in the parametrization above we get the curve $\gamma(t)$, as expected.

Proof of Theorem 1.3 We follow closely the proof of Theorem 6.1 in [4] and consider all the possible cases, depending on the causal character of the base curve and director vector field.

Case 1 Let X be a non-cylindrical ruled surface of type X^3_+, parametrized by (19) such that $\langle \beta, \beta \rangle = -1$ and $\langle \alpha', \beta' \rangle = 0$. In this case α is the striction curve and the parameter is the arc-length on the curve β. We define the distribution parameter as

$$\lambda(t) = \frac{\langle \alpha' \times \beta, \beta' \rangle}{\langle \beta', \beta' \rangle}, \tag{26}$$

since $\langle \alpha' \times \beta, \beta' \rangle \neq 0$. In fact, $\alpha' \times \beta = \lambda \beta'$ and $X_t \times X_s = \lambda \beta' + s\beta' \times \beta$. Moreover $||X_t \times X_s||^2 = (\lambda^2 + s^2) \langle \beta', \beta' \rangle$. So the striction curve is a curve on the surface, obtained by setting $s = 0$. The Gauss map on the ruled surface is

$$N(t,s) = \frac{\lambda(t)\beta'(t) + s\beta'(t) \times \beta(t)}{\sqrt{\lambda^2(t) + s^2}\ ||\beta'(t)||}.$$

Hence $\langle N(t_o, 0), N(t_o, s) \rangle = \frac{1}{\sqrt{1+c^2 s^2}}$, where $t_o \in J_1$ and $c = \frac{1}{\lambda(t_o)}$. Let us assume $N(t_o, 0) = (0, 0, 1)$ and $L_s := X(t_o, s)$, $s \in J_2$ parametrizing the x_1-axis. Since $\langle N, N \rangle = 1$ we can assume

$$N(t_o, s) = \frac{(0, -cs, 1)}{\sqrt{1 + c^2 s^2}}$$

and that L_s is parametrized as $\gamma(s) = (s, 0, 0)$. So the minimal timelike ruled surface is the solution to the Björling problem with the data: $\gamma(s) = (s, 0, 0), W(s) = \frac{1}{\sqrt{1+c^2 s^2}}(0, -cs, 1)$. Hence the surface is a piece of the timelike helicoid of the 3rd kind, according to Example 5.1.

Case 2 In this case the vector β' is assumed to be non-null, so we must consider two subcases depending on whether β' is spacelike or timelike. In any subcase we have the parametrized surfaces given by (19), with $\langle \beta, \beta \rangle = 1$ and $\langle \alpha', \beta' \rangle = 0$ and α is the striction curve. We define the distribution parameter by (26) and conclude that $||X_t \times X_s||^2 = (\lambda^2 - s^2) \langle \beta', \beta' \rangle$. It follows that if β' is spacelike the striction curve α is on the surface. If β' is timelike, we cannot have $s = 0$.

a) If β' is spacelike, one gets that

$$N(t,s) = \frac{\lambda(t)\beta'(t) + s\beta'(t) \times \beta(t)}{\sqrt{\lambda^2(t) - s^2}\ ||\beta'(t)||}.$$

Hence $\langle N(t_o,0), N(t_o,s)\rangle = \frac{1}{\sqrt{1-c^2s^2}}$ where $t_o \in J_1$ and $c = \frac{1}{\lambda(t_o)}$. Now we assume $N(t_o,0) = (0,0,1)$ and that $L_s := X(t_o,s)$ for $s \in J_2$ parametrizes the x_2-axis. Since $\langle N, N\rangle = 1$, we can assume

$$N(t_o,s) = \frac{(cs,0,1)}{\sqrt{1-c^2s^2}}$$

and L_s is parametrized by $\gamma(s) = (0,s,0)$. Hence the timelike minimal ruled surface is the solution to the Björling problem with the data: $\gamma(s) = (0,s,0), W(s) = \frac{1}{\sqrt{1-c^2s^2}}(cs,0,1)$. According to Example 5.2, the ruled surface is a piece of the timelike helicoid of the 1st kind.

b) If β' is timelike, one gets that

$$N(t,s) = \frac{\lambda(t)\beta'(t) + s\beta'(t)\times\beta(t)}{\sqrt{s^2-\lambda^2(t)}\sqrt{|\,\langle\beta'(t),\beta'(t)\rangle\,|}}.$$

Hence $\langle N(t_o,s_o), N(t_o,s)\rangle = \frac{-\lambda^2(t_o)+s_os}{\sqrt{s_o^2-\lambda^2(t_o)}\sqrt{s^2-\lambda^2(t_o)}}$, where $t_o \in J_1$ and $s_o \in J_2$ fixed. Now we assume $N(t_o,s_o) = (0,0,1)$ and $L_s := X(t_o,s)$, $s \in J_2$, parametrizes the x_2-axis. Since $\langle N, N\rangle = 1$ we can assume

$$N(t_o,s) = \frac{1}{\sqrt{s_o^2-\lambda^2(t_o)}\sqrt{s^2-\lambda^2(t_o)}}(|\lambda(t_o)|(s_o-s), 0, -\lambda^2(t_o)+s_os)$$

and L_s is parametrized by $\gamma(s) = (0,s,0)$.

Now we take $s_o = \sqrt{2}|\lambda|$ and, substituting in $N(t_o,s)$, one gets

$$N(t_o,s) = \frac{1}{\sqrt{s^2-\lambda^2}}(-s+\sqrt{2}|\lambda|, 0, -|\lambda|+\sqrt{2}s),$$

where λ is calculated at t_o. Now composing with the orthogonal transformation of $\mathbb{L}^3$, written in the canonical coordinates as:

$$\begin{pmatrix} -\sqrt{2} & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -\sqrt{2} \end{pmatrix}$$

we obtain

$$N(t_o,s) = -\frac{(1,0,cs)}{\sqrt{c^2s^2-1}},$$

where $c = \frac{1}{|\lambda|}$. Hence the timelike ruled surface is the solution of the Björling problem with respect to the data: $\gamma(s) = (0,s,0), W(s) = -\frac{1}{\sqrt{c^2s^2-1}}(1,0,cs)$. According to Example 5.3, it is a piece of the timelike helicoid of the 2nd kind.

Case 3 Let X be a non-cylindric ruled surface of type X^2_-, which may be parametrized by (19) where $\langle \alpha', \alpha' \rangle = -1$, $\langle \alpha', \beta \rangle = 0$, $\langle \beta, \beta \rangle = 1$ and $\langle \beta', \beta' \rangle = 0$, $(\beta' \neq 0)$.

Consider the non-zero smooth functions

$$-||X_t||^2(t_o, s) = 1 - 2s \left\langle \alpha'(t_o), \beta'(t_o) \right\rangle$$

and $\langle \beta' \times \beta, \alpha' \times \beta \rangle (t_o) = - \langle \beta', \alpha' \rangle (t_o)$. As $\beta \times \beta' = \beta'$, we have

$$N(t_o, s) = \frac{1}{\sqrt{1 - 2sc}} (\alpha' \times \beta - s\beta')(t_o),$$

where $c = \langle \alpha'(t_o), \beta'(t_o) \rangle$. Moreover $\langle N(t_o, s), N(t_o, 0) \rangle = \frac{1-sc}{\sqrt{1-2sc}}$. So, one may assume that $N(t_o, 0) = (0, 0, 1)$ and $X(t_o, s)$, $s \in J_2$, parametrizes the x_2-axis. Since $\langle N, N \rangle = 1$, it follows that

$$N(t_o, s) = \frac{1}{\sqrt{1 - 2sc}} (cs, 0, 1 - sc).$$

So this timelike ruled surface is a solution of the spacelike Björling problem with respect to the data: $\gamma(s) = (0, s, 0), W(s) = \frac{1}{\sqrt{1-2sc}}(cs, 0, 1 - cs)$. It corresponds to a piece of the conjugate of Enneper's timelike surface of the 2nd kind, just as in Example 5.4.

B-scrolls Each of the cases above has essentially one surface, but the class of B-scrolls is larger, so we must use a different proof, which is similar to the proof found in [26]. We will find a simple representation for the B-scroll using the Björling procedure.

Begin with a timelike ruled surface $f(t, s) = \alpha(s) + t\beta(s)$, where $\langle \alpha'(s), \alpha'(s) \rangle = 0$ and $\langle \beta(s), \beta(s) \rangle = 0$. This gives:

$$\begin{aligned} f_s &= \alpha' + t\beta' \\ f_t &= \beta. \end{aligned}$$

Since the surface is timelike we must have $\langle \alpha'(s), \beta(s) \rangle \neq 0$. We can first find a multiple of $\beta(s)$ so that $\langle \alpha'(s), \beta(s) \rangle = 1$. We construct a pseudo-orthonormal frame along $\alpha(s)$ using $\{\alpha', \beta, m = \alpha' \times \beta\}$. From the inner products we see there are functions $\{x_1(s), x_2(s), x_3(s)\}$ along the curve $\alpha(s)$ so that

$$\alpha'' = x_1\alpha' + x_3 m \tag{27}$$

$$\beta' = -x_1\beta + y_3 m \tag{28}$$

$$m' = -y_3\alpha' - x_3\beta. \tag{29}$$

The surface unit normal is $f_s \times f_t = m + t\beta' \times \beta$. We note that $\beta' \times \beta = y_3 m \times \beta$ is a multiple of β, say $d(s)\beta$. Thus the surface normal is $N(s, t) = m + td(s)\beta$. $N_s = m' + td'\beta + td\beta' = -y_3\alpha' - x_3\beta + td'\beta + td\beta'$ and must be a linear combination

of f_s and f_t. Thus $d = -y_3$. The shape operator has the form: $\begin{pmatrix} y_3 & 0 \\ * & y_3 \end{pmatrix}$, so by minimality $y_3 = 0$. Finally we see that α is a pre-geodesic, and, by reparametrizing the curve we get $x_1 = 0$. Thus our surface is a B-scroll as in Example 5.5. We can find its (simple) spacelike Björling represention using $\gamma(s) = \alpha(s) + s\beta$ and $W(s) = m(s)$. $W \times \gamma' = (\alpha' \times \beta) \times (\alpha' + \beta) = (\alpha' - \beta)$.

$$\begin{aligned} X(w) &= Re\left(\gamma(w) + k' \int_{s_o}^{w} (\alpha' - \beta) d\zeta\right) \\ &= Re(\alpha(w)) + Im(\alpha(w)) + Re(\beta)(s-t) + Im(\beta)(t-s). \ \square \end{aligned}$$

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