METHOD OF GENERATING STATIONARY
EINSTEIN-MAXWELL FIELDS*

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We describe a method of generating stationary asymptotically flat solutions of the
Einstein-Maxwell equations starting from a stationary vacuum metric. As a simple example,
we derive the Kerr-Newman solution.

Recently a number of new stationary solutions was found [1–3] and new methods
of generating stationary Einstein–Maxwell fields were discovered [4–6]. In this note I
would like to describe another method of generating asymptotically flat solutions of
the Einstein–Maxwell equations starting from stationary vacuum metrics.

The general stationary metric can be written in the form

$$ds^2 = f(dt + w_i dx^i)^2 - f^{-1} h_{ij} dx^i dx^j,$$

where $i, j = 1, 2, 3$ and the function $f$, $w_i$ and $h_{ij}$ do not depend on $t$. This notation closely
follows that of Kinnersley [6].

The electromagnetic field is very conveniently described by the complex electromag­
netic tensor $\mathcal{F}_{\mu\nu}$

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + i*F_{\mu\nu},$$

where $F_{\mu\nu}$ is the Maxwell tensor and $*F_{\mu\nu}$ is its dual. The source free Maxwell equations
could be written as

$$\mathcal{F}_{[\mu\nu;\rho]} = 0,$$

which assures the existence of the electromagnetic potential $a_\mu$ such that

$$\mathcal{F}_{\mu\nu} = a_{\nu;\mu} - a_{\mu;\nu}.$$
The coupled Einstein–Maxwell field equations may be written as equations in a 3-space \( H \) with metric tensor \( h_{ij} \). Let \( \nabla \) denote the covariant derivative in \( H \). We define a twist vector

\[
\tilde{\tau} = f^2 \nabla \times \tilde{\omega} + i(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*),
\]

where \( \Psi \) is a complex function describing uniquely the electromagnetic field and * denotes complex conjugation.

Using part of the Einstein equations,

\[
G_{jo} = 8\pi T_{jo},
\]

one can show that

\[
\nabla \times \tilde{\tau} = 0,
\]

implying the existence of a real scalar potential \( \chi \) such that

\[
\tilde{\tau} = \nabla \chi.
\]

Let us now define a complex scalar potential for gravitation

\[
\varepsilon = f - \Psi \Psi^* + i\chi.
\]

Given \( h_{ij}, \varepsilon \) completely determines the metric and hence the gravitational field.

The Maxwell equations (3) and the remaining Einstein equations may now be written in terms of \( \varepsilon \) and \( \Psi \). They assume the form

\[
f \nabla^2 \varepsilon = (\nabla \varepsilon + 2\Psi^* \nabla \Psi) \nabla \varepsilon,
\]

\[
f \nabla^2 \Psi = (\nabla \varepsilon + 2\Psi^* \nabla \Psi) \nabla \Psi.
\]

The curvature tensor of \( H \) is also determined by \( \varepsilon \) and \( \Psi \) through the relation,

\[
f^2 R_{kj}^{(3)} = \frac{1}{2} \varepsilon_{,i(j}^* \varepsilon_{,k)}^* + \Psi \varepsilon_{,i(j}^* \varepsilon_{,k)}^* + \Psi^* \varepsilon^*_{,i(j} \varepsilon_{,k)} - (\varepsilon + \varepsilon^*) \varepsilon^*_{,i(j} \varepsilon_{,k)}.
\]

The field equations in empty space where \( \Psi \) vanishes can be compactly written in the form

\[
(\xi^* \xi - 1) \nabla^2 \xi = 2\xi^* \nabla \xi \cdot \nabla \xi,
\]

where \( \xi \) is a complex Ernst potential defined by the relation,

\[
\frac{\xi - 1}{\xi + 1} = f + i\chi.
\]

Equation (13) possesses a number of invariant properties. Taking the complex conjugate, we see that if \( \xi \) is a solution of (13), so is \( \xi^* \). Ehlers [7] some time ago noticed that one can replace \( \xi \) by \( e^{i\chi} \xi \) without altering the form of the equation. It is also invariant with respect to the following fractional transformation,

\[
\xi \rightarrow \frac{(1 + \beta) \xi + \beta^*}{1 + \beta^* + \beta \xi},
\]
where $\beta$ is an arbitrary complex constant. When $\beta = -1$ (15) reduces to the inversion transformation $\xi \to \xi^{-1}$.

We shall now show that the Ernst potential $\xi$ for the stationary vacuum spacetime could be treated as a complex electromagnetic potential in some stationary electrovac gravitational field. Let us assume that $\psi = \sqrt{\kappa} \xi$ where $\xi$ is any solution of (13), $\kappa$ is a positive constant and

$$f = \kappa(\xi \xi^* - 1), \quad \chi = \alpha,$$

where $\alpha$ is a real constant. In this case $\varepsilon = -\kappa + i\alpha = \text{const.}$. It is now apparent that Equation (10) is trivially satisfied and Equation (11) reduces to Equation (13). Therefore (16) describes a solution of coupled Einstein–Maxwell equations. In order to assure the asymptotic flatness of the gravitational field, $f$ should tend to 1 at spatial infinity, implying that $\xi \to \sqrt{1 + 1/\kappa}$ asymptotically. Using the transformation (15), we can always satisfy this condition.

The remaining metric coefficients one obtains from Equation (12), which now simplifies to

$$f^2 R_{ik}^{(3)} = 2\kappa \Psi_{,(j} \Psi^*_{,k)};$$

(17)

and Equation (5), which now reduces to

$$f^2 \nabla \times \bar{w} = i(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi).$$

(18)

Solutions of those equations provide us with $w_i$ and $h_{ij}$.

As an example, let us consider the Kerr metric, which is described by the complex function $\xi = px - igy$, where $x$ and $y$ are oblate spheroidal coordinates and $p^2 + g^2 = 1$. Using the transformation (15) with $\beta = \kappa \pm \sqrt{\kappa(\kappa + 1)}$ we obtain

$$\xi = \frac{(1 + \beta)(px - igy) + \beta}{1 + \beta + \beta(px - igy)},$$

(19)

which satisfies the required boundary condition at $x \to \infty$.

The metric we shall take in the form,

$$ds^2 = f(dt + wd\varphi)^2 - f^{-1} \left[ e^{2\gamma} \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2)d\varphi^2 \right],$$

(20)

where

$$f = \frac{p^2 x^2 + g^2 y^2 - 1}{[px + 1 + \beta^{-1}]^2 + g^2 y^2}.$$ 

(21)

Equation (17) leads to

$$e^{2\gamma} = p^2 x^2 + g^2 y^2 - 1,$$

(22)

and from (18) we obtain

$$w = -\frac{g(1 - y^2) [2(1 + \beta^{-1})px + 1 + (1 + \beta^{-1})^2]}{p(p^2 x^2 + g^2 y^2 - 1)}.$$ 

(23)
Introducing the spherical coordinates $r$ and $\theta$, which are related to $x$ and $y$ by

$$x = \frac{1 + \beta^{-1}}{mp} (r - m), \quad y = \cos \theta,$$

and identifying $g^2$ with $(1 + \beta^{-1})^2 a^2/m^2$, where $a$ is the Kerr parameter, we obtain the Kerr-Newman solution with $e^2 = m^2(1 + 2\beta)/(1 + \beta)^2$.

This procedure, when applied simultaneously with Kinnersley's method, leads to a new class of exact, stationary, asymptotically flat Einstein-Maxwell solutions. It also throws some light on the structure of the space of stationary Einstein-Maxwell solutions and indicates that there is a new relation between vacuum stationary solutions and Einstein-Maxwell solutions.

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REFERENCES