APPLIED MECHANICS
DYNAMICS

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PREFACE

The present edition incorporates a number of revisions and additions which should improve its usefulness as a textbook without changing the basic organization or the general philosophy of presentation of the subject matter. The experience of the past few years at the California Institute of Technology and other schools indicates that the book has been useful to engineering students who wish to prepare for more advanced studies and applications of dynamics, and hence a new edition was felt to be justified.

Among the additions and modifications the following may be mentioned to indicate the scope of the revision. The section on dimensional analysis has been rewritten and a brief treatment of the theory of models has been added. The section on impact problems has been revised, and a more extensive treatment of variable mass systems has been included. A more general discussion of the moment of momentum equations for systems of particles has been added, and the general momentum and energy equations for rigid bodies have been more completely developed. The discussion of rotation about a fixed point and gyroscopic motion has been expanded and somewhat more complex systems have been considered, including problems on the stability of rolling motion. The problem of longitudinal waves in an elastic bar is discussed, and a comparison is made between wave propagation techniques and vibration methods for such problems. The discussion of generalized coordinates and Lagrange's equations has been revised, and a general treatment of the problem of small oscillations of a conservative system has been added. The sections on the Calculus of Variations and Hamilton's Principle have been rewritten with some expansion.

Over one hundred new problems have been added to increase the total number to some four hundred. All of the new problems have been thoroughly tested in classroom use. The number of illustrative examples has been increased and many of the original examples have been modified.

As in the first edition, the main emphasis of the book is on particle and rigid-body dynamics, although some other aspects of the subject have been included to show how the methods of classical mechanics are applied to the various branches of engineering science. Some of these topics, such as fluid dynamics and the kinetics of gases, have been treated in a very brief fashion. Although the student will make a more complete analysis of these subjects in specialized courses, it is believed that the brief discussions will help him to acquire a broader view of the applied sciences. In all such instances care has been taken to use methods that can be extended later for more complete treatments, and the student has been informed of the limitations of the analyses.

As a textbook the main emphasis has been on method and on development of fundamental principles. The problems form an essential part of the presentation, and important conclusions are sometimes given in problems and illustrative examples. The student should examine such problems and note the results, even if the details of the proofs are not carried through.

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Pasadena, California
May, 1959

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Chapter 1

THE GENERAL PRINCIPLES OF DYNAMICS

. . . the whole burden of philosophy seems to consist in this, from the phenomena of motions to investigate the forces of nature, and from these forces to demonstrate the other phenomena.—I. Newton, *Principia Philosophiae* (1686).

The science of mechanics has as its object the study of the motions of material bodies, and its aim is to describe the facts concerning these motions in the simplest way. From this description of observed facts, generalizations can be formulated which permit valid predictions as to the behavior of other bodies.

The motions occurring in nature are the result of interactions between the various bodies which make up the system under consideration. That portion of the subject of mechanics which describes the motion of bodies, without reference to the causes of the motion, is called *kinematics*, while that portion which studies the relationship between the mutual influences and the resulting motions is called *kinetics*. These two subjects are usually combined under the name *dynamics*, and it is this general problem that is to be treated in this book.

1.1 The Laws of Motion. The principles of dynamics are founded upon extensive experimental investigations. The first noteworthy experiments were performed by Galileo (1564–1642). Other investigators followed Galileo, among them being Newton (1642–1727), who, after carrying out a large number of experiments, formulated the statements which are now known as Newton's Laws of Motion:

(1) Every body persists in a state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

(2) The time rate of change of momentum is equal to the force producing it, and the change takes place in the direction in which the force is acting:

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v})\text{; or, for constant } m, \ \boldsymbol{F} = m\boldsymbol{a}$$

(3) To every action there is an equal and opposite reaction, or the mutual actions of any two bodies are always equal and oppositely directed.

These statements may be regarded partly as definitions and partly as experimental facts. As a description of the behavior of bodies of ordinary size moving with velocities which are small compared with the velocity of light, these statements have remained to this day the most convenient expression of the fundamental principles of dynamics.

1.2 Definitions. The intuitive concepts which arise concerning such basic quantities of dynamics as force, mass, and time must be put into a precise form before they can serve as a foundation for the development of the subject. The following definitions prescribe the sense in which these words will be used in this book.*

Force and Mass. The primary objective of the science of mechanics is the study of the interactions which occur between material bodies. These interactions are of various types and might be, for example, impacts, electrical or gravitational attractions, mechanical linkages, etc. Experience shows that during these interactions, the velocities of the interacting bodies are changed. We define *force*, by Newton's first law, as an action which tends to change the motion of a body. The fact that forces arise from mutual interactions and thus occur in equal and opposite pairs forms

* A considerable difference of opinion has existed amongst various eminent authorities as to the most logical form in which to cast the basic definitions and principles of classical mechanics. Controversy has continued to the present day, and books and papers appear regularly which aim to give a final clarification to the matter. For a critical survey of this aspect of the subject, the classic book of E. Mach, *Science of Mechanics* (1893), is still of great interest.

the empirical content of Newton's third law. The concept of force is made quantitatively precise by the definition that a unit force produces a unit acceleration of a specified standard body.

The *mass* of a body may now be defined by Newton's second law as the ratio of the force acting on the body to the resulting acceleration. By international agreement, the unit of mass is defined as the mass of a particular platinum-iridium cylinder, called the *international prototype kilogram,* which is in the possession of the International Committee of Weights and Measures at Sèvres, France.

The force exerted upon a body by the earth's gravitational field is called the *weight* of the body. The weight of a body is thus variable, depending upon the location of the body with respect to the earth. The magnitude of the earth's gravitational field is specified by the *acceleration of gravity* (g) which is the acceleration of an otherwise unrestrained body attracted to the earth. The gravitational acceleration has been determined experimentally and is given at a latitude ϕ and an elevation h ft by the empirical formula:

$$g = 32.089(1 + 0.00524 \sin^2 \phi)(1 - 0.000000096h) \text{ ft/sec}^2$$

At sea level the maximum variation of g with latitude is of the order of 0.5%, while the variation from sea level to an altitude of 24,000 ft is of the order of 0.25%. In engineering it is customary to use a constant value of g equal to 32.2 ft/sec^2.

The mass of any body can be determined by comparing the body with the standard kilogram. In practice the mass of a body is usually determined by means of the ordinary balance. The unknown mass m_1 is balanced with a known mass m_2 so that the weights W_1 and W_2 are equal. Since $m_1 = W_1/g$ and $m_2 = W_2/g$, it follows that $m_1 = m_2$.

Experiment shows that for the bodies and motions with which the engineer is usually concerned, the mass of a body is a constant within the limits of accuracy of measurement. Experiments in atomic physics, however, show that at sufficiently high velocities the mass of a particle is not a constant, but, as predicted by the theory of relativity, is given by

$$m = \frac{m_0}{\sqrt{1 - (v/c)^2}}$$

where c is the velocity of light, v the velocity of the particle, and m_0

the mass of the particle at zero velocity.* Because of the large magnitude of c, it is impossible to detect the variability of mass at even the highest velocities encountered in engineering practice. It is important to note that Newton's Second Law refers to a specific body and does not refer directly to a system which is losing or gaining material.

A number of different standards of mass have been defined in terms of the prototype kilogram. In the United States the *pound-mass avoirdupois* has been defined legally by Congress as the 1/2.2046 part of the international prototype kilogram. The *pound-force* is defined as the gravitational force exerted on a standard pound-mass when g has the "standard" value of 32.174 ft/sec^2. In engineering the pound-force is taken as the unit of force, but the pound-mass is *not* taken as the unit of mass. The engineering unit of mass is that mass which is given an acceleration of 1 ft/sec^2 by a force of 1 pound-force. This unit of mass is called a *slug* and is equal to 32.174 pound-mass. Since the multiplicity of standards of force and mass sometimes leads to confusion, a summary of definitions of a number of the commonly encountered terms is given in Appendix II.

Time. The unit of time is the *second*, which is defined as the 1/86,400 part of a mean solar day. The mean solar day is the yearly average of the time intervals between successive transits of the sun past a meridian of the earth.

Length. The international standard of length is the *standard meter*, which is defined as the distance, at zero degrees centigrade, between two lines on a platinum-iridium bar in the possession of the International Committee of Weights and Measures. The *United States Yard* is defined legally by act of Congress as the 3600/3937 part of the standard meter, and the *foot* is defined as one-third of a yard.

1.3 Frames of Reference. In the preceding discussion of acceleration, force, and mass, it has been implied that there exists a frame of reference with respect to which measurements can be made. In engineering, unless stated to the contrary, it is always understood that measurements are to be made with respect to a coordinate

* Some physicists prefer to suppose that the mass is constant and that at high velocities a suitable transformation of the space-time coordinates must be introduced. The resulting equations of motion will of course be the same as those obtained on the above supposition of varying mass. (See refs. 6 and 16.)

system which is fixed at the earth's surface. In astronomy, distances may be measured with respect to certain stars. In any event it is always necessary to perform measurements in some coordinate system which is located with respect to some physical object.

The fact that the coordinate system may be located in various ways naturally raises the question as to the effect its position might have on the equations of motion and the solution of problems. It is possible to select a coordinate system with respect to which it is not permissible to write simply $F = ma$. An example is a coordinate system fixed with respect to an airplane which is making a turn. It then would be necessary to apply a force to a body in order to keep its observed acceleration equal to zero. The equation $F = ma$ would not give a correct description of the motion if the acceleration were measured with respect to an accelerating coordinate system, for it would be necessary to add correction terms which take into account the motion of the coordinate system. It is clearly an advantage to locate the coordinate system so that such correction terms are not required. It was formerly customary to define an absolute space and to refer all measurements to a coordinate system fixed with respect to absolute space. It is now recognized that all measurements are relative, and the concepts of absolute space and time have been discarded. The location of the coordinate system is now based upon experience. We locate the system so that the equation $F = ma$ describes the motion within the required limits of accuracy.

The difficulties associated with the ideas of absolute space, absolute time, and the location of coordinate systems might seem to be chiefly problems of philosophy. It was just these difficulties, however, which led to the formulation of the Theory of Relativity, which has been of such importance in the development of modern physics.

1.4 Fundamental and Derived Units. A physical quantity can be measured by a comparison of a sample with a known amount of the same quantity. The known amount which is taken as a reference is called a unit, and the specification of any physical quantity requires both an indication of the units used and a numerical value which gives the number of units contained in the sample.

For the measurement of the physical quantities with which we shall be concerned in mechanics, it is found convenient to use three independent units. In engineering it is customary to use the unit of *length* (foot), the unit of *force* (pound), and the unit of *time* (second), as the three *fundamental units.* All other quantities can be expressed in terms of the three fundamental units. The unit of acceleration, for example, is written (ft/sec^2), and the unit of mass, which, from the equation $F = ma$ is seen to be equal to force divided by acceleration, is written (lb sec^2/ft). Such units are called *derived units* to indicate the fact that they are expressed by combinations of the fundamental units. As a matter of convenience the derived units are sometimes given special names. For example, the foregoing derived unit of mass is called a slug. Many of the derived units have no special names, however, velocities being referred to as so many ft/sec, accelerations as so many ft/sec^2, etc.

The system of units described in the preceding paragraph, in which length-force-time are the fundamental units, is called the (L-F-T) or *gravitational* system of units. In physics it is customary to take length-mass-time as the fundamental units. This system is called the (L-M-T) or *absolute* system of units. The words "gravitational" and "absolute" are merely the names of the systems, and it should not be inferred that there is anything absolute about a system of units.

The (L-M-T) system differs from the (L-F-T) system only in that the unit of mass instead of the unit of force is taken as the third fundamental unit. In the (L-M-T) system the fundamental units are named the centimeter, the gram, and the second. The derived unit of acceleration is cm/sec^2, and the unit of force is that force which gives a mass of one gram an acceleration of one cm/sec^2. This derived unit of force, the gram cm/sec^2, is called the dyne.

1.5 Dimensions. Quantities which are measured in terms of derived units are often called *secondary quantities,* as distinguished from *primary quantities,* which are measured in fundamental units. The measured values of such secondary quantities can always be expressed as products and quotients of certain numbers, as in the following examples:

$$\text{velocity} = v = \frac{n_1 \text{ ft}}{n_2 \text{ sec}} = \frac{(n_1)(12 \text{ in.})}{(n_2)(\frac{1}{60} \text{ min})} = \left(\frac{n_1}{n_2}\right)\left(\frac{a}{b}\right)\left[\frac{L}{T}\right]$$

$$\text{mass} = m = \frac{n_1 \text{ lb}}{n_2 \text{ ft}/(n_3 \text{ sec})^2} = \frac{(n_1)(16 \text{ oz})}{(n_2)(12 \text{ in.})}(n_3 \cdot \tfrac{1}{60} \text{ min})^2$$

$$= \left(\frac{n_1 n_3{}^2}{n_2}\right)\left(\frac{ac^2}{b}\right)\left[\frac{FT^2}{L}\right]$$

in which the constants a, b, c refer to the size of the units, L, F, T, and the constants n_1, n_2, n_3, refer to the number of such fundamental units contained in a particular sample of the quantity.*

A consequence of the particular form of the secondary quantities is the fact that the ratio of two measured values of a secondary quantity is independent of the sizes of the fundamental units used, for if:

$$m_1 = \left(\frac{n_1 n_3^2}{n_2}\right)\left(\frac{ac^2}{b}\right)\left[\frac{FT^2}{L}\right]$$

$$m_2 = \left(\frac{N_1 N_3^2}{N_2}\right)\left(\frac{ac^2}{b}\right)\left[\frac{FT^2}{L}\right]$$

then:

$$\frac{m_1}{m_2} = \frac{n_1 n_3^2 N_2}{N_1 N_3^2 n_2}$$

since the constants a, b, c which define the size of the fundamental units cancel. This type of independence is of course very convenient for any practical system of units which is to be used to describe physical situations.

Generalizing from the above examples, we may say that a secondary quantity q will have the form:

$$q = (n_1 a)^\alpha (n_2 b)^\beta (n_3 c)^\gamma [F^\alpha L^\beta T^\gamma]$$

where $[F^\alpha L^\beta T^\gamma]$ is the derived unit, and the exponents α, β, γ are called the *dimensions of q*. Some common secondary quantities and their dimensions are listed in the following tables.

Quantity	*(L-F-T) System*	*(L-M-T) System*
Length	L	L
Time	T	T
Force	F	MLT^{-2}
Mass	FT^2L^{-1}	M
Velocity	LT^{-1}	LT^{-1}
Acceleration	LT^{-2}	LT^{-2}

* So-called *dimensional constants* are sometimes encountered. For example, the area of a rectangle could be written as $A = clw$, where A is the area in acres, l the length in feet, w the width in feet, and $c = 2.296 \times 10^{-5}$ acres/ft². By a proper selection of units, the dimensional constant can always be given a value of unity; in the above example, if the unit of area is square feet, $c = 1$. It will be noted that secondary quantities, according to the definition given above, involve systems of units for which $c = 1$.

Quantity	*(L-F-T) System*	*(L-M-T) System*
Area	L^2	L^2
Volume	L^3	L^3
Density	FT^2L^{-4}	ML^{-3}
Momentum	FT	MLT^{-1}
Work, Energy, Heat	FL	ML^2T^{-2}
Power	FLT^{-1}	ML^2T^{-3}
Pressure, Stress	FL^{-2}	$ML^{-1}T^{-2}$
Moment	FL	ML^2T^{-2}
Viscosity	$FL^{-2}T$	$ML^{-1}T^{-1}$
Angle	dimensionless	

The number of fundamental units is to a certain extent arbitrary. In the field of thermodynamics a fourth fundamental unit is commonly added, which is usually taken as temperature θ. In the field of electricity, a fourth fundamental unit is also commonly added, which is often taken as the electric charge Q. The dimensions of some common thermodynamic and electrical quantities follow:

Quantity	*(L-F-T) System*	*(L-M-T) System*
Temperature	θ	θ
Thermal Conductivity	$FT^{-1}\theta^{-1}$	$MLT^{-3}\theta^{-1}$
Entropy	$FL\theta^{-1}$	$ML^2T^{-2}\theta^{-1}$
Gas Constant	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$
Electric Charge	Q	Q
Current	QT^{-1}	QT^{-1}
Voltage	$FQ^{-1}L$	$MQ^{-1}L^2T^{-2}$
Resistance	$FQ^{-2}LT$	$MQ^{-2}L^2T^{-1}$
Inductance	$FQ^{-2}LT^2$	$MQ^{-2}L^2$
Capacitance	$F^{-1}Q^2L^{-1}$	$M^{-1}Q^2L^{-2}T^2$

We have shown above that secondary quantities having the form of products of powers of the primary quantities have the proper independence of unit size. The converse statement is also true, and it can be shown in general terms that the secondary quantity must have this particular form. Consider, for example, two secondary quantities:

$$q_1 = f(n_1a, n_2b, n_3c)$$
$$q_2 = f(N_1a, N_2b, N_3c)$$

and impose the condition that:

$$\frac{\partial\left(\dfrac{q_1}{q_2}\right)}{\partial a} = 0, \text{ etc.}$$

The solution of these equations will give the form of the function which will satisfy the condition of independence of size of units and which will be found to involve products of powers as indicated above.

1.6 Dimensional Homogeneity. Every equation that describes a physical process should be dimensionally correct, that is, the dimensions on one side of the equation should be the same as the dimensions on the other side. In the dimensional equations that follow, we shall indicate the fact that it is the dimensions only which are equated by enclosing the dimensional expression in square brackets. For example, the equation for the radial force F_r, acting upon a mass m which moves in a circle of radius r with velocity v is:

$$F_r = \frac{mv^2}{r}$$

Dimensionally,

$$F_r = [F]$$

$$\frac{mv^2}{r} = \frac{[FT^2L^{-1}][LT^{-1}]^2}{[L]} = [F]$$

The equation is thus dimensionally correct. If the dimensions of such an equation do not check, then we know that an error of some kind exists.

As a second example, consider the equation describing the velocity of a particle falling through a resisting medium:

$$v = \frac{W}{k}(1 - e^{-\frac{k}{m}t})$$

where:

v = velocity $[LT^{-1}]$

W = weight of the body $[F]$

k = resistance factor $[FTL^{-1}]$

m = mass of the body $[FT^2L^{-1}]$

t = time $[T]$

The dimensions of $\left(\frac{k}{m}t\right)$ are:

$$\frac{[FTL^{-1}]}{[FT^2L^{-1}]}[T] = [F^0T^0L^0]$$

and the dimensions of $\left(\frac{W}{k}\right)$ are:

$$\frac{[F]}{[FTL^{-1}]} = [LT^{-1}]$$

Thus the equation is dimensionally correct.

In the second example it will be noted that the exponent of the term $e^{-(k/m)t}$ is dimensionless. In any expression of the type $\log x$, e^x, $\sin x$, $\cosh x$, etc., the argument x can always be written as a dimensionless quantity. This follows from the fact that dimensional homogeneity of an equation involving transcendental functions can be maintained only if the arguments are dimensionless. This may be seen, for example, by examining the series expansion for a typical function of this kind:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \cdots$$

Having the arguments in a dimensionless form has the advantage that the terms are independent of the size of the particular fundamental units used.

It is always possible to write a dimensionally homogeneous equation in the form:

$$\pi_1 = \phi(\pi_2, \pi_3, \ldots)$$

where $\pi_1, \pi_2, \ldots$ represent dimensionless quantities and ϕ indicates that π_1 can be written as a function of the other dimensionless quantities. For example, taking the equation used in the second example above,

$$v = \frac{W}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

we see that this can be put in the form:

$$\left(\frac{vk}{W}\right) = 1 - e^{-\frac{k}{m}t} = \phi\left(\frac{k}{m}t\right)$$

where the quantities $\left(\frac{vk}{W}\right)$ and $\left(\frac{k}{m}t\right)$ are both dimensionless.

1.7 Dimensional Analysis. A study of the dimensions of the quantities involved in a physical problem may yield some useful information about the form of the solution. The brief treatment of this subject which follows is intended to indicate the application of the method to mechanics problems, and it should be realized that a rigorous development of the various theorems and a thorough examination of the foundations and limitations of the method would require a much more extended discussion.*

An example has been given of the way in which a dimensionally homogeneous equation can be expressed in terms of dimensionless parameters. We shall now indicate the principle behind this transformation.

If the variables which enter into a particular problem are $(v_1, v_2 \ldots v_n)$, the solution of the problem may be written as:

$$f(v_1, v_2, \ldots v_n) = 0$$

We shall show that under certain conditions this equation can be written as:

$$\phi(\pi_1, \pi_2, \ldots) = 0$$

where the $\pi_1, \pi_2, \ldots$ terms are dimensionless, and we shall show how to determine the number and the composition of these π-terms.

A general requirement which we shall impose on our dimensional system is that the function $f(v_1, v_2, \ldots v_n)$ should remain unchanged as the sizes of the fundamental units entering into the v's are changed. In mechanics the equations describing the basic laws are by definition independent of the size of the fundamental units, and it thus follows that any equations derived from the basic laws will also have this same independence. Equations that satisfy this condition are called *physical equations.*

The variables entering the problem can be written in terms of their dimensions as:

$$\begin{aligned}[v_1] &= [L^{\alpha_1}F^{\beta_1}T^{\gamma_1}] \\ [v_2] &= [L^{\alpha_2}F^{\beta_2}T^{\gamma_2}] \\ &\cdot \\ &\cdot \\ &\cdot\end{aligned}$$

* The classic book on the subject is P. W. Bridgman, *Dimensional Analysis* (1922). It is interesting to note that this subject has led to many differences of opinion amongst eminent scientists and engineers. A good summary of many of these discussions is given in the book by C. M. Focken, *Dimensional Methods*, and the other books mentioned in the bibliography may be consulted for other proofs and examples.

One of the dimensions, as for example α_1, can be eliminated from the equations by the following change of variables:*

$$
\begin{aligned}
v_1' &= v_1 &&= [L^{\alpha_1}F^{\beta_1}T^{\gamma_1}] \\
v_2' &= v_2 v_1^{-\frac{\alpha_2}{\alpha_1}} &&= [L^0 F^{\beta_2'} T^{\gamma_2'}] \\
v_3' &= v_3 v_1^{-\frac{\alpha_3}{\alpha_1}} &&= [L^0 F^{\beta_3'} T^{\gamma_3'}] \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}
$$

The original equation in terms of these new variables becomes:

$$f(v_1,\ v_2' v_1^{\frac{\alpha_2}{\alpha_1}},\ v_3' v_1^{\frac{\alpha_3}{\alpha_1}},\ \ldots) = 0$$

Note that the new variables ($v_2', v_3', \ldots v_{n-1}'$) have zero α-dimension, since the transformation equations have been chosen so that the α-dimension cancels from all terms except v_1. If the function f is to be independent of a change in size of the fundamental unit L, then the variable v_1 which still contains α must have been eliminated by the above change of variables so that the original equation becomes:

$$f_1(v_2', v_3', \ldots v_{n-1}') = 0$$

In this new function f_1 the α dimension has disappeared, and the number of variables has been reduced by one.

By repeating the above process the β dimension can be eliminated through the following change of variables:

$$
\begin{aligned}
v_2'' &= v_2' &&= [L^0 F^{\beta_2'} T^{\gamma_2'}] \\
v_3'' &= v_3' v_2'^{-\frac{\beta_3'}{\beta_2'}} &&= [L^0 F^0 T^{\gamma_3'}] \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}
$$

in terms of which f_1 becomes:

$$f_1(v_2'',\ v_3'' v_2'^{\frac{\beta_3'}{\beta_2'}},\ v_4'' v_2'^{\frac{\beta_4'}{\beta_2'}},\ \ldots) = 0$$

But f_1 is to be independent of changes of size of the fundamental unit F, which appears only in the v_2''. The equation must then be independent of v_2'' and can be written:

$$f_2(v_3'', v_4'', \ldots v_{n-2}'') = 0$$

* This procedure follows that given in E. B. Wilson's excellent book, *An Introduction to Scientific Research*.

If this process is repeated to eliminate the third dimension γ, the terms left in the equation will be dimensionless, and the number of such terms will be $(n - 3)$ so that the equation becomes:

$$\phi(\pi_1, \pi_2, \ldots, \pi_{n-3}) = 0.$$

It will be seen that if in general the number of fundamental units is N and the number of variables in the problem is n, the above procedure will reduce the number of variables to $(n - N)$ dimensionless terms.*

The nature of the above transformations of the variables also indicates the general form of the dimensionless π factors. The π factors are built up out of products of powers of the variables v. Since these v's must be formed of products of powers of the fundamental units, it follows that the dimensionless π factors must have the form of products of powers of the fundamental units.

The values of the exponents of the variables appearing in the dimensionless factors can be found by expressing each variable in terms of the fundamental units, and by equating to zero the sums of the exponents for each fundamental unit. N algebraic equations will be obtained from which the ratios of the exponents for $(N + 1)$ terms can be computed. It is in this way possible to determine the exponents for a π term containing not more than $(N + 1)$ variables. If a π term is to be unique it is necessary that it contain a variable that does not appear in any of the other π terms for otherwise it would be possible to express this term by combinations of other terms. The first π term requires $(N + 1)$ variables, thus leaving $n - (N + 1)$ variables to form the remaining π's. The total number of unique π terms that can be formed is therefore $1 + n - (N + 1) = (n - N)$, in agreement with the previous development.

The foregoing discussion can be summarized in the form of the so-called π theorem,† which may be stated as follows:

* An exception to this $(n - N)$ rule occurs, for example, if a statics problem, in which time does not appear, were to be analyzed using an M-L-T system of fundamental units in which force has the dimensions $[MLT^{-2}]$. The number of π terms would be found to be $(n - 2)$, which would indicate that in this particular problem it would have been more appropriate to use only two fundamental units L and F.

† The basic ideas of the π-theorem are usually attributed to Buckingham, who first carried through a systematic development of the subject. See

A physical equation involving n variables expressed by N fundamental units can be put into the form

$$\phi(\pi_1, \pi_2, \ldots \pi_{n-N}) = 0$$

where the π's are dimensionless parameters formed of the products of powers of the variables. The number of such independent π terms will in general be $(n - N)$.

The application of the above principles is illustrated by the specific examples to follow. To use the method we must first be in a position to formulate the basic equations of motion in sufficient detail so that all of the variables which enter into the problem are known. The method of dimensional analysis will then give some information about the form which the solution must take, without the necessity of actually solving the equations. This information will, however, never be complete, and dimensional analysis can in no sense be looked upon as a substitute for the complete analytical formulation and solution of problems.

EXAMPLE 1. A mass m moves in a circle of radius r with a constant velocity v. What can be concluded as to the force F which causes the motion?

Solution. The variables which enter this problem, along with their dimensions, are:

$$\begin{aligned} F &= [L^0F^1T^0] \\ m &= [L^{-1}F^1T^2] \\ r &= [L^1F^0T^0] \\ v &= [L^1F^0T^{-1}] \end{aligned}$$

There are 4 variables and 3 fundamental units so that there is $4 - 3 = 1$ dimensionless term which can be formed. This term is:*

$$\begin{aligned} Fm^av^br^c &= [L^0F^1T^0]^1[L^{-1}F^1T^2]^a[L^1F^0T^{-1}]^b[L^1F^0T^0]^c \\ &= [L^{-a+b+c}F^{1+a}T^{2a-b}] \end{aligned}$$

E. Buckingham, "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations", *Physical Review* Vol. 2, (1914), pp. 345.

* Note that we can either use the four exponents a, b, c, d and then solve the algebraic equations for the ratios of these exponents, or we can assign a value of unity to one of the exponents, and thus determine numerical values directly. In this case the unity exponent should be assigned to the variable for which the equation is to be solved. If one should happen to give this unity exponent to a variable which should not have appeared in that particular π term, the algebraic equations would give the anomalous answer $1 = 0$.

For the term to be dimensionless it is required that:

$$1 + a = 0 \qquad a = -1$$
$$2a - b = 0 \qquad b = -2$$
$$-a + b + c = 0 \qquad c = 1$$

The dimensionless term is therefore Fr/mv^2 and the equation describing the motion is:

$$\phi\left(\frac{Fr}{mv^2}\right) = 0$$

Since m, v, and r are constant, F must also be constant and the functional form of the equation must be:

$$\frac{Fr}{mv^2} - C = 0$$

where C is a dimensionless constant (pure number). We may therefore write

$$F = C\frac{mv^2}{r}$$

If the problem were to be solved completely by using the principles of dynamics it would be found that $C = 1$, but we cannot deduce this from dimensional considerations alone.

EXAMPLE 2. The twist per unit length θ of a circular shaft depends upon the shearing modulus of elasticity G, the diameter d and the twisting moment M_t. (a) Using dimensional analysis, express θ as a function of the other variables. (b) If it is found experimentally that θ varies linearly with M_t, find from the results of the dimensional analysis of part (a) the way in which θ varies with the diameter d.

Solution. (a) Since time does not appear in this problem there are only two fundamental units, so that we would expect $(4 - 2) = 2$ π-factors. π_1 can be formed by inspection; $\pi_1 = \theta d$. π_2 will involve (G, d, M_t) so:

$$[F^1L^{-2}]^a[F^0L^1]^b[F^1L^1]^1 = [0]$$
$$a + 1 = 0; \quad a = -1$$
$$-2a + b + 1 = 0; \quad b = 2a - 1 = -3$$

Thus $\pi_2 = M_t/Gd^3$ and

$$\phi\left[(\theta d), \left(\frac{M_t}{Gd^3}\right)\right] = 0$$

from which

$$\theta = \frac{1}{d} f(M_t/Gd^3).$$

(b) If it is known that $\theta = CM_t$, then we know the form of the function of part (a) to be linear, and we have $f = C(M_t/Gd^3)$, so:

$$\theta = \frac{1}{d} \cdot C \frac{M_t}{Gd^3} = C \frac{M_t}{Gd^4}$$

Part (b) illustrates the way in which additional information, in this case obtained from experiments, can be used to supplement the results of the dimensional analysis.

Example 3. A problem of considerable practical importance is that of determining the drag force F_d acting upon a body moving through a fluid. Consider a body of specified shape whose size is defined by some characteristic length l, moving with a constant velocity v through a fluid of density ρ and viscosity μ. Apply dimensional analysis to this problem.

Solution. The solution will be of the form $f(F_d, v, l, \rho, \mu) = 0$. There are five variables and three fundamental units, so we expect two π-terms. We are particularly concerned with the force F_d so we shall select the π terms so that F_d will appear in only one. Choosing the viscosity μ as the other unique variable, the π terms become:

$$\pi_1 = F_d l^a \rho^b v^c; \quad \pi_2 = \mu l^a \rho^b v^c$$

Determining the exponents as in the preceding examples, we obtain:

$$\pi_1 = \frac{F_d}{l^2 \rho v^2}; \quad \pi_2 = \frac{\mu}{v l \rho}$$

and we have:

$$\phi\left[\left(\frac{F_d}{l^2 \rho v^2}\right), \left(\frac{v l \rho}{\mu}\right)\right] = 0$$

from which:

$$F_d = l^2 \rho v^2 f\left(\frac{v l \rho}{\mu}\right)$$

This is as far as one can proceed with dimensional analysis alone. The π_2 term will be recognized as Reynolds number, which plays a prominent part in many fluid mechanics problems. The form of the function f would in general have to be determined experimentally, and the above expressions would be helpful in suggesting the way in which the experiments should be carried out and in organizing the results of the tests.

1.8 The Theory of Models. The testing of a model for the purpose of predicting the behavior of the prototype is a useful technique in many fields of engineering. The principles on which such models should be designed and tested may be established by a direct application of dimensional analysis.

As a starting point it will be necessary to know all of the variables which are important for the particular problem. The safest way to be sure that all of the proper variables have been included is to derive general equations that will apply to both model and prototype, but this cannot always be done without more knowledge of the problem than is readily available. Using the π theorem, the variables can be arranged in dimensionless groups so that the equation assumes the form:

$$\phi(\pi_1, \pi_2, \ldots) = 0$$

As shown above, each of the dimensionless terms will contain one unique variable which does not appear in the other terms. The equation can then be solved for the π factor involving the variable which is of primary importance in the particular problem:

$$\pi_1 = \theta(\pi_2, \pi_3, \ldots)$$

This is a general relationship which will apply both to the prototype and to the model:

$$\pi_{1p} = \theta(\pi_{2p}, \pi_{3p}, \ldots)$$
$$\pi_{1m} = \theta(\pi_{2m}, \pi_{3m}, \ldots)$$

If now the model is designed and tested in such a way that:

$$\begin{aligned} \pi_{2m} &= \pi_{2p} \\ \pi_{3m} &= \pi_{3p} \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned} \tag{1.1}$$

then

$$\theta(\pi_{2p}, \pi_{3p}, \ldots) = \theta(\pi_{2m}, \pi_{3m}, \ldots)$$

and thus:

$$\pi_{1p} = \pi_{1m} \tag{1.2}$$

Equations (1.1) can be thought of as the conditions which specify the design and method of testing of the model, whereas Equation (1.2) permits a determination of prototype behavior in terms of the model behavior.

EXAMPLE 1. Show how to design and interpret tests on a model to study the problem of the deflections of a beam under (a) its own weight and (b) under a concentrated load, assuming that the weight of the beam is negligible.

Solution. (a) From our knowledge of beam deflection theory we know that the important variables are: the deflection δ, the weight per unit volume γ, the modulus of elasticity E, some characteristic length l, such as the length of the beam, various linear dimensions $(a, b, c, \ldots)$ to specify the point at which the deflection is desired and the shape of the cross-section of the beam. Thus we have:

$$f(\delta, \gamma, E, l, a, b, c, \ldots) = 0$$

or in terms of the π-factors:

$$\phi\left(\frac{\delta}{l}, \frac{\gamma l}{E}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right) = 0$$

solving for the term involving the deflection:

$$\left(\frac{\delta}{l}\right) = \theta\left(\frac{\gamma l}{E}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right)$$

From the right hand side of this equation we see that the model must be geometrically similar in all respects to the prototype, so that:

$$\left(\frac{a}{l}\right)_m = \left(\frac{a}{l}\right)_p; \quad \left(\frac{b}{l}\right)_m = \left(\frac{b}{l}\right)_p; \quad \text{etc.}$$

and in addition we must have:

$$\left(\frac{\gamma l}{E}\right)_m = \left(\frac{\gamma l}{E}\right)_p$$

Then the prototype behavior may be calculated from the model:

$$\left(\frac{\delta}{l}\right)_p = \left(\frac{\delta}{l}\right)_m$$

Note that it will not be possible to make a model with a scale factor different from unity with the same material as the prototype. For a given scale factor, a model material must be found which has a γ and E such that the $\gamma l/E$ condition is satisfied.

If we introduce the additional information that we are to be concerned only with small deflections for which there is a linear relationship between deflection and load, the general equation can be written as :

$$\left(\frac{\delta}{l}\right) = \left(\frac{\gamma l}{E}\right)\theta\left(\frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \ldots\right)$$

In this case the only scale factors are the geometric conditions, and it will no longer be necessary to limit the model material to satisfy a $\gamma l/E$ requirement.

(b) If the deflections are caused by a concentrated load P instead of by the uniformly distributed gravity load of part (a), the variables are:

$$f(\delta, P, E, l, a, b, c, \ldots) = 0$$

where the location of the load is specified by one of the linear factors $(a, b, c, \ldots)$. Then, proceeding as in part (a):

$$\phi\left(\frac{\delta}{l}, \frac{P}{El^2}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \ldots\right) = 0$$

$$\left(\frac{\delta}{l}\right) = \theta\left(\frac{P}{El^2}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \ldots\right) = 0$$

The model must again be geometrically similar, and in addition:

$$\left(\frac{P}{El^2}\right)_m = \left(\frac{P}{El^2}\right)_p$$

If the model is made of the same material as the prototype, then the load to be placed on the model would be:

$$P_m = \frac{l_m^2}{l_p^2}\cdot P_p$$

and the deflection of the model would be:

$$\delta_m = \frac{l_m}{l_p}\cdot\delta_p$$

As in part (a), if it is known that there is a linear relationship between δ and P the problem is simplified, and the equation becomes:

$$\left(\frac{\delta}{l}\right) = \frac{P}{El^2}\,\theta\left(\frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right)$$

and only the geometrical scale factors are required.

EXAMPLE 2. A sinusoidally varying force of amplitude F and frequency ω cyc/sec is applied to a beam of density ρ, modulus of elasticity E, characteristic length l, and linear dimensions $(a, b, c, \ldots)$ specifying location of loads, deflections, and geometrical shape. Show how to design and use a model to study the dynamic deflection δ of the beam.

Solution. The solution can be expressed as:

$$f(\delta, F, E, \rho, \omega, l, a, b, c, \cdots) = 0$$

or, in terms of the dimensionless π factors:

$$\left(\frac{\delta}{l}, \frac{F}{El^2}, \frac{1}{\omega l}\sqrt{\frac{E}{\rho}}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right) = 0$$

Solving for the deflection term we obtain:

$$\left(\frac{\delta}{l}\right) = \theta\left(\frac{F}{El^2}, \frac{1}{\omega l}\sqrt{\frac{E}{\rho}}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right)$$

Thus, the conditions for designing and loading the model are:

(a) Geometric similarity, so that $\left(\frac{a}{l}\right)_m = \left(\frac{a}{l}\right)_p$, etc.

(b) $\left(\frac{1}{\omega l}\sqrt{\frac{E}{\rho}}\right)_m = \left(\frac{1}{\omega l}\sqrt{\frac{E}{\rho}}\right)_p$

(c) $\left(\frac{F}{El^2}\right)_m = \left(\frac{F}{El^2}\right)_p$

and the prototype deflection is obtained from:

$$\delta_p = \frac{l_p}{l_m}\cdot\delta_m$$

If the model is made of the same material as the prototype condition (b) becomes:

$$\omega_m = \frac{l_p}{l_m}\,\omega_p$$

and the model load is given by:

$$F_m = \frac{l_m{}^2}{l_p{}^2}\,F_p.$$

If additional information is available which indicates that there is a linear relationship between the deflection and the magnitude of the force, the conditions become somewhat simpler, for the term (F/El^2) can then be taken out of the function θ so that:

$$\left(\frac{\delta El}{F}\right) = \theta\left(\frac{1}{\omega l}\sqrt{\frac{E}{\rho}}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right)$$

Besides the geometrical conditions there is now only one modeling condition:

$$\left(\frac{1}{\omega l}\sqrt{\frac{E}{\rho}}\right)_m = \left(\frac{1}{\omega l}\sqrt{\frac{E}{\rho}}\right)_p$$

and the prototype deflection is determined from:

$$\delta_p = \left(\frac{E_m}{E_p}\right)\left(\frac{F_p}{F_m}\right)\left(\frac{l_m}{l_p}\right)\delta_m.$$

EXAMPLE 3. A rigid body of a given shape moves with a uniform velocity v through an infinite incompressible fluid medium of density ρ and viscosity μ. Model studies are to be made of the resisting forces F acting on the body. Establish suitable modeling conditions for this problem.

Solution. Taking l as some characteristic length, and $(a, b, c, \ldots)$ as various linear dimensions, we may write the solution as:

$$f(F, v, \rho, \mu, l, a, b, c, \ldots) = 0$$

Or, in terms of the π-factors:

$$\phi\left(\frac{F}{\rho l^2 v^2}, \frac{\rho l v}{\mu}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right) = 0$$

$$\frac{F}{\rho l^2 v^2} = \theta\left(\frac{\rho l v}{\mu}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right)$$

Thus the model should be scaled geometrically, and the Reynolds number $(\rho lv/\mu)$ should be the same for the model test and for the prototype.

If the fluid is to be the same for the model and prototype, the Reynolds number equality becomes:

$$l_m v_m = l_p v_p$$

and the prototype resisting force is:

$$F_p = F_m.$$

Note that the Reynolds number equality requires that a small model should have a larger velocity than the prototype. In aerodynamic model investigations this may lead to difficulties, since at the high velocity demanded by the Reynolds number equality the nature of the physical phenomena may change. To obtain the correct Reynolds number without a large increase in velocity, model tests are sometimes made in wind tunnels in which the density of the air has been increased by pressurizing.

PROBLEMS

1.1. The secant formula gives the bending stress σ in an eccentrically loaded column in terms of the area A, the eccentricity e, the distance to the extreme fiber c, the radius of gyration of the cross-section area about the neutral axis r, and the axial load P as:

$$\sigma = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\frac{l}{2r}\sqrt{\frac{P}{EA}}\right)$$

Check the dimensions of this physical equation.

1.2. A certain problem in dynamics leads to the equation:

$$m\frac{d^2x}{dt^2} + \left[k - m\left(\frac{d\theta}{dt}\right)^2\right]x = my\frac{d^2\theta}{dt^2}$$

where m is the mass of a body, x and y are the coordinates of the displacement, θ is an angle measured in radians, t is the time, and k is a stiffness modulus measured in pounds per foot. Check the dimensions of this equation.

1.3. A body falling through a certain resisting medium with a velocity v is subjected to a drag force that is proportional to the velocity squared,

$F_d = kv^2$. If the body has a weight W and starts from rest at a time $t = 0$, its velocity at any subsequent time is:

$$v = \sqrt{\frac{W}{k}} \frac{\left(e^{2g\sqrt{\frac{k}{W}}t} - 1\right)}{\left(e^{2g\sqrt{\frac{k}{W}}t} + 1\right)}$$

Check the dimensions of this equation.

1.4. The equation describing the flow of a viscous fluid through a pipe may be put in the form:

$$\frac{\partial v}{\partial t} = F_{B_z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v$$

where v is the axial velocity, F_{B_z} is the body force per unit mass in the z direction, ρ is the density, p is the pressure, μ is the viscosity, r is a radial distance, and t is time. Check the dimensions of this equation.

1.5. A jet of water of cross-sectional area A and velocity v impinges normally on a stationary flat plate. The mass per unit volume of the water is ρ. By dimensional analysis, determine an expression for the force F exerted by the jet against the plate.

1.6. (a) A gun shoots a projectile with an initial velocity v, which makes an angle of θ with the vertical. The projectile has a mass m and a range R. The gravitational acceleration is g. Find the expression for R by dimensional analysis. Assume there is no air resistance.

(b) Suppose the projectile of part (a) were subjected to an air resistance proportional to the velocity squared; that is, the magnitude of the retarding force is kv^2 lb. Apply dimensional analysis to obtain an expression for the range.

1.7. Using the method suggested in the text, show that if the ratio of two measured values of a secondary quantity is independent of the sizes of the fundamental units used, the secondary quantity must have the form of the product of powers of the primary quantities.

1.8. (a) Ocean waves have a wave length from crest to crest of l ft and a height from trough to crest of h ft. The density of the fluid is ρ and the acceleration of gravity is g. Find an expression for the velocity of the waves by dimensional analysis.

(b) If it has been determined experimentally that the geometrical shape of the waves is constant, that is, if l/h is a constant, show how the analysis of part (a) can be further simplified.

1.9. (a) Experiment shows that for laminar flow of a fluid through a circular pipe the significant variables are D the diameter of the pipe, v the mean velocity of the fluid, $\frac{dp}{dx}$ the rate of change of pressure along the pipe and μ the coefficient of viscosity. Derive an expression for the velocity v by means of dimensional analysis.

(b) If the flow through the pipe of part (a) is turbulent, then experiment shows that the density of the fluid is also a significant variable. By dimensional analysis derive an expression for the velocity v for turbulent flow.

1.10. A journal bearing has a diameter d and a radial clearance c. The bearing is loaded so that a mean pressure of p lb/in.^{2} is set up. The viscosity of the lubricant is μ, and the speed of the rotating shaft is N rpm. By dimensional analysis alone, what can be said about the relationship between the coefficient of friction f of the bearing and the above variables?

1.11. The frequency of transverse vibrations (f cycles per second) of a stretched uniform string depends on the tension force T in the string, the mass per unit length μ, the length l, and the acceleration of gravity g.

(a) Using dimensional analysis, find an expression for the frequency f. (Note: choose the π factors so that the unique factor in one of them is f, and in the other l.)

(b) If it is experimentally determined that the frequency is inversely proportional to the length, find by using the results of (a) above, the way in which the frequency depends on the tension T.

1.12. In a certain problem in fluid mechanics it is determined that surface tension σ is an important factor. The other significant variables in the problem are velocity v, force F, density ρ, and cross-sectional area A. By dimensional analysis derive an expression for the force F in terms of the other quantities. The dimensionless quantity $v^2A^{\frac{1}{2}}/(\sigma/\rho)$ which appears in the analysis is called Weber's number.

1.13. When a body falls through a resisting medium, such as air, its velocity increases until the drag force counterbalances the gravity force. The resultant force on the body is then zero, the velocity does not change, and the body is said to have reached its terminal velocity. Consider a spherical body of radius r falling through a very rarefied atmosphere. Let the density of the body be ρ_b, and the density of the medium ρ_m. From dimensional analysis what can be concluded as to the effect of the size of the body on the terminal velocity?

1.14. Model tests are to be made to study the stresses in a statically loaded beam. Find the design and loading conditions which must be satisfied and the relationship between the model and prototype stresses when (a) concentrated loads act on the beam and (b) when the only load is the weight of the beam.

1.15. The buckling load of a steel column is to be determined from tests on a 1/5 scale model made of aluminium. It is assumed that the end conditions and loading conditions are the same for both model and prototype and that the form of the force-deflection curve is the same for both. If the model buckles at a load of 90 lb find the load at which the prototype column would buckle, using the results of the analysis of Example 1 above.

1.16. A study of the resisting forces acting on a ship moving on the free surface of the water shows that the most important factor is associated with wave resistance, which is a gravity effect. Show that for ship model studies an important modeling law is defined by Froude's number, v^2/lg, and show how the force on the model is related to the prototype force.

1.17. Referring to Example 3 and to Prob. 1.16, show that if both wave drag and viscous friction forces are important in a fluid problem, both Froude's number and Reynolds number must be modeled, and indicate the practical consequences of this requirement.

Chapter 2

KINEMATICS: THE DESCRIPTION OF MOTION

The circumstances of mere motion, considered without reference to the bodies moved, or to the forces producing the motion, or to the forces called into action by the motion, constitute the subject of a branch of Pure Mathematics, which is called Kinematics.—W. Thomson and P. G. Tait, *Elements of Natural Philosophy* (1872).

For the development of dynamics a concise and consistent notation is required for the description of the displacements, velocities, and accelerations of a body. The vector notation for these quantities will be presented first, and then various scalar components of these vectors will be developed.

2.1 Displacement, Velocity, and Acceleration. The displacement of a point P (Fig. 2.1) is described by the magnitude and direction of the radius vector $\boldsymbol{r}$ which extends from the origin of a coordinate system to the point P. At time t let the displacement be $\boldsymbol{r}$ then at time $t + \Delta t$ the displacement is $\boldsymbol{r} + \Delta\boldsymbol{r}$ where $\Delta\boldsymbol{r}$ is the vector from P to P'. Between P and P' the average change of $\boldsymbol{r}$ per unit time is $\Delta\boldsymbol{r}/\Delta t$ and the *velocity* at P is obtained by taking the limiting value of $\Delta\boldsymbol{r}/\Delta t$ as Δt approaches zero:

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} \tag{2.1}$$

The direction of $\boldsymbol{v}$ is tangent to the path of motion at P. At point P the velocity is $\boldsymbol{v}$, and at the point P' it is $\boldsymbol{v} + \Delta\boldsymbol{v}$. The change of velocity with time may be illustrated by a diagram in which the velocity $\boldsymbol{v}$ is drawn as a radius vector as in Fig. 2.2. The curve described by the endpoint of $\boldsymbol{v}$ in this figure is called the hodograph

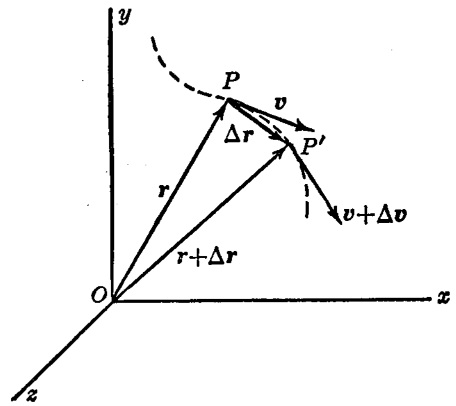


FIG. 2.1

of the motion. Let the velocity be $\boldsymbol{v}$ at time t and $\boldsymbol{v} + \Delta\boldsymbol{v}$ at time $t + \Delta t$. In this interval the average change of velocity per unit time is $\Delta\boldsymbol{v}/\Delta t$, and the acceleration at time t is obtained by taking the limiting value of $\Delta\boldsymbol{v}/\Delta t$.

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} \tag{2.2}$$

The direction of the acceleration vector $\boldsymbol{a}$ coincides with the direction of the tangent to the hodograph, since the velocity of the endpoint of a vector in the hodograph plane is the time derivative of the vector. It should be noted that the acceleration is equal to zero

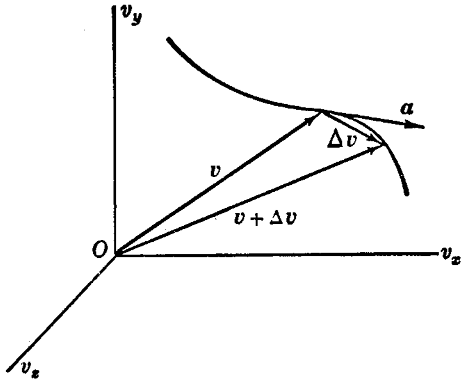


FIG. 2.2

only when both the magnitude and direction of $\boldsymbol{v}$ are constant. For example, a particle moving on a circular path can never have zero acceleration since the direction of $\boldsymbol{v}$ is always changing.

In vector notation the equation of motion is written:*

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{r}}{dt^2} = m\ddot{\boldsymbol{r}} \tag{2.3}$$

It is often convenient to resolve the displacement, velocity and acceleration vectors into components. These components are usually taken in the principal directions of the coordinate system which is most appropriate for the particular problem involved. Three commonly used sets of components will now be discussed.

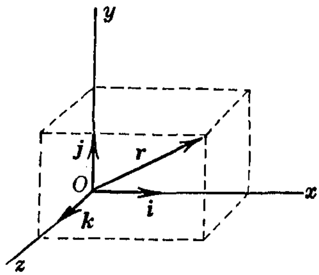


FIG. 2.3

(a) *Rectangular coordinates* (Fig. 2.3) are used to describe the displacement, velocity and acceleration vectors when they are resolved into components parallel to the orthogonal xyz axes. In terms of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, which are constant:

$$\begin{aligned} \boldsymbol{r} &= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \\ \dot{\boldsymbol{r}} &= \frac{d\boldsymbol{r}}{dt} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \\ \ddot{\boldsymbol{r}} &= \frac{d^2\boldsymbol{r}}{dt^2} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k} \end{aligned} \tag{2.4}$$

where $\dot{x} = \dfrac{dx}{dt} = v_x$, etc., $\ddot{x} = \dfrac{d^2x}{dt^2} = a_x$, etc.

In rectangular coordinates the equation of motion is written:

$$F_x = m\ddot{x}; \quad F_y = m\ddot{y}; \quad F_z = m\ddot{z} \tag{2.5}$$

where F_x, F_y, F_z are the x, y, z components of the resultant force acting on the particle.

(b) *Cylindrical coordinates* z, r, ϕ are used when suited to the geometry of the problem. In this system there are three mutually

* We shall use a dot placed above a letter to indicate the derivative with respect to time, and two dots to indicate the second derivative. This is the notation originally adopted by Newton.

perpendicular components; one parallel to the z-axis, one with a direction parallel to the radius vector in the xy plane, and the third with the direction of increasing ϕ as shown in Fig. 2.4 (a).

The unit vectors specifying these directions are designated by $\boldsymbol{e}_z$, $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$. The unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ are not constant, but change direction with time. The time derivative of a unit vector is perpendicular to the vector since the length of the vector is constant. As may be seen from Fig. 2.4 (b):

$$\Delta \boldsymbol{e}_r = (\Delta\phi)(1)\boldsymbol{e}_\phi \quad \text{and} \quad \Delta \boldsymbol{e}_\phi = (\Delta\phi)(1)(-\boldsymbol{e}_r)$$

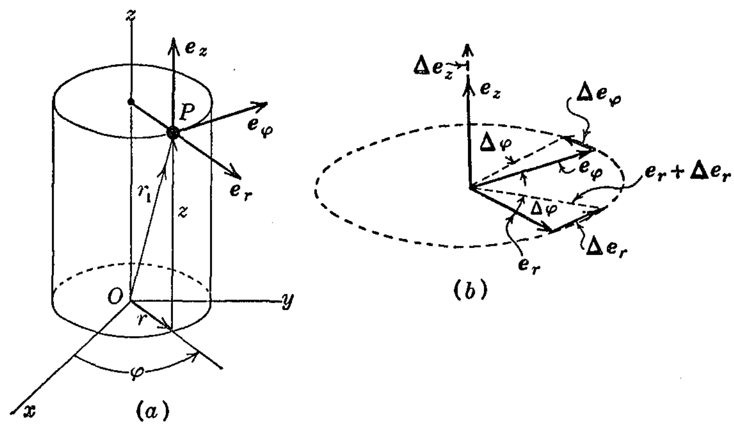


FIG. 2.4

so that the derivatives are:

$$\dot{\boldsymbol{e}}_r = \dot\phi \boldsymbol{e}_\phi; \quad \dot{\boldsymbol{e}}_\phi = -\dot\phi \boldsymbol{e}_r$$

The displacement of point P is:

$$\boldsymbol{r}_1 = r\boldsymbol{e}_r + z\boldsymbol{e}_z$$

The velocity is obtained by taking the derivative with respect to t:

$$\begin{aligned} \boldsymbol{v} &= \dot r\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r + \dot z\boldsymbol{e}_z + z\dot{\boldsymbol{e}}_z \\ &= \dot r\boldsymbol{e}_r + r\dot\phi\boldsymbol{e}_\phi + \dot z\boldsymbol{e}_z \end{aligned} \tag{2.6}$$

The components of the velocity in the r, ϕ and z directions are respectively $\dot r$, $r\dot\phi$, and $\dot z$. The acceleration is obtained by a second differentiation:

$$\begin{aligned} \boldsymbol{a} &= \ddot r\boldsymbol{e}_r + \dot r\dot{\boldsymbol{e}}_r + \dot r\dot\phi\boldsymbol{e}_\phi + r\ddot\phi\boldsymbol{e}_\phi + r\dot\phi\dot{\boldsymbol{e}}_\phi + \ddot z\boldsymbol{e}_z + \dot z\dot{\boldsymbol{e}}_z \\ &= (\ddot r - r\dot\phi^2)\boldsymbol{e}_r + (r\ddot\phi + 2\dot r\dot\phi)\boldsymbol{e}_\phi + \ddot z\boldsymbol{e}_z \end{aligned} \tag{2.7}$$

The equation of motion is written:

$$\begin{aligned} F_r &= m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \\ F_z &= m\ddot{z} \end{aligned} \tag{2.8}$$

where F_r, F_ϕ, F_z are the components of the resultant force on the particle in the r, ϕ, z directions, and $(\ddot{r} - r\dot{\phi}^2)$, $(r\ddot{\phi} + 2\dot{r}\dot{\phi})$ and $\ddot{z}$ are the components of acceleration. Since the expressions for the acceleration components are not as simple as for rectangular components, it is not desirable to use cylindrical coordinates unless the geometry of the problem is particularly suited to their use.

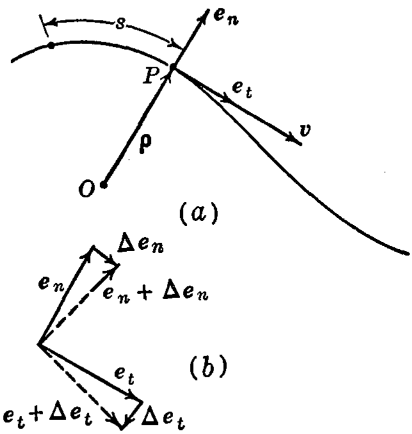


FIG. 2.5

(c) *Tangential and normal components* are used chiefly because they give a simple representation of acceleration in curvilinear motion. Let s be the arc length measured along the path of motion [Fig. 2.5 (a)] and let ρ be the principal radius of curvature at the point P. The velocity is:

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t \tag{2.9}$$

where the unit vector has the direction of $\boldsymbol{v}$, that is, tangent to the path of motion. The acceleration is obtained by differentiating the velocity with respect to the time:

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \dot{s}\dot{\boldsymbol{e}}_t$$

To evaluate the time derivative of the unit vector $\boldsymbol{e}_t$, note from Fig. 2.5 (b) that since this vector changes direction but not length, $\Delta\boldsymbol{e}_t$ is perpendicular to $\boldsymbol{e}_t$, so that:

$$\Delta\boldsymbol{e}_t = -\frac{\Delta s}{\rho}\boldsymbol{e}_n; \quad \dot{\boldsymbol{e}}_t = \lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{e}_t}{\Delta t} = -\frac{1}{\rho}\boldsymbol{e}_n \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = -\frac{1}{\rho}\dot{s}\boldsymbol{e}_n$$

where the minus sign indicates that $\dot{\boldsymbol{e}}_t$ is opposite in direction to $\boldsymbol{e}_n$.

Substituting this value of $\dot{\boldsymbol{e}}_t$ in the foregoing expression for $\boldsymbol{a}$ gives:

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t - \frac{\dot{s}^2}{\rho}\boldsymbol{e}_n \tag{2.10}$$

The acceleration vector $\boldsymbol{a}$ may thus be resolved into two perpendicular components, a tangential acceleration of magnitude $\ddot{s}$ and a normal acceleration of magnitude $-\dot{s}^2/\rho$. The minus sign indicates that the direction of the normal acceleration is toward the center of curvature.

The equations of motion in terms of tangential and normal components of acceleration are:

$$\begin{aligned} F_t &= m\ddot{s} \\ F_n &= -\frac{m\dot{s}^2}{\rho} \end{aligned} \tag{2.11}$$

Coordinate systems of other types may be found to be convenient for certain problems. The spherical coordinate system discussed in problems 2.3 and 2.4 below is often employed. It is often wise to compare the possibilities of several coordinate systems for a particular problem.

EXAMPLE 1. A particle moves along the parabolic path $y = ax^2$ in such a way that the x-component of the velocity of the particle remains constant. Find the acceleration of the particle.

Solution. Since the conditions of the problem are stated in terms of a rectangular coordinate system, we shall probably find rectangular coordinates most convenient for this problem. Since $\dot{x} = c$, we have $\ddot{x} = 0$. Also:

$$\begin{aligned} y &= ax^2 \\ \dot{y} &= 2ax\dot{x} = 2acx \\ \ddot{y} &= 2ac\dot{x} = 2ac^2 \end{aligned}$$

so that the motion of the particle is:

$$\begin{aligned} \boldsymbol{r} &= x\boldsymbol{i} + ax^2\boldsymbol{j} \\ \dot{\boldsymbol{r}} &= c\boldsymbol{i} + 2acx\boldsymbol{j} \\ \ddot{\boldsymbol{r}} &= 2ac^2\boldsymbol{j} \end{aligned}$$

EXAMPLE 2. A particle P moves in a plane in such a way that its distance from a fixed point O is $r = a + bt^2$ and the line connecting

O and P makes an angle $\phi = ct$ with a fixed line OA, as shown in Fig. 2.6. Find the acceleration of P.

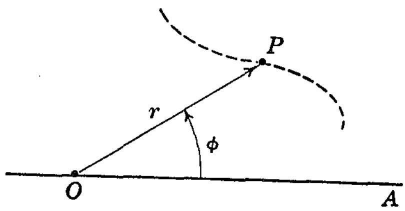


FIG. 2.6

Solution. The data for this problem are given in such a way that a plane polar coordinate system is convenient for the description of the motion. The acceleration of the point P in plane polar coordinates is found from Equation (2.7):

$$\boldsymbol{a} = (\ddot{r} - r\dot{\phi}^2)\boldsymbol{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\boldsymbol{e}_\phi$$

In this problem:

$$\begin{aligned} r &= a + bt^2 & \phi &= ct \\ \dot{r} &= 2bt & \dot{\phi} &= c \\ \ddot{r} &= 2b & \ddot{\phi} &= 0 \end{aligned}$$

So that:

$$\boldsymbol{a} = [2b - c^2(a + bt^2)]\boldsymbol{e}_r + 2bct\,\boldsymbol{e}_\phi$$

EXAMPLE 3. A particle moves along a path composed of two straight lines connected by a circular arc of radius r, as shown in Fig. 2.7. The speed along the path is given by $\dot{s} = at$. Find the maximum acceleration of the particle.

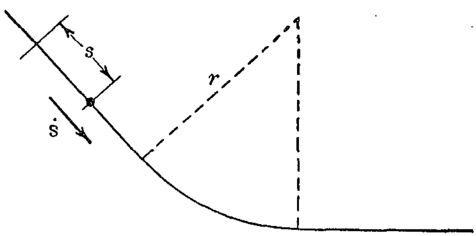


FIG. 2.7

Solution. The form of the data in this problem makes the use of radial and tangential components of acceleration suitable. Using Equation (2.10):

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t - \frac{\dot{s}^2}{\rho}\boldsymbol{e}_n$$

we note that the normal component of acceleration is zero along the straight portion of the path, and $\frac{\dot{s}^2}{r}$ along the curved portion. The maximum acceleration will thus occur when $\dot{s}$ is a maximum on the curved path, that is, just at the end of the curve:

$$\boldsymbol{a}_{\text{max}} = a\boldsymbol{e}_t - \frac{a^2t^2}{r}\boldsymbol{e}_n$$

PROBLEMS

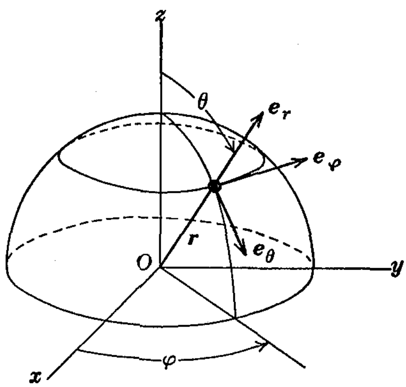
2.1. Derive the r, ϕ components of acceleration in a plane polar coordinate system without using unit vectors. This may be done by showing that:

$$\begin{aligned} a_r &= a_x \cos\phi + a_y \sin\phi \\ a_\phi &= -a_x \sin\phi + a_y \cos\phi \end{aligned}$$

and by finding a_x and a_y by differentiating the relations $x = f_1(r, \phi)$, $y = f_2(r, \phi)$.

2.2. Derive the expressions for the tangential and normal components of acceleration without using unit vectors. First determine the normal and tangential components of $\Delta\boldsymbol{v}/\Delta t$ and then let $\Delta t \to 0$.

2.3. Referring to the figure, consider the effect of increments in ϕ and θ upon the unit vectors, $\boldsymbol{e}_r$, $\boldsymbol{e}_\phi$, $\boldsymbol{e}_\theta$ of a spherical coordinate system. $\boldsymbol{e}_r$ is radial, positive out from the pole O, $\boldsymbol{e}_\phi$ is tangential to the circle of latitude, and $\boldsymbol{e}_\theta$ is tangential to the meridian circle as shown. Derive the expressions for $\dot{\boldsymbol{e}}_r$, $\dot{\boldsymbol{e}}_\phi$, and $\dot{\boldsymbol{e}}_\theta$.



PROB. 2.3

2.4. Using the results of Problem 2.3, derive the r, ϕ, and θ components of velocity and acceleration in spherical coordinates.

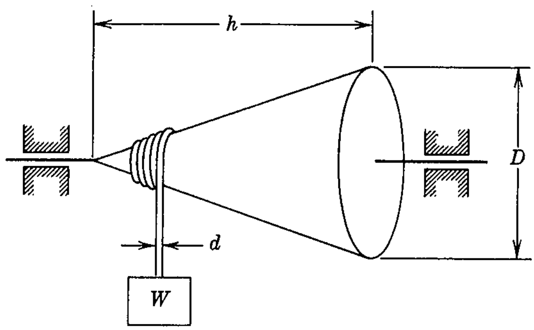
2.5. A body moves in a straight line parallel to and at a distance a from the x-axis with a constant velocity $\boldsymbol{v}$. Using Equation (2.7) evaluate the acceleration term by term, and show that the coefficients of $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ vanish.

2.6. A mass m slides under the action of gravity inside a frictionless tube which is bent in the shape of a helix of radius a and pitch angle α.

The geometric axis of the helix is vertical. If the mass starts from rest at $t = 0$ from the point $z = 0$ and $\phi = 0$, find the $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$ components of the force exerted by the particle on the helix as a function of time.

2.7. The third derivative of displacement with respect to time, the time rate of change of acceleration, is sometimes called the "jerk" and is used to evaluate the riding comfort of vehicles. Find the components of jerk in the directions of the unit vectors in a cylindrical coordinate system.

2.8. A weight is lifted by a flexible cable wound on a cone as shown in the figure. The diameter of the cable is d, and the diameter of the large



PROB. 2.8

end of the cone is D. If the cone is rotated at a constant angular velocity ω, find the acceleration of the weight, neglecting the small horizontal motion of the weight.

2.2 Angular Velocity. Consider a rigid body rotating about a fixed axis OA, as shown in Fig. 2.8. By the definition of rotation, this means that all points of the body are moving in circular paths about centers on the axis of rotation. The *angular velocity* of the body is described by the vector $\boldsymbol{\omega}$, which has the direction of the axis of rotation as given by the right-hand screw rule and which has a magnitude equal to the time rate of change of the angular displacement of any line in the body which is normal to the axis of rotation. Thus in Fig. 2.8, $\boldsymbol{\omega}$ would have the direction of OA and the magnitude:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \dot{\theta}$$

There is a simple relation between the angular velocity of a rigid body rotating about a fixed axis and the linear velocity of any point in the body. Referring to Fig. 2.8 and the definition of the angular velocity, it will be seen that the velocity $\boldsymbol{v}$ of a point P located by the radius vector $\boldsymbol{r}$ is:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{2.12}$$

since $\boldsymbol{v}$ is perpendicular to the plane of $\boldsymbol{\omega}$ and $\boldsymbol{r}$ and

$$v = \lim_{\Delta t \to 0} \frac{a\Delta\theta}{\Delta t} = \omega r \sin \alpha$$

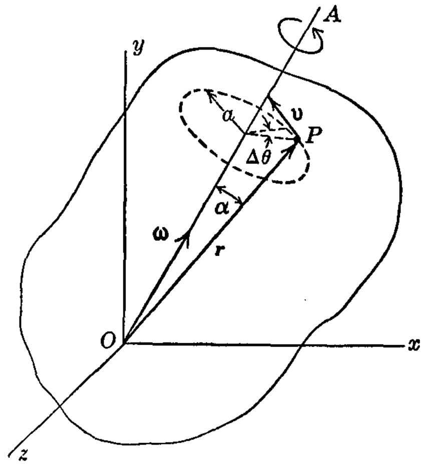


Fig. 2.8

It should be noted that the angular velocity vector is a free vector, in the sense that it is not associated with a particular line of action, as is a force vector.

An expression involving angular velocity which will frequently be found useful is that which gives the time derivatives of the unit vectors in a rotating coordinate system. In Fig. 2.9 the xyz coordinate system rotates with respect to a fixed XYZ

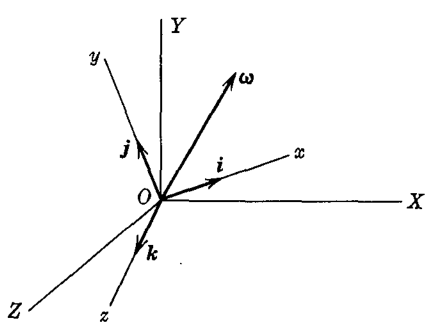


Fig. 2.9

system with an angular velocity $\boldsymbol{\omega}$. The unit vectors in the xyz directions are $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, and we are to determine the time derivatives $\dot{\boldsymbol{i}}, \dot{\boldsymbol{j}}, \dot{\boldsymbol{k}}$

of these unit vectors. Referring to Equation (2.12) which we may write as $\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$, we see that we may write:

$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix}$$

with similar expressions for $\dot{\boldsymbol{j}}$ and $\dot{\boldsymbol{k}}$. In this way the result is found to be:

$$\begin{aligned} \dot{\boldsymbol{i}} &= \boldsymbol{\omega} \times \boldsymbol{i} = \omega_z \boldsymbol{j} - \omega_y \boldsymbol{k} \\ \dot{\boldsymbol{j}} &= \boldsymbol{\omega} \times \boldsymbol{j} = \omega_x \boldsymbol{k} - \omega_z \boldsymbol{i} \\ \dot{\boldsymbol{k}} &= \boldsymbol{\omega} \times \boldsymbol{k} = \omega_y \boldsymbol{i} - \omega_x \boldsymbol{j} \end{aligned} \tag{2.13}$$

A summary of the algebraic properties of vector products is given in Appendix III, for those who wish to review the subject.

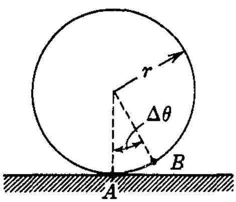


FIG. 2.10

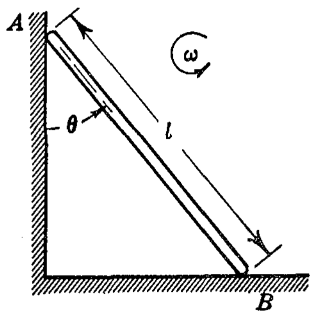


FIG. 2.11

EXAMPLE 1. A cylinder of radius r rolls without slipping on a horizontal plane. Find the relationship between the linear velocity of the center of the cylinder and the angular velocity of the cylinder.

Solution. Referring to Fig. 2.10 it will be seen that as the point B rolls into contact with the plane, the center of the cylinder will move through a horizontal distance $(r\Delta\theta)$. If this motion occurs in a time Δt, we have:

$$v_0 = \lim_{\Delta t \to 0} \frac{(r\Delta\theta)}{\Delta t}; \quad \text{but} \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \omega$$

so:

$$v_0 = r\omega$$

EXAMPLE 2. A rigid straight bar of length l slides down a vertical wall and along a horizontal floor as shown in Fig. 2.11. The end A has a constant downward vertical velocity v_A. Find the angular velocity and the angular acceleration of the bar as a function of θ.

Solution. Let

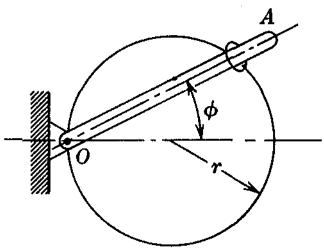
$$y = l \cos \theta$$
$$\dot{y} = -l (\sin \theta) \dot{\theta}$$

but $\dot{\theta} = \omega$ and $\dot{y} = v_A$, so $v_A = -l\omega \sin \theta$ and the angular velocity is $\omega = \dfrac{v_A}{-l \sin \theta}$. An additional differentiation gives the angular acceleration $\ddot{\theta}$:

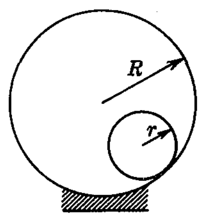
$$\ddot{y} = -l (\sin \theta)\ddot{\theta} - l (\cos \theta)\dot{\theta}^2 = 0$$
$$\ddot{\theta} = -\frac{\omega^2 \cos \theta}{\sin \theta} = \frac{v_A{}^2}{l^2 \sin^2 \theta \tan \theta}$$

PROBLEMS

2.9. A small ring moves on a circular hoop of radius r. A rod OA passes through the ring and rotates about the fixed point O on the circumference



PROB. 2.9



PROB. 2.12

of the ring with a constant angular velocity $\dot{\phi}$. Find the absolute acceleration of the ring.

2.10. A rigid body is rotating with an angular velocity of magnitude 500 rpm about a fixed axis which has the direction and location of the radius vector $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$. Find the linear velocity of the point $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ ft in the body.

2.11. Check the magnitudes and signs of the individual components of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of Equations (2.13) by considering rotations of the unit vectors separately about each axis.

2.12. A cylinder of radius r rolls without slipping on the inside of a larger cylindrical surface of radius R, as shown in the figure. Find the relationship between the linear velocity of the center of the rolling cylinder and its absolute angular velocity.

2.3 Motion Referred to a Moving Coordinate System. Suppose that the position of a point P (Fig. 2.12) is determined with respect to an xyz coordinate system, while at the same time this coordinate system moves with a translational velocity $\dot{\boldsymbol{R}}$ and an angular velocity $\boldsymbol{\omega}$ with respect to a "fixed" XYZ coordinate system. This is the type of coordinate system which might become necessary, for example, in a long range ballistics problem for which the motion of the earth would have to be taken into account. In such a problem

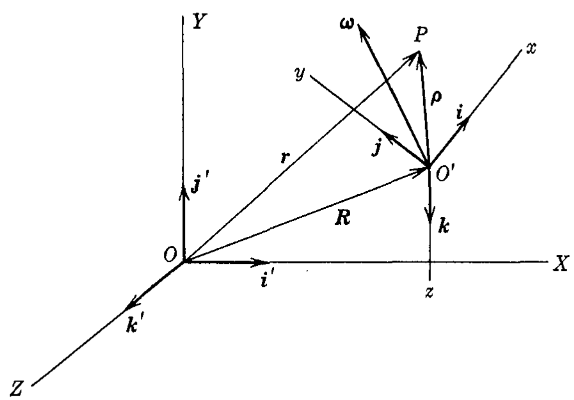


FIG. 2.12

the measurements would be made with respect to the earth, and the motion of the earth relative to some coordinate system fixed with respect to certain stars would be considered. We shall now derive a general expression for the acceleration of a point referred to a coordinate system which itself is moving.

In the analysis to follow, we shall always measure the vectors $\boldsymbol{R}$ and $\boldsymbol{r}$ in the fixed XYZ system. The unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ always have the direction of the moving coordinate axes, while the unit vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ always have the direction of the fixed coordinate axes.

By the *absolute displacement* $\boldsymbol{r}$ of the point P is meant the displacement measured with respect to the fixed XYZ system. By differentiating this absolute displacement we obtain the *absolute velocity* $\dot{\boldsymbol{r}}$ and the *absolute acceleration* $\ddot{\boldsymbol{r}}$.

$$\begin{aligned}\boldsymbol{r} &= X\boldsymbol{i}' + Y\boldsymbol{j}' + Z\boldsymbol{k}' \\ \dot{\boldsymbol{r}} &= \dot{X}\boldsymbol{i}' + \dot{Y}\boldsymbol{j}' + \dot{Z}\boldsymbol{k}' \\ \ddot{\boldsymbol{r}} &= \ddot{X}\boldsymbol{i}' + \ddot{Y}\boldsymbol{j}' + \ddot{Z}\boldsymbol{k}'\end{aligned} \tag{2.14}$$

During these differentiations, the unit vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ are treated as constants, since neither their magnitudes nor their directions change with time.

If we wish to express the absolute motion in terms of motion measured in the moving xyz system, we have:

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{\rho} = \boldsymbol{R} + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

where the directions of the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ unit vectors are known with respect to the fixed system. However, the unit vectors are changing direction with time, since they rotate with the xyz system. In taking the derivatives $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$, therefore, the time derivatives of these unit vectors must be included:

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + \dot{\boldsymbol{\rho}} = \dot{\boldsymbol{R}} + \dot{x}\boldsymbol{i} + x\dot{\boldsymbol{i}} + \dot{y}\boldsymbol{j} + y\dot{\boldsymbol{j}} + \dot{z}\boldsymbol{k} + z\dot{\boldsymbol{k}}$$

The derivatives of the unit vectors are given by Equations (2.13):

$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i}; \quad \dot{\boldsymbol{j}} = \boldsymbol{\omega} \times \boldsymbol{j}; \quad \dot{\boldsymbol{k}} = \boldsymbol{\omega} \times \boldsymbol{k}$$

so that

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) + \boldsymbol{\omega} \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

The quantity $(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k})$ represents the velocity of the point P, measured relative to the moving coordinate system, which we shall call the *relative velocity* $\dot{\boldsymbol{\rho}}_r$. Using this notation, the expression for $\dot{\boldsymbol{r}}$ becomes:

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + \dot{\boldsymbol{\rho}}_r + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{2.15}$$

and it is seen that $\dot{\boldsymbol{\rho}} = \dot{\boldsymbol{\rho}}_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$.

The acceleration of P may be found by a second differentiation:

$$\begin{aligned}\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \ddot{\boldsymbol{\rho}} = \ddot{\boldsymbol{R}} &+ (\ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}) + (\dot{x}\dot{\boldsymbol{i}} + \dot{y}\dot{\boldsymbol{j}} + \dot{z}\dot{\boldsymbol{k}}) \\ &+ \dot{\boldsymbol{\omega}} \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) + \boldsymbol{\omega} \times (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) \\ &+ \boldsymbol{\omega} \times (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}} + z\dot{\boldsymbol{k}})\end{aligned}$$

Writing $(\ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}) = \ddot{\boldsymbol{\rho}}_r$ which we call the *relative acceleration* of the point P, the expression for $\ddot{\boldsymbol{r}}$ can be written:

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r \qquad (2.16)$$

The first three terms in this expression for $\ddot{\boldsymbol{r}}$ represent the absolute acceleration of a point attached to the moving coordinate system, coincident with the point P at any given time. This may be seen by noting that for a point fixed in the moving system $\dot{\boldsymbol{\rho}}_r = \ddot{\boldsymbol{\rho}}_r = 0$. The fourth term $\ddot{\boldsymbol{\rho}}_r$ represents the acceleration of P relative to the moving system. The last term $2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$ is sometimes called the *acceleration of Coriolis*, after G. Coriolis (1792–1843), a French engineer who first called attention to this term.

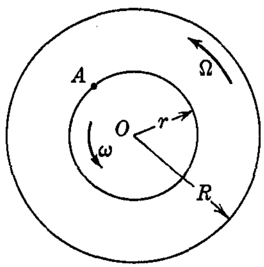


FIG. 2.13

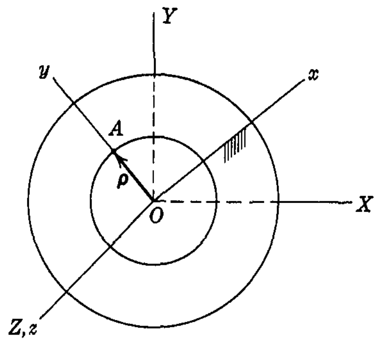


FIG. 2.14

The equation of motion in terms of the moving coordinate system may thus be written:

$$\boldsymbol{F} = m\ddot{\boldsymbol{R}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + m\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + m\ddot{\boldsymbol{\rho}}_r + 2m\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r \qquad (2.17)$$

Some applications of this equation will be given in the chapter on rigid body dynamics.

EXAMPLE 1. Two concentric disks of radius r and R rotate about the same fixed center O, as shown in Fig. 2.13. The constant angular velocity of the large disk, measured in a fixed system, is Ω. The constant angular velocity of the small disk relative to the large disk, measured in a coordinate system attached to the large disk, is ω. Find the acceleration of the point A, on the circumference of the small disk, and indicate the Coriolis acceleration term in this problem.

Solution. We shall solve this simple problem in two ways, as a means of gaining a thorough insight into the kinematics of relative motion. We shall first use Equation (2.16), taking the coordinate systems as shown in Fig. 2.14. The XY system is fixed, with the origin at the center of rotation. The moving xy system is attached to the large disk with its origin at the center of rotation, and hence has an angular velocity $\Omega\boldsymbol{k}$. Then we have:

$$\ddot{\boldsymbol{r}}_A = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

where in this problem:

$$\ddot{\boldsymbol{R}} = 0, \quad \boldsymbol{\omega} = \Omega\boldsymbol{k}, \quad \boldsymbol{\rho} = r\boldsymbol{j}, \quad \dot{\boldsymbol{\omega}} = 0, \ \dot{\boldsymbol{\rho}}_r = -r\omega\boldsymbol{i}, \text{ and } \ddot{\boldsymbol{\rho}}_r = -r\omega^2\boldsymbol{j}$$

Thus we obtain:

$$\ddot{\boldsymbol{r}}_A = -r\Omega^2\boldsymbol{j} - r\omega^2\boldsymbol{j} - 2\Omega(r\omega)\boldsymbol{j} = -r(\omega + \Omega)^2\boldsymbol{j}$$

As a second method of solution, we may note that if the angular velocity of the small disk with respect to the large disk is $\boldsymbol{\omega}$, and if the large disk has an absolute angular velocity $\boldsymbol{\Omega}$, then the absolute angular velocity of the small disk is $(\boldsymbol{\omega} + \boldsymbol{\Omega})$. The absolute velocity of the point A is thus $r(\omega + \Omega)$. Since the angular velocities are constant, the only acceleration component is the normal v^2/r component, as given by Equation (2.10). Thus we have:

$$\ddot{\boldsymbol{r}}_A = -\frac{[r(\omega + \Omega)]^2}{r}\boldsymbol{j} = -r(\omega + \Omega)^2\boldsymbol{j}$$

as in the first solution.

It will be evident from this solution that the Coriolis acceleration term $-2r\omega\Omega\boldsymbol{j}$ is just the cross-product term which is the consequence of squaring a sum. Since the expressions for velocity do not involve squares, no term corresponding to the Coriolis acceleration term appears in them.

EXAMPLE 2. A small body of mass m slides on a rod which is the chord of a circular wheel, as in Fig. 2.15. The wheel rotates about its center O with a clockwise angular velocity $\omega = 4$ rad/sec, and a clockwise angular acceleration $\dot{\omega} = 5$ rad/sec^2. The body m has a constant velocity of 6 ft/sec to the right, relative to the wheel. Find the absolute acceleration of m when $\rho = 1.5$ ft if $R = 3$ ft.

Solution. We shall fix the moving xyz coordinate system to the wheel as shown in Fig. 2.15. The angular velocity of the coordinate system is $\boldsymbol{\omega} = 4$ rad/sec $\boldsymbol{k}$ and the angular acceleration $\dot{\boldsymbol{\omega}} = 5$ rad/sec² $\boldsymbol{k}$. Applying Equation (2.16):

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

the terms may be evaluated as follows:

$$\begin{aligned}
\ddot{\boldsymbol{R}} &= -(3\text{ ft})(4\text{ rad/sec})^2\boldsymbol{i} \\
&\qquad + (3\text{ ft})(5\text{ rad/sec}^2)\boldsymbol{j} &&= -48\boldsymbol{i} + 15\boldsymbol{j}\text{ ft/sec}^2 \\
\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) &= (4\text{ rad/sec})(4\text{ rad/sec})(1.5\text{ ft})\boldsymbol{j} &&= \qquad\quad + 24\boldsymbol{j} \\
\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} &= (5\text{ rad/sec}^2)(1.5\text{ ft})\boldsymbol{i} &&= +\ 7.5\boldsymbol{i} \\
\ddot{\boldsymbol{\rho}}_r &= 0 \\
2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r &= -2(4\text{ rad/sec})(6\text{ ft/sec})\boldsymbol{i} &&= -48\boldsymbol{i} \\
\hline
\ddot{\boldsymbol{r}} &= &&\ -88.5\boldsymbol{i} + 39\boldsymbol{j}\text{ ft/sec}^2
\end{aligned}$$

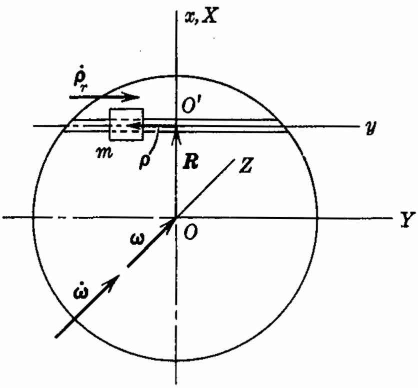


FIG. 2.15

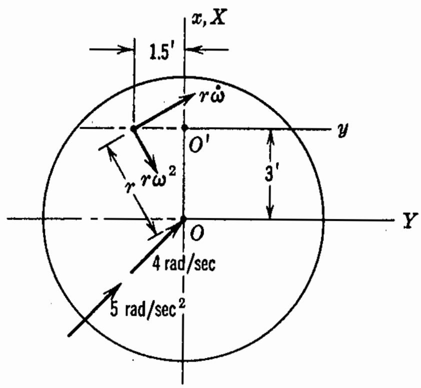


FIG. 2.16

In some problems of this type the sum of the first three terms ($\ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}$) can be computed more directly by noting that this vector sum represents the absolute acceleration of a fixed point on the moving coordinate system which is coincident with the moving point. In the present problem, for example, this coincident-point acceleration is (Fig. 2.16):

$$\begin{aligned}
&\left[(r\dot{\omega})\left(\frac{1.5}{r}\right) - (r\omega^2)\left(\frac{3}{r}\right)\right]\boldsymbol{i} + \left[(r\dot{\omega})\left(\frac{3}{r}\right) + (r\omega^2)\left(\frac{1.5}{r}\right)\right]\boldsymbol{j} \\
&\quad = [(1.5)(5) - (3)(4)^2]\boldsymbol{i} + [(3)(5) + (1.5)(16)]\boldsymbol{j} \\
&\quad = -40.5\boldsymbol{i} + 39\boldsymbol{j}\text{ ft/sec}^2
\end{aligned}$$

which will be seen to be equal to the sum of the first three vectors of the solution by the other method.

EXAMPLE 3. A simplified picture of the mechanism of a helicopter blade is shown in Fig. 2.17. The blade oscillates about the horizontal axis P–P', which is carried on a rotating disk OB so that the whole assembly rotates with a constant angular velocity $\boldsymbol{\omega}$. The blade has a mean position defined by the line BC, which makes an angle θ with the horizontal. At any time t the blade makes an angle ϕ with the line BC, where ϕ is given by the equation $\phi = \phi_0 \sin pt$, p being the angular frequency of the "flapping" oscillation of the

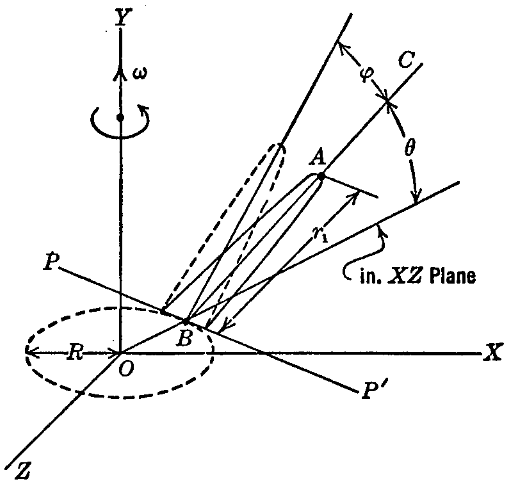


FIG. 2.17

blade. Find the velocity and acceleration of A, the tip of the blade, when $\phi = 0$ and the blade is moving upwards.

Solution. We shall first find the velocity and acceleration of A relative to a coordinate system which rotates with the disk OB with an angular velocity ω, and whose origin is located at B. We orient the system so that $\boldsymbol{R}$ lies along X-axis (Fig. 2.18). Since:

$$\phi = \phi_0 \sin pt$$

then:

$$\dot{\phi} = \phi_0 p \cos pt$$
$$\ddot{\phi} = -\phi_0 p^2 \sin pt$$

Thus, the magnitudes of the relative velocity and acceleration, when $\phi = 0$, are:

$$\begin{aligned} \dot{\rho}_r &= r_1\dot{\phi} = r_1\phi_0 p \\ (\ddot{\rho}_r)_t &= r_1\ddot{\phi} = 0 \\ (\ddot{\rho}_r)_n &= r_1\dot{\phi}^2 = r_1\phi_0{}^2 p^2 \end{aligned}$$

Now, using Equation (2.15):

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + \dot{\boldsymbol{\rho}}_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

with the following values of the terms:

$$\begin{aligned} \dot{\boldsymbol{R}} &= -R\omega\boldsymbol{k} \\ \dot{\boldsymbol{\rho}}_r &= -r_1\phi_0 p \sin\theta\boldsymbol{i} + r_1\phi_0 p \cos\theta\boldsymbol{j} \\ \boldsymbol{\omega} \times \boldsymbol{\rho} &= -r_1\omega\cos\theta\boldsymbol{k} \end{aligned}$$

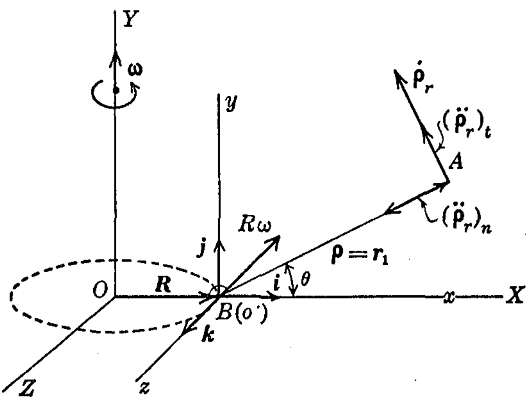


FIG. 2.18

So that we obtain:

$$\dot{\boldsymbol{r}} = (-r_1\phi_0 p \sin\theta)\boldsymbol{i} + (r_1\phi_0 p \cos\theta)\boldsymbol{j} + (-R\omega - r_1\omega\cos\theta)\boldsymbol{k}$$

To find the acceleration we use Equation (2.16):

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

where in this problem the terms become:

$$\begin{aligned} \ddot{\boldsymbol{R}} &= -R\omega^2\boldsymbol{i} \\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) &= -r_1\omega^2\cos\theta\,\boldsymbol{i} \\ \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} &= 0 \\ \ddot{\boldsymbol{\rho}}_r &= -r_1\phi_0{}^2p^2\cos\theta\,\boldsymbol{i} - r_1\phi_0{}^2p^2\sin\theta\,\boldsymbol{j} \\ 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r &= 2\phi_0 r_1 p\omega\sin\theta\,\boldsymbol{k} \end{aligned}$$

So that

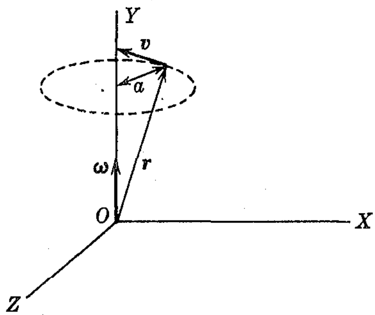
$$\ddot{\mathbf{r}} = (-R\omega^2 - r_1\omega^2 \cos\theta - r_1\phi_0^2 p^2 \cos\theta)\mathbf{i} + (-r_1\phi_0^2 p^2 \sin\theta)\mathbf{j} + (2\phi_0 r_1 p\omega \sin\theta)\mathbf{k}$$

PROBLEMS

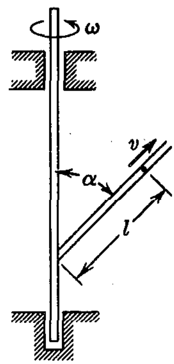
2.13. A particle moves in a circular path of radius a with a constant angular velocity $\boldsymbol{\omega}$ as shown in the diagram. (a) Show that the acceleration term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ has a radial direction and a magnitude of

$$a\omega^2 = \frac{v^2}{a}$$

(b) The magnitude of the angular velocity of the particle varies with



PROB. 2.13



PROB. 2.14

time according to the equation $\omega = \alpha t$ where α is a constant angular acceleration. The acceleration of the particle is given by

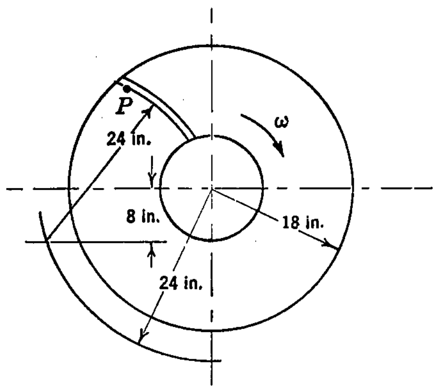
$$\mathbf{a} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega \times \dot{\mathbf{r}}.$$

Find the magnitude and direction of each of the two terms $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ and $\boldsymbol{\omega} \times \dot{\mathbf{r}}$.

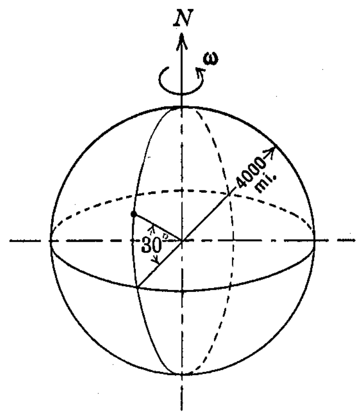
2.14. A straight tube is attached to a vertical shaft at a fixed angle α as shown in the figure. The shaft rotates with a constant angular velocity ω. A particle moves along the tube with a constant velocity v relative to the tube. Find the magnitude of the acceleration of the particle when it is at a distance l along the tube from the center. Do this problem in two ways: (a) by setting up a spherical coordinate system and using the known expressions for the components of acceleration in a spherical system, and (b) by setting up a moving and a fixed coordinate system and using the known expression for the acceleration of a particle which moves in a moving coordinate system.

2.15. A particle of water P moves outward along the impeller of a centrifugal pump with a constant tangential velocity of 100 ft/sec relative to the impeller. The impeller is rotating with a uniform speed of 1800 rpm in a clockwise direction. What is the acceleration of the particle at the point where it leaves the impeller?

2.16. A river is flowing directly south along the surface of the earth at a uniform speed of 5 mph relative to the earth. What is the acceleration of a particle of water in the river when it crosses the 30° North latitude line?



PROB. 2.15

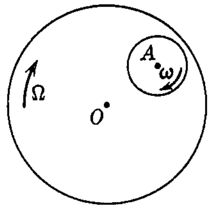


PROB. 2.16

2.17. Referring to the helicopter blade of Example 3 above, find the acceleration of the blade tip A when the angle ϕ has one-half of its maximum value $\phi_0 = 6°$ and is increasing. The "coning angle" θ is 7°, the radius to the tip of the blade is $r_1 = 15$ ft, the radius of the disk is $R = 1$ ft, and the assembly rotates at 225 rpm. The blades flap once per revolution of the rotor, that is, $p = \omega$.

2.18. A horizontal track for experimental studies of high velocity missiles is to be designed. Because of the rotation of the earth, there would be, for certain orientations of the track, accelerations at right angles to the rails in a horizontal plane, and hence lateral forces between the missile carriage and the track would be present and might be a factor in the design of the structure. Find this lateral acceleration for a N–S orientation and for an E–W orientation, assuming that the missile has a constant velocity of 2000 ft/sec and that the track is located at a 30° N latitude. Assume that the E–W track follows a circle of latitude.

2.19. A large disk rotates with a constant angular velocity Ω about a central shaft O. A small disk is attached to the large disk by a shaft at A, as shown in the figure. The small disk rotates with a constant angular velocity ω with respect to the large disk. Find the relationship between ω and Ω for which the absolute acceleration vector of any point on the circumference of the small disk passes through the point O.



Prob. 2.19

2.20. An airplane flies in a vertical arc of 1000 ft radius with a constant speed of 300 mph. The propeller, which has a 12 ft diameter, rotates at an angular speed of 1200 rpm, clockwise viewed from the rear. Find the absolute acceleration of the propeller tip when both the plane and the propeller blade are horizontal. Consider the left hand blade, looking from the rear and assume that the plane is about to climb.

2.21. The center of mass of a single engine airplane is travelling along a horizontal straight line with a velocity $\boldsymbol{v}$ and an acceleration $\boldsymbol{a}$. The angle of pitching of the airplane about a horizontal axis through its center of mass perpendicular to the flight path is $\phi = \phi_0 \sin pt$. The radius of the propeller is r_0 and its center is a distance d ahead of the center of mass. The propeller rotates at ω rad/sec clockwise as viewed from the rear. (a) Find the velocity of the tip of the propeller blade at the time $t = \dfrac{2\pi}{p}$ if the blade is horizontal and is moving upwards at that time. (b) Find the acceleration of the tip of the propeller under the conditions of part (a).

Chapter 3

DYNAMICS OF A PARTICLE

Newton admits nothing but what he gains from experiments and accurate observations. From this foundation, whatever is further advanced, is deduced by strict mathematical reasoning.—William Emerson, *The Principles of Mechanics* (1754)

The equation of motion as given in Chapter 1 is theoretically sufficient for the solution of any of the solvable problems of classical mechanics. There are several other ways, however, of presenting the basic information contained in this equation. Each of these has advantages for the solution of certain types of problems. In the present chapter we shall first show, in some simple examples, how the equation of motion can be integrated directly to give the solution of certain types of problems, and we shall then discuss some other forms in which the equation of motion can be expressed.

The problems treated in this chapter will be restricted to the dynamics of a particle. If rotational effects can be neglected for a particular body, then that body can be treated as though it were a single particle with the mass of the body concentrated at one point.* If the rotational effects need to be considered, then the problem must be treated by the more general methods of rigid body dynamics. It should be noted that the same body might in one problem behave as a particle, while in another problem it might have to be treated as a rigid body. For example, a cannon ball shot through the air could be treated as a particle; the same ball rolled along the ground would have to be considered as a rigid body of a given radius.

* See Section 6.2, Chapter 6.

In general, we shall consider any body as being made up of a number of particles, so that, once the basic laws describing the motion of a particle have been established, the theory may be extended to any body without the introduction of new principles. By first studying the behavior of a single particle, the various laws of dynamics can be exhibited in their simplest form, unencumbered with the purely mathematical difficulties involved in the description of complex motions.

3.1 Integration of the Equation of Motion for Particular Problems. In many problems the known quantities and the information desired are such that a direct integration of the equation $F = m\ddot{r}$, expressed in an appropriate coordinate system, will give the solution.

EXAMPLE 1. Consider a particle, of mass m, which at time $t = 0$ is projected horizontally with an initial velocity $\dot{x}_0$, and is subsequently acted upon by gravity and by air resistance. Find the position and velocity of the particle at any subsequent time.

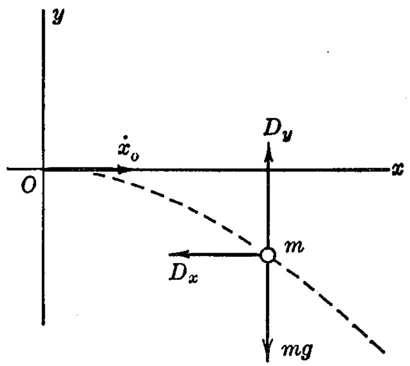


FIG. 3.1

Solution. Fig. 3.1 shows a free-body diagram of the particle with all the forces acting. The drag force produced by air resistance has been resolved into two rectangular components, and the gravity force is shown as a downward force mg. To describe the motion we choose a rectangular xy coordinate system with the xy plane coinciding with the plane of motion. In this system the equation $F = m\ddot{r}$ becomes:

$$F_x = m\ddot{x} = -D_x$$

$$F_y = m\ddot{y} = D_y - mg$$

In general, the drag forces D_x and D_y will be functions of the velocities $\dot{x}$ and $\dot{y}$, and these functions must be known before the equations can be integrated. In a later section we shall consider the nature of these functions and methods of integrating the resulting equations.

For the present, as an illustration of the general method in its simplest form, let us suppose that the motion is taking place in a vacuum, so that $D_x = D_y = 0$. Then:

$$\ddot{x} = 0$$
$$\ddot{y} = -g$$

Integrating once:

$$\dot{x} = C_1$$
$$\dot{y} = -gt + C_2$$

The constants of integration can be determined from the initial conditions $\dot{x} = \dot{x}_0$, $\dot{y} = 0$ when $t = 0$; hence:

$$C_1 = \dot{x}_0$$
$$C_2 = 0$$

Performing a second integration:

$$x = \dot{x}_0 t + C_3$$
$$y = -\frac{gt^2}{2} + C_4$$

Also, when $t = 0$; $x = 0$, $y = 0$, so $C_3 = C_4 = 0$, and we have the result:

$$x = \dot{x}_0 t$$
$$y = -\frac{1}{2} g t^2$$

By eliminating t, the relationship between x and y is found:

$$y = \frac{1}{2} \frac{g}{\dot{x}_0^2} x^2$$

and the path of motion is found to be a parabola.

EXAMPLE 2. J. V. Poncelet (1829) concluded on the basis of tests that when a projectile of mass m is fired into earth or masonry it experiences a retarding force $F_D = -C_1 - C_2 v^2$, where the constants C_1 and C_2 depend on the properties of the material and the shape of the projectile. If the impact velocity of the projectile is v_0, find the total penetration.

Solution. The equation of motion is:

$$m \frac{dv}{dt} = -C_1 - C_2 v^2$$

In this problem we wish to have a relationship between velocity and distance, so we shall change the variable in the differential equation as follows:

$$m\frac{dv}{dt} = m\frac{dx}{dt}\cdot\frac{dv}{dx} = mv\frac{dv}{dx}$$

so:

$$mv\frac{dv}{dx} = -C_1 - C_2v^2$$

The variables in this differential equation can be separated to give:

$$\frac{mvdv}{-C_1 - C_2v^2} = dx$$

which integrates directly to:

$$\frac{-m}{2C_2}\log(-C_1 - C_2v^2) = x + C$$

when $x = 0$, $v = v_0$, so $C = -\frac{m}{C_2}\log(-C_1 - C_2v_0^2)$

and:

$$x = \frac{m}{2C_2}\log\left(\frac{-C_1 - C_2v_0^2}{-C_1 - C_2v^2}\right)$$

At maximum penetration $v = 0$, so finally:

$$x_{\max} = \frac{m}{2C_2}\log\left(1 + \frac{C_2}{C_1}v_0^2\right)$$

EXAMPLE 3. A particle is attracted toward the origin of a coordinate system by a radial force. Show that the angular velocity of the particle is inversely proportional to the square of its distance from the origin.

Solution. A polar coordinate system will be convenient for the solution of this problem. The equations of motion are:

$$F_r = m(\ddot{r} - r\dot{\phi}^2);\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

Since the force is radial, we have $F_\phi = 0$, so:

$$(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}) = 0$$

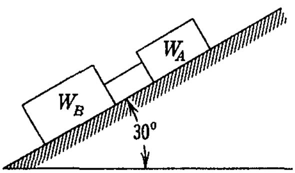
Thus:

$$r^2\dot{\phi} = \text{constant} = C,\ \dot{\phi} = \frac{C}{r^2}$$

PROBLEMS

3.1. Given a particle of mass m moving in a circular path of radius r with a velocity of constant magnitude v. Find the force required to maintain this motion. Do this problem in two ways: (a) by describing the motions in a rectangular coordinate system whose origin is at the center of the circular path and (b) by using a plane polar coordinate system to describe the motion. Compare the two procedures.

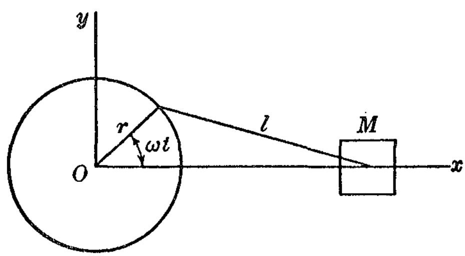
3.2. A block W_A weighing 15 lbs and a block W_B weighing 25 lbs slide down a 30° inclined plane as shown in the figure. The blocks are connected by a link of negligible mass. The coefficient of sliding friction between W_B and the plane is 0.20, and between W_A and the plane is 0.40. Find the force in the connecting link, and the acceleration of the blocks.



PROB. 3.2

3.3. A weight rests on the top of a frictionless sphere of radius r. The weight slides down the side of the sphere in a vertical plane under the action of gravity. Find both the point at which the weight leaves the sphere and the velocity at that instant.

3.4. A wheel of radius r rotates with a uniform angular velocity, ω. A massless connecting rod of length l is fastened to the wheel, and moves a piston of mass M back and forth along the x-axis as shown in the figure.



PROB. 3.4

Show that the x-component of the resultant force acting upon M is:

$$R_x = -Mr\omega^2\left[\cos\omega t + \frac{rl^2\cos 2\omega t + r^3\sin^4\omega t}{(l^2 - r^2\sin^2\omega t)^{3/2}}\right]$$

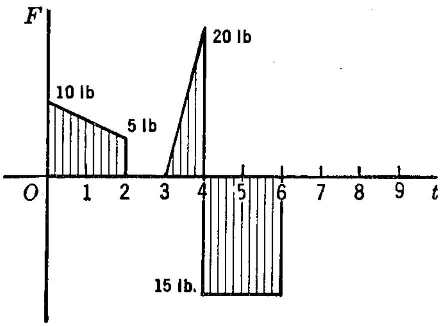
Find an approximate solution for R_x for the particular case in which the ratio $\left(\frac{r}{l}\right)$ is small compared to unity.

3.5. The drag force exerted by the water on a ship weighing 12,000 tons varies with speed according to the following approximate formula:

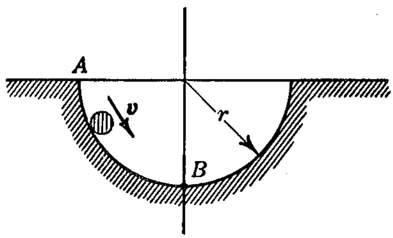
$$F_D = -kv^n$$

For the range of speeds to be considered in the present problem, n may be taken as 3. If the drag force for a particular ship has been determined to be 80 tons at a speed of 16 ft/sec, find the distance which the unpowered ship would travel as its speed decreases from 15 ft/sec to 12 ft/sec. What is the time required for this decrease?

3.6. The force acting on a body which weighs 150 lb and which moves in a straight line is given at any time by the accompanying graph. After $t = 6$ sec, $F = 0$. If the velocity of the particle is 25 ft/sec when $t = 0$, find the velocity of the particle and the distance that it has traveled at $t = 8$ sec.



PROB. 3.6



PROB. 3.7

3.7. A particle of mass m starts from rest and slides down the side of a hemispherical surface of radius r under the action of gravity. If there is no friction, and the particle starts from A, what force will be exerted on the surface by the particle at the instant when it is located at B?

3.2 The Equation of Impulse and Momentum. If the force acting upon a particle is specified as a function of time, the direct integration of the equation of motion, as illustrated in the preceding section, will give the solution of the problem. If a complete specification of the forces is not available, it may still be possible to obtain some information about the motion by obtaining a general first integral of the equation of motion. Two such general integrals can be obtained and we shall first derive the time integral.

Beginning with the equation of motion in the form:

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}$$

and multiplying both sides by dt and integrating, we obtain:

$$\int_{t_1}^{t_2} \boldsymbol{F}\,dt = \int_{t_1}^{t_2} m\ddot{\boldsymbol{r}}\,dt = m\dot{\boldsymbol{r}}\Big|_1^2 = m\dot{\boldsymbol{r}}_2 - m\dot{\boldsymbol{r}}_1$$

$$\int_{t_1}^{t_2} \boldsymbol{F}\,dt = m\boldsymbol{v}_2 - m\boldsymbol{v}_1 \tag{3.1}$$

The term $\int_{t_1}^{t_2} \boldsymbol{F}\,dt$ is called the *impulse* of the force $\boldsymbol{F}$ and the term $(m\boldsymbol{v})$ is called the *momentum* of the particle. Equation (3.1) thus states that the impulse is equal to the *change* in the momentum.

Both impulse and momentum are vector quantities and hence can be written in terms of components in various coordinate systems. In rectangular components, for example, the equations are:

$$\int_{t_1}^{t_2} F_x\,dt = m\dot{x}_2 - m\dot{x}_1; \text{ etc.}$$

It should be noted that the impulse-momentum equation is but another form of the equation of motion and that it furnishes no new information, although its use may simplify the solution of certain problems. In some problems the impulse applied to the system may be known whereas the forces are unknown. In such a problem the impulse-momentum equation gives the change of velocity directly. If no force acts upon a particle, the equation of impulse-momentum is:

$$m\boldsymbol{v}_2 - m\boldsymbol{v}_1 = 0$$

$$m\boldsymbol{v}_2 = m\boldsymbol{v}_1$$

If no impulse is acting there is no change in momentum, and the momentum of the system may be said to be conserved.

Consider two particles of mass m_a and m_b which exert a mutual action upon each other as, for example, in a collision. From the Third Law of Motion we know that the forces, and hence the impulses experienced by m_a and m_b during this mutual interaction are equal and opposite. The total impulse for the system of two particles is thus zero, and hence the total change in momentum of the system must be zero. We may thus state that the total momentum is a constant:

$$m_a\mathbf{v}_a + m_b\mathbf{v}_b = \text{constant}$$

If there are more than two particles involved and all the forces acting upon the particles are due to mutual interactions, that is, there are no external forces applied to the system, we can say:

$$\Sigma m_i\mathbf{v}_i = \text{constant}$$

This is a vector equation and in rectangular coordinates its components are:

$$\Sigma m_i\dot{x}_i = \text{constant, etc}$$

This is the *Principle of the Conservation of Momentum,* which holds for any system upon which no external forces are acting.

An extension of the above concept of the momentum vector leads to the definition of the moment of momentum vector, and to the principle of the conservation of moment of momentum. These ideas will be discussed in detail in Chapter 6.

EXAMPLE 1. Rain falls steadily at the rate of 1 in./hr. If the terminal velocity of the raindrops is 20 ft/sec, find the average force per unit area on a horizontal surface caused by vertically falling rain.

Solution. The average force will be equal to the average rate of change of momentum, which can be calculated if it is assumed that the amount of water that is splashed upwards is negligible:

$$\left(\frac{F}{A}\right)_{\text{avg}} = \frac{(1\text{ in./hr})(62.4\text{ lb/ft}^3)(20\text{ ft/sec})}{(32.2\text{ ft/sec}^2)(3600\text{ sec/hr})(12\text{ in./ft})} = 8.97 \times 10^{-4}\text{ lb/ft}^2$$

EXAMPLE 2. A uniform chain is coiled in a small pile on the floor. A man picks up one end of the chain and raises it vertically with a uniform velocity v. Find the force acting on the man's hand at any height x above the floor.

Solution. Suppose that a link of mass (Δm) is picked up in a time (Δt); then, by the impulse-momentum equation:

$$F_1(\Delta t) = (\Delta m)v; \text{ and } F_1 = \left(\frac{\Delta m}{\Delta t}\right)v$$

If the mass of the chain per unit length is μ, then $m = \mu x$ and $\frac{dm}{dt} = \mu v$ so that $F_1 = \mu v^2$. The total force acting on the man's hand will be F_1 plus the gravity force $\mu g x$, thus:

$$F = \mu(gx + v^2)$$

EXAMPLE 3. A gun whose barrel makes an angle α with the horizontal fires a shell having muzzle velocity v_r with respect to the barrel. The whole gun is mounted on a frictionless horizontal track, so that recoil takes place with no resistive forces. The total mass of the gun is M and the mass of the shell is m. Find the recoil velocity V of the gun, and the magnitude of the absolute velocity of the shell as it leaves the gun.

Solution. Since there are no horizontal forces acting on the system of the gun, shell, and explosive gases, the horizontal momentum of the system is conserved. This horizontal momentum is initially zero, so, neglecting the small momentum of the explosive gases, we have:

$$m(v_r \cos\alpha - V) - MV = 0$$

from which:

$$V = \frac{m}{(m+M)} v_r \cos\alpha$$

The components of the absolute velocity of the shell are $(v_r \cos\alpha - V)$ and $v_r \sin\alpha$; thus:

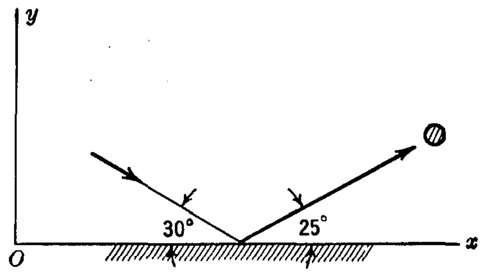
$$|v_s| = \sqrt{(v_r \cos\alpha - V)^2 + (v_r \sin\alpha)^2}$$

$$= v_r\sqrt{1 - \frac{m(m+2M)}{(m+M^2)}\cos^2\alpha}$$

PROBLEMS

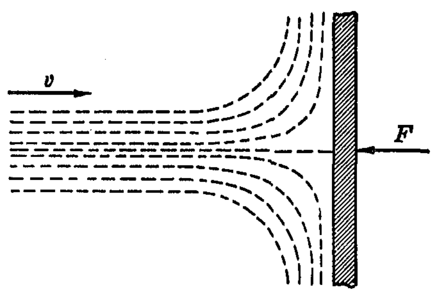
3.8. A ball weighing 1 lb is thrown vertically upward; neglecting air resistance, find: (a) the velocity at $t = 1$ sec, if the velocity at $t = 0$ is 30 ft/sec and (b) the velocity at $t = 0$, given that the ball reaches its maximum height after 2.5 sec.

3.9. A particle weighing 5 lb bounces against a surface as shown in the diagram. If the approach velocity is 20 ft/sec and the velocity of departure is 15 ft/sec, find the magnitude and direction of the impulse to which the mass is subjected.



PROB. 3.9

3.10. A jet of water impinges against a flat plate as shown in the diagram. The velocity of the water is v ft/sec, the density is ρ lb sec^2/ft^4. What is the force exerted by the jet against the plate? Taking the weight of water as 62.4 lb/ft^3, find the force for a jet having an area of 6 in.^{2} and a velocity of 30 ft/sec.



PROB. 3.10

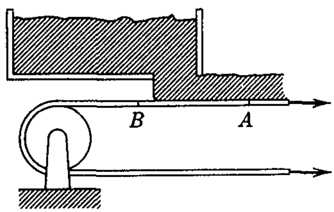
3.11. Two men A and B of equal weight hold on to the free ends of a massless rope which passes over a frictionless pulley of negligible weight. A climbs up the rope with a velocity v_r relative to the rope, while B hangs on without climbing. Find the absolute velocity of B.

3.12. A body weighing 10 lbs is projected up an inclined plane which makes an angle of 20° with the horizontal. The coefficient of sliding friction between the body and the plane is $\mu = 0.3$. At time $t = 0$, the velocity up the plane is 20 ft/sec. What will be the velocity at the end of 3 sec?

3.13. A projectile weighing 100 lbs strikes the concrete wall of a fort with an impact velocity of 1200 ft/sec. The projectile comes to rest in

0.01 sec, having penetrated 6 ft of the 8-ft thick wall. What is the time average force exerted on the wall by the projectile?

3.14. A machine gun fires a steady stream of bullets into a stationary target in which the bullets come to rest. Each bullet weighs 1 oz and on impact has a velocity of 1200 ft/sec. Find the average force exerted on the target if the rate of fire is 800 per minute.



PROB. 3.15

3.15. Dry material is discharged from a bin to a belt conveyor at the rate of 3000 lb/min, as shown in the diagram. The speed of the belt is 5 ft/sec. Find the difference between the belt tension at A and B.

3.3 The Equation of Work and Energy.

The first integral of the equation of motion with respect to time gives the equation of impulse and momentum. We shall now derive the first integral of the equation of motion with respect to displacement.

We start with the equation of motion in the form:

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}$$

Forming the dot product of each side with the displacement $d\boldsymbol{r}$, and integrating, we obtain:

$$\int_{r_1}^{r_2} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{r_1}^{r_2} m\ddot{\boldsymbol{r}}\cdot d\boldsymbol{r} = \int_{t_1}^{t_2} m\ddot{\boldsymbol{r}}\cdot\frac{d\boldsymbol{r}}{dt}\,dt$$

$$= \frac{1}{2}m\int_{t_1}^{t_2}\frac{d}{dt}(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\,dt = \frac{1}{2}m\int_{t_1}^{t_2}\frac{d}{dt}(v^2)\,dt$$

$$\int_{r_1}^{r_2} \boldsymbol{F}\cdot d\boldsymbol{r} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{3.2}$$

The integral on the left side of this equation is called the *work* done by the force $\boldsymbol{F}$, and the term $\frac{1}{2}mv^2$ is called the *kinetic energy* of the particle. Thus the equation states that the work done upon the mass m by the force $\boldsymbol{F}$ is equal to the *change* in the kinetic energy of the mass.

The vector displacement $d\boldsymbol{r}$ is tangent to the path of motion of the particle, so that the scalar product $\boldsymbol{F}\cdot d\boldsymbol{r}$ represents the component of the force in the direction of the displacement multiplied by the displacement. The total work done by the force in moving along a path from A to B (Fig. 3.2) is given by the line integral $\int_A^B \boldsymbol{F}\cdot d\boldsymbol{r}$. Expanding the dot product in terms of rectangular coordinates we have:

$$\text{Work} = \int_A^B (F_x\,dx + F_y\,dy + F_z\,dz)$$

The rate of work, $\boldsymbol{F}\cdot d\boldsymbol{r}/dt = \boldsymbol{F}\cdot\boldsymbol{v}$, is called the *power*. It should be noted that work, kinetic energy, and power are scalar quantities, as

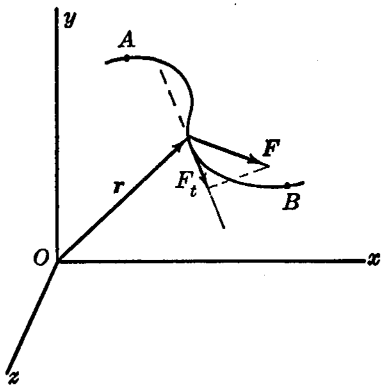


FIG. 3.2

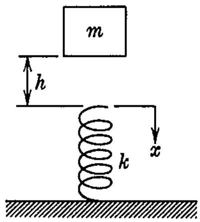


FIG. 3.3

defined by the dot-products, and are completely specified by their magnitudes.

Since the equation of work and energy is simply a restatement of the original law of motion, it cannot furnish any new information. In many problems, however, the work-energy equation leads directly to simple solutions.

EXAMPLE. A mass m falls through a distance h and strikes the end of a linear spring, as shown in Fig. 3.3. The spring constant is k lb/in., that is, it requires a force of kx lb to compress or extend the spring x in. Find the maximum compression of the spring.

Solution. The forces acting on the mass are the gravity force mg and the spring force $(-kx)$. If δ is the maximum compression of the spring, the total work done by the forces acting on the mass from its initial position to the final fully compressed position is:

$$\text{Work} = mg(h + \delta) - \int_0^\delta kx\,dx$$

Since the velocity of the mass is initially zero, and is zero again at the instant of maximum spring compression, the change in kinetic energy is zero, and hence the work done is also zero, and we have:

$$mg(h + \delta) - \tfrac{1}{2}k\delta^2 = 0$$

from which:

$$\delta = \frac{mg}{k} + \sqrt{\left(\frac{mg}{k}\right)^2 + \frac{2mgh}{k}}$$

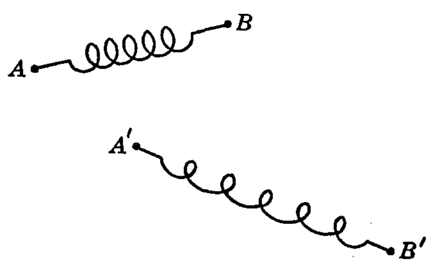
The deflection corresponding to a static force equal to the weight of the body is (mg/k). Calling this the *static deflection*, δ_{st}, we have:

$$\delta = \delta_{st}\left[1 + \sqrt{1 + \frac{2h}{\delta_{st}}}\right]$$

PROBLEMS

3.16. Integrate the equation of motion $F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} = m\ddot{x}\boldsymbol{i} + m\ddot{y}\boldsymbol{j} + m\ddot{z}\boldsymbol{k}$ term by term to obtain the equation of work-energy expressed in rectangular coordinates.

3.17. A spring which has been initially stretched into the position AB is elongated and displaced into the position $A'B'$ as shown in the diagram.

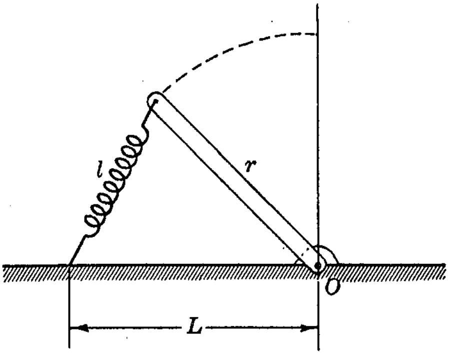


PROB. 3.17

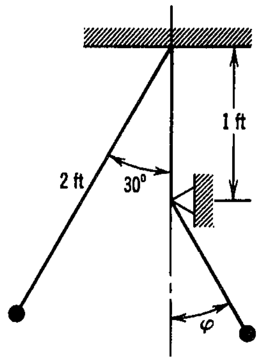
Show that the total work done by the forces which elongate the spring depends only on the change in the length of the spring and on the average force in the spring:

$$\text{Work} = (F_{\text{avg}})(A'B' - AB)$$

3.18. A spring of spring constant k whose unstretched length l is fixed at one end, while the other end is fastened to a rigid bar of length r, as shown in the diagram. How much work will be done by the force exerted by the spring on the bar as the bar is rotated about O into a vertical position?



Prob. 3.18

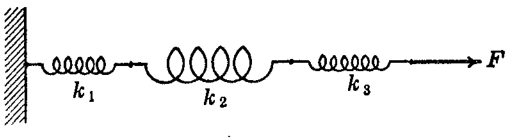


Prob. 3.19

3.19. A particle weighing 1 lb is supported as a simple pendulum by a massless inextensible string 2 ft in length. The pendulum is released from rest at an angle of 30° as shown in the diagram. When the string is just vertical, it strikes a rigid support and the particle continues to swing as a pendulum of shorter length. (a) Find the maximum value of the angle ϕ. (b) Find the force exerted on the weight by the string when $\phi = 30°$.

3.20. A particle of mass m is acted upon by a force whose components are $F_x = At$, $F_y = Bt$, $F_z = 0$. At time $t = 0$, the velocity of the mass is zero. What is the work done by the force in the first T sec?

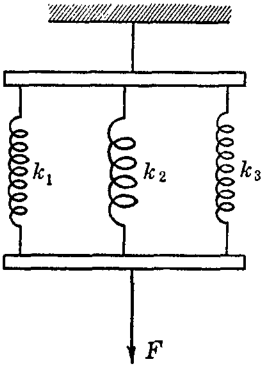
3.21. A spring whose unstretched length is l requires a force of kx lb to elongate it x ft. If three such springs having spring constants k_1, k_2, and



Prob. 3.21

k_3 are hooked together end-to-end, how much work would be done by a force F as it elongates the system of springs through a total distance δ? In such a system the springs are said to be in series. What is the equivalent spring constant for a system of springs in series in terms of the spring constants of the individual springs?

3.22. The three springs of Problem 3.22 are arranged in parallel as shown in the figure. How much work is done by the force F as the assembly is stretched through a distance δ? The springs are initially in an unstretched position and the plates remain parallel. What is the equivalent spring constant for a system of springs in parallel in terms of the spring constants of the individual springs?

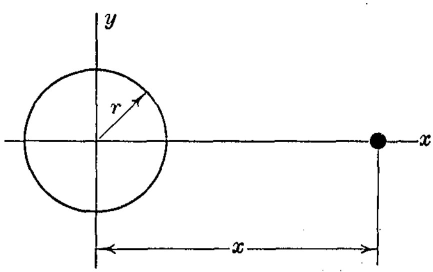


PROB. 3.22

3.23. Prove that the total work done by all of the gravity forces acting on a system is given by the product of the total weight and the vertical distance moved by the center of gravity.

3.24. A 10-ton airplane lands on the deck of a motionless aircraft carrier at a speed of 90 mph. Upon landing it catches the center of a horizontal elastic cable which is perpendicular to the direction of motion of the plane. The cable is 200 ft long, is fixed at the two ends, and has zero initial tension. At contact the propeller pitch is reversed to exert a constant braking thrust of 2000 lb. The force in the cable is $F = kx$, where x is the change in length of the cable. Find the value of k for which the plane will be stopped in 100 ft.

3.25. The force of gravity varies inversely as the square of the distance from the center of the earth. A projectile in space is thus acted upon by a gravitational force $F_x = -W\dfrac{r^2}{x^2}$, where W is the weight of the projectile at the earth's surface and r is the radius of the earth. How much work



PROB. 3.25

must be done against the gravitational force if the projectile is to reach a distance of $(x - r)$ from the earth's surface? Neglecting air resistance, what initial velocity must the projectile have in order to reach that distance? What initial velocity must the projectile have to escape from the earth's gravitational field? Take the radius of the earth as 4000 mi.

3.4 Potential. The equation of work-energy for a particle of mass m acted upon by a force $\boldsymbol{F}$ is:

$$\int_A^B \boldsymbol{F}\cdot d\boldsymbol{r} = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2$$

The right side of this equation depends only upon the velocities of the particle at the two end-points A and B. The value of the left side, however, will depend in general upon the path of integration followed between A and B. The value of the line integral $\int_A^B \boldsymbol{F}\cdot d\boldsymbol{r}$ will depend only upon the limits of integration and not upon the path only if $(\boldsymbol{F}\cdot d\boldsymbol{r})$ is an exact differential. If this is so, there exists some function Φ such that $d\Phi = \boldsymbol{F}\cdot d\boldsymbol{r}$ and:

$$\int_A^B \boldsymbol{F}\cdot d\boldsymbol{r} = \int_A^B d\Phi = \Phi\Big|_A^B = \Phi(B) - \Phi(A)$$

If the function Φ exists, the force $\boldsymbol{F}$ is said to be derivable from a *potential,* for:

$$\boldsymbol{F}\cdot d\boldsymbol{r} = F_x\,dx + F_y\,dy + F_z\,dz$$

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

From which:

$$F_x = \frac{\partial\Phi}{\partial x};\ F_y = \frac{\partial\Phi}{\partial y};\ F_z = \frac{\partial\Phi}{\partial z} \tag{3.3}$$

This potential function Φ is called a *force function.*

When a force is derivable from a potential, the work done by the force is independent of the path of motion and depends only upon the end-points of the path. Since Φ is a function of the space coordinates only, the magnitude and direction of the force are completely determined when its point of application is known. This will be true if the force is a function of the displacement only. If the force is a function of velocity, it cannot be derived from a potential, and the line integral representing the work is not independent of the path of integration.

The concept of a potential function has much wider application than is suggested by the force potential. In fluid mechanics, for example, it is customary to define a velocity potential whose derivatives give the components of velocity, and in thermodynamics several potential functions are defined whose derivatives give certain thermodynamic variables.

3.5 Potential Energy. Suppose that a force $\boldsymbol{F}$, which is derivable from a potential, acts upon a particle that moves from point A to point B. We define the *change in the potential energy of the force*, $(V_B - V_A)$, as the negative of the work done by the force as it moves from A to B.

$$V_B - V_A = -\int_A^B \boldsymbol{F}\cdot d\boldsymbol{r} \tag{3.4}$$

To specify the potential energy at a point it is necessary to select a datum point at which the potential energy is arbitrarily set equal to zero. Taking some point D as the datum point, we have $V_D = 0$ and:

$$V_A = \int_A^D \boldsymbol{F}\cdot d\boldsymbol{r} \tag{3.5}$$

The datum point D is selected at any point that is convenient for the particular problem being considered.

From the definition it is seen that the potential energy is the negative of the force function since $\boldsymbol{F}\cdot d\boldsymbol{r} = -dV$. The components of the force may thus be expressed in terms of the potential energy in the same way in which they were expressed in terms of the force function, and we have:

$$F_x = -\frac{\partial V}{\partial x}; \quad F_y = -\frac{\partial V}{\partial y}; \quad F_z = -\frac{\partial V}{\partial z}$$

The only difference between the potential energy V and the force function Φ, other than sign, is that the potential energy usually involves an additive constant, since it is defined with respect to an arbitrarily chosen datum point; $V = -\Phi + C$. The advantage of using a potential as a description of a force is that it permits an analysis of the force without bringing into the picture the mechanism causing the force or the bodies upon which the force acts. This advantage is particularly useful for forces which act at a distance, such as gravitational and electrical forces.

If a force is not derivable from a potential function, as, for example, frictional forces or forces proportional to velocity, it is not possible to define a potential energy.

3.6 The Conservation of Energy. If a particle is acted upon by a force which has a potential energy V, the equation of work-energy gives:

$$V_B - V_A = -\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = -(\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2)$$

or:

$$V_A + \tfrac{1}{2}mv_A^2 = V_B + \tfrac{1}{2}mv_B^2$$

This equation states that *the sum of the potential and the kinetic energy remains constant.* The energy is said to be conserved and we have:

$$V + \tfrac{1}{2}mv^2 = \text{constant} \tag{3.6}$$

This is the *principle of the conservation of mechanical energy.* It is valid for any system for which a potential energy can be defined. Any system to which the principle of conservation of energy applies is said to be a *conservative system,* and the forces are said to be *conservative forces.* It should be noted that the principle of the conservation of energy is a direct consequence of Newton's Laws and the definitions of the terms involved. It introduces no new physical facts into the science of mechanics.

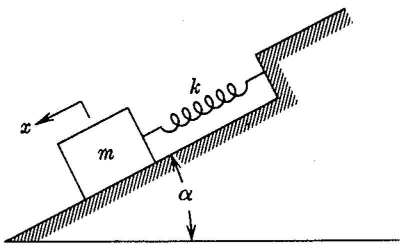


FIG. 3.4

The principle of conservation of energy is applicable only when the forces of a system have potential energies. If this is not true, for example, if frictional forces are acting, the system is said to be non-conservative and the equation of work-energy must be used. The equation of work-energy is thus more general. The use of the principle of conservation of energy is, however, very convenient where conservative systems are involved.

EXAMPLE 1. A mass m is supported on a frictionless inclined plane which makes an angle α with the horizontal by a linear spring of spring constant k as shown in Fig. 3.4. Find the potential energy of the system as a function of the displacement x of the mass along the plane, where x is measured from the position of static equilibrium.

Assume that the potential energy is zero at the position of static equilibrium.

Solution. The potential energy is the negative of the work done by the forces acting on the mass as it moves from the static equilibrium position. These forces are the component of the gravity force parallel to the plane, and the spring force F_s. The initial force in the spring at the static equilibrium position is $(mg \sin \alpha)$, and so the spring force at any displacement x is:

$$F_s = (mg \sin \alpha + kx)$$

Thus we have:

$$V = -\left[(mg \sin \alpha)(x) - \int_0^x (mg \sin \alpha + kx)\, dx\right] = \tfrac{1}{2}kx^2$$

Note that the angle α drops out of the final expression for the potential energy. Thus if x is measured from the position of static equilibrium, the potential energy of the spring-mass system will always be just $\frac{1}{2}kx^2$, whether the spring is horizontal, vertical, or inclined at an angle.

EXAMPLE 2. The mass of Fig. 3.4 is moved a distance A along the plane from the position of static equilibrium and is then released from rest. Write the equation of conservation of energy for the system and apply this equation to a discussion of the motion of the mass.

Solution. Since there is no friction and the gravity force and the spring force are conservative, the principle of conservation of energy applies:

$$V + T = C$$

As was shown in Example 1, $V = \frac{1}{2}kx^2$, so:

$$\tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = C$$

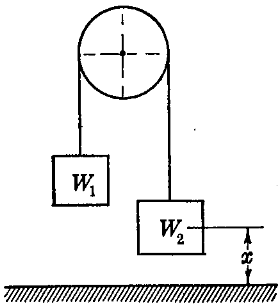
In this problem, when $x = A$, $v = 0$, so $C = \frac{1}{2}kA^2$ and we have finally:

$$\tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}kA^2$$

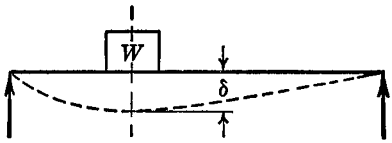
When the body is at $x = 0$, the kinetic energy is equal to $\frac{1}{2}kA^2$, so that the body oscillates between $x = +A$ and $x = -A$. The energy is all kinetic energy at $x = 0$ and all potential energy at $x = \pm A$. The sum of the energies is always a constant, but there is a transfer of energy between kinetic and potential.

PROBLEMS

3.26. Two weights W_1 and W_2 are connected by a cable of length l which passes over a smooth shaft as shown. W_2 is larger than W_1. W_2 starts from rest and moves downward under the action of gravity. Assuming no energy loss during the motion, find the velocity of W_2 after it has moved x ft. If there is a constant friction force F_D between the cable and the shaft, what would be the velocity of W_2 after it has moved x ft?



PROB. 3.26



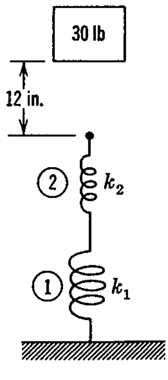
PROB. 3.27

3.27. A beam is found to deflect δ in. under the point of application of a static load W. It is also found that the magnitude of the deflection is proportional to the load. If the weight W is raised a distance of h ft and is dropped on the beam, what is the maximum deflection of the beam under the load? Neglect the mass of the beam.

3.28. A 30 lb weight falls 12 in. before striking two springs (1) and (2) connected in series as shown in the figure. (a) If spring (1) is compressed one inch and spring (2) is compressed two inches, find the values of the spring constants k_1 and k_2.

(b) How high will the weight rebound above the unstretched length of the springs if it is assumed that the weight becomes attached to the top of the spring when contact is first made?

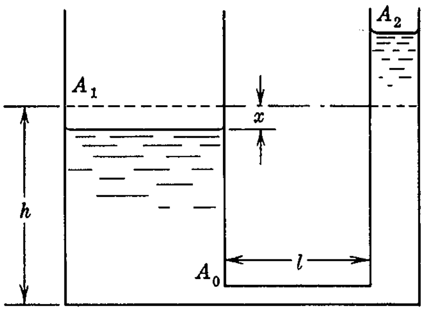
(c) Find the velocity of the weight on the rebound when the springs have stretched a total of 2 in.



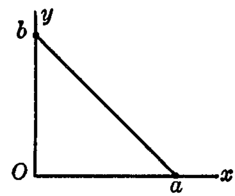
PROB. 3.28

3.29. Two vertical cylindrical tanks of area A_1 and A_2 are connected by a horizontal pipe of length l and area A_0. The tanks are filled with a fluid to a height h above the horizontal pipe as shown in the figure. The fluid in tank A_1 is depressed a distance x, with a corresponding rise of the fluid in tank A_2.

Find the potential energy of the system as a function of x, the density of the fluid ρ, and the areas.



PROB. 3.29



PROB. 3.30

3.30. The force acting upon a particle is given by:

$$F_x = Ax$$
$$F_y = Bx + Cy^2$$

(a) Calculate the work done by this force as the particle is moved from O around the triangular path $OabO$. Is this a conservative force system?

(b) The condition that an expression of the form $Mdx + Ndy$ is an exact differential is that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Applying this to the present problem, determine whether the system is conservative or non-conservative.

(c) Repeat parts (a) and (b) if the force components are:

$$F_x = Ax + By$$
$$F_y = Bx + Cy^2$$

(d) What is the potential function Φ for the system of part (c)?

3.7 The Solution of Problems in Dynamics. The solution of any problem in dynamics involves, in some form or another, the integration of the equation $F = m\ddot{r}$. For problems in which the forces are specified and the velocities and displacements are required as a function of time, the direct integration of this equation may be the most convenient method of procedure. In other problems, some labor may be saved by using the work-energy equation or the impulse-momentum equation. The decision as to which of these forms to use for a particular problem will depend upon the given data and the desired result. As the first step in solving a dynamics problem it is well to review the given data, the required result, and the various dynamic equations from the standpoint of determining which method will be the most suitable.

Some of the characteristics of the methods which should be considered in this connection are:

(1) The impulse-momentum equation involves the velocity and force at specified times. Displacements do not appear in the expression.

(2) The impulse-momentum equation is useful for problems involving large forces of indeterminate magnitudes acting for short times. A consideration of time-average values of the forces may suffice to solve such problems.

(3) The principle of the conservation of momentum can be used only for systems not acted upon by external forces. This principle is most useful when it can be recognized that by treating several bodies together as a system certain unknown forces will occur as equal and opposite pairs and will thus cancel.

(4) The work-energy equation involves velocity and force at specified displacements. Time does not appear in the expression.

(5) The principle of the conservation of energy is applicable only to systems for which a potential energy can be defined.

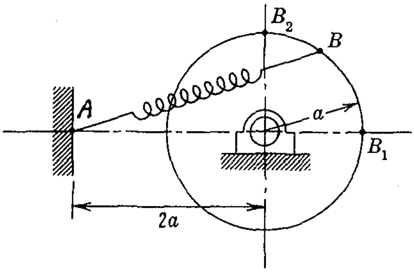
(6) Potential energy is defined only for forces which can be derived from potential functions. Forces, such as friction, which cause a dissipation of energy, have no potential.

(7) The work-energy equation is more general than the equation of the conservation of energy in that it applies to non-conservative systems as well as conservative systems.

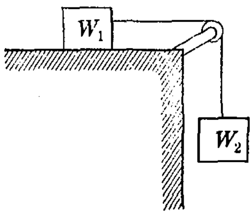
PROBLEMS

3.31. A 120-ton freight car on a level track hits a spring-type bumper with a velocity of 4 mph. The bumper has a spring constant of 12,000 lb per in. of compression. (a) What is the maximum compression of the spring? (b) If the brakes on the car are operated so that a constant braking force of 25 tons is set up, what is the maximum compression of the bumper spring?

3.32. A spring of unstretched length $2a$ and spring constant k is connected to a fixed point A and to a point B on the edge of a wheel as shown in the figure. Find the total work done by the force exerted on the wheel by the spring as the wheel is rotated from B_1 to B_2.



PROB. 3.32



PROB. 3.36

3.33. An automobile with a total weight of 3000 lb runs into a heavy metal power-pole. After the accident it is observed that the pole is undamaged but that the front bumper of the car is bent. Experiment shows that it requires 30,000 ft-lb of work to put such a bend in the bumper. What was the impact velocity of the automobile?

3.34. A particle carrying an electric charge e_1 is fixed at the origin of a coordinate system. A second particle of charge e_2 is placed at a distance r from the origin. The potential of the system is:

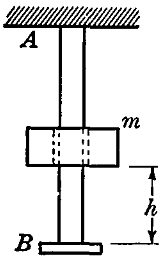
$$\Phi = -\frac{e_1 e_2}{r}$$

Find the radial force between the particles and the x-component of the force exerted on the second particle.

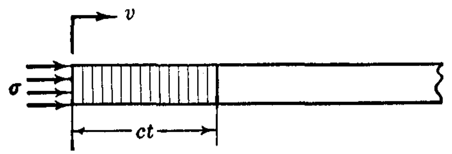
3.35. A spider is suspended from the ceiling on a thread of negligible mass having a length l. Supposing that the thread is linearly elastic with spring constant k, and that l is the stretched length, calculate the total work which the spider would have to do to climb to the ceiling. Compare this with the work required to climb an inextensible thread of length l.

3.36. Two weights, $W_1 = 10$ lbs, and $W_2 = 20$ lbs, are connected by an inextensible rope as shown in the diagram. W_1 moves on a smooth horizontal surface. If the system starts from rest, what will be the velocity of W_2 after it has fallen 10 ft? What change of momentum has taken place during this motion? What time is required for this displacement to take place?

3.37. The bar AB shown in the diagram has dimensions and elastic properties such that it requires a force of kx lb to elongate it x ft. A mass m drops through a distance of h ft and strikes the end of the bar. Find the maximum elongation of the bar and the maximum force produced in the bar. Neglect the mass of the bar. Assume that the mass remains in contact with the end of the bar and that no energy is lost during the motion. If the mass is dropped from $h = 0$, compare the elongation with that which would be produced by m acting statically.



PROB. 3.37



PROB. 3.38

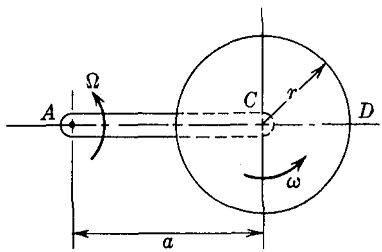
3.38. A long straight rod of uniform cross-sectional area is initially at rest. One end of the rod is suddenly given a velocity v, by the application of a load which sets up a uniformly distributed stress $\sigma \dfrac{\text{lb}}{\text{in.}^2}$ over the end of the rod. At a time t later, a length ct of the bar will be compressed, where c is the velocity of propagation of the stress wave along the rod. It will be assumed that the stress in the rod is below the elastic limit of the material so that Hooke's Law can be used; hence, $\sigma = E\epsilon$, where E is the modulus of elasticity of the material, and ϵ is the strain, or unit deformation, of the rod.

By applying the principle of impulse and momentum to the strained element of the rod, find the velocity of propagation of the elastic wave in the rod. Find also the relationship between the velocity of the end of the rod and the applied stress. Examine Problem 3.37 from this point of view.

3.39. An airplane of mass m_a lands horizontally with a speed v_a on the deck of an aircraft carrier. The mass of the carrier is m_c and it has a velocity v_c in the same direction as the airplane. Neglecting friction at wheels, resistance of air and water, and assuming that all of the energy dissipation occurs in the carrier arresting gear assembly, how much energy is absorbed by the arresting gear? Compare the energy absorbed by the arresting gear with the change of kinetic energy of the airplane.

3.40. Consider a system consisting of two point masses m_1 and m_2 a distance r apart. The gravitational attractive force between the masses is $F = \frac{Gm_1m_2}{r^2}$, which may be described by saying that the gravitational potential of the system is $\Phi = \frac{Gm_1m_2}{r}$. Suppose that m_1, instead of being concentrated at a point, is uniformly distributed in the form of a thin shell and that m_2 = unity. The function Φ can then be called the potential of the shell. Show that: (a) The potential of a spherical shell at any point outside the shell is $\frac{Gm}{r}$. (b) The potential of a spherical shell at any point inside the shell is constant, and hence the gravitational attraction force on a mass inside a shell is zero.

PROB. 3.43



PROB. 3.45

3.41. Using the result of Problem 3.40, determine the gravitational attractive force between a point mass m and a uniformly dense sphere of mass M and radius R. Find this force for m outside of the sphere and for m inside of the sphere.

3.42. A small hole is drilled diametrically through the earth. A particle is dropped down the hole, starting with zero velocity at the surface of the earth. With what velocity would the particle reach the center of the earth?

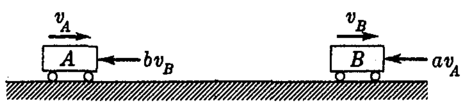
3.43. A flexible rope of length l and weight W rests on a horizontal table with a vertical overhang a as shown in the figure. The coefficient of static and kinetic friction between the rope and the table is μ. (a) What is the maximum length of overhang for which the rope can be in static equilibrium?

(b) Assuming that motion occurs, with what velocity will the rope leave the table?

3.44. In Problem 3.43 assume that the friction between the rope and the table is zero, and find the time required for the rope to slide off the table if it starts from rest with overhang a.

3.45. The bar AC of length a shown in the figure rotates counterclockwise about a vertical axis through A with a constant angular velocity Ω. The disk of radius r rotates counterclockwise about a vertical axis through C with a constant relative angular velocity ω with respect to the bar. (a) A fly starts at rest at C and crawls along a radial groove CD in the disk, with a constant speed relative to the disk of $\frac{2}{\pi} r\omega$. When the fly arrives at the point D, it stops with respect to the disk. Find the total work done by all of the forces acting on the fly as it moves from C to D. (b) Find the magnitude of the forces acting on the fly at the instant it passes the midpoint of the radius of the disk.

3.46. A chain lies in a small pile at the edge of a table, except for a short piece which hangs over the edge. The links of the chain start falling one *at a time over the edge of the table, so that at any given time a total* length of chain x is off the table. (a) Show that the differential equation of motion of the chain is $x\ddot{x} + \dot{x}^2 = gx$. (b) Making the transformation $x = z^{1/2}$, show that this differential equation becomes $\ddot{z} = 2gz^{1/2}$. (c) Noting that $\ddot{z} = \dot{z}\frac{d\dot{z}}{dz}$, integrate the equation and find x as a function of t, if $x = \dot{x} = 0$ when $t = 0$.



PROB. 3.47

3.47. Two identical vehicles move on a horizontal plane with velocities v_A and v_B. By means of a radio control system each vehicle is subjected to a braking force proportional to the velocity of the other vehicle as shown in the diagram. At time $t = 0$ the velocities are respectively v_{0A} and v_{0B}. Show that if $av_{0A}^2 > bv_{0B}^2$ the vehicle B will have its velocity reduced to zero, whereas vehicle A will retain some velocity, i.e., that A will overtake B.

This is the mechanical analog of a fundamental problem in combat tactics. A "Blue" force of B elements battles a "Red" force of R elements, and in each engagement Blue destroys bB Reds and Red destroys rR Blues. The Blue force must win if $bB_0^2 > rR_0^2$. This inequality is called *Lanchester's Square Law*, after F. W. Lanchester (1868–1946), a famous British engineer.

3.48. High speed missiles are frequently ground tested by running them under their own power on a test track. One of the problems involved is that of decelerating the missile and bringing it safely to rest at the end of the run. One way this is done is to extend a scoop from the bottom of the test carriage so that it runs into a long trough of water and turns a jet of water of area A up through 90° with respect to the carriage. Find the length of trough required to reduce the speed of the test missile from V_1 to V_2. Neglect the drag forces on the scoop and missile and the friction at the rails.

Chapter 4

APPLICATIONS OF PARTICLE DYNAMICS

An intelligent being who knew for a given instant all the forces by which nature is animated and possessed complete information on the state of matter of which nature consists—providing his mind were powerful enough to analyze these data—could express in the same equation the motions of the largest bodies of the universe and the motion of the smallest atoms. Nothing would be uncertain for him, and he would see the future as well as the past at one glance.—Marquis de Laplace, *Théorie analytique des Probabilités* (1820).

The principles of particle dynamics as developed in the preceding chapter can be applied to the solution of a large number of interesting and important problems. In the present chapter the solution of such problems as the motion of a particle in a resisting medium, projectile motion, planetary motion, impact, jet propulsion, and electron dynamics will be given. These solutions will illustrate the application of the general principles to particular problems. It should be noted, however, that, although the principles involved in most problems in mechanics are relatively simple, the differential equations that are obtained may be of a type which cannot be integrated by elementary methods.

4.1 The Motion of a Body Falling Through a Resisting Medium. In the preceding chapter the equations of motion were integrated for a body falling in a vacuum. It was also indicated at that point that such a solution is approximate for a body falling through a resisting medium. Experiment shows that frictional forces exert a drag which depends upon the shape of the body, the velocity of the body, and the density and viscosity of the medium.*

* The drag force depends upon Reynolds number; see Example 3 of Section 1.7, Chapter 1.

In general there is a certain optimum shape for which the drag is a minimum.

We shall suppose for the present example that the drag force is proportional to the velocity, $\boldsymbol{F_D} = -k\boldsymbol{v}$. Experimentally it is found that this expression is satisfactory for small velocities. The factor k must be determined experimentally. Choosing the positive z-axis in the vertical downward direction (Fig. 4.1), we have for a particle of mass m and weight W:

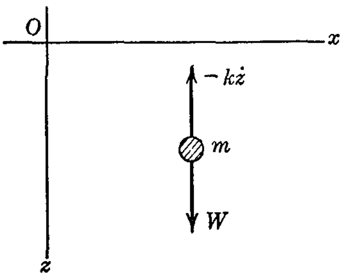


FIG. 4.1

$$W - k\dot{z} = m\ddot{z}$$

Letting $\dot{z} = v$, the equation of motion may be written:

$$\dot{v} + \frac{k}{m}v = \frac{W}{m}$$

The solution is of the form:

$$v = Ce^{-\frac{k}{m}t} + \frac{W}{k}$$

If the velocity $\dot{z}$ is zero when $t = 0$, $C = -W/k$, and:

$$\dot{z} = \frac{W}{k}(1 - e^{-\frac{k}{m}t})$$

Plotting this equation in dimensionless form in Fig. 4.2, we note that as $\left(\frac{kt}{m}\right)$ increases, the quantity $\left(\frac{k}{W}\dot{z}\right)$ also increases, but that it

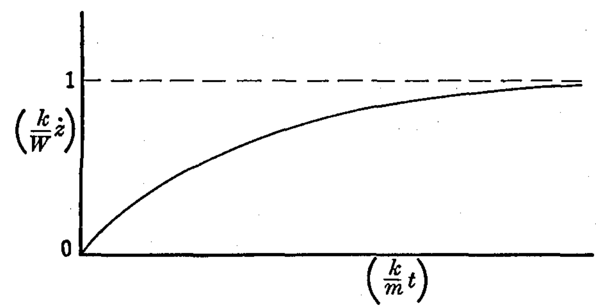


FIG. 4.2

approaches a limiting value of $\left(\frac{k}{W}\dot{z}\right) = 1$. When the limiting value $\left(\frac{k}{W}\dot{z}\right) = 1$ is reached, $k\dot{z} = W$; that is, the drag force has become equal to the weight of the body. The resultant force acting upon the body is then equal to zero; hence, there is no acceleration and consequently no further increase in velocity. It will be seen that whenever the drag increases with velocity, a point will be reached at which the drag is equal to the weight and there will be no subsequent increase in velocity. This limiting value of velocity is called the *terminal velocity* of the body.

We have supposed, in this example, that the falling body moves through a medium of uniform density. Since the density of the atmosphere decreases with altitude, the drag force must also be a function of the altitude, and this additional factor would have to be included in the analysis if a more accurate solution were required for a large altitude range.

PROBLEMS

4.1. A body of weight W falls through a resisting medium in which the drag is proportional to the velocity. Find the displacement at any time, assuming that $z_0 = \dot{z}_0 = 0$ when $t = 0$.

4.2. For velocities above approximately 100 ft/sec, the drag force is approximately described by taking it as proportional to the square of the velocity, $F_D = -kv^2$. Proceeding in the same way as above for drag proportional to velocity, integrate the equation of motion and find the velocity and displacement of a falling body at any time t, if $z_0 = \dot{z}_0 = 0$ when $t = 0$.

4.3. What is the terminal velocity for a falling body subjected to a drag proportional to velocity squared?

4.4. Determine the velocity with which a rain drop would strike the ground falling from a height of 1 mi. if air resistance is neglected. If measurements show that the terminal velocity of the rain drop is approximately 20 ft/sec, find the drag constant k, assuming $F_D = -kv$. How far does the rain drop fall before its velocity is within 0.1% of the terminal velocity?

4.5 The relation $\ddot{s} = -k(\dot{s})^n$ is assumed to describe the motion of a certain body in a viscous medium.

(a) If the body has an initial velocity, v_0, find the highest value of n for which it is brought to rest within a finite period of time. Find the time required.

(b) Find the highest value of n for which the total distance traversed by the body before it comes to rest is finite.

(c) Find the distance traveled by the body for $n = 1$.

4.6. An airplane weighing 20,000 lbs starts from rest and accelerates along a horizontal runway. Acting on the airplane is a constant propulsion force of 2500 lbs, and a drag force $F_d = -0.04v^2$ lb where v is in ft/sec. How long a run must the plane make if it takes off at 150 mph?

4.2 Projectile Motion. The preparation of ballistic tables requires precise calculations of the trajectories, velocities, and times of flight of projectiles. The chief difficulty in making such calculations arises from the fact that the drag is a complicated function of the velocity. For purposes of illustration some simplified problems that are amenable to a mathematical treatment will be considered. We shall suppose that the projectile remains sufficiently close to the earth so that g may be taken as a constant, and we shall neglect the rotation of the earth and any spin or other motion of the projectile as a rigid body. All these factors would have to be taken into account in a precise calculation of the trajectory of a long-range projectile or guided missile.

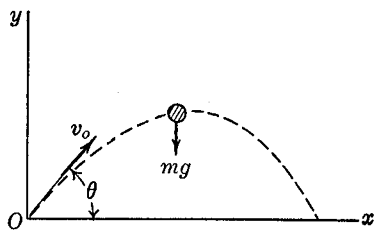


FIG. 4.3

Consider first the two-dimensional motion of a projectile with zero drag. Let the projectile have an initial velocity v_0 making an angle θ with the x-axis of the rectangular coordinate system of Fig. 4.3. The equations of motion are:

$$m\ddot{x} = 0$$
$$m\ddot{y} = -mg$$

Integrating these equations and evaluating the constants in terms of the initial conditions, we obtain:

$$x = (v_0 \cos\theta)t$$
$$y = -\tfrac{1}{2}gt^2 + (v_0 \sin\theta)t$$

The equation for the trajectory is obtained by eliminating t from these equations:

$$y = -\tfrac{1}{2}g\left(\frac{x}{v_0 \cos\theta}\right)^2 + x\tan\theta$$

which shows that the trajectory is a parabola. All the significant features of the motion can be determined from these equations.

A more practical approach to the ballistic problem must take into account the effect of air resistance. This leads to a complex mathematical problem which is usually solved by numerical integration or by special computing machines. A few special problems can be treated by simpler mathematical procedures, and it is one of these which we shall treat in the following example. For an intermediate range of velocities, of approximately 100 to 1000 ft/sec, it may be assumed that the drag force is approximately proportional to the square of the velocity. If we further assume that the variation in

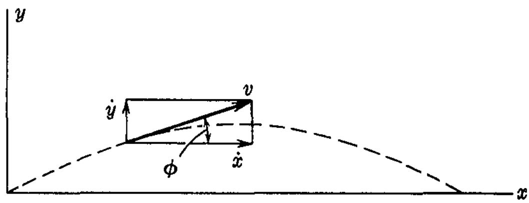


FIG. 4.4

altitude of the projectile is small, so that variations in air density may be neglected, we may write the equations of motion as (Fig. 4.4):

$$m\ddot{x} = -kv^2 \cos\phi = -k\dot{x}^2\left[1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{1}{2}}$$

$$m\ddot{y} = -kv^2 \sin\phi - mg$$

$$= -k\dot{x}\dot{y}\left[1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{1}{2}} - mg$$

Since the problem of solving these equations is a complex one, we shall simplify them by restricting the problem to a consideration of relatively flat trajectories for which the ratio $\left(\frac{\dot{y}}{\dot{x}}\right)$ is small. This is consistent with the assumption that the variation in altitude of the projectile is small. It may be seen from Fig. 4.4, that small values of $\left(\frac{\dot{y}}{\dot{x}}\right)$ mean that the slope of the trajectory is small everywhere.

With this assumption we have $\left[1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{1}{2}} \approx 1$, and the differential equations become:

$$m\ddot{x} = -k\dot{x}^2$$

$$m\ddot{y} = -k\dot{x}\dot{y} - mg$$

From these equations a relatively simple solution can be obtained which is satisfactory over a limited range of trajectories. The first equation can readily be solved by separating the variables:

$$\frac{d\dot{x}}{\dot{x}^2} = -\frac{k}{m}\,dt$$

$$-\frac{1}{\dot{x}} = -\frac{k}{m}t + C_1$$

When $t = 0$, $\dot{x} = \dot{x}_0$ so that $C_1 = -\frac{1}{\dot{x}_0}$, thus:

$$\dot{x} = \frac{\dot{x}_0}{\frac{k\dot{x}_0}{m}t + 1}$$

Substituting this value of $\dot{x}$ into the second differential equation gives:

$$m\ddot{y} + \frac{k\dot{x}_0}{\frac{k\dot{x}_0}{m}t + 1}\dot{y} = -mg$$

or:

$$\ddot{y} + \frac{1}{t + \frac{m}{k\dot{x}_0}}\dot{y} = -g$$

This equation is a linear differential equation for which the principle of superposition is valid, that is, if two expressions are found each of which satisfies the equation, then the sum of the two expressions will satisfy the equation. The equation may thus be solved in two steps. Consider first the homogeneous equation, with the right side equal to zero instead of $-g$. The variables can then be separated and we have:

$$\frac{d\dot{y}}{\dot{y}} = -\frac{dt}{t + \frac{m}{k\dot{x}_0}}$$

$$\log \dot{y} = \log \left(t + \frac{m}{k\dot{x}_0}\right)^{-1} + \log C_2$$

$$\dot{y} = \frac{C_2}{t + \frac{m}{k\dot{x}_0}}$$

This expression is not the complete solution, however, because it gives zero instead of $-g$ when substituted into the original differential equation. We must therefore add to this solution a term which will give $-g$ when substituted into the differential equation.

If, on the basis of an inspection of the differential equation, we try an expression of the form:

$$\dot{y} = C_3\left(t + \frac{m}{k\dot{x}_0}\right)$$

we obtain upon substituting this into the differential equation:

$$C_3 + C_3 = -g; \quad C_3 = -\frac{g}{2}$$

So that the complete solution of the equation is:

$$\dot{y} = \frac{C_2}{t + \frac{m}{k\dot{x}_0}} - \frac{g}{2}\left(t + \frac{m}{k\dot{x}_0}\right)$$

If $\dot{y} = \dot{y}_0$ when $t = 0$, then:

$$\dot{y}_0 = \frac{C_2}{\frac{m}{k\dot{x}_0}} - \frac{mg}{2k\dot{x}_0}$$

and:

$$C_2 = \frac{m}{k\dot{x}_0}\left(\dot{y}_0 + \frac{mg}{2k\dot{x}_0}\right)$$

So that:

$$\dot{y} = \frac{\dot{y}_0 + \dfrac{mg}{2k\dot{x}_0}}{\left(\dfrac{k\dot{x}_0}{m}t + 1\right)} - \frac{mg}{2k\dot{x}_0}\left(\frac{k\dot{x}_0}{m}t + 1\right)$$

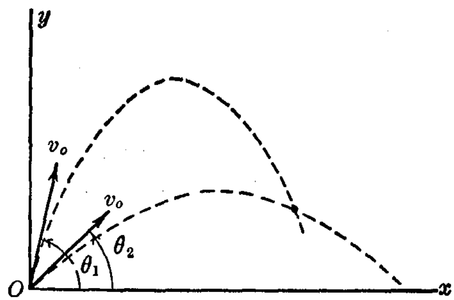
This expression may now be checked by substituting it into the differential equation and verifying that the equation is satisfied. The equations for $\dot{x}$ and $\dot{y}$ can be integrated to obtain expressions for x and y.

It must be kept in mind that these equations are satisfactory only for trajectories which satisfy the assumed condition that $\left(\dfrac{\dot{y}}{\dot{x}}\right)$ is small. The method of obtaining approximate solutions by dropping small terms from a differential equation is often a convenient procedure. The justification for it is that an analytical expression is obtained for the solution which is approximately correct over a particular range of interest in the variables. It might otherwise be necessary to perform a numerical or graphical integration which would not only be very laborious, but which would probably not exhibit the answer in a general form. Simplifications of this type will always have a physical interpretation which should be studied carefully, so that the exact nature of the limitations on the solution will be known. In the above example, the approximation is deduced by noting that there is only a small angle between the resisting force and the x-axis. From the differential equations in their simplified form we can see that this is equivalent to saying that the small vertical velocity has no effect upon the horizontal drag, but that the large horizontal velocity does have an effect upon the vertical drag. Such solutions must, of course, be used with caution.

PROBLEMS

4.7. Two particles are projected from the same point with the same magnitude of velocity but with different angles of elevation, as shown in the diagram. The second particle is fired a time Δt later than the first particle. What is the relation between v, θ_1, θ_2, and Δt for which the two particles will collide? Assume zero drag.

4.8. Find the maximum range and the angle θ for the maximum, if the range is measured along a 45° slope, as shown in the diagram, and zero drag is assumed.



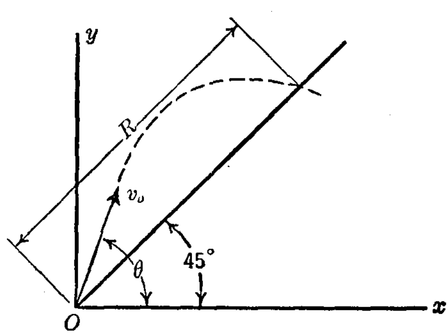
PROB. 4.7

4.9. In section 4.2 the equations are given for the x and y components of the velocity of a projectile which is subjected to a drag force proportional to the square of the velocity. These components were worked out for a a flat trajectory for which the ratio $\left(\frac{\dot{y}}{\dot{x}}\right)$ is small. From these expressions find the x and y coordinates of the projectile as a function of time, under the same assumption of a flat trajectory.

4.10. Assume that the total drag on an airplane is proportional to the square of the velocity. With a feathered propeller, the plane is put into a straight glide making an angle α with the horizontal. What is the expression for the velocity of the plane? What is the expression for the terminal velocity?

4.11. For a relatively slow-speed projectile, the air drag force can be assumed to be proportional to the velocity. Find the horizontal distance which such a projectile must travel before the tangent to the trajectory becomes horizontal.

4.12. A body weighing 300 lb is projected with an initial speed of 120 mph up a straight track sloping 60° from the horizontal. The



PROB. 4.8

coefficient of friction between the body and the track is 0.15, and it is subjected to an aerodynamic drag force $F_d = -0.09v^2$ lb (v in ft/sec). How far will the body travel?

4.3 Planetary Motion. As an example of a two-dimensional motion of a particle under the action of a central force, we shall consider the problem of planetary motion. This problem is particularly interesting as an example of the method of deducing general laws from experimental observations. By studying a large amount

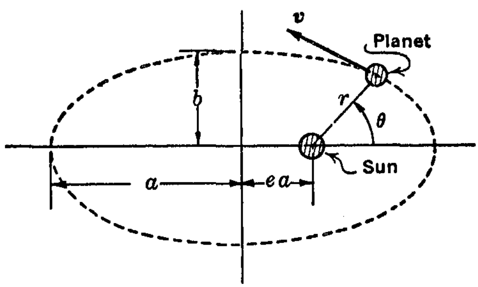


FIG. 4.5

of experimental data, Kepler determined the following three facts about the motions of the planets:*

(1) *The orbit of each planet is an ellipse with the sun at a focus.*
(2) *The radius vector drawn from the sun to the planet sweeps over equal areas in equal times.*
(3) *The squares of the periods of the planets are proportional to the cubes of the semi-major axes of the elliptical orbit.*

It will be of interest to see how, from these statements of empirical facts, Newton was able to deduce the law of gravitation.†

Using the notation of Fig. 4.5, Kepler's three statements may be written analytically in the form:

$$(1) \qquad r = \frac{p}{1 + e\cos\theta} \quad \text{where} \quad p = b\sqrt{1 - e^2} = \frac{b^2}{a}$$

* J. Kepler (1571–1630). The first two statements were published in 1609, and the third in 1619.

† Certain letters of Newton indicate the methods he used.

This is the equation of an ellipse in polar coordinates.

(2) $$A = \int \frac{r^2}{2}\,d\theta = \int \frac{r^2}{2}\frac{d\theta}{dt}\,dt = \frac{k}{2}t$$

(3) $T^2/a^3 =$ constant, where T is the period of a complete revolution.

In plane polar coordinates, the equations of motion of the planet are:

$$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$
$$F_r = m(\ddot{r} - r\dot{\theta}^2)$$

Differentiating (2) above gives:

$$r^2\dot{\theta} = k$$

A second differentiation gives:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

It thus appears that $F_\theta = 0$, and we conclude that the force on the planet must be radial.

The radial force can be determined by differentiating (1), proceeding as follows:

$$r = \frac{p}{1 + e\cos\theta}$$

$$\dot{r} = \frac{pe\sin\theta}{(1 + e\cos\theta)^2}\,\dot{\theta}$$

In this expression for $\dot{r}$, substitute $r^2\dot{\theta} = k$ and $(1 + e\cos\theta) = \frac{p}{r}$ obtaining:

$$\dot{r} = \frac{ek}{p}\sin\theta$$

Differentiating this expression again, and substituting $r^2\dot{\theta} = k$, we have:

$$\ddot{r} = \frac{e}{p}\frac{k^2}{r^2}\cos\theta$$

From the equation of the orbit $\cos\theta = \frac{1}{e}\left(\frac{p}{r} - 1\right)$, therefore:

$$\ddot{r} = \frac{k^2}{pr^2}\left(\frac{p}{r} - 1\right) = \frac{k^2}{r^3} - \frac{k^2}{pr^2}$$

The expression for the radial force may thus be written:

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = m\left(\frac{k^2}{r^3} - \frac{k^2}{pr^2} - \frac{k^2}{r^3}\right)$$

$$F_r = -\frac{mk^2}{pr^2}$$

and the magnitude of the radial force is *inversely proportional to the square of the distance.*

The force is now completely determined except for the factor k^2, which may depend upon the mass of the planet and the mass of the sun. From Kepler's second statement, we have for one complete period T:

$$\frac{k}{2}T = \pi ab, \quad k = \frac{2\pi ab}{T}$$

So that:

$$F_r = -\frac{m}{pr^2}\cdot\frac{4\pi^2a^2b^2}{T^2} = -4\pi^2\left(\frac{a^3}{T^2}\right)\frac{m}{r^2}$$

From Kepler's third statement, that (a^3/T^2) is the same for all of the planets, it is clear that its value depends only upon the sun. Since the force is directly proportional to the mass of the planet, we assume it is also directly proportional to the mass of the sun. Writing $\gamma m_2 = -4\pi^2(a^3/T^2)$, where m_2 is the mass of the sun, we have:

$$F_r = \gamma\frac{m_1m_2}{r^2}$$

where m_1 is the mass of the planet, and γ is a gravitational constant. Newton tested this result by computing, from the motion of the moon about the earth, the gravitational acceleration at the earth's surface. He then was able to check the computation against observed values.

PROBLEMS

4.13. Compute the value of the acceleration of gravity g from the motion of the moon about the earth, and compare this with values experimentally determined at the earth's surface. Take the radius of the earth as $R = 3950$ miles, the radius of the moon's orbit as $60R$, and the period of the moon revolving about the earth as 39,000 minutes. The gravitational attraction on a body is $W\frac{R^2}{a^2}$, where W is the weight of the body at the surface of the earth, and a is the distance of the body from the center of earth. This attractive force is also given by the expression $4\pi^2\frac{a^3}{T^2}\frac{m}{r^2}$. The value of g can be computed from the fact that these two expression when equated should give $W = mg$.

4.14. How much energy is required to establish motion of a rocket ship of mass m in a stable circular orbit of radius r about the earth? Take the radius of the earth as R, and neglect air resistance. Find the velocity of the rocket if $r = 4500$ miles.

4.15. Given that the radius of the earth is 3950 miles, the radius of the moon is 1080 miles, the distance from the center of the earth to the center of the moon is 240,000 miles, and that the mass of the moon is 0.0122 times the mass of the earth. (a) Find the acceleration of gravity on the moon. (b) Find the gravitational force on a rocket which is at a distance x from the center of the earth on the straight line joining the centers of the earth and the moon. (c) If the rocket of part (b) leaves the earth with a velocity of 30,000 mph, find its speed on arrival at the moon.

4.16. A particle is attracted to a point by a central force, and it is observed that the orbit of the particle is the spiral $r = e^\theta$. Determine the force that is acting.

4.17. A particle of mass m is attracted to a fixed point by a force μ/r^3, where μ is a constant and r is the distance from the point. The particle starts from rest at a distance r_0. Find the time required to reach the center.

4.18. The orbit of a particle acted upon by a central force is the spiral $r\theta =$ constant. Find the force which would produce this motion.

4.19. A particle of mass m is attracted to the origin of a coordinate system by a force which is inversely proportional to the square of the distance; $F_r = -\frac{mK}{r^2}$.

(a) Show that the equation for the conservation of energy of the system becomes:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{mK}{r} = \text{constant} = E$$

(b) From the fact that $F_\theta = 0$, we have $r^2\dot{\theta} = k$. Substituting this into the energy expression, and making the transformation $u = \frac{1}{r}$, show that the differential equation of the orbit is:

$$\left(\frac{du}{d\theta}\right)^2 = \frac{e^2}{p^2} - \left(u - \frac{1}{p}\right)^2$$

where:

$$\frac{1}{p} = \frac{K}{k^2}$$

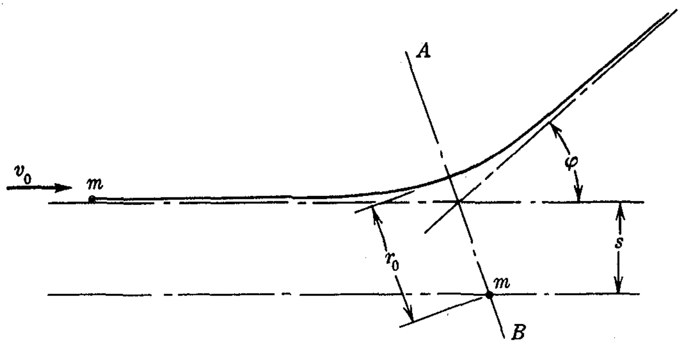
$$e^2 = 1 + \frac{2Ek^2}{mK^2}$$

(c) Integrating directly the differential equation of the orbit, show that the equation of the orbit is:

$$r = \frac{p}{1 + e \cos \theta}$$

This is the equation of a conic having one focus at the origin. If $e < 1$, the conic is an ellipse; for $e = 1$, the conic is a parabola; and for $e > 1$, the conic is a hyperbola.

(d) Show that the form of the orbit, that is, the type of conic, depends on the total energy of the system and hence on the magnitude of the initial velocity, and not on the direction of the initial velocity.



PROB. 4.20

4.20. A particle of mass m is projected with a velocity v_0 along a line which is at a distance s from a fixed identical particle. There is a repulsive force $F = K_0/r^2$, where r is the distance between the particles at any instant. Find the closest approach r_0 between the two particles. The velocity v_0 is the velocity of the particle at a large distance from the fixed particle, i.e., the trajectory shown in the figure is really asymptotic to the lines shown at the angle ϕ. Note that the trajectory is symmetrical about the line AB. This problem may be considered as a type of impact or collision, and will be discussed from this point of view in Section 4.5 of this chapter.

4.4 Impact. The problem of impact between two bodies is characterized by the presence of forces of large magnitude and short time duration. Because of these forces sudden changes occur in the velocities of the bodies, and it is these velocity changes which are ordinarily observed and measured in impact experiments. If the forces acting on the bodies were known, the solution of an impact problem would require only the integration of the equation of motion. Experimental difficulties, however, make the precise measurement of impact forces difficult, so that a different method of solution of the problem is usually required. The motion of the bodies during impact must always satisfy the momentum equation, the energy equation, and the third law of motion, that action is equal and opposite to reaction. These are sufficient to determine all the features of the motion except those occurring during the time of impact. Since the impact time interval is usually very short, of the order of milliseconds, only small errors are introduced by assuming an instantaneous impact. If the time interval of impact is not short, the approximate nature of the solution must be kept in mind.

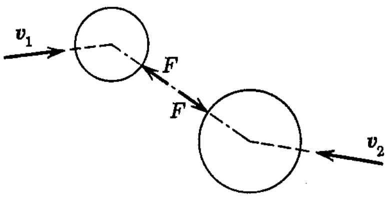


FIG. 4.6

As an illustration of the methods used in solving impact problems, consider two smooth spherical bodies colliding with velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as in Fig. 4.6. At impact two equal and opposite forces act normal to the surface of each sphere at the point of contact. The location and the direction of the forces can be determined from the geometry of the problem, and hence the locations and directions of the impulses $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ acting on the spheres are known. These impulses will be equal and opposite:

$$\boldsymbol{I}_1 = -\boldsymbol{I}_2 = \boldsymbol{I}$$

and the impulse-momentum relations for the two masses are:

$$m_1(\boldsymbol{V}_1 - \boldsymbol{v}_1) = \boldsymbol{I}$$
$$m_2(\boldsymbol{V}_2 - \boldsymbol{v}_2) = -\boldsymbol{I}$$

where $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ are the velocities after impact. Each of these vector equations is equivalent to 3 scalar equations, and we thus have 6 equations available. There are, however, 7 unknowns in the problem: the three unknown components of the two velocities after impact, and the magnitude of the impulse. The additional equation that is required can be obtained from energy considerations. If, for example, the impact is perfectly elastic with no energy loss, then the equation of the conservation of energy must also be satisfied, so that:

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2$$

In this discussion rotation of the spheres was not involved, so that the spheres were treated as particles. If rotation existed, as in the collision of rough spheres, the angular velocities of the spheres would be involved, and the momentum and energy equations for the angular motions would have to be considered.

Whenever there is an impact between actual bodies, there is always some loss of energy. If the impact velocity is small, this energy loss, for many purposes, may be neglected and the equation of conservation of energy may be used as above. If, however, the impact forces are sufficiently large to produce permanent deformations of the bodies, the work done in producing these deformations represents an energy loss which may be too large to be neglected. The energy equation then becomes:

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 - E = \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2$$

where E is the energy loss during impact. It was pointed out by Newton that the information in this equation could be stated in a more useful form by the following method. Consider an impact for which the changes in velocities are parallel to the x-axis. Write first the energy and momentum equations for no energy loss:

$$\tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 = \tfrac{1}{2}m_1\dot{X}_1^2 + \tfrac{1}{2}m_2\dot{X}_2^2$$
$$\tfrac{1}{2}m_1\dot{x}_1 + \tfrac{1}{2}m_2\dot{x}_2 = \tfrac{1}{2}m_1\dot{X}_1 + \tfrac{1}{2}m_2\dot{X}_2$$

Recombining the terms gives:

$$\tfrac{1}{2}m_1(\dot{x}_1 - \dot{X}_1)(\dot{x}_1 + \dot{X}_1) = -\tfrac{1}{2}m_2(\dot{x}_2 - \dot{X}_2)(\dot{x}_2 + \dot{X}_2)$$
$$\tfrac{1}{2}m_1(\dot{x}_1 - \dot{X}_1) = -\tfrac{1}{2}m_2(\dot{x}_2 - \dot{X}_2)$$

Dividing the first equation by the second gives:

$$\dot{x}_1 + \dot{X}_1 = \dot{x}_2 + \dot{X}_2$$

or:

$$\dot{X}_1 - \dot{X}_2 = -(\dot{x}_1 - \dot{x}_2)$$

This equation states that the relative rebound velocity after impact is equal and opposite to the relative velocity of approach. This is equivalent to stating that there is no energy loss during the impact. If there is a loss of energy, the two relative velocities will not be equal, but $(\dot{X}_1 - \dot{X}_2)$ will be smaller than $(\dot{x}_1 - \dot{x}_2)$. The energy loss, therefore, can be determined by measuring the relative velocities, and we may say:

$$(\dot{X}_1 - \dot{X}_2) = -e(\dot{x}_1 - \dot{x}_2)$$

where e is a number less than unity. The quantity e, called the *coefficient of restitution*, is thus a measure of the energy loss. When $e = 1$, with no energy loss, the impact is said to be perfectly elastic. If $e = 0$, the impact is said to be plastic, and the two colliding bodies remain in contact after impact with zero relative rebound velocity.

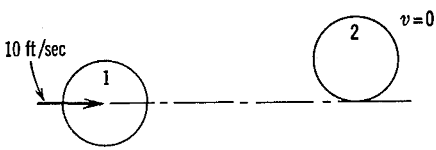


FIG. 4.7

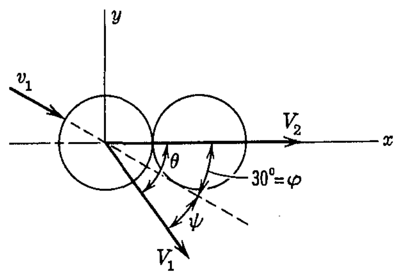


FIG. 4.8

As a specific example of the above ideas, consider the problem shown in Fig. 4.7. The center of gravity of a smooth sphere 1 which has a weight of 2 lb and a radius of 1 in. moves along a straight line with a velocity of 10 ft/sec. The line along which the center of gravity moves is tangent to an identical sphere which is at rest. The coefficient of restitution for a direct central impact between the spheres is 0.80. Find the angle between the original and the final directions of the initially moving sphere.

Taking the coordinate system as in Fig. 4.8, we have:

For conservation of momentum in the x-direction:

$$mv_1 \cos\phi = mV_1 \cos\theta + mV_2$$

For conservation of momentum in the y-direction:

$$mv_1 \sin \phi = mV_1 \sin \theta$$

From the definition of the coefficient of restitution:

$$(V_1 \cos \theta - V_2) = - e(v_1 \cos \phi)$$

Substituting values for V_1 and V_2 from the first two equations into the third equation, we obtain:

$$\tan \theta = \frac{2}{(1 - e)} \tan \phi$$

From the geometry of the impact we have:

$$\tan \phi = \frac{1}{\sqrt{3}}$$

So, for $e = 0.80$ we find that:

$$\psi = \theta - 30° = 50.2°$$

PROBLEMS

4.21. Show that for direct central impact, that is, direction of rebound same as direction of approach, the velocities after impact are given by the following equations if there is no energy loss:

$$V_1 = \frac{2m_2v_2 + (m_1 - m_2)v_1}{m_1 + m_2}$$

$$V_2 = \frac{2m_1v_1 + (m_2 - m_1)v_2}{m_1 + m_2}$$

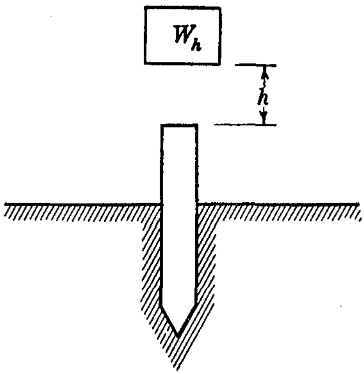
4.22. Show that for direct central impact with coefficient of restitution e, the velocities after impact are given by the following equations:

$$V_1 = \frac{m_2v_2(1 + e) + (m_1 - em_2)v_1}{m_1 + m_2}$$

$$V_2 = \frac{m_1v_1(1 + e) + (m_2 - em_1)v_2}{m_1 + m_2}$$

4.23. Compute the percentage loss in kinetic energy which takes place in a direct central impact if $m_1 = m_2$, $v_1 = -v_2$, and the coefficient of restitution is e.

4.24. In a pile driving operation, a hammer of weight W_h falls through a height h and makes a plastic impact with the pile of weight W_p. The penetration of the pile is resisted by a constant force R, which is chiefly due to the friction between the earth and the pile. Show that if the pile penetrates a distance x after impact then $R = \frac{W_h{}^2 h}{(W_h + W_p)x}$. This expression neglects the work done by gravity forces after impact and also assumes instantaneous impact.



PROB. 4.24

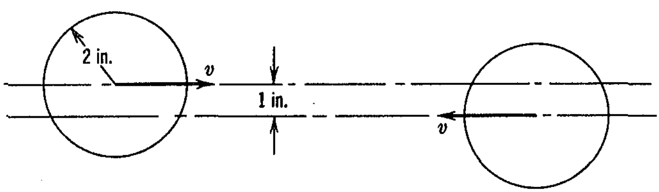
4.25. A golf ball dropped from rest from a height h rebounds from a steel surface to a height $0.85h$. What is the coefficient of restitution?

4.26. An impact may be considered as consisting of two stages, a first stage during which the relative velocity between the two bodies is being reduced, and a second stage in which the rebound velocities are being acquired. If I_1 and I_2 are the magnitudes of the impulses applied to the masses m_1 and m_2 during these two stages, show that for a direct central impact $I_2 = eI_1$, where e is the coefficient of restitution.

4.27. A particle rebounds from a flat surface. If the coefficient of restitution is e and the coefficient of sliding friction is μ, find the relationship between the angle of incidence and the angle of rebound.

4.28. A particle is projected against a horizontal surface with a velocity v at an angle θ with the vertical. The coefficient of restitution for the impact is e, and the coefficient of friction between the particle and the surface is μ. For what angle of incidence θ will the rebound be vertical?

4.29. A particle strikes a horizontal flat surface at an angle θ with the normal to the surface. If the coefficient of friction between the particle and the surface is μ and the coefficient of restitution for normal impact is e, find the energy ΔE lost during the impact. Show that for $\mu = 0$, $\Delta E = \frac{1}{2}mv_n{}^2(1 - e^2)$, where v_n is the component of the initial velocity normal to the surface..



PROB. 4.30

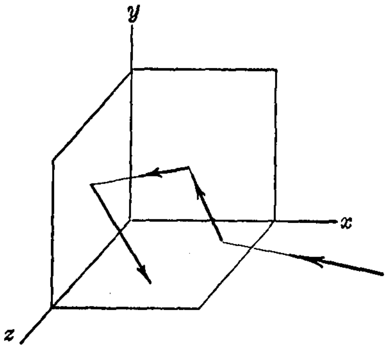
4.30. The centers of mass of two identical smooth steel spheres move along two straight parallel lines a distance 1 in. apart, as shown in the figure.

The spheres are approaching each other with equal speeds of $v = 10$ ft/sec. The radii of the spheres are 2 in., and the coefficient of restitution is 5/9. Find the velocity of the spheres after impact.

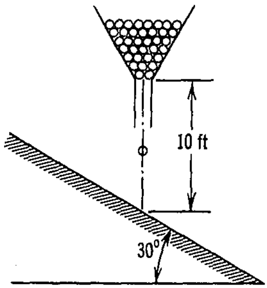
4.31. A particle of mass m rebounds from the corner of a smooth box.

(a) What is the relation between the direction of approach and the direction of departure if no energy is lost during the motion?

(b) Find the relation between the direction of approach and the direction of departure taking account of the energy loss during impact, assuming that the coefficient of restitution is the same for all surfaces.



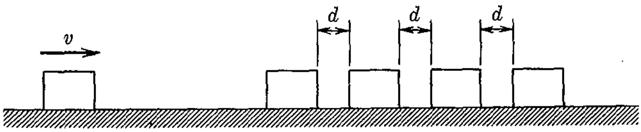
PROB. 4.31



PROB. 4.32

4.32. Steel balls, each weighing 1 oz, drop from a funnel onto an inclined plane at the rate of 12 per second. Each ball falls 10 ft before striking the plane. If the coefficient of restitution is 0.8, find the average force on the plane.

4.33. Four identical bodies each of mass m are set up in a straight line on a smooth horizontal plane with distances d between them. A fifth body, identical to the other four, approaches with a velocity v and makes

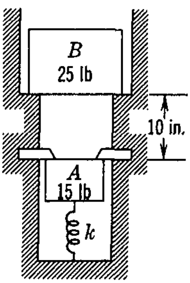


PROB. 4.33

a perfectly elastic impact with the first body. (a) Describe the motion of the bodies. (b) If the distances d approach zero, what is the resulting motion?

4.34. A body A, weighing 15 lb, compresses a spring having a spring constant $k = 25$ lb/in. and is held in compressed position by two latches, as shown in the diagram. The total upward force on the two latches is 200 lb. The spring is not attached to the body A. The coefficient of restitution for impact between body A and body B, which weighs 25 lb, is 0.80. Find the maximum height above its original position attained by B after the latches are released.

4.5 The Scattering of Particles. Another type of impact problem which has in recent years become of great importance in physics may be illustrated by the collision of two charged particles that exert an inverse square repulsive force upon each other. In this problem the nature of the "physical contact" between the two interacting bodies must be studied in considerably more detail than for



PROB. 4.34

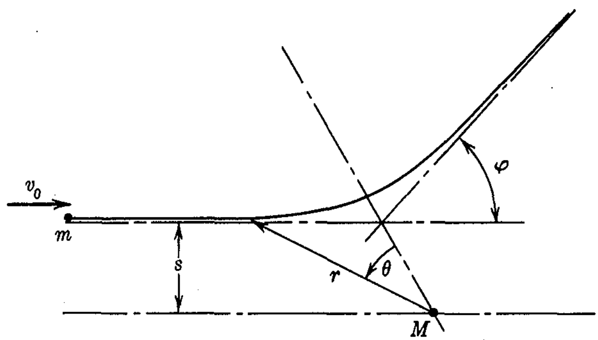


FIG. 4.9

the impact problems previously discussed. The interaction of the particles will be a central force problem, and the results previously derived for planetary motion may be directly applied.

Consider a light particle of mass m and electric charge q_1 which approaches a heavy particle of mass M and electric charge q_2, as shown in Fig. 4.9. The mass m has at a large distance from M a velocity v_0 along a line which is at a distance s, the *impact parameter*, from the heavy particle. It will be assumed that M is so large compared to m that the heavy particle remains at rest during the interaction. The angle ϕ through which the trajectory of m is deflected is to be determined.

The electrostatic repulsive force between the two charged particles is a radial force of magnitude:

$$F_r = \frac{q_1 q_2}{r^2} = \frac{K}{r^2}$$

The problem is thus the same as that discussed in the section in planetary motion, except for the sign of the force, and the results obtained in that section and in Probls. 4.19 and 4.20 can be applied. The orbit under the action of a central inverse square force is a conic which for the repulsion force of the present problem can be described by the equation:

$$r = \frac{p}{1 - e\cos\theta}$$

as $r \to \infty$, $(1 - e\cos\theta) \to 0$, and $\theta \to \frac{\pi}{2} - \frac{\phi}{2}$,

thus:

$$1 - e\cos\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = 0$$

$$\sin\frac{\phi}{2} = \frac{1}{e}$$

The sum of the kinetic and potential energies of the system is a constant E, as in Problem 4.19. Using K_0 instead of mK as in Problem 4.20, we have:

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{K_0}{r} = E$$

Since the force is radial, $r^2\dot{\theta} = \text{constant} = k$ and, following the method outlined in Prob. 4.19, the eccentricity of the hyperbola can be written as:

$$e^2 = 1 + \frac{2Emk^2}{K^2}$$

For this particular problem, the constant $k = r^2\dot{\theta}$ can be evaluated in terms of the known velocity v_0 at large r, which from Kepler's second law is:

$$k = sv_0$$

From the conditions at large r it is seen that $E = \frac{1}{2}mv_0^2$, so that the expression for eccentricity may be written:

$$e^2 = 1 + \left(\frac{msv_0^{\,2}}{K_0}\right)^2$$

Substituting this into the above relationship between the deflection angle and the eccentricity, we finally obtain:

$$\sin^2 \frac{\phi}{2} = \frac{1}{1 + \left(\frac{msv_0^2}{K_0}\right)^2}$$

or:

$$\tan \frac{\phi}{2} = \frac{K_0}{msv_0^2}$$

This equation gives the *scattering angle* ϕ in terms of the initial conditions of the problem.

The impact parameter s is not a quantity that can be directly determined experimentally, whereas the scattering angle ϕ can be directly observed. It is therefore desirable to put the above relation-

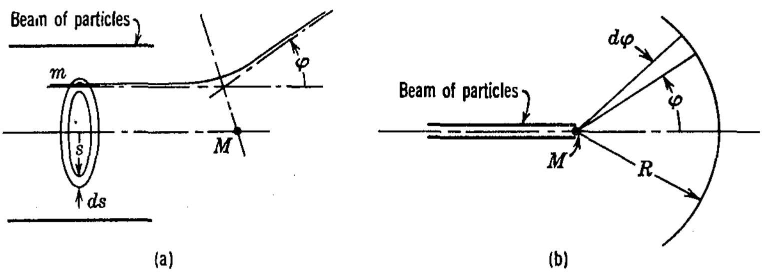


FIG. 4.10

ship in such a form that s does not appear. Suppose, for example, that a large number of light particles forming a parallel beam approach a heavy target particle as in Fig. 4.10. Fig. 4.10 (a) shows a detailed view of the process, while the reduced scale in Fig. 4.10 (b) indicates that the size of the region in which scattering occurs is so small that those light particles close to M can be considered to be scattered to the surface of a sphere of radius R with M as a center. This supposes that R is large compared to the impact parameter s.

Referring now to Fig. 4.10 (a) it will be seen that all of the particles which pass through the annular area of radius s to $s + ds$ are deflected through angles of ϕ to $\phi + d\phi$. If the number of particles passing through a unit area of the beam in a unit time is N, then the number passing through the annular area in unit time is

$$dN = N \cdot 2\pi s ds$$

and this is also the number of particles per unit time that are deflected through the angles ϕ to $\phi + d\phi$.

From the equation derived above relating s and ϕ we obtain:

$$ds = -\frac{K}{2mv_0^2}\frac{1}{\sin^2\frac{\phi}{2}}\,d\phi$$

and hence we can write:

$$dN = N\cdot 2\pi\left(\frac{K}{mv_0^2}\frac{1}{\tan\frac{\phi}{2}}\right)\left(-\frac{K}{2mv_0^2}\frac{1}{\sin^2\frac{\phi}{2}}\,d\phi\right)$$

$$|\,dN\,| = N\pi\left(\frac{K}{mv_0^2}\right)^2\frac{\cos\frac{\phi}{2}}{\sin^3\frac{\phi}{2}}\,|\,d\phi\,|$$

These particles are distributed over a spherical zone on the sphere of radius R shown in Fig. 4.10 (b), the area of the zone being:

$$dA = (2\pi R\sin\phi)R\,d\phi = 4\pi R^2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\,d\phi$$

Thus the number of particles deflected per unit time onto a unit area of the sphere of radius R is:

$$\frac{dN}{dA} = \frac{N\pi\left(\frac{K}{mv_0^2}\right)^2\frac{\cos\frac{\phi}{2}}{\sin^3\frac{\phi}{2}}\,d\phi}{4\pi R^2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\,d\phi} = \frac{N}{R^2}\left(\frac{K}{2mv_0^2}\right)^2\frac{1}{\sin^4\frac{\phi}{2}}$$

This is the famous Rutherford scattering formula, which puts the parameters of the problem in such a form that experimental verification of system behavior may be made.

4.6 The Pressure in a Gas. As another illustration of an impact problem we shall consider a simple example from the kinetic theory of gases. A gas at low density may be considered as a system containing a very large number of particles, each one of which is so far from its neighbors that it is not influenced by them. The behavior of such gases thus becomes a problem in particle dynamics.

Consider a rectangular box of volume V containing N molecules of a gas. Each molecule has certain velocity components v_x, v_y, v_z, as shown in Fig. 4.11. We suppose that the gas has reached a steady-state condition, that is, the center of mass of the system of molecules is at rest, and the average density is the same throughout the volume. Considering now the x-components of velocity, we shall distinguish between the molecules which travel in the positive x-direction and those which travel in the negative x-direction. On the average $N/2$ of the molecules will have the positive direction and $N/2$ will have the negative direction. We also note that the total momentum of the molecules moving in the $(+\ x)$ direction

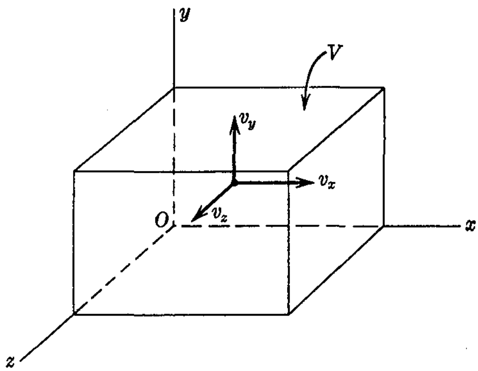


FIG. 4.11

must equal the total momentum of those moving in the $(-\ x)$ direction, since the mass center of the system is at rest.

We now examine the molecules which have velocities in the $(+\ x)$ direction. Such particles will eventually collide with the end of the box and, after the impact, will rebound with reversed velocities. Let us first consider all the molecules in the box which have a specific velocity v_i in the $(+\ x)$ direction. If there is a total of n_i of such molecules in the box, then on the average during a time Δt the number of molecules colliding with the end wall will be $n_i \dfrac{v_i \Delta t A}{V}$. This may be seen from Fig. 4.12, since any molecules which are further from the wall than $v_i \Delta t$ will not reach the wall in the time Δt. The total number reaching the wall is therefore n_i

multiplied by the ratio of the volume $v_i\Delta tA$ to the total volume V. As each particle collides with the wall and rebounds, the velocity of the molecule is reversed, and the change in momentum, assuming no energy loss during impact, is $(-2mv_i)$ per molecule, where m is the mass of the molecule. The total change in momentum during the time Δt is:

$$-2m\frac{n_iv_i\Delta tA}{V}v_i = -\frac{2m}{V}n_iv_i^2A\,\Delta t$$

From the fact that the total impulse equals the total change in momentum, we have:

$$-\frac{F}{A}\Delta t = -\frac{2m}{V}n_iv_i^2\Delta t$$

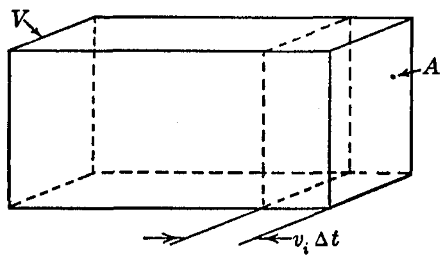


FIG. 4.12

If we let p_i be the *average force per unit area* or *pressure* exerted by the molecules against the end wall, we have:

$$p_i = \frac{2m}{V}n_iv_i^2$$

A similar expression will be obtained for the molecules which have a different velocity, say v_j; summing up the p_i over all the different velocities, we obtain:

$$\Sigma p_i = \frac{2m}{V}\Sigma n_iv_i^2$$

where $\Sigma p_i = p =$ the total pressure on the end of the box, and $\Sigma n_iv_i^2$ is equal to $\frac{N}{2}(v_x^2)_{\text{avg}}$, where $(v_x^2)_{\text{avg}}$ is the average value of

v_x^2 taken over all $\frac{N}{2}$ molecules which have velocity components $+ v_x$. We may therefore write:

$$p = \frac{2m}{V}\frac{N}{2}(v_x^2)_{\text{avg}}$$

or:

$$pV = Nm\,(v_x^2)_{\text{avg}}$$

We now note that the kinetic energy due to the $+\,x$ components of velocity is equal to:

$$\frac{1}{2}\frac{N}{2}m(v_x^2)_{\text{avg}}$$

and that the kinetic energy of translation of all the components of velocity ($\pm\,v_x$, $\pm\,v_y$, $\pm\,v_z$) is equal to:

$$\frac{3}{2}Nm(v_x^2)_{\text{avg}}$$

since in a steady state $(v_x^2)_{\text{avg}} = (v_y^2)_{\text{avg}} = (v_z^2)_{\text{avg}}$. We may thus write the above expression for pV as:

$$pV = \tfrac{2}{3}E$$

where E is the total kinetic energy of translation of the molecules. It will be noted that this expression is just a statement of Boyle's law, which states that the product of the pressure and volume of a gas is constant at a constant temperature. From thermodynamics we know that the gas law for varying temperatures is:

$$pV = RT$$

where V is the volume of one mole of gas, T is the absolute temperature, and R is the universal gas constant.* We thus obtain the following relation between temperature and the kinetic energy of a gas:

$$\tfrac{2}{3}E = RT$$

* $R = 8.314 \times 10^7$ ergs per degree Centigrade per mole of gas. One mole is that volume which has a mass in grams equal to the molecular weight of the gas, and which contains 6.02×10^{23} molecules (Avogadro's Number).

$$R = \frac{8.314}{6.02} \times 10^{-16}N = kN,$$

where k = Boltzmann's Constant.

For example, at a temperature of 100°C or 373° abs, the kinetic energy of one mole of gas is:

$$E = (\tfrac{3}{2})(8.314)(10^7)(373) = 4.65 \times 10^{10} \text{ ergs} = 3430 \text{ ft lb}$$

For oxygen molecules (32 grams per mole), the average value v^2 per molecule may be computed as follows:

$$\tfrac{1}{2}(32 \text{ grams})(v^2)_{\text{avg}}\, cm^2/\text{sec}^2 = (\tfrac{3}{2})(8.314)(10^7)(373) \text{ ergs}$$

$$(v^2)_{\text{avg}} = 2.91 \times 10^9$$

$$\sqrt{(v^2)_{\text{avg}}} = 5.4 \times 10^4 \text{ cm/sec} = 1770 \text{ ft/sec}$$

The root-mean-square velocity is somewhat larger than the mean velocity.

In the preceding derivation of Boyle's law it was assumed that the gas molecules traveled from wall to wall without mutual interference. This assumption seems reasonable for a gas with small density, but for a gas of high density it would be expected that there would be many collisions between molecules. The impacts against the walls would then be different, and if this is taken into account certain correction terms must be included in Boyle's law. Experimental evidence confirms the fact that Boyle's law is not satisfactory for high gas densities. Another factor must be considered if the gas consists of complex molecules. If, for example, each molecule is formed of several atoms, the molecule can have an appreciable kinetic energy of rotation, which must be included in the analysis. A more complete analysis of the problem, as developed in the kinetic theory of gases, shows that in the steady-state condition the total kinetic energy is divided equally among the degrees of freedom. If the molecule has the form of a rigid multi-atom body it has six degrees of freedom—three in translation and three in rotation. In a steady state $\frac{1}{6}$ of the total kinetic energy would be associated with each degree of freedom.

PROBLEMS

4.35. How many ft lb of energy are required to give one gram of oxygen a temperature increase of one degree centigrade?

4.36. Show that the work that is done by a gas as it expands under a variable pressure is given by $\int_a^b p\, dV$.

4.37. Determine the vertical distribution of pressure in an isothermal atmosphere. This is the problem of finding the pressure distribution in a vertical column of gas under the action of gravity. It is assumed that the temperature is constant throughout the column, and that the pressure and density are related by the perfect gas law. If M is the mass of one mole of gas and V is its volume, show that the pressure p at any height h is:

$$p = p_0 e^{-\left(\frac{Mgh}{RT}\right)}$$

where p_0 is the pressure at the base of the column.

4.38. Differences in altitude can be computed from measurements of barometric pressure if it is assumed that the atmosphere is a perfect gas of uniform temperature. Using the results of Prob. 4.37, calculate the difference in altitude corresponding to a change of barometric pressure from 28 to 26 in. of mercury.

4.7 Variable Mass Systems. There are two types of variable mass problems which should be carefully distinguished. In the first type, the variable mass is a consequence of the particular way in which the system is defined. If, for example, we take as a system a rocket, excluding the exhaust gases, then the mass of the system changes because material is being expelled from the system. The individual particles of the rocket and exhaust gases themselves are of constant mass, but the number of particles being considered changes with time. For such problems the motion of each particle is described by the equation $\boldsymbol{F}_i = \frac{d}{dt}(m_i\boldsymbol{v}_i) = m_i\boldsymbol{a}_i$, but the total mass of the system $M = \sum_{i=1}^{n} m_i$ is changing because the number of particles n being considered is changing.

The second type of variable mass problem involves an actual variation in the mass of the individual particle. This is the situation encountered in the relativistic variation of mass with velocity, which will be considered in more detail in later sections of this chapter. In this case, the motion of the particle is described by the equation:

$$\boldsymbol{F}_i = \frac{d}{dt}(m_i\boldsymbol{v}_i) = m_i\frac{d\boldsymbol{v}_i}{dt} + \boldsymbol{v}_i\frac{dm_i}{dt}$$

4.8 Jet Propulsion Problems. As an example of a system whose total mass is changing because material is being ejected we shall consider a simple problem of jet propulsion. In Chapter 8 a more general treatment will be given, and new forms of Newton's Laws will be derived that will be particularly appropriate for such problems.

Consider a rocket with velocity v and mass $M = M_0 - \mu t$, where μ is the rate at which mass is ejected and M_0 is the initial mass of the rocket. The exhaust gases have a velocity v_e relative to the rocket. We wish to derive the equation of motion for the rocket, neglecting gravity and air resistance. There are two methods of approach that may be used.

Method 1. We consider as the system to which we shall first apply Newton's Law the mass (Δm) that is exhausted with relative velocity v_e from the rocket in a time Δt, which is shown within the

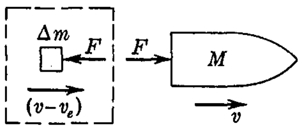


Fig. 4.13

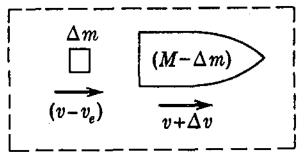


Fig. 4.14

dashed area of Fig. 4.13. Writing the equation of impulse-momentum for this element, we have:

$$F\Delta t = \Delta m v_e$$

$$F = \frac{\Delta m}{\Delta t} v_e$$

As $\Delta t \to 0$, we may say that the force F on the ejected material is given by:

$$F = \frac{dm}{dt} v_e$$

This force is equal in magnitude to the propulsive force acting on the rocket. Considering now the rocket itself, we may write the equation of motion:

$$\frac{dm}{dt} v_e = M \frac{dv}{dt}$$

The rate at which the total amount of ejected material gains mass is equal in magnitude to the rate at which the rocket loses mass, that is:

$$\frac{dm}{dt} = -\frac{dM}{dt}$$

Since $M = M_0 - \mu t$, $\dfrac{dM}{dt} = -\mu$ and $\dfrac{dm}{dt} = \mu$. The final equation of motion thus becomes:

$$(M_0 - \mu t)\frac{dv}{dt} = \mu v_e$$

Method 2. In this method the system under consideration will include the rocket and the ejected mass, as shown in the dashed area of Fig. 4.14.

Let p_t = total momentum of the system just before the ejection of Δm, when the rocket has a velocity v.

$p_{t+\Delta t}$ = total momentum of the system a time Δt after ejection of Δm.

Since there are no external forces acting on the system, the total impulse is zero and there is no change in momentum, so $p_t = p_{t+\Delta t}$. The momenta are given by

$$\begin{aligned} p_t &= Mv \\ p_{t+\Delta t} &= (M - \Delta m)(v + \Delta v) + (\Delta m)(v - v_e) \\ &= Mv + M(\Delta v) - (\Delta m)v_e \end{aligned}$$

where the second order term $(\Delta m)(\Delta v)$ has been dropped. Thus equating momenta:

$$\begin{aligned} Mv &= Mv + M(\Delta v) - (\Delta m)v_e \\ M(\Delta v) &= (\Delta m)v_e \end{aligned}$$

Dividing by Δt, and taking the limit as $\Delta t \to 0$:

$$M\frac{dv}{dt} = \frac{dm}{dt}v_e$$

as in Method 1 above.

Other examples of variable mass problems of the same type are given in Section 3.2, Example 2; and Probls. 3.46 and 4.43.

PROBLEMS

4.39. A rocket having a total weight of 50 lb contains 2 lb of propellant which is burned at a uniform rate in one second. The propellant has a "specific impulse" of 200 lb sec per lb of propellant; that is, a thrust force of 200 lb is produced by burning one pound of propellant in one second. Assuming that the rocket moves horizontally with negligible frictional resisting forces, find the velocity of the rocket at the end of the burning time. If the propellant were burned in two seconds instead of one, would the velocity be different?

4.40. A rocket travels with a velocity v. The exhaust gas issues from the rocket with a velocity v_e relative to the rocket. If the mass of gas exhausted per second is m', then the thrust force is $m'v_e$, and the propulsion power, which is the rate at which the thrust force does work, is given by $m'v_ev$. The kinetic energy lost in the exhaust gas per unit time, which represents a power loss, is $\frac{1}{2}m(v - v_e)^2$. Show that the propulsion efficiency is given by:

$$\text{eff} = \frac{2\left(\dfrac{v}{v_e}\right)}{1 + \left(\dfrac{v}{v_e}\right)^2}$$

For typical propellants, $v_e = 4000$–6000 ft/sec. What rocket velocities must be obtained in order to have 50% efficiency?

4.41. A small compressed air rocket runs along a horizontal wire. The initial weight of the rocket and charge is W_0, and the final weight of the empty rocket is W_F. The air is discharged at a uniform rate of c lb/sec, with a relative exhaust velocity v_e. The air resistance and the wire friction can be approximated by a drag force proportional to the velocity, $F_D = -kv$. (a) Find the speed of the rocket as a function of time, assuming that the rocket starts from rest at $t = 0$. (b) If $W_0 = 12$ lb, $W_F = 2$ lb, $c = 2$ lb/sec, $v_e = 480$ ft/sec, and $k = 0.01$ lb sec/ft, find the maximum velocity of the rocket.

4.42. A ram-jet missile flies at a constant altitude with a forward velocity v. The engine operates steadily by taking in $15w$ pounds per second of air from the atmosphere, and burning it with w pounds per second of fuel which is carried on board the missile. The engine ejects the products of combustion at a constant speed v_e with respect to the missile. If the device is launched with an initial horizontal velocity v_0, an initial weight W_0, and is subject to negligible external drag forces, find its velocity as a function of time.

4.43. A spherical raindrop falls through air saturated with water vapor. Because of condensation, the mass of the raindrop increases at a rate proportional to the surface area, that is, $dm = 4\pi r^2 \cdot k\rho \cdot dt$, where k is a constant and r is the radius of the drop at any time. Assuming negligible air resistance, show that the equation of motion can be written as:

$$kr^3 \frac{dv}{dr} = r^3 g - 3kr^2 v$$

or:

$$k \frac{d}{dr}(r^3 v) = r^3 g$$

Find the velocity of the drop as a function of time if $r = 0$ and $v = 0$ when $t = 0$.

4.9 Electron Dynamics. For our present purpose, the electron may be considered to be a particle having a mass $m_0 = 9.1 \times 10^{-28}$ grams and carrying a negative electric charge $e = -4.80 \times 10^{-10}$ electrostatic units. If the electron is subjected to a known force, its motion can be computed from Newton's law written in the form $\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v})$. The equation of motion is written in this form since it has been observed that when an electron is accelerated to a high velocity the apparent mass of the electron increases according to the equation $m = m_0/\sqrt{1 - v^2/c^2}$, where c is the velocity of light in vacuum (299,800 km/sec) and m_0 is the mass of the electron when $v = 0$, the *rest mass*. Since v must exceed 40,000 km/sec for the

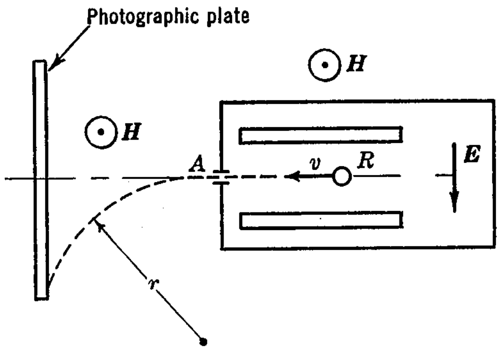


FIG. 4.15

mass to be increased by 1%, it is only for very high velocities that the effect becomes appreciable.

As a first example of a problem in electron dynamics we shall consider a simplified version of one of the basic experiments which first permitted a direct experimental verification of the variation of mass with velocity. That the apparent mass of a charged particle should increase with speed had been shown from theoretical considerations by J. J. Thomson in 1881, and a direct experimental confirmation was obtained in 1901 by Kaufmann. In 1909 experiments of Bucherer made possible the accurate determination of the relation between mass and velocity. The experimental method used by Bucherer is shown, in a simplified way, in Fig. 4.15. Electrons are emitted, with all directions and speeds, from a radioactive source R, located between two plates. Only those electrons having velocities along the axis of the apparatus are used in this experiment. An electric field $\boldsymbol{E}$, directed downward, is maintained between the plates, and the whole apparatus is placed in a magnetic field $\boldsymbol{H}$ directed perpendicularly out of the paper. While the electron is between the two plates, it is subjected to a vertical upward force of magnitude (Ee) due to the electric field. The force acting on an electron moving with a velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{H}$ is $e\boldsymbol{v} \times \boldsymbol{H}$. In the present experiment $\boldsymbol{v}$ is perpendicular to $\boldsymbol{H}$, so that a vertical downward force of magnitude (evH) acts on the electron. If the forces due to the electric and magnetic field are just equal in magnitude, the resultant vertical force on the electron will be zero, and the electron will move horizontally with a velocity given by:

$$Ee = evH; \quad v = \frac{E}{H}$$

The apparatus is arranged with a small hole at A, so that only electrons which have this horizontal velocity v can emerge from the box. We thus have a method of producing specified electron velocities.

After leaving the electric field $\boldsymbol{E}$, in the region between the two plates, the electron is subjected only to the force of the magnetic field. Since the force on the electron and hence the acceleration of the electron are always perpendicular to the velocity, the magnitude of the velocity does not change but the electron moves in a circular path of radius r, where:

$$evH = \frac{mv^2}{r}$$

from which:

$$\frac{e}{m} = \frac{v}{rH}$$

From this equation an experimental value of $\left(\frac{e}{m}\right)$ for various velocities can be determined by measuring the radius of curvature r of the electron path. As far as this experiment is concerned, the change in $\left(\frac{e}{m}\right)$ with velocity could be attributed to changes in either the charge or the mass, or both, with velocity. On various theoretical grounds, however, it is supposed that it is the mass that varies and not the charge, and this supposition leads to a consistent theoretical and experimental picture. Both the Lorentz theory of electromagnetic mass and the theory of relativity predicted that the variation of mass with velocity should be of the form $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ and hence, according to these theories, the results of the above experiments should be

$$\left(\frac{e}{m}\right) = \left(\frac{e}{m_0}\right)\sqrt{1 - \frac{v^2}{c^2}}$$

If this expression is confirmed by the experiment, the same value of $\left(\dfrac{e}{m_0}\right)$ should be computed from the experimental measurements for different values of v.

Some typical results of Bucherer's experiments are given in the following table.*

$\left(\frac{v}{c}\right)$	$\left(\frac{e}{m_0}\right)$ (emu/gram)
0.3173	1.752×10^7
0.3787	1.761
0.4281	1.760
0.5154	1.763
0.6870	1.767

Since these values of $\left(\dfrac{e}{m_0}\right)$ are constant within the limits of experimental error, it appears that the experimental results are in good agreement with the expression for the variation in mass, which is derived on the basis of the special theory of relativity.

4.10 The Acceleration of Electrons. A number of frequently used instruments employ a stream of high-speed electrons, so that the problem of producing high-velocity electrons is of practical importance. Consider the apparatus shown schematically in Fig. 4.16. A potential difference ΔV is maintained between a cathode C and a plate anode A. By definition, the electric field $\boldsymbol{E}$ at a point is the

* Bucherer, A. H., *Annalen der Physik* 28, 513 (1909). Bucherer's value for $\left(\dfrac{e}{m_0}\right)$ was $(1.763 \pm 0.008) \times 10^7$ and the modern accepted value (1941) is $(1.7592 \pm 0.0005) \times 10^7$.

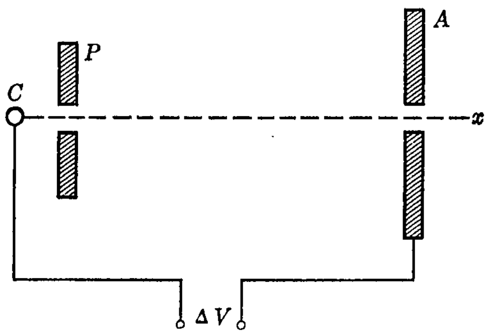


FIG. 4.16

force on a unit charge at the point, and the potential function V is related to the electric field $\boldsymbol{E}$ as follows:

$$\boldsymbol{E} = - \text{grad } V = - \left(\frac{\partial V}{\partial x} \boldsymbol{i} + \frac{\partial V}{\partial y} \boldsymbol{j} + \frac{\partial V}{\partial z} \boldsymbol{k} \right)$$

Any free electron of charge e in the field will thus be accelerated toward A by a force whose components are:

$$F_x = - e \frac{\partial V}{\partial x}$$

$$F_y = - e \frac{\partial V}{\partial y}$$

$$F_z = - e \frac{\partial V}{\partial z}$$

If the cathode C is heated, as for example, by a resistance filament, electrons will be "boiled" off and will then be accelerated by the electric field. If a plate P with a small opening is placed in front of the cathode as shown in the diagram, only those electrons traveling along the x-axis will be free to move to the anode. Such electrons will move toward the anode under the action of a force F_x. If now a hole is arranged in the anode at the x-axis, the electrons can pass out of the electric field and continue on with the constant velocity v which they had attained at the anode. Knowing the potential difference between the cathode and the anode, the velocity may be computed by using the principle of work and energy, where the work can be obtained by integration of the above expression for F_x:

$$e(V_A - V_C) = \tfrac{1}{2}mv^2$$

hence:

$$v = \sqrt{\frac{2e}{m}(V_A - V_C)}$$

We can compute the velocity for a potential difference of 1 volt $= \frac{1}{300}$ esu for the electron as follows:

$$v = \left[\frac{(2)(-4.8 \times 10^{-10})}{(9.1 \times 10^{-28})}\left(-\frac{1}{300}\right)\right]^{\frac{1}{2}} = 5.94 \times 10^7 \text{ cm/sec}$$

This velocity is of the order of 133,000 mph, which is sufficiently small compared to the velocity of light so that the variability of mass with velocity does not need to be taken into account. It is relatively easy, however, to accelerate electrons to high velocities because the mass of the electron is small compared to its charge.

4.11 The Cathode-Ray Oscilloscope. An electronic device which has wide application is the cathode-ray oscilloscope, represented in simplified form in Fig. 4.17. By an arrangement of cathode

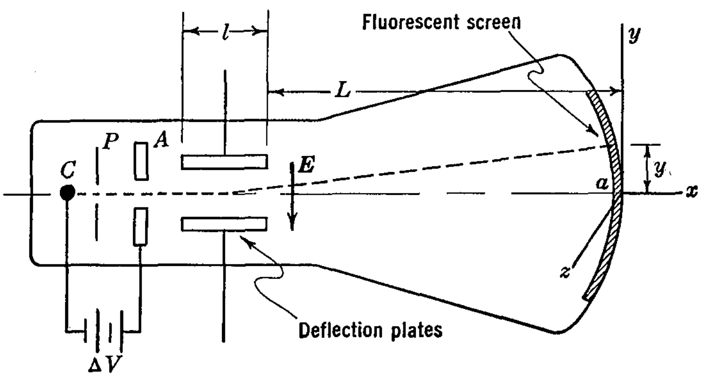


FIG. 4.17

and anode as previously described, electrons are accelerated to a suitable velocity and are focused into a narrow beam through A along the x-axis. Two plates are arranged parallel to the x-axis so that an electric field $\boldsymbol{E}$ can be established over a length l of the beam.

The whole apparatus is enclosed in an evacuated glass tube. If the electric field $\boldsymbol{E}$ is zero, the electron stream continues along the x-axis and impinges against the end of the tube at a, where a bright spot is formed on the fluorescent screen. If an electric field is set up, the electrons, as they pass between the plates, will be subjected to a force which will deflect the beam and so change the position of the bright spot on the screen.

An enlarged view of the two plates with a single electron between them is shown in Fig. 4.18. As a simplifying assumption, we suppose that $\boldsymbol{E}$ is zero outside the plates and uniform between the plates.

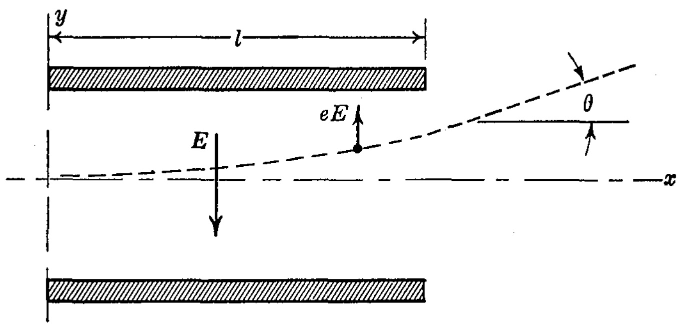


FIG 4.18

Then an upward force of (eE) is exerted on the electron and the equations of motion are:

$$m\ddot{x} = 0$$
$$m\ddot{y} = eE$$

subject to the conditions that $x_0 = y_0 = \dot{y}_0 = 0$ and $\dot{x}_0 = v$ when $t = 0$. After the electron has passed through the parallel plates, an integration of these equations shows that the velocity components are:

$$\dot{x} = v$$
$$\dot{y} = \frac{eEl}{mv}$$

As the electron leaves the plates, its path makes an angle θ with the x-axis, where:

$$\tan\theta = \frac{\dot{y}}{\dot{x}} = \frac{eEl}{mv^2}$$

The effect of the two plates, therefore, is to deflect an electron beam through an angle θ. After the beam leaves the plates, no force is exerted on the electrons, and the beam continues in a straight line at the angle θ. The luminous spot will thus be deflected from its zero position a distance y, where:

$$y = \frac{eEl^2}{2mv^2} + \frac{eEl}{mv^2}L$$
$$= \left[\frac{el}{mv^2}\left(L + \frac{l}{2}\right)\right]E = C_1E$$

The displacement of the luminous spot is thus directly proportional to E and can be taken as a measure of the potential difference between the plates.

If a second pair of deflecting plates, oriented at 90° to the first pair, is added to the cathode-ray tube of Fig. 4.17, the luminous spot will be deflected in the z-direction with a displacement:

$$z = C_2E_z$$

where E_z is the electric field set up between the second pair of plates. The motion of the luminous spot on the face of the tube is thus given by the two equations:

$$y = C_1E_y; \quad z = C_2E_z$$

If, for example, $E_y = A \sin \omega t$ and $E_z = B \cos \omega t$, the path of the spot is the ellipse:

$$\frac{y^2}{C_1{}^2A^2} + \frac{z^2}{C_2{}^2B^2} = 1$$

which appears on the screen as a luminous line. In general, if E_y is known, E_z can be determined from the picture on the tube. The oscilloscope can thus be used to measure any quantity which can be converted into a potential difference.

4.12 The Equivalence of Mass and Energy. We shall now investigate some of the consequences of the fact that the apparent mass of a particle increases with velocity. We shall start with Newton's law in the form $\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v})$ and shall take m as a variable, $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$. To simplify the analysis we shall consider only one-dimensional motion starting from rest, with F and v always parallel to the x-axis.

The impulse-momentum equation will be derived first. For the particular conditions specified above, we have:

$$I = \int_0^t F\,dt = mv = \frac{m_0 v}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

Solving for v in terms of the impulse I:

$$v = c\left[\frac{1}{\sqrt{1 + \dfrac{m_0{}^2c^2}{I^2}}}\right]$$

It is seen that as $I \to \infty$, $v \to c$, so that no matter how large the impulse, the velocity can never exceed the velocity of light.

The equation of work and energy is obtained in the usual way except that m is now a variable:

$$\int_0^x F\,dx = \int_0^x \frac{d}{dt}(mv)\,dx = \int_0^v vd(mv)$$

Integrating by parts the right side of this expression, and writing E_w for the work done by F, we obtain:

$$E_w = mv^2 - \int_0^v mv\,dv$$

Substituting the expression for m:

$$E_w = \frac{m_0 v^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} - \int_0^v \frac{m_0 v}{\sqrt{1 - \dfrac{v^2}{c^2}}}\,dv$$

Evaluating the integral, this becomes:

$$E_w = m_0 c^2\left[\frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} - 1\right]$$

which is the relativistic expression for the kinetic energy of the particle. This expression can also be written in the form:

$$E_w = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0c^2$$

$$= mc^2 - m_0c^2$$

Since m_0 and c are constants, it appears from this equation that an increase in the work done on the particle requires a corresponding increase in the mass of the particle. This unexpected result gave rise to much speculation regarding our fundamental concepts of mass and energy. The Theory of Relativity, which Einstein postulated in 1905, states that what we measure as mass is equivalent to energy, and that the term m_0c^2 represents the energy equivalent of the particle when it is at rest. If we write $E_w + m_0c^2 = mc^2$ and call $E_w + m_0c^2$ the total energy E, we have:

$$E = mc^2$$

This is the famous expression for the equivalence of mass and energy. To obtain an idea of the magnitudes involved, we shall compute the rest energy equivalent of a mass weighing one pound.

$$E = m_0c^2 = \frac{1 \text{ lb}}{32.2 \text{ ft/sec}^2}\left(\frac{2.998 \times 10^{10} \text{ cm/sec}}{2.54 \text{ cm/in.} \times 12 \text{ in./ft}}\right)^2 \text{ ft-lb}$$

$$E = 3.0 \times 10^{16} \text{ ft-lb}$$

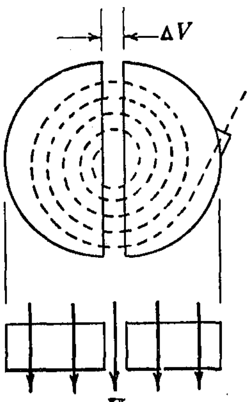
This is roughly equivalent to the energy which would be obtained from the combustion of 1,500,000 tons of coal or 300,000,000 gallons of gasoline.

The first approximate experimental verification of the equivalence of mass and energy was obtained in 1932 by J. D. Cockroft and E. T. Walton by particle bombardment of lithium. The fact that large quantities of energy can be released by nuclear fission was demonstrated by the atomic bomb in 1945. Measurements have shown that the difference in mass between the fission products and the original nucleus is just equivalent to the energy released.

PROBLEMS

4.44. An electron is accelerated from rest through a potential drop of 100,000 volts. Compute its velocity assuming that its mass remains constant, and compare with the velocity obtained when the variability of mass with velocity is taken into account. This potential drop is small compared with the several million volts used in modern particle accelerators.

4.45. The basic elements of a cyclotron are shown in the accompanying diagram. The device consists of two halves of a cylindrical box, placed in a uniform magnetic field $\boldsymbol{H}$ as shown. If a particle having a charge e and a velocity $\boldsymbol{v}$ in the plane of the box is introduced into the box, it is subjected to the force of the magnetic field which is given by $\boldsymbol{F} = e\boldsymbol{v} \times \boldsymbol{H}$. The two halves of the box are maintained at a potential difference ΔV, so that as the particle travels from one half to the other it experiences a velocity change corresponding to ΔV. By means of an oscillator this potential is varied periodically in such a way that the particle always experiences a potential drop. Show that the particle will move in a circular path whose radius increases with v, and that the time required for one-half a revolution is independent of v. In this way the particle can be accelerated to a high velocity and can then be drawn off and used as a bombarding particle. The foregoing analysis is based on constant mass. If the velocity is so high that the variability of mass must be considered, the time of revolution is not independent of v and difficulties are encountered in synchronizing the potential drop ΔV.



PROB. 4.45

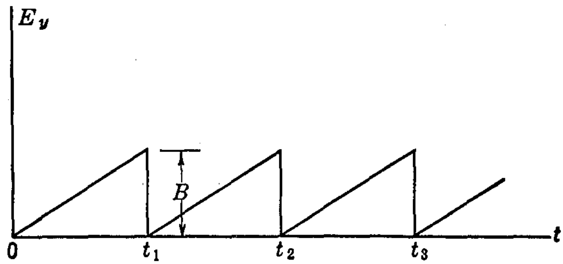
4.46. The electric field between the y deflecting plates of a cathode-ray oscilloscope tube varies as shown in the diagram:

(a) If a field $E_z = E_0 \sin \omega t$ is set up between the z deflecting plates, what picture would be traced out on the screen if $t_1 = \dfrac{2\pi}{\omega}$, and the time intervals are equal?

(b) If in part (a) $t_1 = \dfrac{4\pi}{\omega}$, what picture would appear?

(c) If $E_z = f(t)$, what picture would appear on the screen during time $t = 0$ to $t = t_1$?

4.47. Work out the steps in the derivation of the expression for the relativistic kinetic energy of a particle, and show that for $v << c$ this



PROB. 4.46

reduces to the familiar expression for the kinetic energy of a slow-speed particle.

4.48. Two electrons approach each other along a straight line, each having a velocity v which is not small compared to the velocity of light. How close together can the electrons come?

4.49. The magnitude of the energy released by the fission of uranium 235 may be estimated by assuming that the products of the fission are nuclei with mass numbers 95 and 139, since these are known to make up the greatest amount of the products. The fission takes place according to the following relation, where the single neutron on the left side initiates the fission which results in the formation of two neutrons:

$$U^{235} + n^1 \rightarrow X^{95} + Y^{139} + 2n^1$$

The mass of the 235 atom is 235.124 amu (amu = atomic mass unit = 1.660×10^{-24} grams) and the mass of the neutron is 1.00897 amu. The masses of the X and Y atoms are determined experimentally to be 94.945 amu and 138.955 amu. Calculate the amount of energy in foot-pounds released per fission.

Chapter 5

DYNAMICS OF VIBRATING SYSTEMS

First of all one must observe that each pendulum has its own time of vibration, so definite and determinate that it is not possible to make it move with any other period than that which nature has given it. On the other hand one can confer motion upon even a heavy pendulum which is at rest by simply blowing against it. By repeating these blasts with a frequency which is the same as that of the pendulum one can impart considerable motion.— G. Galilei, *Discorsi a Due Nuove Scienze* (1638).

The analysis of mechanical vibrations is a problem in dynamics which is often encountered by the engineer. Such problems arise in connection with the design of almost every type of machine or structure. The vibration of high-speed machinery, aircraft flutter, the vibration of buildings during earthquakes, and the design of dynamic measuring instruments are current problems which indicate the wide scope of the subject. The same mathematical theory which is used for the study of mechanical vibrations is also applicable to certain problems of oscillations in electrical circuits. The similarity between the basic equations of mechanical and electrical systems has led to several useful methods whereby the results of analysis or experimental investigations in one field have been applied to the other.

In the present chapter we shall consider only the motion of systems having one degree of freedom. Such problems are excellent examples of the methods of particle dynamics and also indicate the theory behind a large number of interesting technical applications.

5.1 The Vibration Problem. We shall first investigate the simplest possible mechanical system which contains all the significant features of a vibration problem. Consider a mass m which has one degree of freedom, that is, its location at any time is specified by the one coordinate x. The mass is restrained by a spring k, and an external force $F(t)$, whose magnitude varies with time, is applied to the mass, as shown in Fig. **5.1** (a). Fig. **5.1** (b) shows a free-body diagram of the mass, when it has a positive velocity and displacement as measured from the position of static equilibrium ($x = 0$). The fact that a friction force opposes the motion of the mass is indicated by the force F_d, which is usually some function of the velocity of the system, depending upon the nature of the contacting surfaces and the conditions of lubrication.

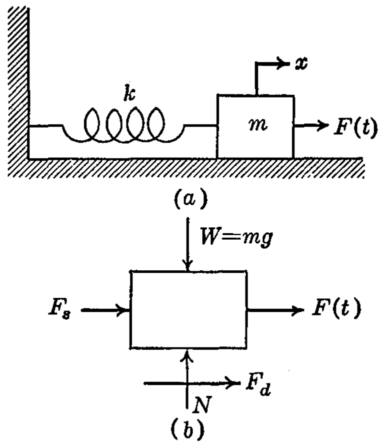


FIG. 5.1

The forces acting upon the mass belong to three general classes. First there is the exciting force $F(t)$, which is the externally applied force that causes the motion of the system. Second, there is the restoring force F_s, which is the force exerted by the spring on the mass and which tends to restore the mass to its original position. Third, there is a damping force F_d, which is always in such a direction that it opposes the motion of the system, and which is thus responsible for a dissipation of energy. The equation of motion can be written:

$$m\ddot{x} = F_s + F_d + F(t) \tag{5.1}$$

Three such forces, along with the equation of motion, characterize the vibration problem. When an analysis of a physical problem leads to Equation (5.1), many of the essential features of the motion can be analyzed as in the following sections.

5.2 The Characteristics of the Forces. From the definition of the restoring force it is known that its direction is always toward the equilibrium position of the system. If the restoring force is produced by a spring as in Fig. 5.1, it is known that:

$$F_s = -kx$$

where the point $x = 0$ is the equilibrium position and k is the spring constant which indicates the stiffness of the spring. This spring force is called a linear restoring force because it is a function of x raised to the first power. In some instances the restoring force is not linear. To obtain the solution for non-linear restoring forces is difficult, so that it is customary to linearize the problem, if possible, by treating only small oscillations. For example, suppose that the restoring force is given by some function of x which can be expanded in a power series:

$$F_s = -\phi(x) = -\left(k_1 x + k_2 \frac{x^2}{2!} + k_3 \frac{x^3}{3!} + \cdots\right)$$

If x is small, the first term of this series is large compared to the sum of the following terms. Thus if the amplitude of the vibration is always sufficiently small, a satisfactory approximate solution can be obtained by taking for the restoring force only the first term of the series:

$$F_s = -k_1 x$$

We shall treat only linear restoring forces. In recent years considerable work has been done on the non-linear problem, but as yet no general solutions of a simple form have been determined.

The most important characteristic of the damping force is that its direction is always opposed to the direction of the motion. The work done by the damping force is thus always negative, and energy is dissipated from the system. In many instances the damping force is directly proportional to the velocity of the mass, so that:

$$F_d = -c\dot{x}$$

Damping which can be described by this equation is called *viscous damping*, and c is called the coefficient of viscous damping. Such a damping force may arise in a number of ways. The frictional force set up between two lubricated surfaces, under the usual conditions of velocity and pressure, is approximately proportional to the velocity, and air resistance at low velocities may also be assumed to be viscous in nature. Damping forces are often intentionally introduced into a system, and this is commonly done by means of a dashpot filled with oil. Such a device can be designed to give viscous damping. In some problems in which the damping is not viscous, the concept of viscous damping may still be used, by defining an *equivalent viscous friction*, for which the coefficient of viscous friction is determined so that the total energy dissipated per cycle is the same as for the actual damping during a steady state of motion. In the analysis to follow, we shall always assume viscous damping forces.

Exciting forces may arise in many different ways. They may, for example, be transient forces such as would be caused by the impact of some external body, or they may be repetitive forces caused by a series of such impacts. Reciprocating or rotating machine parts often produce unbalanced alternating forces that have a sinusoidal variation. Consider the rotation of an unbalanced disk as shown in Fig. 5.2. This arrangement represents a typical vibration isolating mount for a rotating machine. The disk of mass m rotates about the center O with an angular velocity ω. The center of mass of the disk is located at a distance r from the center of rotation. The rotating system is mounted on a larger mass M which can move only in a vertical direction. M is supported on a spring having a spring constant k, and a dashpot having a coefficient of viscous damping c connects the mass to the fixed support. If we assume that the motion of M is small compared to r, then the motion of m can be taken as circular, and the acceleration of the center of mass of the disk is $r\omega^2$. There is thus a force of magnitude $mr\omega^2$ acting in a radial direction upon the large mass M. The component of the force in the y direction, that is, the component of force which causes motion of the system is $mr\omega^2 \sin \theta$. Assuming that the disk rotates with a constant speed we have for the exciting force:

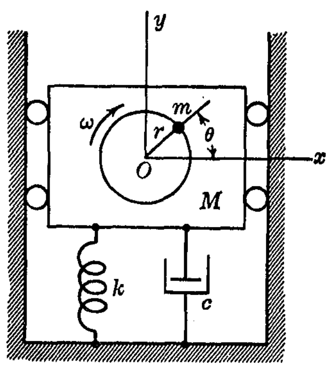


FIG. 5.2

$$F(t) = mr\omega^2 \sin \omega t$$

Since small amounts of unbalance are inevitably present in any rotating machine, sinusoidal exciting forces play an important part in vibration theory.* A more fundamental reason for the importance

* If the motion of M is not small, m performs two-dimensional simple harmonic motion during a steady state and the resulting exciting force is again sinusoidal (see Prob. 5.32).

of the sinusoidal force is the fact that any periodic force can be represented analytically as a series of sine and cosine terms, by a Fourier series expansion. Thus, if the behavior of the system is known for a sinusoidal force, the behavior of the system can be determined for any periodic force.

5.3 The Differential Equation of the Vibration Problem. For the basic vibration problem we shall consider a system which consists of a linear restoring force, a viscous damping force, and a sinusoidal exciting force:

$$F_s = -kx$$
$$F_d = -c\dot{x}$$
$$F(t) = F_0 \sin \omega t$$

Substituting these terms into the equation of motion gives:

$$m\ddot{x} = -kx - c\dot{x} + F_0 \sin \omega t$$

We shall write this equation in the standard form:

$$\ddot{x} + 2n\dot{x} + p^2 x = \frac{F_0}{m} \sin \omega t \tag{5.2}$$

where:

$$\frac{k}{m} = p^2 \quad \text{and} \quad \frac{c}{m} = 2n$$

The term n is called the damping factor. A system described by this equation is said to be a single degree of freedom harmonic oscillator with viscous damping. In the following sections we shall derive the solution of the equation, and we shall examine its physical significance.

5.4 Free Vibrations of an Undamped System. Of the three forces mentioned above, only the restoring force is necessary for the existence of a vibration problem. It may be that energy dissipation is so small that the damping force may be neglected, and the motion of the system may be started by initial displacements or velocities rather than by exciting forces. In this section we shall consider the solution of this simplest type of vibration problem as an illustration of method, the nomenclature to be used, and of the physical interpretation of the results.

Setting the damping force and the exciting force equal to zero, the differential equation becomes:

$$\ddot{x} + p^2 x = 0$$

The solution of this equation is:

$$x = C_1 \sin pt + C_2 \cos pt$$

where C_1 and C_2 are constants of integration which must be evaluated from the initial conditions. That this expression is a solution of the differential equation may be verified by direct substitution.

When $t = 0$ let the initial displacement be x_0 and the initial velocity $\dot{x}_0$. From these two initial conditions the constants C_1 and C_2 may be found, and the solution of the differential equation becomes:

$$x = \frac{\dot{x}_0}{p} \sin pt + x_0 \cos pt \tag{5.3}$$

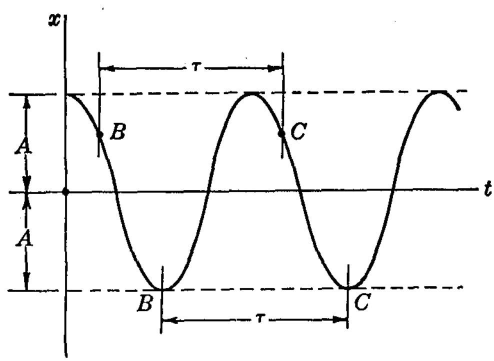


FIG. 5.3

We shall investigate the physical significance of this solution for $x_0 = A$ and $\dot{x}_0 = 0$. This means that the mass is moved a distance A from its position of equilibrium and is then released, at time $t = 0$, with zero initial velocity. The displacement is then given by:

$$x = A \cos pt$$

The motion of the mass as a function of time is shown in Fig. 5.3, where it is seen that the mass performs oscillations about the position of equilibrium. Since there is no energy loss in this ideal system, the oscillation continues indefinitely with the same amplitude A. The portion of the motion included between two points at which the mass has the same position, as B and C in Fig. 5.3, is called one cycle of the vibration. The time required for the completion of one cycle is called the *period*, τ, of the vibration. The number of cycles which occur in one second is called the *frequency*, f, of the vibration. To find the period, consider two displacements of the mass which are one cycle apart, as B and C in Fig. 5.3. Then:

$$A \cos pt = A \cos p(t + \tau)$$

$$\tau = \frac{2\pi}{p} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \tag{5.4}$$

$$f = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The period can also be found by an energy method in the following way. The potential energy of the system at any position is $V = \frac{1}{2} kx^2$ and the kinetic energy is $\frac{1}{2} m\dot{x}^2$. Since the motion is known to be harmonic, the displacement and velocity can be written:

$$x = A \sin \omega t$$
$$\dot{x} = A\omega \cos \omega t$$

When $x = A$, the potential energy is equal to $\frac{1}{2}kA^2$ and the kinetic energy is zero. When $x = 0$, the kinetic energy is $\frac{1}{2}mA^2\omega^2$ and potential energy is zero. Since energy is conserved:

$$\tfrac{1}{2}mA^2\omega^2 = \tfrac{1}{2}kA^2$$

$$\omega = \sqrt{\frac{k}{m}} = p$$

This energy method is useful for obtaining approximate frequencies in more complicated problems when the motion can be assumed to be approximately harmonic.

PROBLEMS

5.1. A pendulum having a mass m and a length l is supported by a string of negligible mass. Write the equation of motion for the pendulum, neglecting air damping, and show that for small oscillations this equation is:

$$\ddot{\phi} + \frac{g}{l}\phi = 0$$

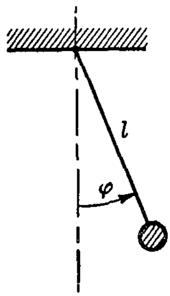
Find the period of small oscillations of the pendulum. If air resistance imposes a damping force proportional to the velocity, what is the differential equation of motion for small oscillations? What force plays the part of the restoring force in this problem?

5.2. Write the equation of the conservation of energy for the pendulum of Problem 5.1. Obtain the differential equation of motion for small oscillations by differentiating the energy equation with respect to t.

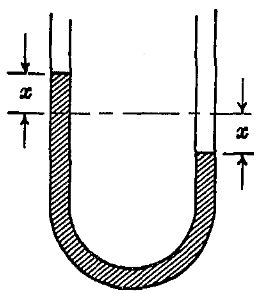
5.3. Show that the natural frequency of free vibrations of an undamped simple harmonic oscillator is given by:

$$f = \frac{3.13}{\sqrt{\delta_{st}}} \text{ cycles per second}$$

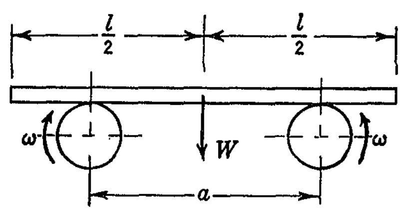
where δ_{st} is the static deflection of the system, in inches. The static



PROB. 5.1



PROB. 5.4



PROB. 5.5

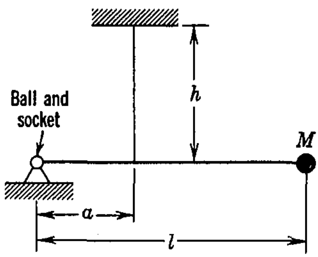
deflection of a system is defined as the deflection caused by a force of mg lb.

5.4. A U-tube is partially filled with mercury and is supported in a vertical position as shown in the diagram. When the system is in equilibrium, the height of the mercury is the same in each arm. The liquid in one arm is depressed a distance x thus raising the liquid a like distance in the other arm. The system is then released. Write the equation of motion for the liquid column, neglecting frictional damping forces and find the frequency of the resulting oscillation.

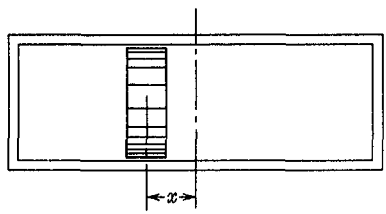
5.5. Two parallel cylindrical rollers rotate in opposite directions as shown in the figure. The distance between the centers of the rollers is a. A straight, uniform horizontal rod of length l and weight W rests on top of the rollers. The coefficient of kinetic friction between the rod and the roller is μ. Taking x as the distance from the center of the rod to the midpoint between the rolls, write the equation of motion of the rod, supposing that it has been initially displaced from the central position. Find the frequency of the resulting vibratory motion.

5.6. A circular cylinder of diameter D and weight W is arranged so that it floats with its axis vertical in a fluid of density ρ. The cylinder is stable against tipping. The cylinder is depressed slightly from its equilibrium position and is then released. Find the frequency of the resulting oscillation, neglecting the effects of the motion of the fluid.

5.7. The mass M is attached to one end of a weightless horizontal rod of length l, and the other end of the rod is supported by a ball-and-socket joint as shown in the figure. An inextensible vertical wire of length h supports the rod at a distance a from the joint. Find the frequency of small oscillations of the system, neglecting friction.



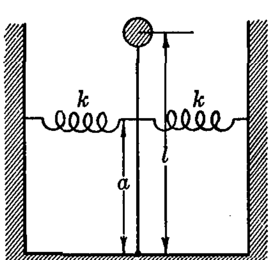
PROB. 5.7



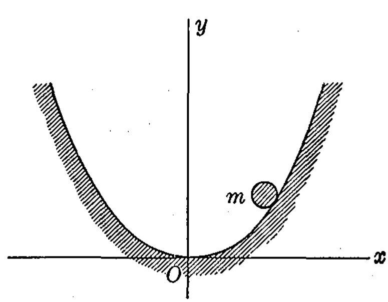
PROB. 5.8

5.8. A piston of mass m fits in a closed cylinder of cross-sectional area A. When the piston is in the central position with $x = 0$, it is in equilibrium, and the pressure on each side is p. The air in the cylinder is assumed to follow Boyle's law, that is, the pressure times the volume is equal to a constant. The piston is moved through a distance x from the position of equilibrium and is then released. Write the differential equation of motion of the system, assuming that there is viscous friction between the piston and the cylinder. Find the frequency of small oscillations of the piston, assuming that the damping force can be neglected.

5.9. Find the frequency of small vibrations of an inverted pendulum restrained by two springs of spring constant k as shown in the diagram.



PROB. 5.9

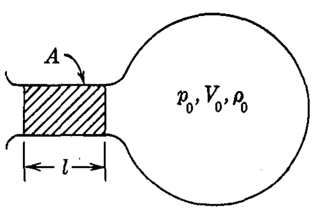


PROB. 5.10

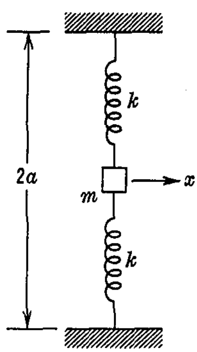
All the mass of the pendulum is assumed to be concentrated at a distance l from the point of support, and the springs are sufficiently stiff so that the pendulum is stable.

5.10. A particle of mass m slides on a smooth surface whose shape is given by the equation $y = ax^2$. The particle is moved along the surface away from the position of equilibrium and is then released. Find the equation of motion of the particle, and find the frequency of small oscillations about the position of static equilibrium.

5.11. A Helmholtz resonator consists of a rigid spherical vessel with a cylindrical neck of length l and cross-sectional area A. The initial pressure of the air in the resonator (and of the surrounding air) is p_0, the spherical volume is V_0, and the density of the air is ρ_0 = constant. A vibrating motion is imparted to the air in the neck of the resonator and the motion is assumed to be confined to the air in the neck, the air in the spherical portion acting as a spring. The system can thus be considered as a single degree of freedom system with the mass shown shaded in the diagram. It may be assumed that during the vibration the air in the spherical cavity obeys the adiabatic law pV^γ = constant, where γ is the ratio of the specific heats. Find the natural frequency of small oscillations in cycles per second of the Helmholtz resonator in terms of p_0, V_0, ρ_0, A, l, and γ.



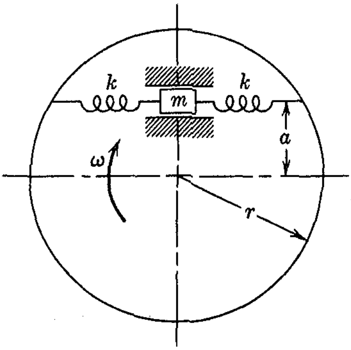
PROB. 5.11



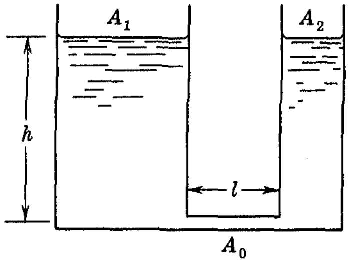
PROB. 5.12

5.12. A small mass m is held by two identical springs of spring constant k. At equilibrium the springs have an initial tension T. Suppose that the system is horizontal, so that gravity forces will not enter the problem. Consider motions of the mass perpendicular to the original spring direction and treat the system as a one degree of freedom system. (a) Derive the exact equation of motion. (b) By assuming small displacement, linearize the equation of motion and find the frequency of small oscillations. Explain what is meant by "small displacements" in terms of a suitable dimensionless parameter.

5.13. A mass m is free to vibrate in the direction of two restraining springs of spring constant k and is guided in frictionless supports. The assembly is attached to a disk which rotates with a uniform angular velocity ω. Find the frequency of free oscillations of the mass relative to the disk.



PROB. 5.13



PROB. 5.14

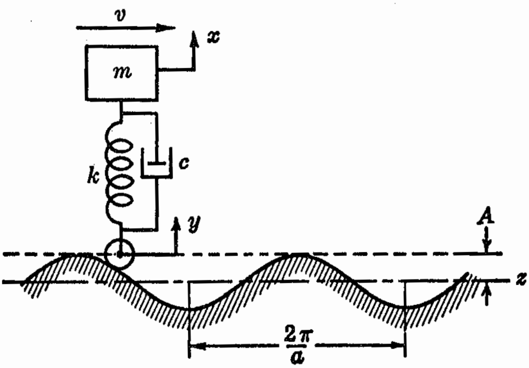
5.14. Two vertical cylindrical tanks of area A_1 and A_2 are connected by a horizontal pipe of length l and area A_0. The tanks are filled with a fluid to a height h above the horizontal pipe as is shown in the figure. If the level of the fluid in one of the cylinders is changed and the system is then left free, oscillations will occur, with the fluid flowing back and forth from one tank into the other. Find the frequency of this oscillation, assuming that all of the particles in any one cylinder have the same velocity at the same time.

5.15. A spring-mounted mass, supported on a wheel, as shown in the diagram, moves with a velocity v along a wavy surface which has a sinusoidal form. The vertical displacement of the wheel y can be found from the fact that:

$$y = A \sin az$$

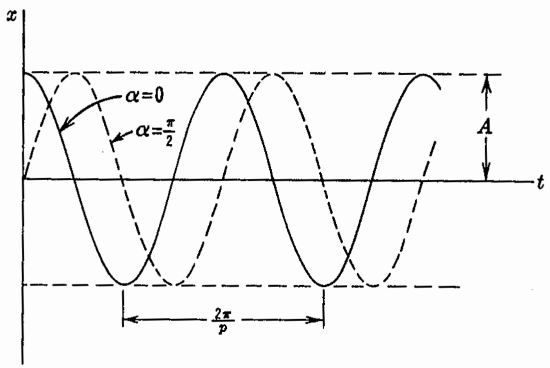
where $z = vt$. The vertical motion of the mass is given by the coordinate x. A dashpot which introduces viscous damping into the system is connected between the wheel and the mass. Write the equation of motion for the vertical movement of the mass, noting that the forces which act upon it are the elastic force, $-k(x - y)$, and the damping force, $-c(\dot{x} - \dot{y})$. Show that this equation reduces to the general form of the vibration equation, with a sinusoidal exciting force.

5.16. Show that the equation $x = C_1 \sin pt + C_2 \cos pt$ can be written in the form $x = A \cos (pt + \alpha)$ where $A = \sqrt{C_1^2 + C_2^2}$ and $\alpha =$

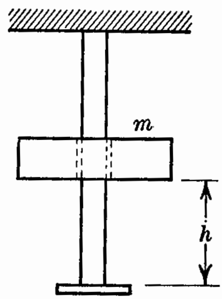


PROB. 5.15

$\tan^{-1}\left(-\frac{C_1}{C_2}\right)$. A is called the amplitude of the vibration, and α is called the phase angle. Changing the phase angle has the effect of shifting the whole curve representing the vibration to the right or left, as shown in the figure.



PROB. 5.16



PROB. 5.17

5.17. A mass m drops from rest through a height h and strikes the bottom of a rod. The rod elongates x feet when acted upon by a force of kx lb. Assuming that the mass remains in contact with the end of the rod after impact, find the motion of the mass after the impact. Neglect the mass of the rod and assume that no energy is lost. If m weighs 20 lb, $k = 100$ lb/ft, and $h = 2$ in., find the amplitude and the frequency of vibration.

5.5 Damped Vibrations. In an actual vibration there will always be some damping present. Let us consider free vibrations with viscous damping, and compare the solution of this problem with that of the undamped oscillation. The differential equation of the free damped vibration is obtained from the general equation by setting the exciting force equal to zero:

$$\ddot{x} + 2n\dot{x} + p^2x = 0 \tag{5.5}$$

The solution of this equation must be a function which has the property that repeated differentiations do not change its form, since the function and its first and second derivatives must be added together to give zero. For such an equation, we take as a trial solution $x = Ce^{\lambda t}$, which, upon substitution into the differential equation gives:

$$C\lambda^2e^{\lambda t} + 2Cn\lambda e^{\lambda t} + p^2Ce^{\lambda t} = 0$$

Cancelling the factor $Ce^{\lambda t}$ we have:

$$\lambda^2 + 2n\lambda + p^2 = 0$$

This equation has two solutions for λ, each of which will make $x = Ce^{\lambda t}$ a solution of the differential equation. The general solution of the equation may thus be written as the sum of the two:*

$$x = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}. \tag{5.6}$$

It should be noted that the superposition of solutions is valid only for linear differential equations, that is, equations which are linear in the dependent variable and its derivatives.

Solving the algebraic equation we obtain the two values:

$$\lambda_1 = -n + \sqrt{n^2 - p^2}$$
$$\lambda_2 = -n - \sqrt{n^2 - p^2}$$

Hence the solution is:

$$x = C_1e^{(-n+\sqrt{n^2-p^2})t} + C_2e^{(-n-\sqrt{n^2-p^2})t}$$

The physical significance of this solution depends upon the relative magnitudes of n^2 and p^2, which determine whether the exponents are real or complex quantities.

Suppose first that $n^2 > p^2$ so that the exponent is a real quantity.

* Since the differential equation is of the second order in its derivatives, the solution with two constants of integration is the complete solution.

Physically this means a relatively large damping, since n is a measure of the damping in the system. The solution is then:

$$x = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t} \tag{5.7}$$

where α_1 and α_2 are real quantities. The motion of the mass, in this event, is not oscillatory, but is an exponential subsidence. Suppose, for example, that the motion is started by giving the mass an initial displacement A and then releasing it from rest. The displacement-time curve is then as shown in Fig. 5.4. Because of the relatively large damping, the mass released from rest never passes the static

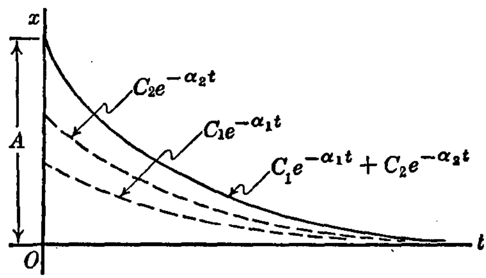


FIG. 5.4

equilibrium position. So much energy is dissipated by the damping force that there is not sufficient kinetic energy left to carry the mass past the equilibrium position. Such a system is said to be *over-damped.*

If the damping is small, so that $n^2 < p^2$, then the term $(n^2 - p^2)$ is negative, and we can write:

$$\begin{aligned} x &= C_1 e^{(-n+i\sqrt{p^2-n^2})t} + C_2 e^{(-n-i\sqrt{p^2-n^2})t} \\ &= e^{-nt}[C_1 e^{i\sqrt{p^2-n^2}t} + C_2 e^{-i\sqrt{p^2-n^2}t}] \end{aligned}$$

Using the trigonometric relation $e^{i\theta} = \cos\theta + i\sin\theta$, the displacement may be written:

$$x = e^{-nt}[(C_1 + C_2)\cos\sqrt{p^2 - n^2}t + i(C_1 - C_2)\sin\sqrt{p^2 - n^2}t]$$

Since the constants C_1 and C_2 are arbitrary and are to be determined by the initial conditions, we may simplify the expression by introducing new constants, $C_1' = i(C_1 - C_2)$ and $C_2' = C_1 + C_2$; dropping the primes:

$$x = e^{-nt}[C_1 \sin\sqrt{p^2 - n^2}t + C_2 \cos\sqrt{p^2 - n^2}t] \tag{5.8}$$

This equation may be checked by setting $n = 0$, thus reducing it to:

$$x_{n=0} = C_1 \sin pt + C_2 \cos pt$$

which was previously derived for the undamped free vibrations.

Comparing the two solutions we see that the effect of the damping is to increase the period of the vibration and to decrease the magnitudes of successive peaks of the vibration, since the amplitude of the vibration decreases exponentially with time. The motion of a typical *underdamped oscillator* is shown in Fig. 5.5.

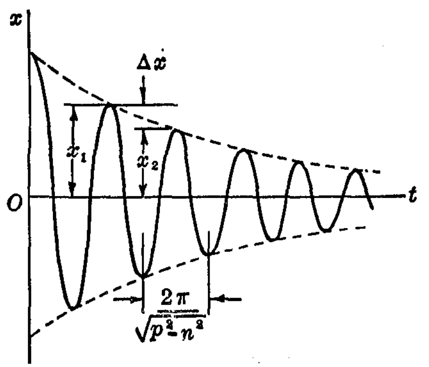


FIG. 5.5

As a convenient measure of the damping we may compute the ratio of the amplitudes of successive cycles of the vibration.

$$\frac{x_1}{x_2} = \frac{e^{-nt_1}}{e^{-n\left(t_1 + \frac{2\pi}{\sqrt{p^2-n^2}}\right)}} = e^{\frac{2\pi n}{\sqrt{p^2-n^2}}}$$

The amount of damping is often specified by giving the *logarithmic decrement* δ, where:

$$\delta = \log \frac{x_1}{x_2} = \log e^{\frac{2\pi n}{\sqrt{p^2-n^2}}} = \frac{2\pi n}{\sqrt{p^2 - n^2}} \tag{5.9}$$

For systems having small damping, a simple way of determining the logarithmic decrement from the free vibration curve is as follows:

$$\delta = \log\left(\frac{x + \Delta x}{x}\right) = \log\left(1 + \frac{\Delta x}{x}\right) = \frac{\Delta x}{x} - \tfrac{1}{2}\left(\frac{\Delta x}{x}\right)^2 + \tfrac{1}{3}\left(\frac{\Delta x}{x}\right)^3 + \ldots$$

for $\left(\frac{\Delta x}{x}\right)$ small, the higher order terms may be dropped, and:

$$\delta \approx \frac{\Delta x}{x} \tag{5.10}$$

Thus the logarithmic decrement is approximately equal to the fractional decrease in amplitude during one cycle of the vibration.

Another important quantity in damped vibration analysis is the energy lost per cycle due to the damping force. The total energy of the system when it is in one of its extreme positions with zero velocity is:

$$W_1 = \tfrac{1}{2}kx^2$$

The energy one cycle later is:

$$W_2 = \tfrac{1}{2}k(x - \Delta x)^2$$

Therefore the energy loss per cycle is:

$$\Delta W = W_1 - W_2 = \tfrac{1}{2}kx^2 - \tfrac{1}{2}kx^2 + kx(\Delta x) - \tfrac{1}{2}(\Delta x)^2$$

Expressing this energy loss as a fraction of the total energy of the system gives:

$$\frac{\Delta W}{W} = 2\left(\frac{\Delta x}{x}\right) - \left(\frac{\Delta x}{x}\right)^2$$

If the damping is small the square term can be dropped, and we have:

$$\frac{\Delta W}{W} \approx 2\left(\frac{\Delta x}{x}\right) \approx 2\delta \tag{5.11}$$

Thus for small damping the fraction of energy lost per cycle is approximately equal to twice the logarithmic decrement.

PROBLEMS

5.18. At time $t = 0$, the initial displacement of a damped harmonic oscillator is x_0, and the initial velocity is $\dot{x}_0$. Show that the free vibrations of the system are described by the equation:

$$x = e^{-nt}\left[x_0 \cos \sqrt{p^2 - n^2}t + \frac{\dot{x}_0 + nx_0}{\sqrt{p^2 - n^2}} \sin \sqrt{p^2 - n^2}t\right]$$

5.19. Critical damping is defined as that damping for which $n = p$. (a) If the damping is less than critical, show that the logarithmic decrement can be written:

$$\delta = \frac{2\pi\left(\frac{n}{n_c}\right)}{\sqrt{1 - \left(\frac{n}{n_c}\right)^2}}$$

where $n_c = p =$ damping factor for critical damping.

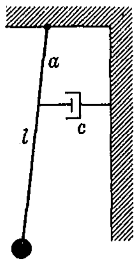
(b) Show that for small damping the logarithmic decrement can be written:

$$\delta \approx 2\pi\left(\frac{n}{n_c}\right) \tag{5.12}$$

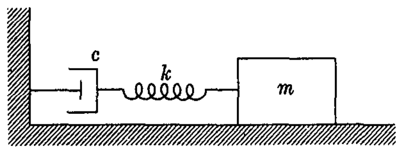
5.20. A single degree of freedom oscillator having an amount of viscous damping just equal to the critical damping value is initially at rest. At time $t = 0$ a velocity V_0 is given to the mass. Find the maximum deflection of the mass in the direction of the initial velocity.

5.21. Solve the differential equation of motion for the critically damped oscillator, $n = p$. Evaluate the constants of integration and determine whether the displacement can change sign during a free vibration.

5.22. A simple pendulum of length l and concentrated mass m is connected to a dashpot with a viscous damping coefficient c at a distance a below the point of support as shown in the figure. Find the logarithmic decrement for small oscillations of the system in terms of the given parameters.



PROB. 5.22



PROB. 5.23

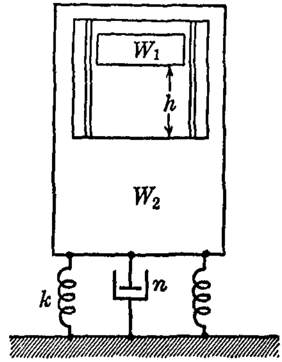
5.23. A mass is constrained to move in a straight line on a horizontal frictionless surface by a spring and viscous dashpot connected in series, as shown in the figure. Find the equation of motion for the mass.

5.24. A mass weighing 10 lb is restrained by a spring which has $k = 15$ lb/ft and is acted upon by a viscous damping force. It is observed that at the end of four cycles of motion the amplitude is reduced by one-half. Find the damping factor n and the period of the vibration.

5.25. A drop hammer is found to transmit an objectionable shock to the surrounding ground. To eliminate this, the machine is mounted on springs, as shown in the diagram. To prevent undue vibration of the system after impact, damping is introduced as shown by the dashpot. The constants of the systems are:

$W_1 = 2000$ lb
$W_2 = 30{,}000$ lb
$h = 8$ ft
k (for all springs) $= 250{,}000$ lb/ft
$n = 0.8\ \text{sec}^{-1}$

The weight W_1 falls through a distance h and makes a plastic (no rebound) impact with W_2. The resulting motion of the system is a free vibration with damping. Find the maximum displacement of W_2, and the displacement three complete cycles after the maximum displacement occurs.



PROB. 5.25

5.6 Forced Vibrations. Vibrations which are maintained by an exciting force are said to be *forced vibrations*. We shall now develop the complete solution for the motion of a damped, simple harmonic oscillator acted upon by the sinusoidal exciting force $F_0 \sin \omega t$. The differential equation of the motion (Equation 5.2) is:

$$\ddot{x} + 2n\dot{x} + p^2 x = \frac{F_0}{m} \sin \omega t$$

The solution of this equation may be written as the sum of two terms:

$$x = e^{-nt}[C_1 \sin \sqrt{p^2 - n^2}t + C_2 \cos \sqrt{p^2 - n^2}t] + f(t)$$

for we have found from the preceding section that the first term, when substituted into the differential equation, gives zero. Therefore, a function $f(t)$ must be added of such a form that it will yield $\frac{F_0}{m} \sin \omega t$ when substituted in the equation. This second term is called the particular solution. Since two arbitrary constants already appear in the first term, no further arbitrary constants need be included.

The particular solution in the present problem may be found by taking a trial solution:

$$x = A \sin \omega t + B \cos \omega t$$

where the values of A and B are to be determined from the condition that the differential equation must be satisfied. Substituting into the equation the expressions:

$$\dot{x} = A\omega \cos \omega t - B\omega \sin \omega t$$
$$\ddot{x} = -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t$$

gives:

$$- A\omega^2 \sin \omega t - B\omega^2 \cos \omega t + 2nA\omega \cos \omega t - 2nB\omega \sin \omega t + p^2A \sin \omega t + p^2B \cos \omega t = \frac{F_0}{m} \sin \omega t$$

or:

$$(- A\omega^2 - 2n\omega B + p^2A) \sin \omega t + (- B\omega^2 + 2nA\omega + p^2B) \cos \omega t = \frac{F_0}{m} \sin \omega t$$

This equation must be identically satisfied, which means that the coefficient of the sin ωt on the left side of the equation must equal the coefficient of the sin ωt term on the right side of the equation, and the coefficient of the cosine term must equal zero. Hence:

$$(p^2 - \omega^2)A + - (2n\omega)B = \frac{F_0}{m}$$
$$(2n\omega)A + (p^2 - \omega^2)B = 0$$

These two algebraic equations determine the proper values of A and B:

$$A = \frac{\begin{vmatrix} \left(\frac{F_0}{m}\right) & (-2n\omega) \\ 0 & (p^2 - \omega^2) \end{vmatrix}}{\begin{vmatrix} (p^2 - \omega^2) & (-2n\omega) \\ (2n\omega) & (p^2 - \omega^2) \end{vmatrix}} = \frac{\frac{F_0}{m}(p^2 - \omega^2)}{(p^2 - \omega^2)^2 + 4n^2\omega^2}$$

$$B = \frac{\begin{vmatrix} (p^2 - \omega^2) & \left(\frac{F_0}{m}\right) \\ (2n\omega) & 0 \end{vmatrix}}{(p^2 - \omega^2)^2 + 4n^2\omega^2} = \frac{-2n\omega \frac{F_0}{m}}{(p^2 - \omega^2)^2 + 4n^2\omega^2}$$

Writing the solution $x = A \sin \omega t + B \cos \omega t$ in the form

$$x = \sqrt{A^2 + B^2} \sin (\omega t - \phi)$$

we have:

$$x = \frac{\frac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}} \sin (\omega t - \phi) \tag{5.13}$$

where:

$$\phi = \tan^{-1}\left(- \frac{B}{A}\right) = \tan^{-1}\left(\frac{2n\omega}{p^2 - \omega^2}\right) \tag{5.14}$$

The complete solution of the differential equation is thus:

$$x = e^{-nt}[C_1 \sin \sqrt{p^2 - n^2}t + C_2 \cos \sqrt{p^2 - n^2}t] + \frac{\frac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}} \sin (\omega t - \phi) \qquad (5.15)$$

Equation (5.15) represents a superposition of two motions. One has a frequency $\frac{1}{2\pi}\sqrt{p^2 - n^2}$ and an exponentially decreasing amplitude, and the other has a constant amplitude and the frequency $\frac{1}{2\pi}\omega$.

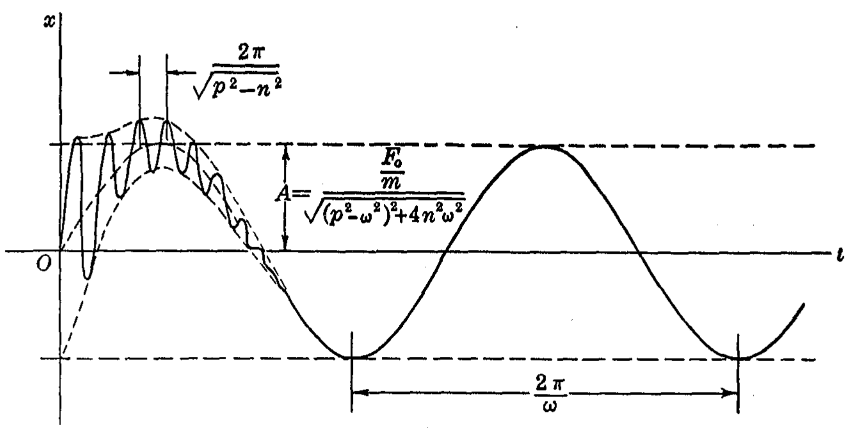


FIG. 5.6

This motion is shown in Fig. 5.6 for $p > \omega$. Because of (e^{-nt}) the first term of the expression decreases with time, and after a sufficient time it can be considered to be damped out, leaving the motion described by the second term. For this reason the first is called the *transient term*, and the second the *steady-state term*. The character of the transient term depends upon the initial conditions of the motion, whereas the steady state vibrations are independent of the initial conditions and depend only upon the forcing function and the parameters of the system.

The most important item in forced vibration problems usually is the amplitude of the steady forced vibration. Calling this amplitude A (Fig. 5.6), we have:

$$A = \frac{\frac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}}$$

Dividing numerator and denominator by p^2, and remembering that $p^2 = \frac{k}{m}$, we obtain:

$$A = \frac{\frac{F_0}{k}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{p}\right)\left(\frac{\omega}{p}\right)\right]^2}}$$

It is customary to express the damping as a fraction of critical damping, where critical damping n_c is defined by $n_c = p$ (see Problem 5.19). We write:

$$\frac{A}{\left(\frac{F_0}{k}\right)} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \tag{5.16}$$

It will be noted that the term $\left(\frac{F_0}{k}\right)$ is the deflection which the system would have under the action of a static load F_0; that is, it is the deflection of the system under a forcing function with zero frequency. The expression on the right side of the equation thus represents a dynamic amplification or magnification factor and gives the ratio between the dynamic and static deflections. The variation of this magnification factor with frequency ratio and damping ratio is shown in Fig. 5.7. The most significant feature of Fig. 5.7 is the fact that, near the frequency ratio $\left(\frac{\omega}{p}\right) = 1$, the magnification factor can become very large if the damping ratio is small. The infinite value indicated at $\left(\frac{n}{n_c}\right) = 0$ would, of course, not exist in practice, since it is impossible to reduce the damping to zero, and since it would require an *infinite time* to reach the *infinite amplitude even* if the damping were zero. The occurrence of large displacements near

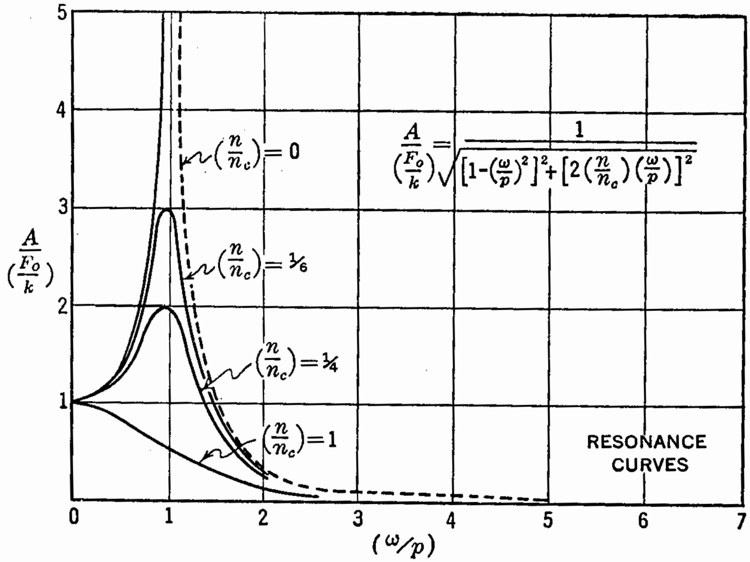


FIG. 5.7

$\left(\dfrac{\omega}{p}\right) = 1$ is called *resonance*, and the frequency for which $\omega = p$ is called the *resonant frequency*.

If the damping is small, the maximum amplitudes occur very near $\left(\dfrac{\omega}{p}\right) = 1$, so that the maximum amplitude may be approximated very closely by setting $\left(\dfrac{\omega}{p}\right) = 1$. Thus we have:

$$A_{\text{res}} = \frac{\dfrac{F_0}{k}}{2\left(\dfrac{n}{n_c}\right)} \tag{5.17}$$

As an example, we might note that the damping ratio $\left(\dfrac{n}{n_c}\right)$ for an aircraft structure, such as a wing, has a magnitude of approximately 0.03; thus the resonant amplitude would be approximately 16 times the static deflection. This illustrates the danger of resonant conditions in structures and machines. However, if resonant vibrations of excessive amplitudes occur, it is possible to improve conditions by changing the frequency ratio or by increasing the damping in the system.

The steady forced vibration is

$$x = A \sin (\omega t - \phi)$$

The angle ϕ gives the phase relation between the motion and the exciting force. The phase angle is given by:

$$\phi = \tan^{-1}\left(\frac{2n\omega}{p^2 - \omega^2}\right) = \tan^{-1} \frac{2n\left(\frac{\omega}{p}\right)}{1 - \left(\frac{\omega}{p}\right)^2} = \tan^{-1}\left[\frac{2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)}{1 - \left(\frac{\omega}{p}\right)^2}\right]$$

If $\left(\frac{\omega}{p}\right) << 1$, that is, if the forcing frequency is relatively low, then ϕ is small, and the motion is nearly in phase with the exciting force. If $\left(\frac{\omega}{p}\right) >> 1$, that is, if the forcing frequency is high, ϕ is nearly 180°, showing that the motion is oppositely directed to the exciting force. At resonance $\left(\frac{\omega}{p}\right) = 1$ and $\phi = 90°$ for all values of the damping so that the exciting force is in the direction of the velocity.

PROBLEMS

5.26. (a) Show that the energy input per cycle, W_i, of the exciting force is equal to:

$$W_i = \int F\,dx = \int F\dot{x}\,dt = \int (F_0 \sin \omega t)[A\omega \cos (\omega t - \phi)]\,dt = \pi F_0 A \sin \phi$$

(b) Show that the energy dissipated by the viscous damping force per cycle is:

$$W_d = c\pi A^2\omega$$

(c) By equating the energy input and the energy dissipation, show that the steady state amplitude of a resonant vibration is:

$$A_{res} = \frac{F_0}{c\omega}$$

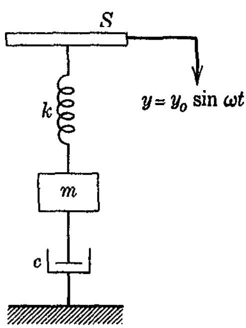
and that this reduces to the same expression as was previously derived for A_{res}. Plot, on a graph of energy per cycle versus amplitude, the energy input and the energy dissipated at resonance and indicate the steady state amplitude.

5.27. A mass m restrained by a spring with a constant k is initially at rest. At time $t = 0$, it is acted upon by an exciting force $F(t) = F_0 \cos \omega t$.

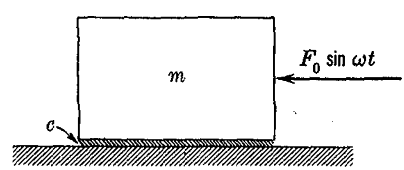
Assuming no damping, and given that $\frac{\omega}{2\pi} = 10$ cycles per second, m weighs 10 lb, $k = 20$ lb/ft, $F_0 = 100$ lb, find (a) the amplitude of the forced vibrations, and (b) the amplitude of the free vibrations.

5.28. An undamped spring-mass system, which under gravity has a static deflection of 1 in., is acted upon by a sinusoidal exciting force which has a frequency of 4 cycles per second. What damping factor n is required to reduce the amplitude of the steady-state forced vibrations to one-half the amplitude of the undamped forced vibrations?

5.29. A support S has a motion $y = y_0 \sin \omega t$ and is attached to a spring and dashpot as is shown in the figure. (a) Find the amplitude of steady forced oscillations of the system. (b) How is this amplitude changed if the dashpot is connected across the spring instead of to ground?



PROB. 5.29



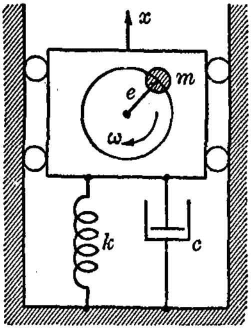
PROB. 5.31

5.30. (a) Find the frequency ratio (ω/p) for which the maximum amplitude of a simple harmonic oscillator with viscous damping and a sinusoidal exciting force is obtained. (b) Show that for small damping this frequency ratio is approximately $1 - (n/n_c)^2$. (c) Find the minimum value of the damping ratio for which the dynamic amplification in the system is never greater than unity.

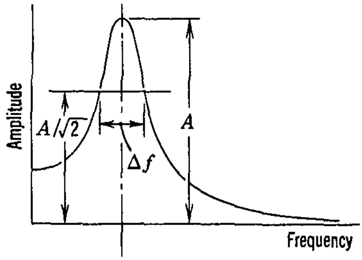
5.31. A mass m rests on a horizontal lubricated surface which may be considered as having viscous friction with a damping constant c. A sinusoidally varying horizontal force $F_0 \sin \omega t$ is applied to the mass as shown. Find the amplitude of the steady state sinusoidal component of the resultant motion.

5.32. An unbalanced rotating mass is supported by a spring of constant k across which a dashpot giving a viscous damping force $c\dot{x}$ is connected. The mass rotates with an angular velocity ω, and the center of mass of the rotating body is at a distance e from the axis of rotation. Find the amplitude A of steady-state forced vibrations and plot a curve of $\left(\frac{\omega}{p}\right)$

versus $\left(\frac{A}{e}\right)$ for several values of the damping ratio $\left(\frac{n}{n_c}\right)$. The system is constrained so that the motion x is rectilinear. Note: The effect of the unbalanced mass is equivalent to a smaller mass m' at a larger distance r from the center of rotation. By assuming that the vibratory amplitude is small compared to r, the motion of the small mass m' can be taken as circular.



PROB. 5.32



PROB 5.34

5.33. A single degree of freedom linear oscillator with viscous damping is acted upon by a harmonic force $F_0 \sin \omega t$. Determine the energy dissipated per unit time, and find under what conditions an increase in damping produces a decrease in the energy dissipated per unit time. How would this be accounted for physically?

5.34. Show that for small amounts of damping the width of the resonance curve Δf at a point where the amplitude is $1/\sqrt{2}$ times the resonant amplitude is given by the relationship:

$$\Delta f = \frac{\delta f}{\pi}$$

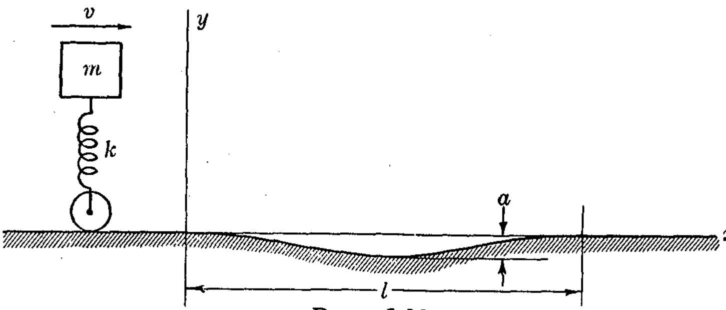
where δ is the logarithmic decrement, f is the resonant frequency of the system.

5.35. An automobile, without shock absorbers, may be represented approximately as a concentrated mass m supported by a spring having a constant k. The automobile runs with a velocity v over a hollow in the road which can be represented by the cosine curve

$$y = -\frac{a}{2}\left(1 - \cos\frac{2\pi x}{l}\right)$$

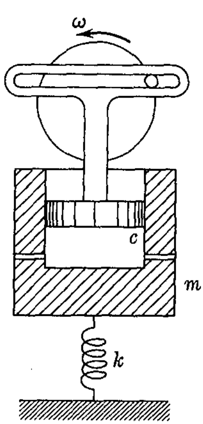
Neglecting damping, find the vertical acceleration of m when $x = l$.

5.36. In the arrangement shown in the figure, a scotch-yoke mechanism drives a piston with a frequency ω and an amplitude y_0. There is

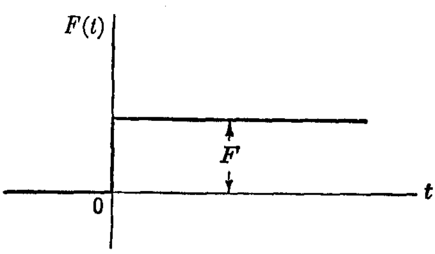


PROB. 5.35

viscous friction with a coefficient c between the piston and a cylinder of mass m, which is constrained to move in a straight line, and which is supported by a spring of spring constant k. Find the amplitude of steady state forced oscillations of the mass, and sketch a typical resonance curve for the motion of the mass. The effects of any compressed air in the cylinder may be neglected.



PROB. 5.36



PROB. 5.37

5.37. A step function as shown in the diagram is applied to an undamped harmonic oscillator; that is, when $t = 0$, a constant force of magnitude F is suddenly applied to the system. If the velocity and displacement of the oscillator are zero at time $t = 0$, find the subsequent motion.

5.38. At time $t = 0$, a step function of the type described in Problem 5.37 is applied to an undamped, simple harmonic oscillator. After a time T the constant force F is suddenly removed, resulting in a forcing function of the type shown in the figure. The velocity and displacement of the mass are zero when $t = 0$.

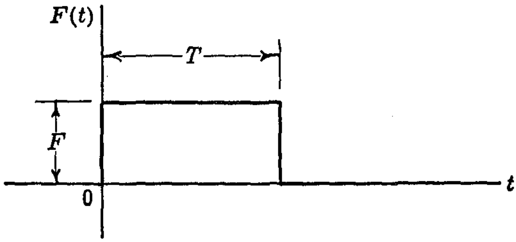
(a) Show that the displacement of the oscillator subsequent to the time T is given by:

$$x = 2\frac{F}{k} \sin\left(\frac{pT}{2}\right) \sin(pt - \psi)$$

(b) The total impulse acting upon the system in this problem is $I = FT$. If I remains constant while T approaches zero, what is the amplitude of the resulting motion? Note that for T sufficiently small:

$$\frac{pT}{2} \approx \sin\left(\frac{pT}{2}\right)$$

Check this answer by treating the problem as a free vibration, with an initial velocity given by the impulse-momentum equation.



PROB. 5.38

5.39. A sinusoidal alternating force is applied to a single degree of freedom harmonic oscillator having viscous friction. When $t = 0$, the displacement and the velocity of the oscillator are zero. If the frequency of the alternating force is just equal to the undamped natural frequency of the system, show that for small amounts of damping the time required to attain an amplitude of vibration equal to 90% of the resonant amplitude is:

$$t = \frac{2.3}{\delta f}$$

where:

t = time to reach 90% of resonant amplitude
δ = logarithmic decrement
f = resonant frequency

How many cycles of the vibration occur in the time t?

5.40. A single degree of freedom vibrating system consists of a mass m supported on a horizontal surface and restrained by a horizontal linear spring of spring constant k. A constant coulomb friction force of magnitude F exists between m and the horizontal surface. A sinusoidal exciting force $F_0 \sin \omega t$ acts on the mass, which may be assumed to be vibrating with a steady state amplitude A. (a) Find the energy dissipated per cycle by the coulomb damping force. (b) Equate this energy dissipation to the energy loss that would occur if the damping force were viscous, with a damping coefficient c, and thus solve for the "equivalent viscous damping constant." (c) Substituting the equivalent damping factor c into the known equation for the resonance curve with viscous friction, solve for the value of the steady state amplitude of vibration of the system containing coulomb friction. This will, of course, be only an approximate solution, but a more detailed analysis shows that the accuracy is satisfactory for many engineering applications.

5.7 Vibration Isolation. One of the useful applications of vibration theory is to the vibration isolation of instruments and machinery. As a first example, we shall consider the problem of mounting an instrument so as to minimize the transmission of vibration from the supporting structure to the instrument. In many applications delicate instruments must be used in structures which have appreciable amplitudes of vibration. Unless the instrument can be isolated from its support it may be impossible to make accurate measurements. Suppose that the support S in Figure 5.8 has a motion $y_0 \sin \omega t$. The mass of the instrument is m, and it is attached to the support by a spring k. The damping in the system is represented by a dashpot having a viscous damping constant c. Letting x be the amplitude of motion of the instrument m, we have as the differential equation of motion:

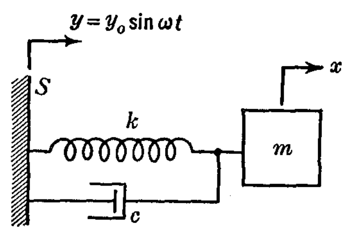


FIG. 5.8

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$
$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = ky_0 \sin \omega t + cy_0\omega \cos \omega t$$

$$\ddot{x} + 2n\dot{x} + p^2x = \frac{y_0}{m}(k \sin \omega t + c\omega \cos \omega t)$$

Writing the right side of the equation as:

$$\frac{y_0}{m}(k \sin \omega t + c\omega \cos \omega t) = \frac{y_0}{m}\sqrt{k^2 + (c\omega)^2} \sin (\omega t - \beta)$$

we see that its effect is the same as a sinusoidal exciting force, so that this is an equation of the same type as Equation (5.2), and the same solution can be used. Putting $F_0 = y_0 \sqrt{k^2 + (c\omega)^2}$ in Equation (5.16), we have for the amplitude of the steady-state forced vibration:

$$A = \frac{\frac{y_0}{k}\sqrt{k^2+(c\omega)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}$$

$$\frac{A}{y_0} = \frac{\sqrt{1+\left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \tag{5.18}$$

Thus the effectiveness of the mounting in reducing the amplitude is measured by the expression on the right side of Equation (5.18). The appearance of this function for various values of $\left(\frac{\omega}{p}\right)$ and $\left(\frac{n}{n_c}\right)$ is shown in Fig. 5.9, where it will be noted that at any frequency ratio greater than $\sqrt{2}$ the amplitude of the mass will be less than the amplitude of the support. The main difference between this resonance curve and that given in Fig. 5.7 is that for $\left(\frac{\omega}{p}\right) > \sqrt{2}$

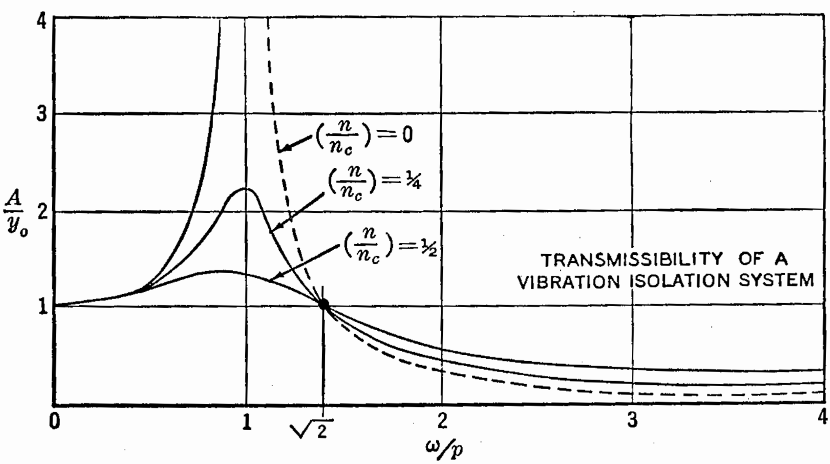


FIG. 5.9

the damped curves are above the undamped curves. This means that the presence of damping decreases somewhat the effectiveness of the mounting. A certain amount of damping, however, is essential in order to maintain stability under transient conditions and to prevent excessive amplitudes should the vibration pass through resonance during the starting or stopping of the motion of the support.

A second type of vibration isolation problem is illustrated in Fig. 5.10. Suppose that a machine, as a result of unbalanced rotating masses, exerts an alternating force of $m'r\omega^2 \sin \omega t$ upon its foundation where the mass of the rotating unbalance is m' and the effective radius r. If the machine is rigidly fastened to the foundation, the force will be transmitted directly to the foundation and may cause objectionable vibrations. It is desirable to isolate the machine from the foundation in such a way that the transmitted force will be reduced. Letting x be the displacement of the total mass m of the machine, we have, from the analysis previously made (Prob. 5.32):

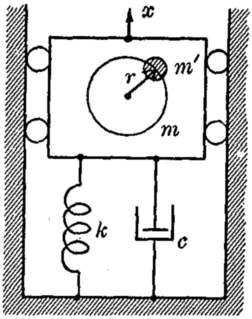


FIG. 5.10

$$x = \frac{\dfrac{m'r\omega^2}{k}}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}} \sin (\omega t - \phi)$$

The only force which can be applied to the floor is the spring force kx and the damping force $c\dot{x}$; hence, the total force acting on the foundation during the steady state forced vibration is:

$$F = kx + c\dot{x} = \frac{m'r\omega^2}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}} \sin (\omega t - \phi)$$

$$+ \frac{\dfrac{cm'r\omega^2}{k}}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}} \omega \cos (\omega t - \phi)$$

The amplitude of the resulting transmitted force is:

$$F_A = m'r\omega^2 \left[\frac{\sqrt{1 + \left(\frac{c\omega}{k}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \right]$$

Since $m'r\omega^2$ is the amplitude of the force which would be transmitted if the springs were infinitely rigid, we have as a measure of the effectiveness of the isolation mounting the expression:

$$\frac{F_A}{(m'r\omega^2)} = \frac{\sqrt{1 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \tag{5.19}$$

This is called the *transmissibility* of the system. It is exactly the same as Equation (5.18) obtained for the vibration isolated instrument, and Fig. 5.9 also represents the solution of the present problem. The frequency ratio and the damping have the same influence on the transmissibility as they had on the vibration isolation.

PROBLEMS

5.41. The amplitude of vibration in an airplane at the point at which it is desired to mount instruments is 0.015 in. and the frequency of the vibration is 1800 cycles per minute. The amplitude of the instruments is to be limited to 0.002 in. The instruments, along with the panel and mounting bracket, weigh 50 lb. Four rubber shock mounts are to be used, spaced in such a way that they are all equally loaded. Find the spring constant required for the rubber mount, assuming that damping can be neglected.

5.42. An instrument panel is mounted on a suspension system having a static deflection under gravity of 1/4 in. It is subjected to vibrations whose frequency corresponds to one-half of the speed of an engine which runs at 2000 rpm. What percentage reduction in amplitude of vibration is to be expected from this suspension system? Neglect the effects of damping.

5.43. An automobile body weighing 3000 lb is mounted on four equal springs which sag 9 in. under the weight of the body. Each of the four shock absorbers exerts a damping force of 7 lb for a velocity of 1 in./sec. The car is placed with all four wheels on a test platform which is moved up and down sinusoidally at resonant speed with an amplitude of 1 in. Find the amplitude of the car body on its suspension system, assuming that the center of gravity is at the center of the wheel base so that the system can be treated as one degree of freedom for vertical motion.

5.44. The characteristics of a railway passenger car and its helical spring suspension system are such that the total static deflection of the system is 10 in. The train travels over tracks that are slightly wavy, with a wavelength of 33 ft. Assuming that the rail waviness can be approximated by a sine wave, what percent of the rail amplitude would be transmitted to the car at speeds of 25 mph and of 90 mph? Discuss the importance of damping in the system for this problem.

5.45. A refrigerating unit consisting of a 1725 rpm electric motor driving a reciprocating compressor at 575 rpm is to be mounted on four equally loaded springs for purposes of vibration isolation. The whole assembly weighs 100 lb. If the principal vibrations occur at a frequency corresponding to the compressor speed and if it is assumed that only vertical forces and motions need be considered, find: (a) The required static deflection of the spring system, in order that only 1% of the compressor shaking force be transmitted to the supporting structure if damping in the system is assumed to be negligible; (b) the spring constant in lb/in. for each spring for the conditions of part (a); (c) the percent of motor unbalance force which would be transmitted to the supporting structure for the springs of part (a).

5.46. A machine having a total weight of 20,000 lb has an unbalance such that it is subjected to a force of amplitude 5000 lb at a frequency of 600 cycles per minute. What should be the spring constant for the supporting springs if the maximum force transmitted into the foundation due to the unbalance is to be 500 lb? Assume that damping may be neglected.

5.47. An instrument whose total weight is 20 lb is to be spring-mounted on a vibrating surface which has a sinusoidal motion of amplitude 1/64 in., and frequency 60 cycles per second. If the instrument is mounted rigidly on the surface, what is the maximum force to which it is subjected? Find the spring constant for the support system which will limit the maximum acceleration of the instrument to one-half the acceleration of gravity. Assume that negligible damping forces have caused the transient vibrations to die out.

5.48. Show that a vibration isolation system is effective only if $(\omega/p) > \sqrt{2}$.

5.8 The Design of Vibration Measuring Instruments. Suppose that the structure S in Fig. 5.11 is vibrating harmonically with an unknown amplitude y_0 and an unknown frequency ω. To measure y_0 and ω we may attach to the structure an instrument which consists of a mass m, a spring k, and viscous damping c.

The output of the instrument will depend upon the relative motion between the mass and the structure, since it is this relative motion which is detected and amplified by mechanical, optical, or electrical means. Taking x as the absolute displacement of the instrument mass, the output of the instrument will be proportional to $z = (x - y)$. The equation of motion of the instrument mass is:

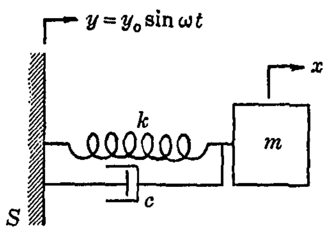


FIG. 5.11

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

Subtracting $(m\ddot{y})$ from each side of the equation gives:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = my_0\omega^2 \sin \omega t$$

This equation is the same as Equation (5.2), so that the solution for steady forced vibrations is:

$$z = \frac{\left(\frac{\omega}{p}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} y_0 \sin (\omega t - \phi)$$

or

$$z = Qy_0 \sin (\omega t - \phi) \tag{5.20}$$

$$\phi = \tan^{-1}\left[\frac{2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)}{1 - \left(\frac{\omega}{p}\right)^2}\right]$$

The instrument will read the displacement of the structure directly if $Q = 1$ and $\phi = 0$. The variation of Q with $\left(\frac{\omega}{p}\right)$ and $\left(\frac{n}{n_c}\right)$ is shown in Fig. 5.12. It is seen that if $\left(\frac{\omega}{p}\right)$ is large, Q is approximately equal to 1, and ϕ is approximately equal to 180°; we conclude, therefore, that, to design a displacement pickup, $\left(\frac{\omega}{p}\right)$ should be large, which means that the natural frequency of the instrument itself should be low compared to the frequency to be measured.

We next consider the region of the diagram where $\left(\dfrac{\omega}{p}\right)$ is small. ϕ is then approximately equal to zero, and the quantity

$$\frac{1}{\sqrt{\left[1-\left(\dfrac{\omega}{p}\right)^2\right]^2+\left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}$$

is approximately equal to 1. The expression $z = Qy_0 \sin(\omega t - \phi)$ then becomes:

$$z = \frac{1}{p^2} y_0 \omega^2 \sin \omega t$$

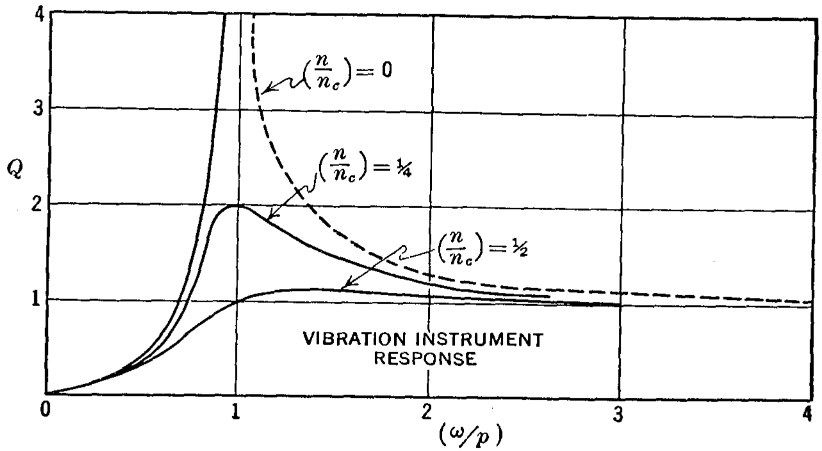


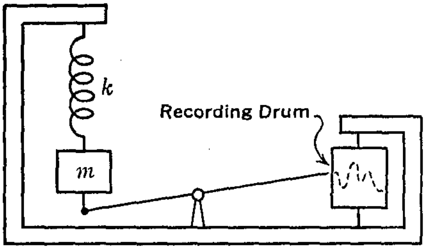
FIG. 5.12

Since $y_0\omega^2 \sin \omega t$ is the acceleration of the structure, the instrument output is proportional to the acceleration. We thus conclude that, to design an accelerometer, $\left(\dfrac{\omega}{p}\right)$ should be small, which means that the natural frequency of the instrument itself should be high compared to the frequency to be measured.

Instruments designed according to the foregoing criteria will have characteristics which are independent of frequency. Such instruments can be used outside of the specified range if the exact curves of Fig. 5.12 are used.

PROBLEMS

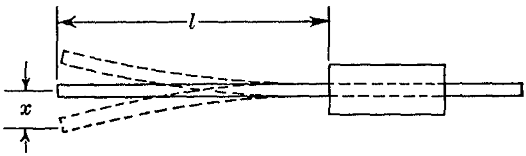
5.49. It is desired to design an instrument to measure the vertical oscillations of the Golden Gate Suspension Bridge. The bridge has a vertical frequency of approximately 1/7 cycles per second and the amplitude may at times reach 4 to 5 ft. An instrument of the type shown in



PROB. 5.49

the diagram has been suggested. Would it be better to design this instrument as an accelerometer or as a displacement meter? What would be a satisfactory frequency for the spring-mass system in the instrument?

5.50. A simple instrument for determining the frequency of vibration is constructed on the principle indicated in the diagram. A flat strip of metal is mounted as a cantilever beam of length l. The free vibrations of the strip are given by $x = A \sin pt$ where $p^2 = \frac{k}{l^4}$ and where k is a constant depending upon the proportions and material of the strip. The instrument is constructed so that the length can be varied. If the instrument

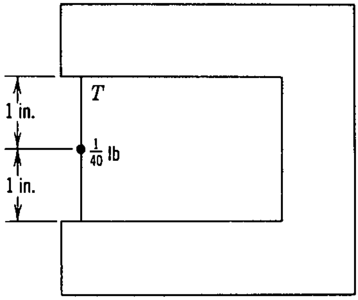


PROB. 5.50

is mounted upon a vibrating body whose frequency is ω, the amplitude will depend upon the ratio of the forced frequency and the natural frequency. At resonance this amplitude will be large, so that by varying l until the amplitude is a maximum the forcing frequency can be determined. Write the expression which gives the frequency of the vibrating body as a function of the length of the strip.

5.51. For measuring the vertical vibrations of a machine foundation, an instrument of the type shown in Fig. 5.11 is used. The spring-mass system of the instrument has been designed so that the static deflection is 3/4 in. The frequency of the vibration corresponds to an engine speed of 1500 rpm. The amplitude of the relative motion between the instrument mass and the foundation is determined, from a dial gage reading, to be 0.008 in. Find the amplitude of the foundation. The damping in the instrument has a magnitude of 70% of critical damping.

5.52. The seismic element of a vibration measuring instrument consists of a concentrated mass weighing 1/40 lb attached at the midpoint of a wire which is 2 in. long. The wire is stretched by a tension force T which can be assumed to be constant for small transverse oscillations of the mass. Find T to give a transverse natural frequency of 30 cycles/sec. Neglect the mass of the wire and gravity forces, and assume negligible damping in the system.



PROB. 5.52

5.9 Vibrations with Non-periodic Forces. The analysis of the preceding sections is sufficient to treat vibrations with sinusoidal exciting forces. Since any periodic forcing function can be represented by a trigonometric series, the analysis can be extended, by using the principle of superposition, to include the solution for a general periodic forcing function. For non-periodic exciting forces, however, it is desirable to develop a different method of approach. We shall limit the following analysis to undamped systems, although it is possible to extend the same method to damped systems (see Prob. 5.60).

We shall consider first the motion of an undamped spring-mass system to which a single impulse is applied. Referring to Fig. 5.13, an impulse $(F_0 \Delta t)$ will produce an initial velocity $\dot{x}_0$ which can be determined by the equation of impulse and momentum:

$$F_0 \Delta t = m\dot{x}_0$$

$$\dot{x}_0 = \frac{F_0 \Delta t}{m}$$

The displacement x of an undamped system performing free vibrations is given by Equation (5.3):

$$x = \frac{\dot{x}_0}{p} \sin pt + x_0 \cos pt$$

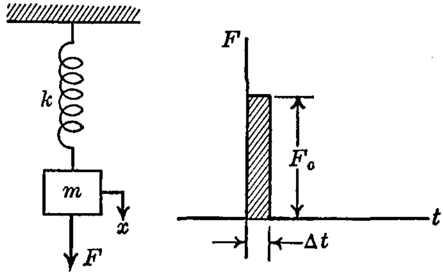


FIG. 5.13

We have $\dot{x}_0 = \dfrac{F_0 \Delta t}{m}$ and $x_0 = 0$ if we measure time from the point of zero deflection, so that:

$$x = \frac{F_0 \Delta t}{mp} \sin pt \tag{5.21}$$

Having found the motion under the action of one impulse, we may now, by the principle of superposition, find the motion under the action of any arbitrary forcing function. It is only necessary to let the arbitrary function be represented by an infinite number of impulses. Suppose that the curve of Fig. 5.14 represents an exciting force, which is applied when $t = 0$, and that it is desired to determine the displacement at time T. Consider the force to be divided into a large number of impulses, of which one, $F(t)\, dt$, is shown in the diagram. The displacement x at the time T due to this impulse can

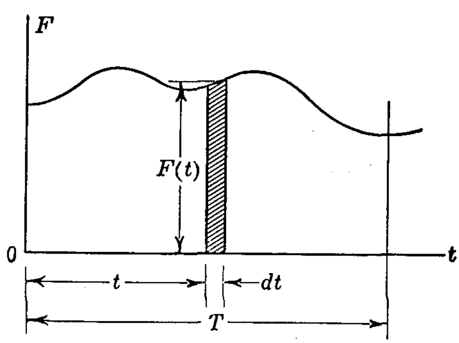


FIG. 5.14

be determined from Equation (5.21). In Equation (5.21), t represents the time which elapses between the application of the impulse and the measurement of the displacement. Thus at T, which is $(T - t)$ after the impulse is applied, we have:

$$dx = \frac{F(t)\,dt}{mp} \sin p(T - t)$$

We use the notation dx because this represents only the contribution of one impulse to the displacement x. To find the total displacement, the effects of all of the impulses from O to T must be added, which means that the expression for dx must be integrated, giving:

$$x = \frac{1}{mp} \int_0^T F(t) \sin p(T - t)\,dt \tag{5.22}$$

With this equation, the motion can be computed for any undamped system which has zero initial velocity and displacement. If $F(t)$ is given as a graph or as numerical data, instead of in analytical form, the integration can be carried out by graphical or numerical methods, and one of the advantages of the equation is its adaptability to solutions of this type.

A more formal derivation of the equation can be obtained in the following way. The differential equation of motion for an undamped system with an exciting force $F(t)$ is:

$$\ddot{x} + p^2 x = \frac{1}{m} F(t)$$

Multiplying through by the integrating factor $\sin p(T - t)$ and integrating, this becomes:

$$\int_0^T \ddot{x} \sin p(T - t)\,dt + \int_0^T p^2 x \sin p(T - t)\,dt = \frac{1}{m} \int_0^T F(t) \sin p(T - t)\,dt$$

Integrating the first term twice by parts reduces this to:

$$\dot{x} \sin p(T - t) \Big|_0^T + px \cos p(T - t) \Big|_0^T = \frac{1}{m} \int_0^T F(t) \sin p(T - t)\,dt$$

Substituting the limits of integration and solving for x gives:

$$x = \frac{1}{mp} \int_0^T F(t) \sin p(T - t)\,dt + \frac{\dot{x}_0}{p} \sin pT + x_0 \cos pT \tag{5.23}$$

If we take as the initial conditions $x_0 = \dot{x}_0 = 0$ when $T = 0$, this expression becomes:

$$x = \frac{1}{mp} \int_0^T F(t) \sin p(T - t)\, dt$$

which is the solution derived by the superposition of impulses.

EXAMPLE. To illustrate the application of the method we shall solve a problem which we have already solved by other methods. Suppose that a sinusoidal exciting force $F_0 \sin \omega t$ is applied; then x is given by:

$$x = \frac{F_0}{mp} \int_0^T \sin \omega t \sin p(T - t)\, dt$$

Making use of the trigonometric relation

$$\sin A \sin B = \tfrac{1}{2}[\cos (A - B) - \cos (A + B)]$$

the displacement may be written:

$$x = \frac{F_0}{2mp} \int_0^T \{\cos[(\omega + p)t - pT] - \cos [(\omega - p)t + pT]\}\, dt$$

Carrying out the integration, we obtain:

$$x = \frac{F_0}{m} \frac{1}{p^2\left[1 - \left(\frac{\omega}{p}\right)^2\right]} \left(\sin \omega T - \frac{\omega}{p} \sin pT\right)$$

This solution represents the superposition of a free vibration of frequency $p/2\pi$ and a forced vibration of frequency $\omega/2\pi$, so that it contains both the transient and the steady-state terms. The amplitude of the steady forced vibration is:

$$A = \frac{F_0}{mp^2} \frac{1}{\left[1 - \left(\frac{\omega}{p}\right)^2\right]} = \frac{F_0}{k} \left[\frac{1}{1 - \left(\frac{\omega}{p}\right)^2}\right]$$

This is the same as Equation (5.16) with the damping set equal to zero.

PROBLEMS

5.53. Carry out the integrations indicated in the preceding example for a sinusoidal exciting force and check the result:

$$x = \frac{F_0}{mp^2} \frac{1}{\left[1 - \left(\frac{\omega}{p}\right)^2\right]} \left(\sin \omega T - \frac{\omega}{p} \sin pT\right)$$

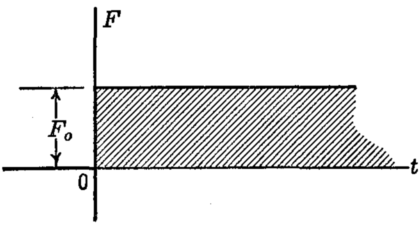
5.54. Show that the solution for damped forced vibrations

$$x = e^{-nt}[C_1 \sin \sqrt{p^2 - n^2}t + C_2 \cos \sqrt{p^2 - n^2}t] + \frac{\frac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}} \sin (\omega t - \phi)$$

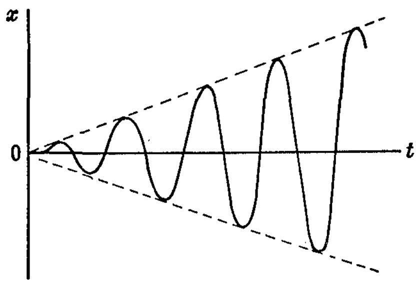
$$\phi = \tan^{-1}\left(\frac{2n\omega}{p^2 - \omega^2}\right)$$

reduces to the expression found in Prob. 5.53 when damping is put equal to zero.

5.55. An undamped vibrating system is at rest until time $t = 0$, when a step function F_0 is applied, as shown in the diagram. Find the resulting motion by the integral method of the preceding section and show that the maximum displacement is twice the static deflection of the system.



PROB. 5.55



PROB. 5.56

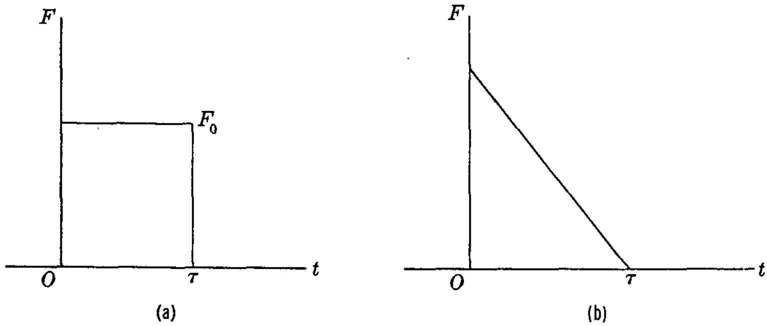
5.56. Suppose that a sinusoidal exciting force $F_0 \sin pt$ having the same frequency as the natural frequency of the undamped oscillator is applied at time $t = 0$. Show that the displacement is given by:

$$x = \frac{F_0}{2mp}\left(\frac{\sin pT}{p} - T \cos pT\right)$$

Thus the resonant amplitude of the system builds up with a linearly increasing term. Use the integral method for this problem.

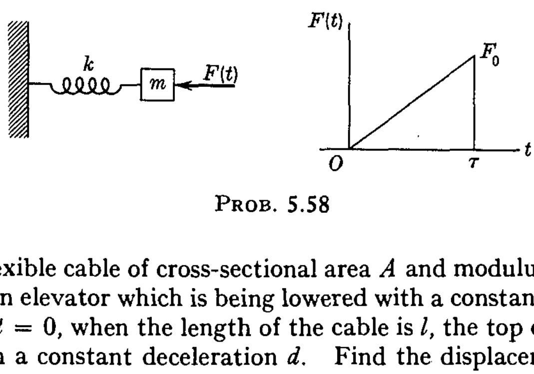
5.57. The motion of a single degree of freedom undamped system under the action of a rectangular pulse as shown in Figure (a) was discussed in Prob. 5.38.

(a) Solve the corresponding problem for $t > \tau$ for the linearly decreasing exciting force shown in Figure (b) under the condition that the total impulse in both cases remains the same. (b) Assuming that τ is small compared with the period of free vibrations of the system, simplify the results of part (a) for the two types of exciting pulses and compare the answers.



PROB. 5.57

5.58. An undamped spring mass system is acted upon by a transient force having the form of a single triangular pulse, as shown in the figure. The force increases linearly to a value F_0 at a time τ, and then drops to zero. Find the displacement of the mass at any time, if it has no initial displacement or velocity.



PROB. 5.58

5.59. A flexible cable of cross-sectional area A and modulus of elasticity E supports an elevator which is being lowered with a constant velocity V_0. At the time $t = 0$, when the length of the cable is l, the top of the cable is stopped with a constant deceleration d. Find the displacement x of the elevator, measured from its position at $t = 0$, as a function of time during the deceleration period. The length of the cable may be considered as constant at l during the deceleration period.

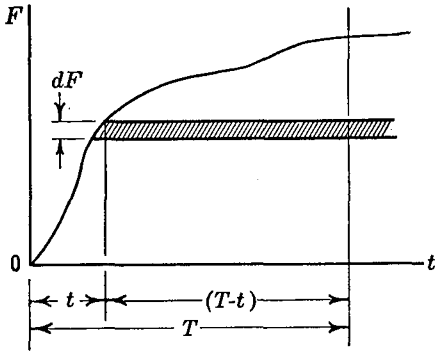
5.60. Show that for a system having viscous damping, the integral solution is:

$$x = \frac{1}{m\sqrt{p^2 - n^2}} \int_0^T F(t)e^{-n(T-t)} \sin \sqrt{p^2 - n^2}(T - t)\, dt$$

5.61. The integral form of the solution for the motion of a spring-mass system can also be derived from the differential equation by use of Lagrange's method of the "variation of parameters." Carry through the solution of the equation $m\ddot{x} + kx = F(t)$ by this method, and show that Equation (5.22) is obtained.

5.62. Show by integrating by parts that the integral solution for the undamped oscillator may be written:

$$\left(\frac{F}{k} - x\right) = \frac{1}{mp^2}\int_0^T \left(\frac{dF}{dt}\right) \cos p(T - t)\, dt$$



PROB. 5.62

where $x_0 = \dot{x}_0 = 0$ and $F(t) = 0$ when $t = 0$. Show that this method of solution is equivalent to cutting F into horizontal slices as shown in the figure, and summing the effect of the successive incremental step functions.

5.10 Oscillations in Electric Circuits. Oscillation problems of the type treated in this chapter are also of frequent occurrence in electrical circuit analysis. Consider an electrical circuit consisting of an inductance L, a capacitance C, and a resistance R as shown in Fig. 5.15. These elements are connected in series with a source of alternating voltage with an amplitude E_0 and a frequency $\frac{\omega}{2\pi}$.

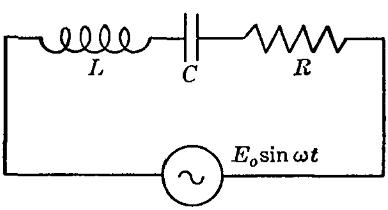


FIG. 5.15

The equation describing the behavior of the system is obtained by equating the applied voltage to the sum of the voltage drops across the three elements. If the current in the circuit is i, then the voltage drop across the inductance is $L\dfrac{di}{dt}$, that across the capacitance is $\dfrac{1}{C}\int i\,dt$, and that across the resistance is Ri. The equation thus is:

$$E_0 \sin \omega t = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt$$

If Q represents the electric charge, then:

$$i = \frac{dQ}{dt}$$

and the equation may be written:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E_0 \sin \omega t \tag{5.24}$$

It will be seen that this equation has exactly the same form as Equation (5.2), which describes the motion of a mechanical vibrating system, with the following analogous quantities:

Electrical System	*Mechanical System*
Inductance, L	Mass, m
Resistance, R	Coefficient of viscous damping, c
Reciprocal of Capacitance, $\frac{1}{C}$	Spring constant, k
Exciting voltage, E	Exciting force, F
Electrical charge, Q	Displacement, x
Current, i	Velocity, $\dot{x}$

The results of the analysis for the mechanical system can therefore be applied to the electrical system, and the solution of the differential equation is:

$$Q = e^{-\frac{R}{2L}t}\left[C_1 \sin\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,t + C_2 \cos\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,t\right] + A \sin(\omega t - \phi) \tag{5.25}$$

where:

$$A = \frac{\dfrac{E_0}{L}}{\sqrt{\left[\dfrac{1}{LC} - \omega^2\right]^2 + \dfrac{R^2}{L^2}\omega^2}}$$

and:

$$\phi = \tan^{-1}\left[\frac{\dfrac{R\omega}{L}}{\dfrac{1}{LC} - \omega^2}\right]$$

Just as in mechanical systems, the solution consists of a transient term and a steady-state term. Because of the resistance in the circuit the electrical transient vibrations die out in time, leaving the forced steady-state oscillations.

To find the steady-state current i in the circuit, we write:

$$i = \frac{dQ}{dt} = \frac{E_0\omega/L}{\sqrt{\left[\frac{1}{LC} - \omega^2\right]^2 + \frac{R^2}{L^2}\omega^2}} \cos(\omega t - \phi)$$

The amplitude of the steady-state current is:

$$i_A = \frac{E_0}{\sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}} \tag{5.26}$$

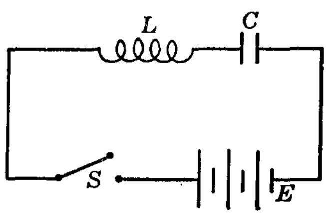
The quantity $\sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}$ is called the electrical impedance of the circuit, $\left(\frac{1}{\omega C}\right)$ is called the capacitive reactance of the circuit, and (ωL) is called the inductive reactance. It will be seen that resonance occurs when $\left(\frac{1}{\omega C}\right) = (\omega L)$ and that the magnitude of the resonant current is limited only by the resistance in the circuit.

Because of the analogy between electrical and mechanical problems, it is often possible to transfer solutions from one field directly to the other, thus saving duplication of work. Such analogies are also often used for experimental solutions. It is usually much easier to build an electrical circuit and to make measurements on it than it is to construct and test the analogous mechanical system. Electrical Analog Computers, which operate on this principle have been constructed so that many different combinations of electrical elements can be set up, and in this way complex electrical, mechanical, and thermal problems have been solved.*

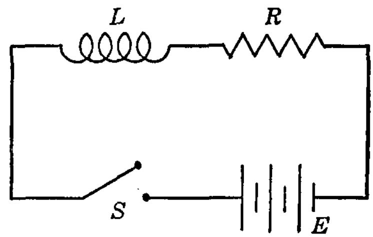
* See, for example, H. E. Criner, G. D. McCann, and C. E. Warren, "A New Device for the Solution of Transient Vibration Problems by the Method of Electrical-Mechanical Analog." *Journal of Applied Mechanics Vol. 12* (1945), p. 135, or the book by W. W. Soroka, *Analog Methods in Computation and Simulation*, New York: McGraw-Hill Book Company, Inc., 1954.

PROBLEMS

5.63. At time $t = 0$, the switch S in the electrical circuit shown in the diagram is closed, applying a voltage E to the series inductance and capacitance. Find the way in which the current in the circuit varies with time, assuming that the resistance in the circuit is negligible. What would be the effect of resistance in the circuit?



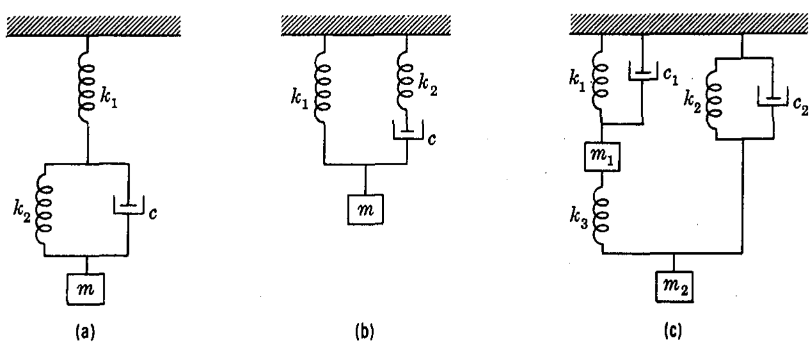
PROB. 5.63



PROB. 5.64

5.64. At time $t = 0$, a switch is closed applying a voltage E to an inductance and a resistance which are in series. Find the relation between the current and time. Show that the time required for the current to reach $\left(1 - \frac{1}{e}\right)$ times its final value is equal to L/R. This is called the time constant of the circuit.

5.65. Show that the single degree of freedom mechanical oscillator can be represented by a direct electric analog using a parallel or nodal type circuit, in which the force in the mechanical system is analogous to the current in the electrical circuit. Draw up a tabular comparison of the analogous quantities in the mechanical system, the force voltage loop



PROB. 5.66

system of the text, and this force-current system. Include in the table masses, viscous damping element, velocity, spring constant, and displacement.

5.66. Construct force-voltage loop electric analog circuits of the type shown above in the text for the mechanical systems shown in the figure.

5.67. Construct force-current nodal electric analog circuits of the type discussed in Prob. 5.65 for the mechanical systems of Prob. 5.66.

Chapter 6

PRINCIPLES OF DYNAMICS FOR SYSTEMS OF PARTICLES

. . . the same law takes place in a system consisting of many bodies as in a single body. For the progressive motion, whether of one single body, or of a whole system of bodies, is always to be estimated from the motion of the center of gravity.—I. Newton, *Principia Philosophiae* (1686).

In most dynamics problems it is not possible to approximate the system by a single particle, but it must be treated as a collection of particles. The system itself may be a solid body, a fluid, or a gas, but in any event it may be thought of as a collection of particles, each of which may be treated by the methods of particle dynamics. The type of interaction between the individual particles will depend upon the system being investigated, but certain general relations may be developed which apply no matter what these interactions may be. In the present chapter these general relations are developed, and in subsequent chapters the consequences of the special characteristics of the systems are treated.

6.1 The Equation of Motion for a System of Particles. The equation of motion for a typical particle of a system is:

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \boldsymbol{f}_i \tag{6.1}$$

The subscript indicates that the equation applies to the i-th particle. The resultant force acting upon the particle is written as the sum of an external force $\boldsymbol{F}$ and an internal force $\boldsymbol{f}$. The external force originates outside of the system, and represents the action of some body or agency upon the system. The internal force originates within the system in the mutual actions and reactions between the particles. The reason for distinguishing between these two types of forces is that when the system as a whole is under consideration, the sum of all the internal forces is equal to zero. This follows from the fact that internal forces always occur in equal, opposite, and collinear pairs and will thus cancel.* An equation of motion for the entire system is obtained by adding the equations for the individual particles and setting $\Sigma \boldsymbol{f}_i = 0$:

$$\Sigma m_i \ddot{\boldsymbol{r}}_i = \Sigma \boldsymbol{F}_i$$

Using the notation $\Sigma \boldsymbol{F}_i = \boldsymbol{F}$, this becomes:

$$\Sigma m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F} \tag{6.2}$$

6.2 The Motion of the Center of Mass. The center of mass of a system of particles is defined as a point located by the vector $\boldsymbol{r}_c$ where:

$$\boldsymbol{r}_c = \frac{\Sigma m_i \boldsymbol{r}_i}{\Sigma m_i}$$

We may introduce this quantity into the equation of motion of the system by writing Equation (6.2) in the form:

$$\boldsymbol{F} = \frac{d^2}{dt^2}(\Sigma m_i \boldsymbol{r}_i) = \frac{d^2}{dt^2}(\boldsymbol{r}_c \Sigma m_i)$$

Setting Σm_i equal to M, the total mass of the system of particles, this becomes:

$$\boldsymbol{F} = M \ddot{\boldsymbol{r}}_c \tag{6.3}$$

Thus we may conclude that the motion of the center of mass is the same as the motion of a particle, having a mass equal to the total mass of the system, acted upon by the resultant external force. The motion of the mass center is therefore a problem in particle dynamics. This is the justification for having treated finite bodies as particles in the preceding chapters.

The equation of motion of the mass center may be integrated with

* Collinearity of the internal forces is not always to be assumed. For example, electromagnetic forces between moving particles are not collinear. It will be found that collinearity is not required for the linear momentum relationships, although it is required for the moment of momentum relationships. See ref. 27, p. 9, 136–140.

respect to time and with respect to displacement, to give the impulse-momentum and the work-energy equations for the motion of the center of mass. These are:

$$\int_1^2 \boldsymbol{F}\,dt = M\dot{\boldsymbol{r}}_c \Big|_1^2 \tag{6.4}$$

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}_c = \tfrac{1}{2}M\dot{r}_c{}^2 \Big|_1^2 \tag{6.5}$$

It should be noted that these equations give only information as to the motion of the center of mass of the system. The term $M\dot{\boldsymbol{r}}_c$ has the magnitude and direction of the total momentum of the system, but the location of the line of action of the total momentum vector is not determined by this expression, for it does not necessarily pass through the center of mass. The term $\frac{1}{2}M\dot{r}_c{}^2$ does not represent the total kinetic energy of the system, since the motion of the parts of the system with respect to the center of mass will contribute additional kinetic energy.

6.3 The Total Kinetic Energy of a System of Particles. The total kinetic energy, T, of a system of particles is the sum of the kinetic energies of the individual particles:

$$T = \Sigma\tfrac{1}{2}m_i v_i{}^2$$

This expression may be put into another form, which is useful for many problems, by referring the motion of each particle to the center of mass of the system. As shown in Fig. 6.1, the vector $\boldsymbol{\rho}_i$ represents

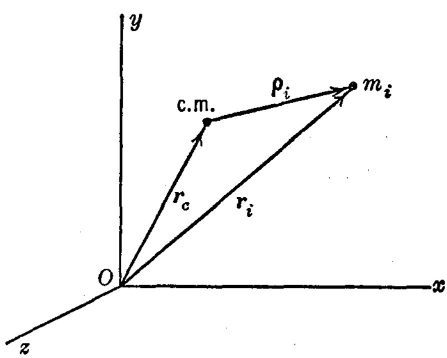


FIG. 6.1

the displacement of the i-th particle with respect to the center of mass. For each particle, $\boldsymbol{r}_i = \boldsymbol{r}_c + \boldsymbol{\rho}_i$, therefore:

$$\begin{aligned} v_i^2 &= (\dot{\boldsymbol{r}}_i)\cdot(\dot{\boldsymbol{r}}_i) = (\dot{\boldsymbol{r}}_c + \dot{\boldsymbol{\rho}}_i)\cdot(\dot{\boldsymbol{r}}_c + \dot{\boldsymbol{\rho}}_i) \\ &= \dot{r}_c^2 + 2\dot{\boldsymbol{r}}_c\cdot\dot{\boldsymbol{\rho}}_i + \dot{\rho}_i^2 \end{aligned}$$

and the kinetic energy may be written:

$$T = \Sigma\tfrac{1}{2}m_i\dot{r}_c^2 + \dot{\boldsymbol{r}}_c\cdot\frac{d}{dt}(\Sigma m_i\boldsymbol{\rho}_i) + \Sigma\tfrac{1}{2}m_i\dot{\rho}_i^2$$

Since ρ_i is measured from the mass-center, we have $\Sigma m_i\boldsymbol{\rho}_i = 0$ and the second term drops out. The first term may be written:

$$\Sigma\tfrac{1}{2}m_i\dot{r}_c^2 = \dot{r}_c^2\Sigma\tfrac{1}{2}m_i = \tfrac{1}{2}M\dot{r}_c^2$$

where M is the total mass of the system. Thus the kinetic energy becomes:

$$T = \tfrac{1}{2}M\dot{r}_c^2 + \Sigma\tfrac{1}{2}m_i\dot{\rho}_i^2 \tag{6.6}$$

The total kinetic energy may thus be said to be the sum of the energy which would be obtained if all the mass were located at the mass-center plus the kinetic energy of the system corresponding to the motion relative to the mass-center.

The work-energy equation for a system of particles may be put into a convenient form by using the same transformation. The total work done by all the forces of the system is:

$$\begin{aligned} \Sigma\int_1^2 (\boldsymbol{F}_i + \boldsymbol{f}_i)\cdot d\boldsymbol{r}_i &= \int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i)\cdot(d\boldsymbol{r}_c + d\boldsymbol{\rho}_i) \\ &= \int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}_c + \int_1^2 \Sigma\boldsymbol{f}_i\cdot d\boldsymbol{r}_c + \int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i)\cdot d\boldsymbol{\rho}_i \end{aligned}$$

The sum of the internal forces is zero, $\Sigma\boldsymbol{f}_i = 0$; hence the second term drops out. Equating the total work done to the change in total kinetic energy, we have:

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}_c + \int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i)\cdot d\boldsymbol{\rho}_i = \tfrac{1}{2}M\dot{r}_c^2\Big|_1^2 + \Sigma\tfrac{1}{2}m_i\dot{\rho}_i^2\Big|_1^2$$

We have already shown that the first term on the left is equal to the first term on the right, so that the second term on the left must equal the second term on the right. We may thus write the two independent equations:

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}_c = \tfrac{1}{2}M\dot{r}_c^2 \Big|_1^2 \tag{6.7}$$

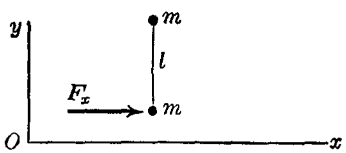
$$\int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i)\cdot d\boldsymbol{\rho}_i = \Sigma\tfrac{1}{2}m_i\dot{\rho}_i^2 \Big|_1^2 \tag{6.8}$$

The first of these equations describes the motion of the center of mass of the system, while the second describes the motion of the system with respect to the center of mass. The fact that these two equations can be written independently of each other simplifies the solution of problems by the energy method.

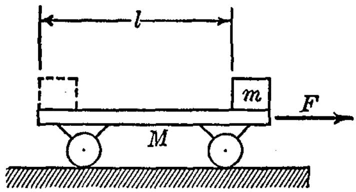
PROBLEMS

6.1. (a) Two particles of mass m, connected by a rigid weightless rod, are acted upon by a force F_x = constant. At time $t = 0$, the system is at rest as shown. Find the displacement of the mass center as a function of time.

(b) Derive the motion of the mass center by setting up the equations of motion for each particle separately and integrating.



PROB. 6.1



PROB. 6.4

6.2. Two barges of weight W and $2W$ are connected by a cable. The cable is shortened to one-half of its original length l by turning a windlass on one of the barges. Neglecting any frictional resistances, find the distance moved by the heavier barge.

6.3. A shell moving through the air is split into two fragments of mass m_1 and m_2 by its explosive charge, which adds an energy E to the fragments. Find the relative velocity between the two fragments after the explosion.

6.4. A cart of mass M, initially at rest, can move horizontally along a frictionless track. When $t = 0$, a force F is applied to the cart as shown. During the acceleration of M by the force F, a small mass m slides along the cart from the front to the rear. The coefficient of friction between m and M is μ, and it is assumed that the acceleration of M is sufficient to cause sliding.

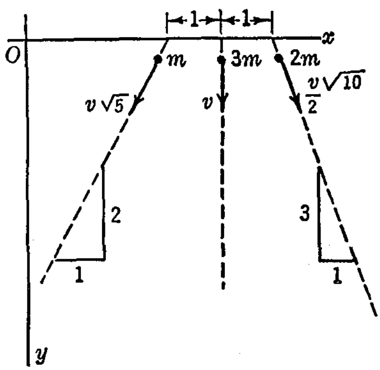
(a) Write two equations of motion, one for m and one for M, and show that they can be combined to give the equation of motion of the mass center of the system of two bodies.

(b) Find the displacement of M at the time when m has moved a distance l along the cart.

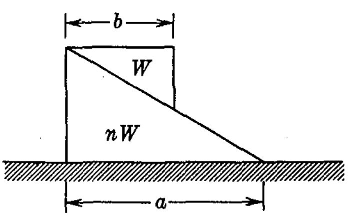
6.5. Three particles of mass, m, $2m$, and $3m$, are moving with constant velocities in the directions shown. The motion takes place in the xy plane.

(a) Find the magnitude and the direction of the total momentum of the system of three particles.

(b) Find the total kinetic energy of the system and compare with the



PROB. 6.5



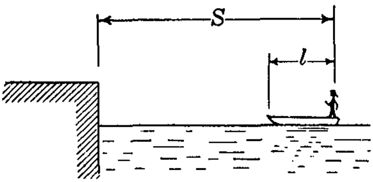
PROB. 6.6

energy which the system would have if all of its mass were concentrated at the mass center.

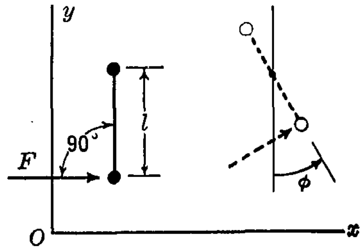
(c) If at time $t = 0$, the particles are all located on the x-axis in the positions indicated, what is the subsequent path of motion of the mass-center?

6.6. Two smooth prisms of similar right-triangular sections are arranged on a smooth horizontal plane as shown in the diagram. The upper prism weighs W lb, and the lower prism weighs nW lb. The prisms are held in an initial position as shown, and are then released, so that the upper prism slides down the lower prism until it just touches the horizontal plane. Find the distance moved by the lower prism during this process.

6.7. A man of mass m stands at the rear of a boat of mass M as shown. The distance of the man from the pier is S ft. What is the distance of the man from the pier after he has walked forward in the boat a distance l? Neglect friction between the boat and the water.



PROB. 6.7



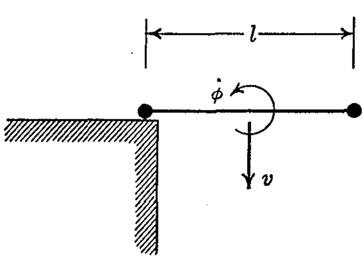
PROB. 6.8

6.8. Two particles each having a mass m are connected by a rigid bar of length l whose mass is negligible. The system is initially at rest in the position shown. At time $t = 0$, a force F, of constant magnitude, acts normal to the bar as shown. Write the work-energy equation for the system with respect to the mass center, and show that: $\dot{\phi} = \sqrt{\dfrac{2F\phi}{ml}}$ and $\phi = \dfrac{Ft^2}{2ml}$.

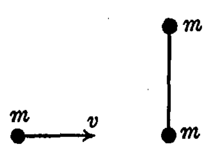
Note that these results are obtained without considering the motion of the mass center.

6.9. Suppose that the system of Prob. 6.8 has an angular velocity $\dot{\phi} =$ constant, the center of mass of the system is initially at rest, and at time $t = 0$ the system is released to fall under the action of gravity. What is the total kinetic energy of the system at a subsequent time?

6.10. The system of Prob. 6.8 moves with angular velocity $\dot{\phi}$ and the linear velocity of the center of mass is v, vertically downward. When the bar is in a horizontal position it makes an elastic impact as shown. Find the subsequent motion of the system, assuming that no energy is lost during the impact and assuming no gravitational force acting. Show that there is an interchange of translational and rotational kinetic energy.



PROB. 6.10



PROB. 6.12

6.11. The system of Prob. 6.8 is initially at rest when an impulse $F\Delta t$, normal to the bar, acts upon one of the masses. If Δt is an infinitesimal, find the total energy imparted to the system and describe the subsequent motion. Assume that no gravitational force is acting.

6.12. A mass m moving with a velocity v in a direction perpendicular to the bar strikes one of the masses in the system of Prob. 6.8. Describe the subsequent motion of the mass-center of the bar, and find its angular velocity at any time, assuming that there is no energy lost during the impact and assuming no gravitational force acting.

6.4 Moment of Momentum. Consider a particle of mass m and momentum $m\dot{\boldsymbol{r}}$, as shown in Fig. 6.2. The *moment of momentum** of the particle about the fixed point O is defined as the moment of the

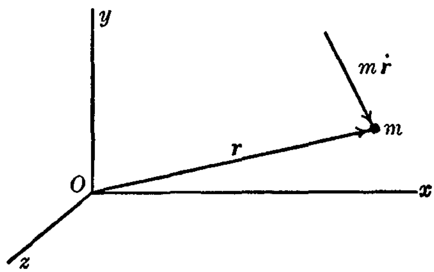


FIG. 6.2

momentum vector about the point O. Calling the moment of momentum vector $\boldsymbol{H}$, we have:

$$\boldsymbol{H} = \boldsymbol{r} \times m\dot{\boldsymbol{r}}$$

The total moment of momentum of a system of particles is the sum of the moments of momentum of all the individual particles:

$$\boldsymbol{H} = \Sigma \boldsymbol{r}_i \times m_i\dot{\boldsymbol{r}}_i \tag{6.9}$$

The concept of the moment of momentum can be used to put the equation of motion into a new form, which is particularly convenient for the treatment of systems of particles. To do this, we differentiate $\boldsymbol{H}$ with respect to time, and find:

$$\dot{\boldsymbol{H}} = \Sigma \dot{\boldsymbol{r}}_i \times m_i\dot{\boldsymbol{r}}_i + \Sigma \boldsymbol{r}_i \times m_i\ddot{\boldsymbol{r}}_i$$

* The quantity moment of momentum is also called the *angular momentum*. Since no angular motion or rotation need be present in order that the moment of momentum should exist, the term *moment of momentum* is preferred.

Since $\dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i = 0$ the first term drops out, giving:

$$\dot{\boldsymbol{H}} = \Sigma \boldsymbol{r}_i \times m_i \ddot{\boldsymbol{r}}_i$$

Taking the cross product of each side of the equation of motion $m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \boldsymbol{f}_i$ by $\boldsymbol{r}_i$ and summing, we obtain:

$$\Sigma \boldsymbol{r}_i \times m_i \ddot{\boldsymbol{r}}_i = \Sigma \boldsymbol{r}_i \times (\boldsymbol{F}_i + \boldsymbol{f}_i)$$

The left side of this equation is $\dot{\boldsymbol{H}}$ and the right side represents the moment of all the forces about the fixed point O. Since the internal forces occur in equal, opposite, and collinear pairs, their moments cancel and the right side of the equation is just the sum of the moments of the external forces. Writing this moment sum as $\boldsymbol{M}$, we have:

$$\dot{\boldsymbol{H}} = \boldsymbol{M} \tag{6.10}$$

$\dot{\boldsymbol{H}}$, *the time rate of change of the moment of momentum of the system about the fixed point O, is equal to* $\boldsymbol{M}$, *the resultant moment of the external forces about the same point O.* This is called the equation of the moment of momentum. It is a restatement of the equation of motion in a form which, as we shall see in the next chapter, is particularly convenient for application to problems of rigid body dynamics.

When there is no external moment acting on the system, the equation of the moment of momentum takes the form:

$$\dot{\boldsymbol{H}} = 0 \quad \text{or} \quad \boldsymbol{H} = \text{constant} \tag{6.11}$$

This is *the principle of the conservation of moment of momentum*, which states that, if there is no external moment of force about the fixed point, the moment of momentum about that point must remain constant. If there is no moment about the x-axis, H_x is conserved even if H_y and H_z are not conserved.

In the preceding paragraphs the moment of momentum was taken with respect to an arbitrary, *fixed* point. It is often convenient to choose a point which is not fixed. We shall now show that an equation of the form of Equation (6.10) will be valid for any point whose velocity is parallel to the velocity of the center of mass of the system.

In Fig. 6.3 the point O is a fixed point and P is the moving point about which the moment of momentum equations are to be expressed:

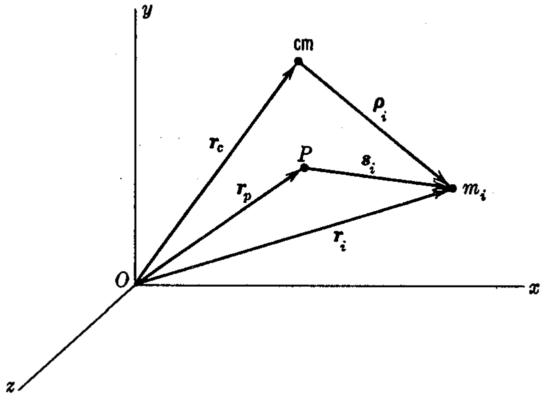


FIG. 6.3

The vector $\boldsymbol{s}_i$ locates a typical mass m_i of the system, and the moment of momentum of the system about P will be:

$$\boldsymbol{H}_p = \Sigma \boldsymbol{s}_i \times m_i \dot{\boldsymbol{r}}_i$$

The time derivative of $\boldsymbol{H}_p$ is:

$$\dot{\boldsymbol{H}}_p = \Sigma \boldsymbol{s}_i \times m_i \ddot{\boldsymbol{r}}_i + \Sigma \dot{\boldsymbol{s}}_i \times m_i \dot{\boldsymbol{r}}_i$$

Since $\boldsymbol{s}_i = \boldsymbol{r}_i - \boldsymbol{r}_p$, the second term of this expression becomes:

$$\Sigma \dot{\boldsymbol{s}}_i \times m_i \dot{\boldsymbol{r}}_i = \Sigma(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{r}}_p) \times m_i \dot{\boldsymbol{r}}_i = - \Sigma \dot{\boldsymbol{r}}_p \times m_i \dot{\boldsymbol{r}}_i$$

We may also put $\boldsymbol{r}_i = \boldsymbol{r}_c + \boldsymbol{\rho}_i$, so that this last term becomes:

$$\begin{aligned} - \Sigma \dot{\boldsymbol{r}}_p \times m_i \dot{\boldsymbol{r}}_i &= - \Sigma \dot{\boldsymbol{r}}_p \times m_i (\dot{\boldsymbol{r}}_c + \dot{\boldsymbol{\rho}}_i) \\ &= - \dot{\boldsymbol{r}}_p \times \dot{\boldsymbol{r}}_c \Sigma m_i - \dot{\boldsymbol{r}}_p \times \Sigma m_i \dot{\boldsymbol{\rho}}_i \end{aligned}$$

since $\boldsymbol{\rho}_i$ is measured from the center of mass, $\Sigma m_i \dot{\boldsymbol{\rho}}_i = 0$, and we have:

$$\dot{\boldsymbol{H}}_p = \Sigma \boldsymbol{s}_i \times m_i \ddot{\boldsymbol{r}}_i - \dot{\boldsymbol{r}}_p \times \dot{\boldsymbol{r}}_c \Sigma m_i$$

Also:

$$\boldsymbol{M}_p = \Sigma \boldsymbol{s}_i \times (\boldsymbol{F}_i + \boldsymbol{f}_i) = \Sigma \boldsymbol{s}_i \times m_i \ddot{\boldsymbol{r}}_i$$

We thus see that if $\dot{\boldsymbol{r}}_p \times \dot{\boldsymbol{r}}_c = 0$, that is if the velocity of the point P is parallel to the velocity of the center of mass, we obtain:

$$\dot{\boldsymbol{H}}_p = \boldsymbol{M}_p$$

If the point P is the center of mass of the system, then the expression

$$\dot{\boldsymbol{H}}_c = \boldsymbol{M}_c \tag{6.12}$$

is always correct.

The equation of moment of momentum in the form $\dot{\boldsymbol{H}} = \boldsymbol{M}$ *can thus be referred to an arbitrary fixed point, to the moving center of mass of the system, or to any moving point whose velocity is parallel to the velocity of the center of mass.*

The equation of moment of momentum is often written in terms of rectangular coordinates as:

$$\dot{H}_x = M_x; \quad \dot{H}_y = M_y; \quad \dot{H}_z = M_z$$

For example, for a single particle acted upon by a force $\boldsymbol{F}$:

$$\boldsymbol{H} = \boldsymbol{r} \times m\boldsymbol{v} \quad \text{and} \quad \boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

Writing $\boldsymbol{r}$, $\boldsymbol{v}$, and $\boldsymbol{F}$ in terms of their rectangular components gives:

$$\boldsymbol{H} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times m(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k})$$
$$\boldsymbol{M} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k})$$

Carrying out the multiplications, remembering that $\boldsymbol{i} \times \boldsymbol{i} = 0$. $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, etc., gives:

$$\boldsymbol{H} = m(y\dot{z} - z\dot{y})\boldsymbol{i} + m(z\dot{x} - x\dot{z})\boldsymbol{j} + m(x\dot{y} - y\dot{x})\boldsymbol{k}$$
$$\boldsymbol{M} = (yF_z - zF_y)\boldsymbol{i} + (zF_x - xF_z)\boldsymbol{j} + (xF_y - yF_x)\boldsymbol{k}$$

The three component equations therefore are:

$$\begin{aligned} m\frac{d}{dt}(y\dot{z} - z\dot{y}) &= yF_z - zF_y \\ m\frac{d}{dt}(z\dot{x} - x\dot{z}) &= zF_x - xF_z \\ m\frac{d}{dt}(x\dot{y} - y\dot{x}) &= xF_y - yF_x \end{aligned} \tag{6.13}$$

These equations can also be obtained by taking moments about the x-, y-, and z-axes respectively. If a system of particles is involved, these equations can be summed over all the particles.

6.5 Summary. It should be emphasized that the principles derived in this chapter are general in application, and that the system of particles need have no special properties. These principles are thus available for use in the analysis of rigid and deformable solid bodies, liquids, and gases. The general conclusions can be summarized in the following statements.

(1) The center of mass of any system of particles moves as though it were a particle, having a mass equal to the total mass of the system, acted upon by the resultant of the external forces applied to the system. All the methods of particle dynamics may thus be applied to the motion of the mass center.

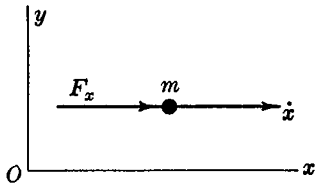
(2) The magnitude and direction of the total momentum of a system of particles are given by the product of the total mass of the system and the velocity of the mass center. The total impulse of all the external forces acting upon the system is equal to the change of the total momentum.

(3) The work-energy principle for a system of particles may be written in the form of two independent equations. One equation describes the motion of the center of mass of the system, and the other equation describes the motion of the particles of the system with respect to the center of mass.

(4) The equation of moment of momentum may be written with respect to: (1) an arbitrary fixed point, (2) the moving center of mass of the system, or (3) any moving point whose velocity is parallel to the velocity of the center of mass.

PROBLEMS

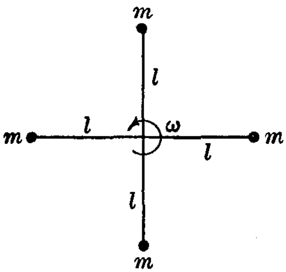
6.13. A particle of mass m is acted upon by a force parallel to the x-axis as shown. The particle has a velocity parallel to the x-axis. Write the equation for the moment of momentum of the system, and show that this equation may be reduced to the equation of motion in the form $F_x = m\ddot{x}$.



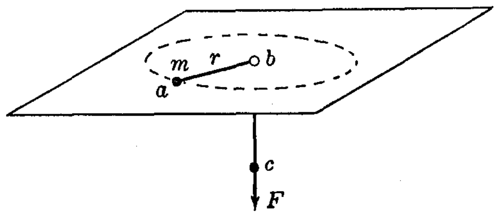
PROB. 6.13

6.14. A system of four particles of equal mass m rotates with an angular velocity ω. The particles are at equal distances from the center of rotation, and they are spaced at equal angles as shown. Find the magnitude and direction of the vector representing the moment of momentum of the system about the point of rotation.

6.15. A particle of mass m is restrained by the string abc to move in a circle of radius r on a horizontal frictionless plane. The particle moves with a constant angular velocity ω. If the radius of the circle is reduced to r_1 by pulling on the string at c, what will be the velocity of the particle?



PROB. 6.14



PROB. 6.15

6.16. Two particles of mass m_1 and m_2 are attached together by an inextensible string of negligible weight and of length l. The two particles and the string move in a vertical plane under the action of gravity only, in such a way that the string is always subjected to a tension force. Show that the angular velocity ω of the string is a constant, and find the tension in the string as a function of m_1, m_2, l, and ω.

6.17. A particle is acted upon by a force which is always directed toward a fixed point. Show that the particle moves in a plane.

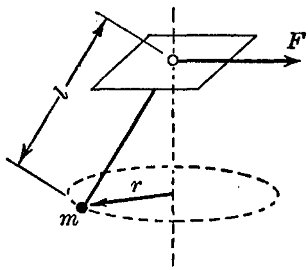
6.18. Using the equation of moment of momentum, show that the area swept out per unit time by the radius vector drawn from the sun to a planet is a constant.

6.19. A particle of mass m fastened to a massless string of length l rotates in a circular path of radius r as a conical pendulum. The force F is gradually increased, thus shortening the length of the pendulum so that finally the particle moves in a circle of radius $\frac{r}{2}$. Find the velocity of the mass after the string has been shortened.

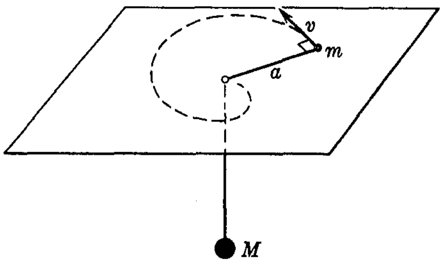
6.20. A flexible chain of mass per unit length μ and length l hangs at rest in a vertical position with the lower end just touching the floor. The chain is released from rest, and falls to the floor. Find the force exerted on the floor by the chain, assuming that the chain always strikes the same point on the floor. Do this problem by considering the motion of the center of the mass of the chain.

6.21. A particle of mass m rests on a smooth table, and is attached to a string which passes through a small hole in the table and supports a mass

M in a vertical position as shown in the diagram. The mass m is given a velocity v perpendicular to the horizontal portion of the string when the mass is at a distance a from the hole, and the system is then released. At some later time it will be found that there is a new position of m, at a distance x from the hole, when the velocity of m is again perpendicular to the string. Find x.



PROB. 6.19



PROB. 6.21

6.22. A car weighing W lb carries a missile of weight w, and travels with a uniform velocity V to the right along a horizontal track. At time $t = 0$ the missile is projected from the car with a velocity v_0 and angle θ with the horizontal, in a direction opposite to the direction of travel of the car. Neglecting any frictional resistances for either the car or the missile, and assuming that the dimensions of the car are very small compared to the flight distances involved, find the distance between the car and the missile when the missile reaches the ground.

Chapter 7

THE DYNAMICS OF RIGID BODIES

It has been long understood that approximate solutions of problems in the ordinary branches of Natural Philosophy may be obtained by a species of abstractions, or rather limitations of the data, such as enables us easily to solve the modified form of the question, while we are well assured that the circumstances (so modified) affect the result only in a superficial manner—W. Thomson and P. G. Tait, *Treatise on Natural Philosophy* (1872).

When applying the principles of dynamics to solid bodies it is usually assumed that the motion of the body is not influenced by the small deformations caused by the applied forces. This is equivalent to the assumption of a rigid body, and so far as the motion of the body is concerned this assumption introduces only negligible errors for the great majority of such problems encountered in engineering practice. The equations of motion for a rigid body may be developed by treating the body as a collection of particles and applying the general principles of dynamics as formulated in the preceding chapter. The condition for a rigid body, that the distances between the particles remain fixed, is then used to simplify the general equations. As the first step in deriving the required equations of motion, it will be necessary to investigate the motion of each point in a rigid body.

7.1 Kinematics of Rigid Body Motion. To describe the motion of a rigid body it is necessary to specify in some way the motion of every point in the body. This may be done analytically by applying the general kinematic equations of Chapter 2.

Consider two points A and B fixed in a rigid body, as in Fig. 7.1. Let the vector $\boldsymbol{r}_A$ describe the position of point A in a fixed (X, Y, Z)

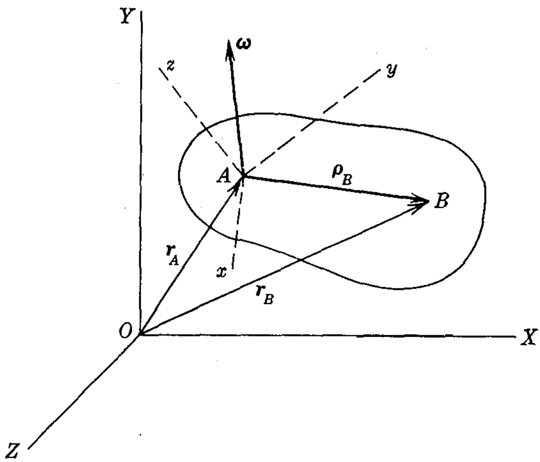


FIG. 7.1

coordinate system, and let A be the origin of a moving (x, y, z) coordinate system which is attached to the rigid body. From Chapter 2, the general kinematic equations are (Fig. 2.12):

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} + \dot{\boldsymbol{\rho}}_r$$
$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

For the present problem the vector $\boldsymbol{\rho}_B$, which locates the point B in the moving (x, y, z) system, is a constant, and the above equations become:

$$\begin{aligned} \boldsymbol{r}_B &= \boldsymbol{r}_A + \boldsymbol{\rho}_B \\ \dot{\boldsymbol{r}}_B &= \dot{\boldsymbol{r}}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_B \\ \ddot{\boldsymbol{r}}_B &= \ddot{\boldsymbol{r}}_A + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_B) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_B \end{aligned} \tag{7.1}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body. From these equations the motion of any point in a rigid body can be determined in terms of the motion of one point in the body and the angular velocity of the body. These equations are thus the analytical expression of the fact that the motion of a rigid body can be described as the sum of a translation and a rotation.*

Equations (7.1) are often written in the form:

$$\begin{aligned} \boldsymbol{r}_B &= \boldsymbol{r}_A + \boldsymbol{r}_{BA} \\ \boldsymbol{v}_B &= \boldsymbol{v}_A + \boldsymbol{v}_{BA} \\ \boldsymbol{a}_B &= \boldsymbol{a}_A + \boldsymbol{a}_{BA} \end{aligned} \tag{7.2}$$

* This is known as the Theorem of Chasle, after M. Chasle (1793–1880), a French mathematician.

where $\boldsymbol{v}$ and $\boldsymbol{a}$ designate velocity and acceleration, and:

$$\begin{aligned} \boldsymbol{r}_{BA} &= \boldsymbol{\rho}_B \\ \boldsymbol{v}_{BA} &= \boldsymbol{\omega} \times \boldsymbol{\rho}_B \\ \boldsymbol{a}_{BA} &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_B) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_B \end{aligned}$$

as may be seen by comparing Equations (7.2) with Equations (7.1).

The quantity $\boldsymbol{r}_{BA}$ is often referred to as the displacement of the point B with respect to the point A. Similarly, $\boldsymbol{v}_{BA}$ and $\boldsymbol{a}_{BA}$ are referred to as the velocity and acceleration of the point B with respect to the point A. It should be noted, however, that displacements, velocities, and accelerations must be defined with respect to coordinate systems, and hence the above phrase "the point B with respect to the point A" really refers to the motion of B with respect to a translating coordinate system whose origin coincides with A. In this way, by the use of the general kinematic relationships, the concept of relative motion can be made precise.

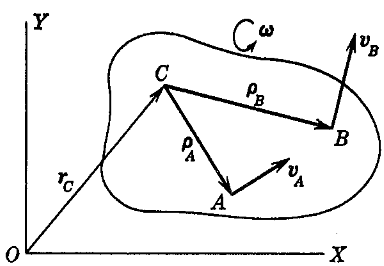


FIG. 7.2

EXAMPLE 1. A rigid body performs plane motion, that is, all points of the body move parallel to a plane. When the body is in the position shown in Fig. 7.2, the velocities of two points A and B are known. Find the angular velocity of the body.

Solution. Erect perpendiculars to $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ through the points A and B and find the point of intersection C, located by the radius vector $\boldsymbol{r}_c$. Then from Equations (7.1) we have

$$\begin{aligned} \boldsymbol{v}_A &= \dot{\boldsymbol{r}}_c + \boldsymbol{\omega} \times \boldsymbol{\rho}_A \\ \boldsymbol{v}_B &= \dot{\boldsymbol{r}}_c + \boldsymbol{\omega} \times \boldsymbol{\rho}_B \end{aligned}$$

But $\boldsymbol{v}_A$ and $\boldsymbol{\omega} \times \boldsymbol{\rho}_A$ have the same direction, so $\dot{\boldsymbol{r}}_c$ cannot have a direction different from $\boldsymbol{v}_A$, and neither can it have a direction different from $\boldsymbol{v}_B$. $\dot{\boldsymbol{r}}_c$ is therefore zero, and we have:

$$\omega = \frac{v_A}{\rho_A} = \frac{v_B}{\rho_B}$$

The point located by $\boldsymbol{r}_c$ has an instantaneous velocity equal to zero.

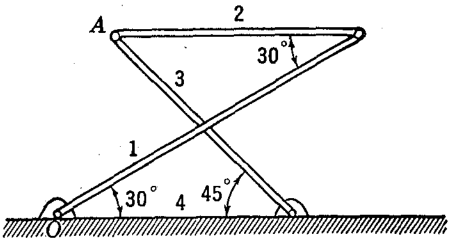


FIG. 7.3

This point is called the *instantaneous center of rotation.* At any particular instant the velocity of every point in the body is the same as if the body were rotating about the instantaneous center.

EXAMPLE 2. A four-bar linkage is shown in Fig. 7.3. Link 1 is 5 ft long and link 4 is 3 ft long. Link 1 has an angular velocity of 2 revolutions per sec and an angular acceleration of 3 revolutions per sec², both clockwise. Determine the velocities and accelerations of links 2 and 3.

Solution. Since each link is a rigid body, the motion of the point A can be expressed in terms of Equations (7.1) or (7.2). The motion of A as determined from the two links 1 and 2 must be the same as that determined from the two links 4 and 3. Thus we may write

$$\mathbf{r}_A = \mathbf{r}_1 + \boldsymbol{\rho}_2 = \mathbf{r}_4 + \boldsymbol{\rho}_3$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_1 + \boldsymbol{\omega}_2 \times \boldsymbol{\rho}_2 = \dot{\mathbf{r}}_4 + \boldsymbol{\omega}_3 \times \boldsymbol{\rho}_3$$

$$\ddot{\mathbf{r}}_A = \ddot{\mathbf{r}}_1 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{\rho}_2 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{\rho}_2)$$
$$= \ddot{\mathbf{r}}_4 + \dot{\boldsymbol{\omega}}_3 \times \boldsymbol{\rho}_3 + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \boldsymbol{\rho}_3)$$

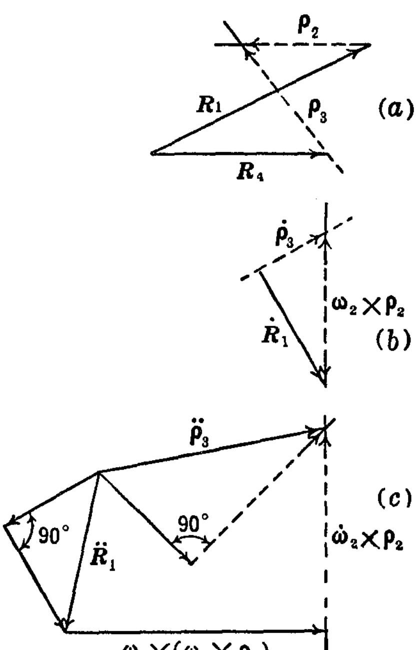
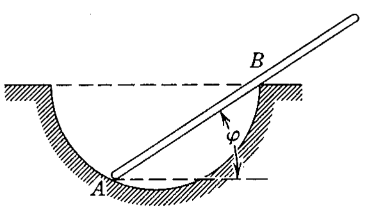


FIG. 7.4

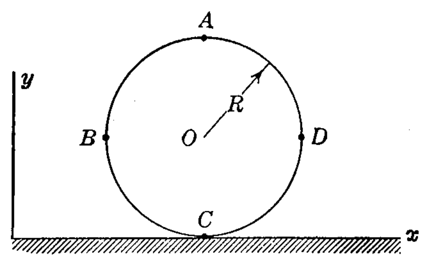
Each of these vector equations is equivalent to two scalar equations and determines two unknowns. From the first equation the lengths of the links 2 and 3 can be found. From the second equation $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ can be determined, and from the third equation the angular accelerations $\dot{\boldsymbol{\omega}}_2$ and $\dot{\boldsymbol{\omega}}_3$ can be found. The actual solutions can be carried out graphically or analytically. Fig. 7.4 shows graphical solutions in which the dotted lines indicate vectors whose directions are known but whose lengths are originally unknown.

PROBLEMS

7.1. The end A of a straight bar moves with a constant tangential velocity v along a semi-cylindrical trough as shown in the diagram. Find the velocity of the point B, at the point of contact of the bar and the edge of the trough, as a function of the angle ϕ between the bar and the horizontal. The bar moves in a plane normal to the axis of the trough.

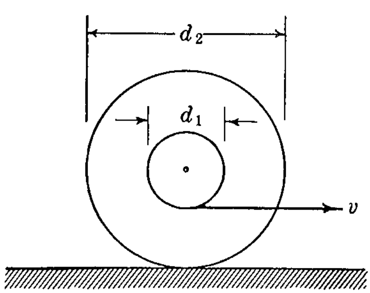


PROB. 7.1



PROB. 7.2

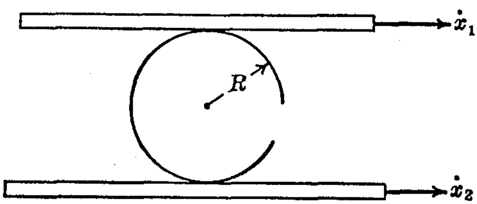
7.2. A circular cylinder of radius R rolls without sliding along a horizontal plane. The horizontal velocity of the center of the cylinder is $\dot{x}_0$, and $\ddot{x}_0 = 0$. Find the velocities of the points A, B, C, and D on the periphery of the cylinder.



PROB. 7.3

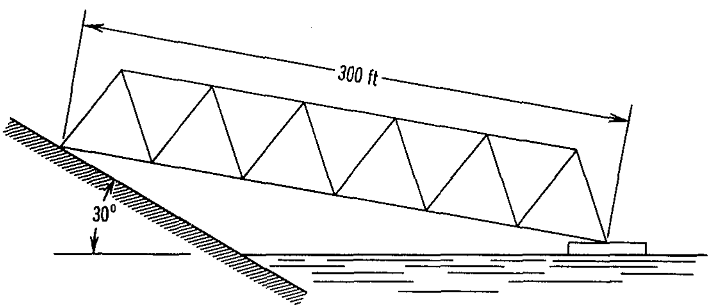
7.3. A pair of wheels of diameter d_2 with a rigidly attached concentric axle of diameter d_1 rolls without sliding along a horizontal plane. A rope wound around the axle is pulled with a constant horizontal velocity v as shown. Find the velocity of the center of the axle.

7.4. A circular cylinder of radius R is supported between two horizontal planks as shown in the diagram. The planks have horizontal velocities $\dot{x}_1$ and $\dot{x}_2$ as shown. Find the velocity of the center of the cylinder.



PROB. 7.4

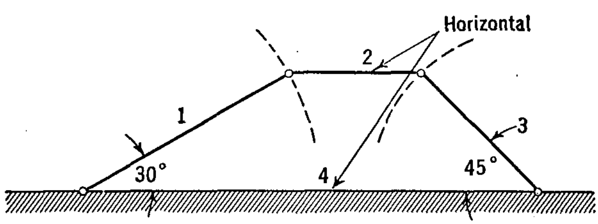
7.5. A test facility for studying the launching of torpedoes into water at various entry angles consists of a large truss structure which supports a compressed air launching tube and is mounted as shown in the figure. The left end of the truss can be moved up or down a track mounted on the



PROB. 7.5

30° slope of a hill, while the other end moves horizontally on a barge floating on the water. The length of the truss is 300 ft. The truss moves in a plane normal to the hill. Find the maximum velocity and acceleration of the barge for a uniform velocity v along the track.

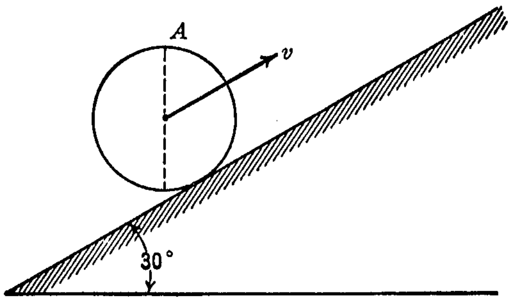
7.6. The mechanism of many useful machines can be reduced in its essentials to that of the four-bar linkage shown. In the particular example



PROB. 7.6

shown link 1 is 3 ft long, and link 4 is 6 ft long. Link 1 has a counterclockwise angular velocity of 100 rpm, and a clockwise angular acceleration of 50 rpm per minute. At the instant when link 2 is horizontal, find the angular velocities and accelerations of links 2 and 3.

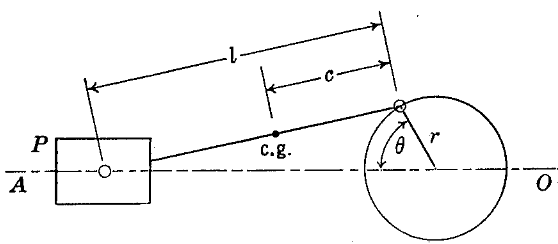
7.7. A circular disk of 1-ft radius rolls without slipping up a 30° inclined plane. The uniform velocity of the center of the disk is 25 ft/sec parallel



PROB. 7.7

to the plane. Find the velocity and acceleration of the point A on the periphery of the disk, where A is the upper end of a vertical diameter of the disk.

7.8. A crank and connecting rod mechanism of the type commonly used in reciprocating engines is shown in the diagram, where r is the radius of the crank, and l is the length of the connecting rod. The piston P is



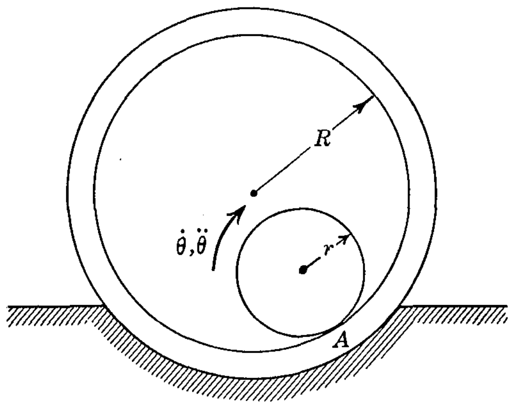
PROB. 7.8

constrained by the cylinder to move along the straight line AO. The center of gravity of the connecting rod is located a distance c from the crank pin as shown. For a particular engine $r = 4$ in., $l = 8$ in., $c = 3$ in., and the crank rotates clockwise with a constant angular velocity of 1000 rpm.

(a) Find the instantaneous center of rotation of the connecting rod at the instant when $\theta = 60°$. Using this instantaneous center, find the velocity of the center of gravity of the connecting rod. Find the required distances in this problem by laying out the diagram to scale and measuring the distances graphically.

(b) Determine the acceleration of the center of gravity of the connecting rod.

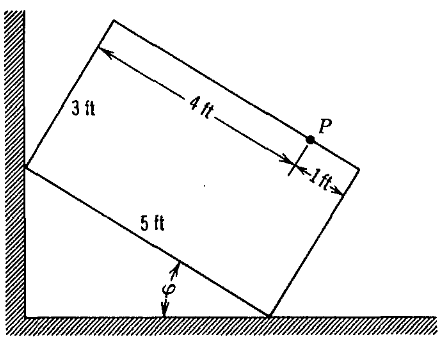
7.9. A small disk of radius r rolls without slipping inside a larger fixed circular ring of radius R as shown. The small disk has an angular velocity



PROB. 7.9

$\dot{\theta}$ and an angular acceleration $\ddot{\theta}$. Find the acceleration of the point on the small disk coinciding with the point of contact A. Find also the velocity and acceleration of the center of the small disk.

7.10. A rectangular object having the dimensions shown in the diagram moves in a plane with one edge following a vertical wall and another edge



PROB. 7.10

following a horizontal floor. The angle ϕ between the long edge of the rectangle and the horizontal varies according to the relationship $\phi = \frac{\pi}{2} + t - t^2$ radians, where t is in seconds. Find the velocity and acceleration of the point P at the time $t = 1.5$ sec.

7.2 The Moment of Momentum of a Rigid Body. For the analysis of the motion of a rigid body the equation of motion of the mass center can always be written:

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$$

This equation describes only the translation of the mass center and

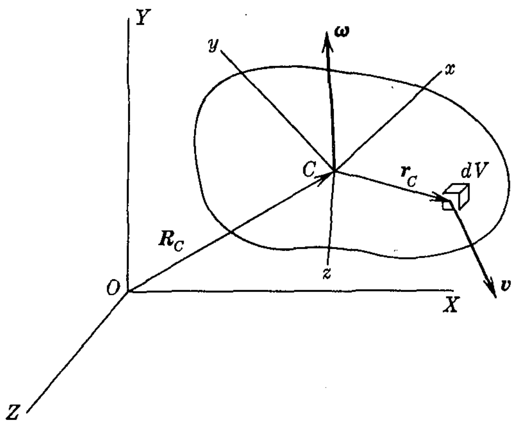


FIG. 7.5

in addition we require equations describing the rotations of the body These can be derived by applying the equation of moment of momentum:

$$\boldsymbol{M} = \dot{\boldsymbol{H}}$$

It has been shown in Chapter 6 that this equation can be written with respect either to a fixed point, or to the center of mass of the system. We shall first derive an expression for the moment of momentum $\boldsymbol{H}_c$ of a rigid body about its center of mass.

In Fig. 7.5, O is the origin of a fixed (X, Y, Z) coordinate system, and C is the origin of a moving (x, y, z) coordinate system which is attached to a rigid body and is located at the center of mass of the body. The density of the body is ρ, dV is a typical volume element located by the radius vector $\boldsymbol{r}_c$ drawn from the center of mass, and the angular velocity of the body is $\boldsymbol{\omega}$. The absolute velocity of the volume element dV is $\boldsymbol{v}$. We may then write:

$$d\boldsymbol{H}_c = \boldsymbol{r}_c \times (\rho dV)\boldsymbol{v}$$
$$\boldsymbol{H}_c = \int \rho(\boldsymbol{r}_c \times \boldsymbol{v})\, dV$$

But also, $\boldsymbol{v} = \dot{\boldsymbol{R}}_c + \boldsymbol{\omega} \times \boldsymbol{r}_c$ so we obtain:

$$\boldsymbol{H}_c = \int \rho[\boldsymbol{r}_c \times (\dot{\boldsymbol{R}}_c + \boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV$$
$$= \int \rho(\boldsymbol{r}_c \times \dot{\boldsymbol{R}}_c)\, dV + \int \rho[\boldsymbol{r}_c \times (\boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV$$

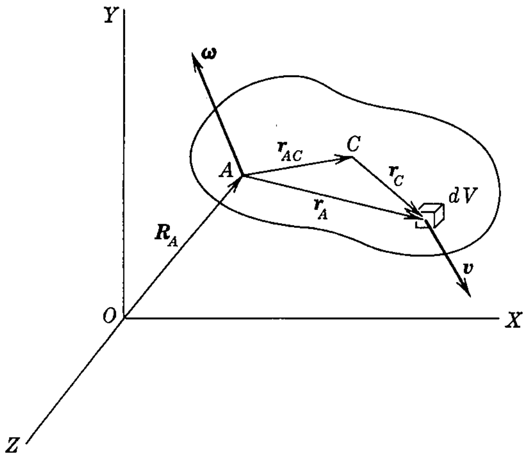


FIG. 7.6

In the first term $\dot{\boldsymbol{R}}_c$ is a constant for all elements of the body, so:

$$\int \rho(\boldsymbol{r}_c \times \dot{\boldsymbol{R}}_c)\, dV = \left(\int \rho \boldsymbol{r}_c\, dV\right) \times \dot{\boldsymbol{R}}_c$$

and this is equal to zero since $\boldsymbol{r}_c$ is measured from the center of mass of the body.

We thus obtain the result:

$$\boldsymbol{H}_c = \int \rho[\boldsymbol{r}_c \times (\boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV \tag{7.3}$$

Having obtained the moment of momentum about the center of mass, the moment of momentum with respect to any other point may be derived as follows.

In Fig. 7.6, O is the origin of a fixed coordinate system; C is the center of mass of a moving rigid body; A is any point in the body, and $\boldsymbol{r}_A$ is the vector locating the volume element dV with respect to A. The absolute velocity of dV is $\boldsymbol{v}$. We thus have for the moment of momentum of the rigid body about the point A:

$$\boldsymbol{H}_A = \int \rho(\boldsymbol{r}_A \times \boldsymbol{v})\, dV$$

since $\boldsymbol{v} = \dot{\boldsymbol{R}}_A + \boldsymbol{\omega} \times \boldsymbol{r}_A$, this becomes:

$$\begin{aligned}\boldsymbol{H}_A &= \int \rho[\boldsymbol{r}_A \times (\dot{\boldsymbol{R}}_A + \boldsymbol{\omega} \times \boldsymbol{r}_A)]\, dV \\ &= \int \rho(\boldsymbol{r}_A \times \dot{\boldsymbol{R}}_A)\, dV + \int \rho[\boldsymbol{r}_A \times (\boldsymbol{\omega} \times \boldsymbol{r}_A)]\, dV\end{aligned}$$

We may then make the substitution $\boldsymbol{r}_A = \boldsymbol{r}_{AC} + \boldsymbol{r}_c$, which leads to:

$$\begin{aligned}\boldsymbol{H}_A = &\int \rho[(\boldsymbol{r}_{AC} + \boldsymbol{r}_c) \times \dot{\boldsymbol{R}}_A]\, dV \\ &+ \int \rho\{(\boldsymbol{r}_{AC} + \boldsymbol{r}_c) \times [\boldsymbol{\omega} \times (\boldsymbol{r}_{AC} + \boldsymbol{r}_c)]\}\, dV\end{aligned}$$

Expanding these expressions, and noting that:

$$\int \rho\, dV = m, \quad \int \rho \boldsymbol{r}_c\, dV = 0$$

we obtain:

$$\boldsymbol{H}_A = (\boldsymbol{r}_{AC} \times \dot{\boldsymbol{R}}_A)m + [\boldsymbol{r}_{AC} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AC})]m + \boldsymbol{H}_c$$

since $\boldsymbol{v}_c = \dot{\boldsymbol{R}}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AC}$, this becomes finally:

$$\boldsymbol{H}_A = \boldsymbol{H}_c + m\boldsymbol{r}_{AC} \times \boldsymbol{v}_c \tag{7.4}$$

Equation (7.4) may be thought of as a general transfer theorem for the moment of momentum of a rigid body. Note that A in Fig. 7.6 can be any point, fixed in space or moving with the body.

7.3 Moments and Products of Inertia. In the preceding section it has been shown that the expression for the moment of momentum of a rigid body has the form:

$$\boldsymbol{H} = \int \rho[\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})]\, dV$$

Writing $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$ we have:

$$\boldsymbol{\omega} \times \boldsymbol{r} = (z\omega_y - y\omega_z)\boldsymbol{i} + (x\omega_z - z\omega_x)\boldsymbol{j} + (y\omega_x - x\omega_y)\boldsymbol{k}$$

So that:

$$\begin{aligned}\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = &[\omega_x(y^2 + z^2) - \omega_y xy - \omega_z xz]\boldsymbol{i} \\ &+ [-\omega_x yx + \omega_y(z^2 + x^2) - \omega_z yz]\boldsymbol{j} \\ &+ [-\omega_x zx - \omega_y zy + \omega_z(x^2 + y^2)]\boldsymbol{k}\end{aligned}$$

The rectangular components of the moment of momentum may thus be written:

$$
\begin{aligned}
H_x &= \omega_x \int \rho(y^2 + z^2)\, dV - \omega_y \int \rho xy\, dV - \omega_z \int \rho xz\, dV \\
H_y &= -\omega_x \int \rho yx\, dV + \omega_y \int \rho(z^2 + x^2)\, dV - \omega_z \int \rho yz\, dV \\
H_z &= -\omega_x \int \rho zx\, dV - \omega_y \int \rho zy\, dV + \omega_z \int \rho(x^2 + y^2)\, dV
\end{aligned}
\tag{7.5}
$$

Introducing the following notation for the integrals which appear in these expressions:

$$
\int \rho(y^2 + z^2)\, dV = I_{xx}
$$
$$
\int \rho xy\, dV = I_{xy}, \text{ etc.,}
$$

the equations become:

$$
\begin{aligned}
H_x &= +I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\
H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\
H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z
\end{aligned}
\tag{7.6}
$$

The terms I_{xx}, I_{yy}, and I_{zz} are called the moments of inertia, and the terms I_{xy}, etc., are called products of inertia. I_{xx} is often written as I_x, etc., and it will be noted from the symmetry of the integrals that $I_{xy} = I_{yx}$, etc.

The inertia integrals are defined with respect to the xyz axes. If the coordinate axes have fixed directions in space, then as the body rotates the numerical values of the inertia integrals will change with time. On the other hand, if the coordinate axes are fixed in the body so that they rotate with it, then the inertia integrals are constants but the components H_x, H_y, and H_z are measured along rotating coordinate axes. Either type of coordinate system may be used, but in most problems the second type is more convenient. In the following sections we shall use a coordinate system that is fixed in the body and rotates with it, unless it is specifically stated to the contrary.

Before treating the general equations of motion of a rigid body we shall consider the problem of determining the inertia integrals.

7.4 The Calculation of Moments and Products of Inertia. Although the computation of moments and products of inertia requires only the evaluation of simple definite integrals it is found that unless the body has a very simple shape and orientation the limits of integration are such as to require an excessive amount of labor. The problem is very much simplified by the following three observations:

(1) If the moments and products of inertia are known for a particular set of axes, they can be found for any parallel set of axes by a transformation of coordinates.
(2) If the moments and products of inertia are known for a particular set of axes, they can be found for a rotated set of axes by a transformation of coordinates.
(3) The moments and products of inertia of a body of complicated shape can be found by subdividing the body into a number of simpler parts, evaluating the integrals for each of these, and then summing them.

The method of calculation is thus to subdivide the body into simple parts, choosing for each part coordinate axes which will make the integration easy. By transformation of coordinates the moments and products of inertia with respect to the desired axes can then be found. It should be noted that this and the following sections deal only with methods of calculation which shorten the labor of evaluating inertia integrals.

Before proceeding with the calculation of inertia integrals, we introduce some commonly used notation. Consider, in Fig. 7.7, any rigid body having a mass per unit volume ρ. Then by definition of the moments of inertia, we have:

$$\begin{aligned} I_x &= \int\rho(y^2 + z^2)\,dV = \int\rho a_x^2\,dV \\ I_y &= \int\rho(x^2 + z^2)\,dV = \int\rho a_y^2\,dV \\ I_z &= \int\rho(x^2 + y^2)\,dV = \int\rho a_z^2\,dV \end{aligned} \tag{7.7}$$

Note that a_x, a_y, a_z are the perpendicular distances from the respective axes to the volume element and are not the components of the radius vector to the element. These expressions are sometimes written as $I_x = mr_x^2$, $I_y = mr_y^2$, and $I_z = mr_z^2$, where m is the total mass of the body, and the quantity r_x is called the *radius of gyration* of the body about the x-axis, etc. The radius of gyration is thus given by:

$$r = \sqrt{\frac{I}{m}}$$

The product of inertia integrals have the form:

$$I_{xy} = \int \rho xy \, dV$$

Since x and y can be either positive or negative, the product of inertia can be either positive or negative. In particular, if the yz plane is a plane of symmetry for the body, there is a negative $\rho xy \, dV$ for each positive $\rho xy \, dV$ and the product of inertia is zero. As will be shown later, the product of inertia may be zero also when there is no plane of symmetry in the body.

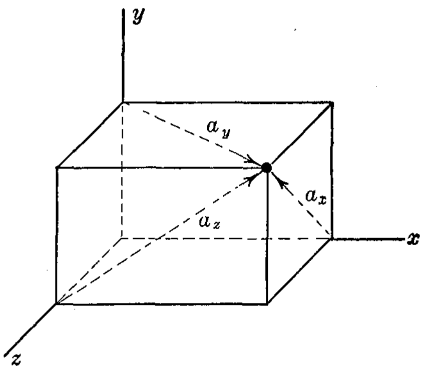


FIG. 7.7

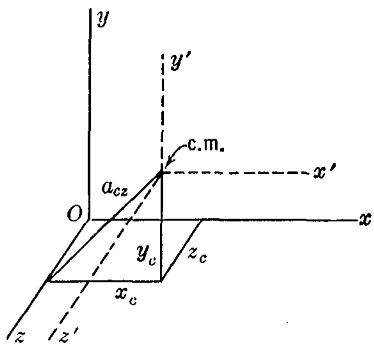


FIG. 7.8

7.5 Translation of Coordinate Axes. Suppose that the moments and products of inertia are known with respect to one set of axes, and we wish to determine the moments and products of inertia with respect to a parallel set of axes. In Fig. 7.8, the x', y', z' system has its origin located at the center of mass of the body. The moments of inertia with respect to this centroidal system are $I_{x'}$, $I_{y'}$, and $I_{z'}$, and the products of inertia are $I_{x'y'}$, $I_{x'z'}$, $I_{y'z'}$. In the parallel xyz system the center of mass is located at the point x_c, y_c, z_c. We have then:

$$\begin{aligned} I_z &= \int \rho(x^2 + y^2) \, dV = \int \rho[(x_c + x')^2 + (y_c + y')^2] \, dV \\ &= \int \rho(x_c^2 + 2x_cx' + x'^2 + y_c^2 + 2y_cy' + y'^2) \, dV \\ &= \int \rho(x'^2 + y'^2) \, dV + (x_c^2 + y_c^2)\int \rho \, dV + 2x_c\int \rho x' \, dV + 2y_c\int \rho y' \, dV \end{aligned}$$

Since the origin of the $x'y'z'$ system is at the center of mass, the integrals $\int \rho x' \, dV$ and $\int \rho y' \, dV$ are equal to zero; also:

$$\int \rho(x'^2 + y'^2) \, dV = I_{z'}$$
$$(x_c^2 + y_c^2) = a^2_{cz}$$
$$\int \rho \, dV = m$$

so:

$$I_z = I_{z'} + ma^2_{cz} \tag{7.8}$$

Thus if the moments of inertia are known with respect to centroidal axes, the moments of inertia with respect to any parallel axes can be obtained.

The transformation of products of inertia for translation of coordinate axes may be derived in the same way. Referring again to Fig. 7.8, we have:

$$\begin{aligned} I_{xy} &= \int \rho xy \, dV = \int \rho(x_c + x')(y_c + y') \, dV \\ &= \int \rho x'y' \, dV + x_c y_c \int \rho \, dV + x_c \int \rho y' \, dV + y_c \int \rho x' \, dV \end{aligned}$$

and the transformation equation is

$$I_{xy} = I_{x'y'} + mx_c y_c \tag{7.9}$$

7.6 Rotation of Coordinate Axes. Suppose that the moments and products of inertia of a body are known with respect to an $x'y'z'$ set of axes. Let us determine the moments and products of inertia

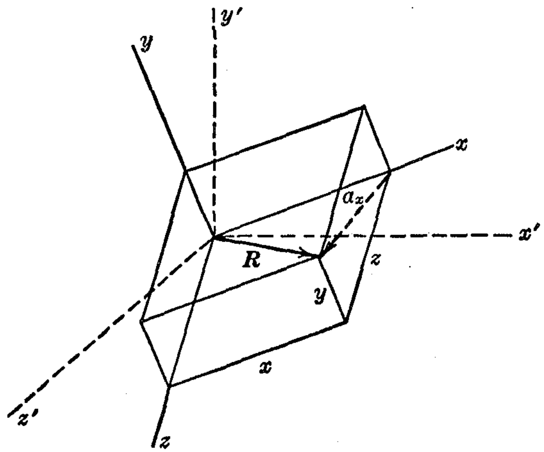


FIG. 7.9

of the body with respect to an xyz set of axes which has been rotated with respect to the $x'y'z'$ axes. In Fig. 7.9, the two coordinate systems are shown, and an element of volume dV of the body is located by the radius vector $\boldsymbol{R}$. Considering first the transformation of a typical moment of inertia, we have:

$$I_{xx} = \int \rho a_x^2 \, dV$$

We shall now express the integral in terms of x', y', and z'. First:

$$a_x^2 = R^2 - x^2 = x'^2 + y'^2 + z'^2 - x^2$$
$$\boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}'$$

Also:

$$x = \boldsymbol{i} \cdot \boldsymbol{R} = \boldsymbol{i} \cdot (x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}') = x'(\boldsymbol{i} \cdot \boldsymbol{i}') + y'(\boldsymbol{i} \cdot \boldsymbol{j}') + z'(\boldsymbol{i} \cdot \boldsymbol{k}')$$

We next note that the term $(\boldsymbol{i} \cdot \boldsymbol{i}')$ is equal to the cosine of the angle between the x-axis and the x'-axis. Denoting the direction cosines by l, we have:

$$l_{xx'} = \boldsymbol{i} \cdot \boldsymbol{i}'; \quad l_{yx'} = \boldsymbol{j} \cdot \boldsymbol{i}'; \text{ etc.}$$

so that:

$$x = x'l_{xx'} + y'l_{xy'} + z'l_{xz'}$$

With this notation the moment of inertia becomes:

$$I_{xx} = \int \rho[(x'^2 + y'^2 + z'^2) - (x'l_{xx'} + y'l_{xy'} + z'l_{xz'})^2] \, dV$$

Since $l^2_{xx'} + l^2_{xy'} + l^2_{xz'} = 1$, we may write:

$$I_{xx} = \int \rho[(x'^2 + y'^2 + z'^2)(l^2_{xx'} + l^2_{xy'} + l^2_{xz'}) \\ - (x'l_{xx'} + y'l_{xy'} + z'l_{xz'})^2] \, dV$$

Multiplying out these expressions, and combining terms gives:

$$\begin{aligned} I_{xx} = {} & l^2_{xx'} \int \rho(y'^2 + z'^2) \, dV + l^2_{xy'} \int \rho(x'^2 + z'^2) \, dV \\ & + l^2_{xz'} \int \rho(x'^2 + y'^2) \, dV - 2l_{xx'}l_{xy'} \int \rho x'y' \, dV \\ & - 2l_{xx'}l_{xz'} \int \rho x'z' \, dV - 2l_{xy'}l_{xz'} \int \rho y'z' \, dV \end{aligned}$$

or:

$$\begin{aligned} I_{xx} = {} & l^2_{xx'}I_{x'x'} + l^2_{xy'}I_{y'y'} + l^2_{xz'}I_{z'z'} - 2l_{xx'}l_{xy'}I_{x'y'} \\ & - 2l_{xx'}l_{xz'}I_{x'z'} - 2l_{xy'}l_{xz'}I_{y'z'} \end{aligned} \tag{7.10}$$

Corresponding expressions are obtained for I_{yy} and I_{zz}.

The products of inertia can be transformed in the same manner, giving:

$$
\begin{aligned}
-I_{xy} = {} & l_{xx'}l_{yx'}I_{x'x'} + l_{xy'}l_{yy'}I_{y'y'} + l_{xz'}l_{yz'}I_{z'z'} \\
& - (l_{xx'}l_{yy'} + l_{xy'}l_{yx'})I_{x'y'} - (l_{xy'}l_{yz'} + l_{xz'}l_{yy'})I_{y'z'} \\
& \qquad - (l_{xz'}l_{yx'} + l_{xx'}l_{yz'})I_{z'x'}
\end{aligned} \tag{7.11}
$$

The large number of terms and subscripts involved in these expressions makes desirable a systematic method of writing the transformations. Let us arrange the moments and products of inertia in rows and columns as:

$$
I' = \begin{array}{c|ccc}
 & x' & y' & z' \\
\hline
x' & +I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\
y' & -I_{y'x'} & +I_{y'y'} & -I_{y'z'} \\
z' & -I_{z'x'} & -I_{z'y'} & +I_{z'z'}
\end{array}
$$

Note the similarity between this group of terms and the terms appearing in Equation (7.6). Such a systematic grouping of rows and columns is called an array, and the purpose of the array is to present a large number of terms in an orderly and easily remembered system. The letters on the outside of the array are usually omitted, it being understood that the terms are arranged in that order. With respect to an xyz coordinate system the array is written:

$$
I = \begin{bmatrix}
+I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & +I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & +I_{zz}
\end{bmatrix} \tag{7.12}
$$

Note that the array is symmetrical about the main diagonal with all products of inertia terms being negative and all moments of inertia being positive.

The procedure for expressing any of the moments or products of inertia of I in terms of the moments and products of inertia I' is as follows. Let $I_{\alpha\beta}$ represent any one of the terms in the I array where both α and β may assume the values x, y, z, depending upon the term under consideration. Similarly, let $I_{\alpha'\beta'}$ represent any term in the I' array. The direction cosines which relate the directions of the two coordinate systems will be written $l_{\alpha\beta'}$. With this notation we observe that each term in the preceding transformation has the form $l_{\alpha\beta'}l_{\beta\alpha'}I_{\alpha'\beta'}$, and the transformation between the two coordinate systems may be written:

$$I_{\alpha\beta} = \sum_{\alpha'}\sum_{\beta'} l_{\alpha\beta'}l_{\beta\alpha'}I_{\alpha'\beta'} \tag{7.13}$$

To illustrate the meaning of this notation, we shall evaluate the term I_{xy} which we can then check with the previously determined expression. We note first that if $\alpha \neq \beta$, the term $I_{\alpha\beta}$ equals $-I_{xy}$, $-I_{yz}$, etc., whereas, if $\alpha = \beta$ the term $I_{\alpha\beta} = +I_{xx}$, $+I_{yy}$, etc. From the preceding equation we have therefore:

$$-I_{xy} = \sum_{\alpha'}\sum_{\beta'} l_{x\beta'}l_{y\alpha'}I_{\alpha'\beta'}$$

Summing first with respect to β':

$$-I_{xy} = \sum_{\alpha'} (l_{xx'}l_{y\alpha'}I_{\alpha'x'} + l_{xy'}l_{y\alpha'}I_{\alpha'y'} + l_{xz'}l_{y\alpha'}I_{\alpha'z'})$$

Then summing with respect to α' we have the nine terms:

$$\begin{aligned} -I_{xy} = &+ l_{xx'}l_{yx'}I_{x'x'} - l_{xy'}l_{yx'}I_{x'y'} - l_{xz'}l_{yx'}I_{x'z'} \\ &- l_{xx'}l_{yy'}I_{y'x'} + l_{xy'}l_{yy'}I_{y'y'} - l_{xz'}l_{yy'}I_{y'z'} \\ &- l_{xx'}l_{yz'}I_{z'x'} - l_{xy'}l_{yz'}I_{z'y'} + l_{xz'}l_{yz'}I_{z'z'} \end{aligned}$$

This is the same as Equation (7.11).

With this transformation, the moments and products of inertia of any rigid body can be computed for any rotated coordinate axes, once the inertia integrals are known for one set of axes in the body.

The array

$$\begin{bmatrix} +I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & +I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & +I_{zz} \end{bmatrix}$$

where it is understood that the terms are defined as above, is called the tensor of inertia, and one may speak of transforming a tensor by means of the tensor transformation:

$$I_{\alpha\beta} = \sum_{\alpha'}\sum_{\beta'} l_{\alpha\beta'}l_{\beta\alpha'}I_{\alpha'\beta'}$$

In general, if under a transformation of coordinates an expression transforms according to this equation, it is called a tensor of the second rank.

The above transformations for the rotated coordinate system can also be derived in a more compact form by a vector method, as follows. Let e_α and e_β be unit vectors in the $\alpha = x, y, z$ and $\beta = x, y, z$ directions. Then, since $r^2 = x^2 + y^2 + z^2$, and since $e_\alpha \cdot e_\beta = 1$ for $\alpha = \beta$ and $e_\alpha \cdot e_\beta = 0$ for $\alpha \neq \beta$, a general expression defining all of the products and moments of inertia is:

$$I_{\alpha\beta} = \int \rho(r^2 e_\alpha \cdot e_\beta - \alpha\beta)\, dV \tag{7.14}$$

A typical transformation equation such as has been written above in the form $x = l_{xx'}x' + l_{xy'}y' + l_{xz'}z'$ becomes in the new notation:

$$\alpha = \sum_{\beta'} l_{\alpha\beta'}\beta'$$

Similarly, the transformation equation for the unit vectors becomes:

$$e_\alpha = \sum_{\beta'} l_{\alpha\beta'} e_{\beta'}$$

Substituting these expressions for α and e_α into Equation (7.14) we obtain:

$$I_{\alpha\beta} = \int \rho\Big[r^2 \sum_{\beta'} l_{\alpha\beta'} e_{\beta'} \cdot \sum_{\alpha'} l_{\beta\alpha'} e_{\alpha'} - \sum_{\beta'} l_{\alpha\beta'}\beta' \sum_{\alpha'} l_{\beta\alpha'}\alpha'\Big]\, dV$$

which becomes:

$$I_{\alpha\beta} = \sum_{\alpha'}\sum_{\beta'} l_{\alpha\beta'} l_{\alpha'\beta} \int \rho[r^2 e_{\alpha'} \cdot e_{\beta'} - \alpha'\beta']\, dV$$

or:

$$I_{\alpha\beta} = \sum_{\alpha'}\sum_{\beta'} l_{\alpha\beta'} l_{\alpha'\beta} I_{\alpha'\beta'}$$

Note that with the notation of equation (7.14) the signs are automatically included in the expressions in the correct way.

7.7 Principal Axes. In the preceding transformations there were three products of inertia and three moments of inertia, so that the transformation formulas involved a correspondingly large number of terms. If, however, the initial set of axes is chosen in a special way, there is a substantial reduction in the number of terms. This is illustrated by the following considerations.

If the unit vectors in the two coordinate systems are $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and

$\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ respectively, then the direction cosines are given by $l_{xx}' = \boldsymbol{i}\cdot\boldsymbol{i}'$, etc. Thus:

$$\begin{aligned} \boldsymbol{i} &= l_{xx}'\boldsymbol{i}' + l_{xy}'\boldsymbol{j}' + l_{xz}'\boldsymbol{k}' \\ \boldsymbol{j} &= l_{yx}'\boldsymbol{i}' + l_{yy}'\boldsymbol{j}' + l_{yz}'\boldsymbol{k}' \\ \boldsymbol{k} &= l_{zx}'\boldsymbol{i}' + l_{zy}'\boldsymbol{j}' + l_{zz}'\boldsymbol{k}' \end{aligned}$$

We have $\boldsymbol{i}\cdot\boldsymbol{i} = 1$, etc., and $\boldsymbol{i}\cdot\boldsymbol{j} = 0$, etc.; carrying out these dot products, using the above expressions for $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, we obtain the following six relations between the direction cosines:

$$\begin{aligned} l_{xx}^2{}' + l_{xy}^2{}' + l_{xz}^2{}' &= 1; & l_{xx}'l_{yx}' + l_{xy}'l_{yy}' + l_{xz}'l_{yz}' &= 0 \\ l_{yx}^2{}' + l_{yy}^2{}' + l_{yz}^2{}' &= 1; & l_{xx}'l_{zx}' + l_{xy}'l_{zy}' + l_{xz}'l_{zz}' &= 0 \\ l_{zx}^2{}' + l_{zy}^2{}' + l_{zz}^2{}' &= 1; & l_{yx}'l_{zx}' + l_{yy}'l_{zy}' + l_{yz}'l_{zz}' &= 0 \end{aligned}$$

Since there are nine direction cosines with these six relations between them which must always be satisfied, there remain three independent relations which are required to specify the orientation of the $x'y'z'$ axes with respect to the xyz axes. We may take as these three additional relations the conditions that the three products of inertia with respect to the $x'y'z'$ axes are to be equal to zero. In this way it is possible to find a coordinate system with respect to which the three products of inertia disappear, so that the inertia tensor becomes:

$$\begin{bmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix}$$

The coordinate axes which satisfy this condition are called *principal axes*. It is customary to use principal axes whenever possible because of the simplifications which they introduce. Since the products of inertia are all zero, the moments and products of inertia can be transformed to any other set of axes which is rotated with respect to the principal axes, by the simplified equations:

$$\begin{aligned} I_{xx} &= l_{xx}^2{}'I_{x'x'} + l_{xy}^2{}'I_{y'y'} + l_{xz}^2{}'I_{z'z'} \\ I_{yy} &= l_{yx}^2{}'I_{x'x'} + l_{yy}^2{}'I_{y'y'} + l_{yz}^2{}'I_{z'z'} \\ I_{zz} &= l_{zx}^2{}'I_{x'x'} + l_{zy}^2{}'I_{y'y'} + l_{zz}^2{}'I_{z'z'} \\ I_{xy} &= -\ (l_{xx}'l_{yx}'I_{x'x'} + l_{xy}'l_{yy}'I_{y'y'} + l_{xz}'l_{yz}'I_{z'z'}) \\ I_{yz} &= -\ (l_{yx}'l_{zx}'I_{x'x'} + l_{yy}'l_{zy}'I_{y'y'} + l_{yz}'l_{zz}'I_{z'z'}) \\ I_{xz} &= -\ (l_{zx}'l_{xx}'I_{x'x'} + l_{zy}'l_{xy}'I_{y'y'} + l_{zz}'l_{xz}'I_{z'z'}) \end{aligned} \tag{7.15}$$

where $I_{x'x'}$, $I_{y'y'}$ and $I_{z'z'}$ are the moments of inertia about the principal axes, the *principal moments of inertia.*

Let us suppose that for a particular body the principal axes have been defined so that $I_{x'x'} > I_{y'y'} > I_{z'z'}$. Then the moment of inertia of the body about some other axis, the x-axis, is:

$$I_{xx} = l^2_{xx'} I_{x'x'} + l^2_{xy'} I_{y'y'} + l^2_{xz'} I_{z'z'}$$

Using the relation $l^2_{xx'} + l^2_{xy'} + l^2_{xz'} = 1$, this becomes:

$$\begin{aligned} I_{xx} &= l^2_{xx'} I_{x'x'} + (1 - l^2_{xz'} - l^2_{xx'}) I_{y'y'} + l^2_{xz'} I_{z'z'} \\ &= [(I_{x'x'} - I_{y'y'}) l^2_{xx'}] - [(I_{y'y'} - I_{z'z'}) l^2_{xz'}] + [I_{y'y'}] \end{aligned}$$

Each of the terms inside of the brackets is positive so that the maximum value of I_{xx} must occur when $l^2_{xx'}$ has its largest value of 1, and when $l^2_{xz'}$ has its smallest value of 0, so that

$$(I_{xx})_{max} = I_{x'x'} - I_{y'y'} + I_{y'y'} = I_{x'x'}$$

Thus it is proved that the largest principal moment of inertia is also the largest moment of inertia that can be obtained by any orientation of the axes. In the same way it can be shown that the smallest principal moment of inertia is the smallest moment of inertia which can be obtained by any orientation of the axes. We thus see that the principal axes have not only the property that the products of inertia about these axes vanish, but in addition the principal moments of inertia correspond to the maximum and minimum moments of inertia for any orientation of the axes. As has been shown above, the moment of inertia of a body may be obtained by adding a term to the moment of inertia about a parallel axis through the center of mass. Therefore, the minimum principal moment of inertia with respect to a coordinate system passing through the center of mass of the body is the minimum moment of inertia for any possible axis.

If a body has two perpendicular planes of symmetry, a set of principal axes can be determined by inspection, since it is only necessary to make two of the coordinate planes coincide with the planes of symmetry in order that the products of inertia become equal to zero. If the body does not have such planes of symmetry, the orientation of the principal axes must be determined from the expressions for the products of inertia, by setting the products of inertia equal to zero.

PROBLEMS

7.11. Show that the transfer equation for the moment of momentum of a rigid body, Equation (7.4), gives the correct expression if A is a fixed point.

7.12. (a) Show that in general the triple vector cross-product can be written as:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$$

(b) Using the results of (a), show that the expression for the moment of momentum of a rigid body, $\boldsymbol{H} = \int\rho\,[\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})]\,dV$, leads to the component form of Equation (7.6).

7.13. Show that the vector Equation (7.14) defines the moments and products of inertia in the same way as Equations (7.5).

7.14. Calculate I_z for a homogeneous right circular cylinder of radius R and total mass m. The z axis coincides with the axis of symmetry of the figure.

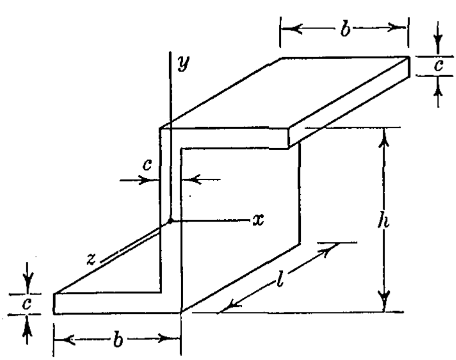
7.15. A rectangular plate of total mass m has a length a, width b, and thickness c. The z-axis is normal to the plane of a and b and passes through the midpoint of the face of the plate. (a) Find I_z for the plate. (b) Find the moment of inertia of the plate about an axis through the corner of the plate parallel to the z-axis.

7.16. Calculate the moment of inertia of a homogeneous circular disk of radius R and thickness l about a diametral axis passing through the center of mass of the disk.

7.17. Determine the moment of inertia of a uniform sphere about a line tangent to the surface of the sphere. Derive any expressions needed for this determination from the basic definition of the moment of inertia.

7.18. Calculate the moment of inertia of a slender rod about a normal axis passing through the midpoint of the rod. The rod has a uniform cross section and a uniform density.

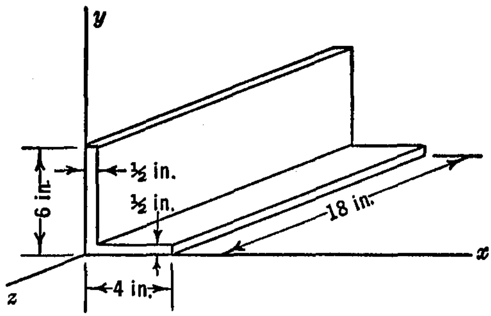
7.19. Calculate the moments of inertia I_y and I_z of the z-shaped body



PROB. 7.19

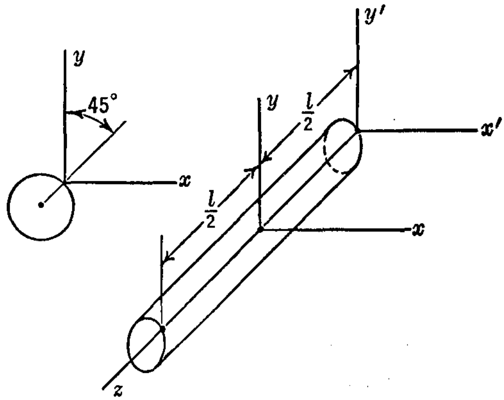
shown. The z-axis passes through the center of mass of the body and is parallel to the faces of the body. The body is homogeneous and is made of aluminum. $b = 3\frac{5}{8}$ in.; $c = \frac{1}{2}$ in.; $h = 6\frac{1}{8}$ in.; $l = 10$ in.

7.20. A solid right circular cone has a height h, a base of radius r, and a total mass m. (a) Calculate the moment of inertia of the body about the axis of symmetry. (b) Calculate the moment of inertia about a diameter of the base.



PROB. 7.21

7.21. A steel angle has the dimensions shown. Find the moment of inertia of the body about the z-axis. Give a numerical answer in units of lb ft sec^2.



PROB. 7.22

7.22. A homogeneous circular cylinder of radius R and length l has a total mass m. The z-axis lies along the surface of the cylinder parallel to its axis. (a) The xy plane passes through the center of the cylinder. Find I_{xy}, I_{yz} and I_{xz} for the cylinder. (b) The $x'y'$ plane coincides with the end of the cylinder. Find $I_{x'y'}$, $I_{y'z'}$, and $I_{x'z'}$.

7.23. Show geometrically that $l_{xx'}^2 + l_{xy'}^2 + l_{xz'}^2 = 1$.

7.24. Derive the transformation for I_{xy} by the first method used in the text to derive I_{xx}.

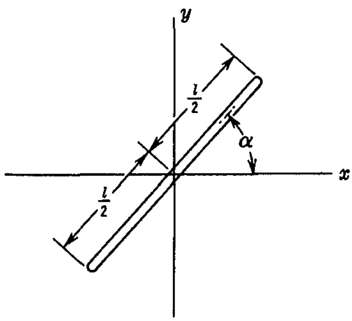
7.25. By using the transformation formula:

$$I_{\alpha\beta} = \sum_{\alpha'} \sum_{\beta'} l_{\alpha\beta'} l_{\beta\alpha'} I_{\alpha'\beta'}$$

derive the expression for I_{xx} which is given in Equation (7.10).

7.26. Write Equations (7.6) for the moment of momentum components in a summation form similar to that of Equation (7.13).

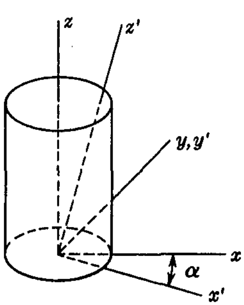
7.27. Compute I_{xy} for a slender rod of length l and mass m. The rod lies in the xy plane and makes an angle α with the x-axis. For what values of α will $I_{xy} = 0$? The rod is homogeneous and of uniform cross section.



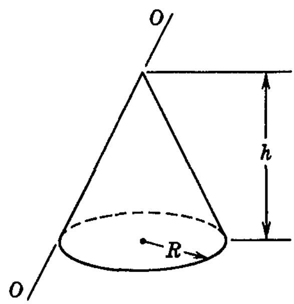
PROB. 7.27

7.28. A circular cylinder of radius R, length l, and total mass m is oriented as shown. The $x'z'$ plane coincides with the xz plane. Find $I_{z'}$.

7.29. Find the moment of inertia of a homogeneous solid right circular cone about an element of the cone (the axis O–O in the diagram). The radius of the base circle is R, the altitude is h, and the total mass is M. Do not derive any expressions that can be taken directly from the book.

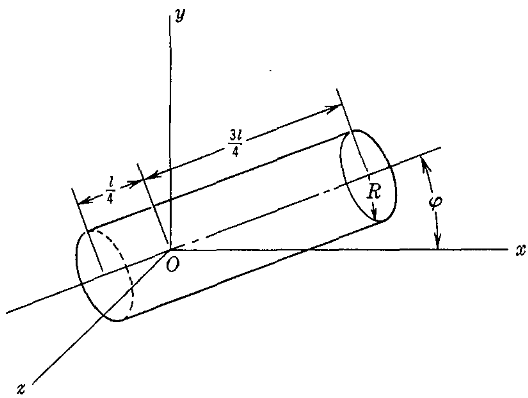


PROB. 7.28



PROB. 7.29

7.30. A solid homogeneous right circular cylinder of radius R and length l is oriented so that the axis of the cylinder, which lies in the x-y plane, makes an angle ϕ with the x-axis as shown. The origin of the coordinate system is 1/4 of the length of the cylinder from the base. Find I_{xy} and I_{xz}.

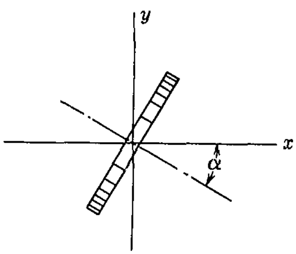


PROB. 7.30

7.31. Compute $I_{x'z'}$ and $I_{y'z'}$ for the cylinder of Prob. 7.28.

7.32. Compute the moment of inertia of a cube about a body diagonal axis passing through two opposite corners.

7.33. Find the products of inertia for a homogeneous cube so oriented that both of the y- and z-axes are face diagonals passing through corners of the cube.



PROB. 7.34

7.34. A thin circular disk of radius R and mass m rotates about the x-axis which passes through the center of mass of the disk. The disk is skewed on the shaft so that the normal to the disk makes an angle α with the axis of rotation. Find I_{xy} for the disk.

7.8 The General Equations of Motion for a Rigid Body. The general equation for the rotational motion of a rigid body is:

$$\boldsymbol{M} = \dot{\boldsymbol{H}}$$

or:

$$\boldsymbol{M} = \frac{d}{dt}(H_x\boldsymbol{i} + H_y\boldsymbol{j} + H_z\boldsymbol{k})$$

where:

$$\begin{aligned} H_x &= + I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= - I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= - I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

The xyz axes are fixed in the body with the origin at the center of mass and are rotating with it so that I_{xx}, I_{xy}, etc., are constants. The unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ rotate with the body so that $\dot{\boldsymbol{i}}, \dot{\boldsymbol{j}}, \dot{\boldsymbol{k}}$ are not zero. The equation of motion is, therefore:

$$\boldsymbol{M} = \dot{H}_x\boldsymbol{i} + H_x\dot{\boldsymbol{i}} + \dot{H}_y\boldsymbol{j} + H_y\dot{\boldsymbol{j}} + \dot{H}_z\boldsymbol{k} + H_z\dot{\boldsymbol{k}}$$

The derivatives of the unit vectors are:

$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i} = (\omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}) \times \boldsymbol{i} = \omega_z\boldsymbol{j} - \omega_y\boldsymbol{k}$$
$$\dot{\boldsymbol{j}} = \omega_x\boldsymbol{k} - \omega_z\boldsymbol{i}; \quad \dot{\boldsymbol{k}} = \omega_y\boldsymbol{i} - \omega_x\boldsymbol{j}$$

Substituting and collecting terms.

$$\boldsymbol{M} = (\dot{H}_x - \omega_z H_y + \omega_y H_z)\boldsymbol{i} + (\dot{H}_y - \omega_x H_z + \omega_z H_x)\boldsymbol{j} + (\dot{H}_z - \omega_y H_x + \omega_x H_y)\boldsymbol{k}$$

This is the general vector equation of motion. The three scalar equations of motion are:

$$\begin{aligned} M_x &= \dot{H}_x - \omega_z H_y + \omega_y H_z \\ M_y &= \dot{H}_y - \omega_x H_z + \omega_z H_x \\ M_z &= \dot{H}_z - \omega_y H_x + \omega_x H_y \end{aligned} \tag{7.16}$$

The appropriate expressions for H_x, H_y and H_z must be substituted in the equations. The resulting expressions are greatly simplified by locating the coordinate axes so that they coincide with the principal axes of the body. The products of inertia are then zero and $H_x = I_{xx}\omega_x$, $H_y = I_{yy}\omega_y$, $H_z = I_{zz}\omega_z$. The equations then become:

$$\begin{aligned} M_x &= I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\ M_y &= I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ M_z &= I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{aligned} \tag{7.17}$$

These are called *Euler's equations of motion of a rigid body.* It should be noted that in Euler's equations the xyz axes are the principal axes of the body, with origin at the center of mass of the body.

An impulse-momentum equation may be obtained to describe the rotation of a rigid body by a direct integration of $\boldsymbol{M} = \dot{\boldsymbol{H}}$ with respect to time:

$$\int_1^2 \boldsymbol{M}\, dt = \boldsymbol{H}_2 - \boldsymbol{H}_1 \tag{7.18}$$

This equation states that the moment of impulse is equal to the change in the moment of momentum.

To obtain the work-energy equation for the rigid body we shall first derive a general expression for the total kinetic energy of a rigid body. Referring to Fig. 7.5, the kinetic energy of the element dV is $1/2(\rho\, dV)v^2$, and hence for the whole body:

$$T = \tfrac{1}{2}\int \rho(\boldsymbol{v}\cdot\boldsymbol{v})\, dV$$

For a rigid body, $\boldsymbol{v} = \boldsymbol{v}_c + \boldsymbol{\omega} \times \boldsymbol{r}_c$, so that the expression for T becomes:

$$\begin{aligned} T &= \tfrac{1}{2}\int \rho[(\boldsymbol{v}_c + \boldsymbol{\omega} \times \boldsymbol{r}_c)\cdot(\boldsymbol{v}_c + \boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV \\ &= \tfrac{1}{2}\int \rho[v_c^2 + 2\boldsymbol{v}_c\cdot(\boldsymbol{\omega} \times \boldsymbol{r}_c) + (\boldsymbol{\omega} \times \boldsymbol{r}_c)\ (\boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV \end{aligned}$$

The first term in this expression can be written as:

$$\tfrac{1}{2}\int \rho v_c^2\, dV = \tfrac{1}{2}v_c^2 \int \rho dV = \tfrac{1}{2}mv_c^2$$

The second term becomes:

$$\int \rho[\boldsymbol{v}_c\cdot(\boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV = \boldsymbol{v}_c\cdot(\boldsymbol{\omega} \times \int \rho \boldsymbol{r}_c\, dV) = 0$$

since $\boldsymbol{r}_c$ is measured from the center of mass. By an interchange of the dot and the cross vector multiplication the third term becomes:

$$\tfrac{1}{2}\int \rho[(\boldsymbol{\omega} \times \boldsymbol{r}_c)\cdot(\boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV = \tfrac{1}{2}\boldsymbol{\omega}\cdot\int \rho[\boldsymbol{r}_c \times (\boldsymbol{\omega} \times \boldsymbol{r}_c)]\, dV = \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{H}_c$$

Thus the complete expression for the total kinetic energy of the rigid body becomes:

$$T = \tfrac{1}{2}mv_c^2 + \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{H}_c \tag{7.19}$$

where m is the total mass of the body, v_c is the velocity of the center of the mass, and $\boldsymbol{H}_c$ is the moment of momentum of the body with respect to the center of mass. The first term represents the kinetic energy of translation, and corresponds to the first term of the general Equation (6.6). The second term represents the kinetic energy of rotation about the mass center, and corresponds to the second term of the general Equation (6.6).

Expanding the dot product, the kinetic energy of rotation can be written as:

$$\begin{aligned} \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{H}_c = \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y \\ - 2I_{yz}\omega_y\omega_z - 2I_{xz}\omega_x\omega_z) \end{aligned} \tag{7.20}$$

where the moments and products of inertia are defined with respect to a coordinate system whose origin is at the center of mass of the body.

The forces acting on the body are equivalent to a resultant force $\boldsymbol{F}$ acting at the center of mass, plus a resultant moment $\boldsymbol{M}_c$ about the center of mass. The rate at which work is done is

$$\boldsymbol{F} \cdot \boldsymbol{v}_c + \boldsymbol{\omega} \cdot \boldsymbol{M}_c$$

The work done is the integral of this with respect to time and thus:

$$\int_1^2 \boldsymbol{F} \cdot \boldsymbol{v}_c \, dt = \tfrac{1}{2} m v_c^2 \Big|_1^2$$

$$\int_1^2 \boldsymbol{\omega} \cdot \boldsymbol{M}_c \, dt = \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_c \Big|_1^2 \qquad (7.21)$$

For a rigid body this equation takes the place of Equation (6.8) and can be used to supplement Equation (6.5), which describes the work energy principle as applied to the motion of the center of mass of any system under the action of external forces.

EXAMPLE 1. The center of mass of a rigid body of mass 50 lb sec²/ft has a velocity of magnitude 40 ft/sec. At a given instant the moment of momentum of the body about the center of mass is $\boldsymbol{H}_c = 1500\boldsymbol{i} + 1000\boldsymbol{j} + 1200\boldsymbol{k}$ lb ft sec. The inertia integrals about the same coordinate axes are:

$$I = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 40 & -20 \\ 0 & -20 & 30 \end{bmatrix} \text{ lb ft sec}^2$$

Find the angular velocity and the kinetic energy of the body at the given instant.

Solution. Substituting in the equations for the rectangular components of $\boldsymbol{H}_c$, Equation (7.6):

$$\begin{aligned} 1500 &= 50\omega_x \\ 1000 &= 40\omega_y - 20\omega_z \\ 1200 &= -20\omega_y + 30\omega_z \end{aligned}$$

from which:

$$\boldsymbol{\omega} = 30\boldsymbol{i} + 67.5\boldsymbol{j} + 85\boldsymbol{k} \quad \text{rad/sec}$$

Then:

$$T = \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{H}_c$$
$$= \tfrac{1}{2}(50)(40)^2 + \tfrac{1}{2}(30\boldsymbol{i} + 67.5\boldsymbol{j} + 85\boldsymbol{k})\cdot(1500\boldsymbol{i} + 1000\boldsymbol{j} + 1200\boldsymbol{k}) \text{ ft lb}$$
$$T = 147{,}250 \text{ ft lb}$$

EXAMPLE 2. In Fig. 7.10, OAB and OAD represent two identical right circular cones of mass M, altitude h and half angle α. The vertical cone OAD is fixed, and OAB rolls around the vertical cone without slipping. The axis OE of the moving cone revolves about

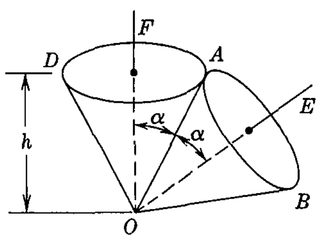


FIG. 7.10

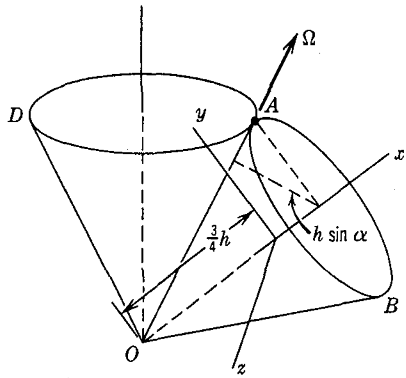


FIG. 7.11

the fixed vertical axis OF with an angular velocity ω_0. Find the moment of momentum of the moving cone about the vertex O, and find the total kinetic energy of the moving cone.

Solution. Take x, y, z as principal axes with the origin at the center of mass of the moving cone as in Fig. 7.11. Let Ω be the angular velocity of rolling of the moving cone about the common element OA.

Then:

$$\omega_0(h \sin 2\alpha) = \Omega(h \sin \alpha); \quad \Omega = 2\omega_0 \cos \alpha$$

and:

$$\omega_x = \Omega \cos \alpha = 2\omega_0 \cos^2 \alpha$$
$$\omega_y = \Omega \sin \alpha = \omega_0 \sin 2\alpha$$
$$\omega_z = 0$$

Also:

$$I_x = \frac{3}{10} MR^2 = \frac{3}{10} Mh^2 \tan^2 \alpha$$

$$I_y = I_z = \frac{3}{80} M(4R^2 + h^2) = \frac{3}{80} Mh^2(4 \tan^2 \alpha + 1)$$

Then from Equation (7.6), noting that all of the products of inertia are zero, $\boldsymbol{H}_c$ is found to be:

$$H_x = \left(\frac{3}{10} Mh^2 \tan^2 \alpha\right)(2\omega_0 \cos^2 \alpha) = \frac{3}{5} Mh^2\omega_0 \sin^2 \alpha$$

$$H_y = \frac{3}{80} Mh^2(4 \tan^2 \alpha + 1)\omega_0 \sin 2\alpha$$
$$= \frac{3}{80} Mh^2\omega_0 \sin 2\alpha(4 \tan^2 \alpha + 1)$$

Knowing $\boldsymbol{H}_c$, the moment of momentum about O can be found from Equation (7.4):

$$\boldsymbol{H}_0 = \boldsymbol{H}_c + M\boldsymbol{r}_{0c} \times \boldsymbol{v}_c$$
$$\boldsymbol{r}_{0c} \times \boldsymbol{v}_c = (\tfrac{3}{4}h)(\tfrac{3}{4}h \sin 2\alpha \cdot \omega_0)\boldsymbol{j}$$

So:

$$H_{0x} = H_{cx}$$

$$H_{0y} = \frac{3}{80} Mh^2\omega_0 \sin 2\alpha(4 \tan^2 \alpha + 1) + \frac{9}{16} Mh^2\omega_0 \sin 2\alpha$$
$$= Mh^2\omega_0 \sin 2\alpha\left[\frac{3}{80} (4 \tan^2 \alpha + 1) + \frac{9}{16}\right]$$

And:

$$\boldsymbol{H}_0 = \frac{1}{5} Mh^2\omega_0[3 \sin^2 \alpha\, \boldsymbol{i} + (\tfrac{3}{4} \tan^2 \alpha + 3) \sin 2\alpha\, \boldsymbol{j}]$$

To find the kinetic energy, we write:

$$T = \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{H}_c = \tfrac{1}{2}M(\tfrac{3}{4}h \sin 2\alpha)^2\omega_0^2 + \tfrac{1}{2}(\omega_x H_{cx} + \omega_y H_{cy})$$

from which:

$$T = \frac{3}{40} Mh^2\omega_0^2 \sin^2 2\alpha(\tan^2 \alpha + 6)$$

7.9 Equations of Motion for a Translating Body. The simplest type of rigid body motion is that of translation. A body having translatory motion moves in such a way that any line in the body always remains parallel to its original position, that is, the angular velocity of the body is always zero. The moment of momentum equation written about the mass center is:

$$\boldsymbol{M}_c = \dot{\boldsymbol{H}}_c$$

and since the angular velocity is zero, $\dot{\boldsymbol{H}}_c = 0$, and therefore:

$$\boldsymbol{M}_c = 0$$

The equation of motion of the mass center is:

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$$

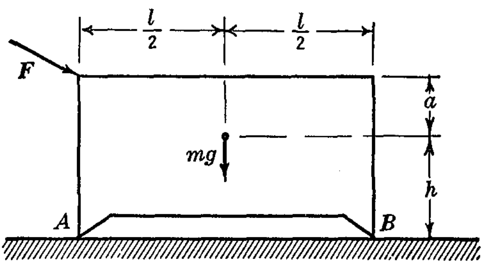


FIG. 7.12

Writing these equations in rectangular coordinates, where x_c, y_c, z_c are the coordinates of the center of mass, we obtain:

$$\begin{aligned} F_x &= m\ddot{x}_c \qquad & M_{cx} &= 0 \\ F_y &= m\ddot{y}_c \qquad & M_{cy} &= 0 \\ F_z &= m\ddot{z}_c \qquad & M_{cz} &= 0 \end{aligned} \tag{7.22}$$

The force equations describe the motion of the mass center and the moment equations describe the reactive forces which prevent rotation of the body.

EXAMPLE. A body of total mass m moves along a horizontal plane under the action of a force F as shown in Fig. 7.12. The coefficient of kinetic friction between the body and the surface is μ. Find the acceleration of the body as a function of the force F, and determine the reactions exerted by the surface on the body at A and B.

Solution. We choose the x-axis in the direction of the motion of the body. The complete free-body diagram is then drawn. The fundamental equations $\boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$ and $\boldsymbol{M}_c = \dot{\boldsymbol{H}}_c$ for this problem become:

$$F_x - \mu(N_A + N_B) = m\ddot{x}_c$$
$$N_A + N_B - mg - F_y = 0$$
$$F_y\left(\frac{l}{2}\right) - F_x a + N_B\left(\frac{l}{2}\right) - N_A\left(\frac{l}{2}\right) - \mu(N_A + N_B)h = 0$$

Eliminating $(N_A + N_B)$ between the first two equations gives the equation of motion of the body:

$$m\ddot{x}_c = F_x - \mu(mg + F_y)$$

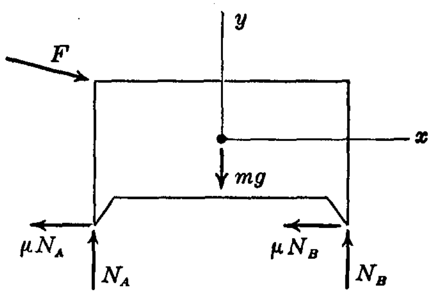


FIG. 7.13

The reactions N_A and N_B are found from the second and third equations:

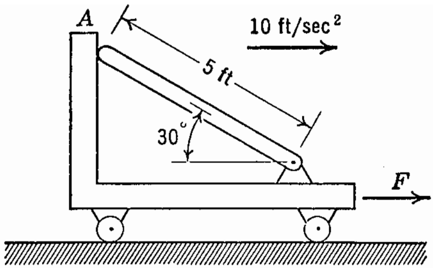
$$N_A = -\frac{a}{l}F_x + \left(1 - \frac{\mu h}{l}\right)F_y + mg\left(\frac{1}{2} - \frac{\mu h}{l}\right)$$
$$N_B = \frac{a}{l}F_x + \frac{\mu h}{l}F_y + mg\left(\frac{1}{2} + \frac{\mu h}{l}\right)$$

PROBLEMS

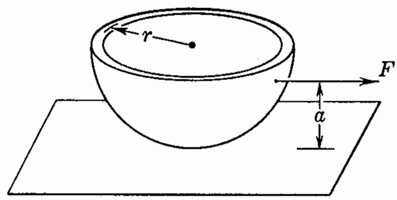
7.35. A uniform, straight bar weighing 50 lb is fastened with a smooth pin at one end and rests at the other end against a smooth vertical surface. The bar is 5 ft long, and the centerline makes an angle of 30° with the horizontal. The whole arrangement is given an acceleration of 10 ft/sec² horizontally to the right by a force F acting on the support, as shown in the diagram. Find all of the forces acting on the bar.

7.36. A thin uniform hemispherical shell has a weight W and a radius r. The shell is pulled along a horizontal surface by a constant horizontal force

F. The coefficient of sliding friction between the shell and the surface is μ which may be assumed to be constant. Find the distance above the plane that the force F should be applied, so that there is no tipping of the shell. What is the acceleration of the shell under these conditions?

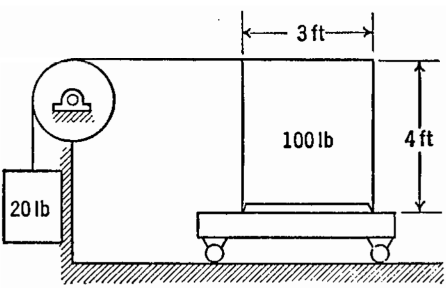


PROB. 7.35

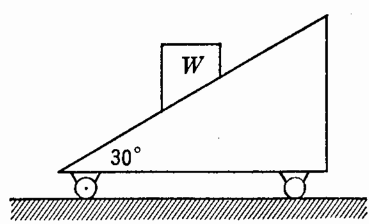


PROB. 7.36

7.37. A homogeneous block having the dimensions shown in the figure weighs 100 lb and rests on a car which can move along a horizontal plane. A 20 lb weight is connected to the block by means of a cable and a frictionless pulley as shown in the diagram. The coefficient of static friction between the block and the car is $\mu = 0.25$. If the car is given an acceleration to the right which starts from zero and gradually increases, will slipping or tipping of the block occur first? At what value of acceleration will this occur?



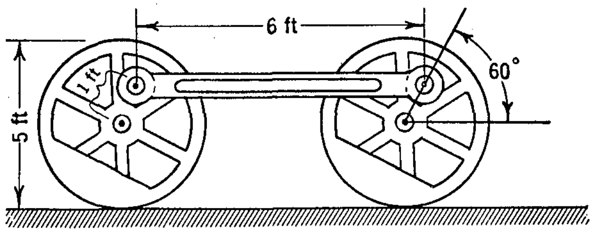
PROB. 7.37



PROB. 7.38

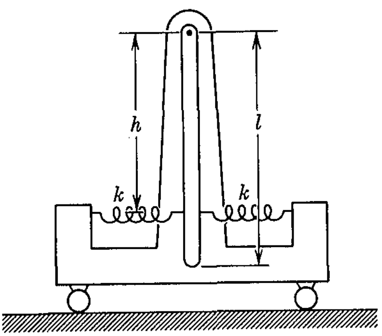
7.38. A body weighing W pounds rests on a 30° inclined plane as shown. The coefficient of static friction between the body and the plane is μ. What is the maximum horizontal acceleration which the whole system can have without causing the body to move on the plane?

7.39. The side-crank connecting rod of a locomotive drive system has the dimensions shown. Assuming that the side-crank is a straight uniform bar weighing 500 lb and that it is fastened by smooth pins at the ends, find the forces acting on the rod when the locomotive is running at 60 mph when the rod is in the position shown.

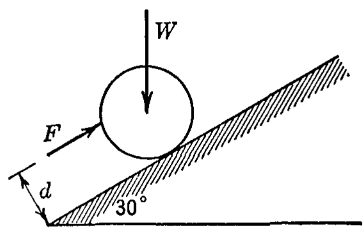


PROB. 7.39

7.40. A uniform straight bar of length l and weight W is suspended on a smooth pin which is mounted on a carriage as shown in the figure. The bar is also connected to the carriage by two equal springs of spring constant k, attached at a distance h below the point of suspension. When the carriage is at rest, the bar hangs vertically with the springs in an unstretched position. Find the angle between the bar and the vertical when the carriage has a small steady horizontal acceleration of magnitude a. It may be assumed that the angle is sufficiently small so that the spring forces are always horizontal.



PROB. 7.40



PROB. 7.41

7.41. A homogeneous circular cylinder of radius r weighing W lb rests on a 30° inclined plane as shown. The coefficient of kinetic friction between the cylinder and the plane is μ. Where should a force F, parallel to the plane be applied, if the cylinder is to slide up the plane without rotation? Find the reaction of the plane on the cylinder under this condition.

7.42. A car moves with a uniform acceleration along a horizontal surface. An instrument is required which will measure the magnitude of this acceleration. It is proposed that a small bar be mounted on a bearing attached to the car, and that the angle made by the bar with the vertical be measured as an indication of the acceleration of the car. Find the relationship between $\ddot{x}$ and ϕ, assuming that the bar is uniform and homogeneous. Would it be better to use a concentrated mass at the end of a light rod?

7.10 The Rotation of a Rigid Body about a Fixed Axis.

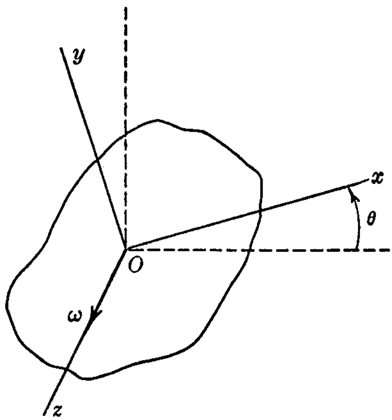


FIG. 7.14

Take the z-axis as the axis of rotation, and let the x and y axes be attached to the body and rotate with it as in Fig. 7.14. Since the origin of this coordinate system is fixed in space, we may write the equation:

$$M = \dot{H}$$

taking the origin as the moment center. In Equations (7.6) for the components of H we may put $\omega_x = \omega_y = 0$, since the rotation is about the z-axis, and hence Equations (7.16) become:

$$M_x = \frac{d}{dt}(-I_{xz}\omega_z) - \omega_z(-I_{yz}\omega_z)$$

$$M_y = \frac{d}{dt}(-I_{yz}\omega_z) + \omega_z(-I_{xz}\omega_z)$$

$$M_z = \frac{d}{dt}(I_{zz}\omega_z)$$

Performing the differentiations, the three moment equations become:

$$\begin{aligned} M_x &= I_{yz}\omega_z^2 - I_{xz}\dot{\omega}_z \\ M_y &= -I_{xz}\omega_z^2 - I_{yz}\dot{\omega}_z \\ M_z &= I_{zz}\dot{\omega}_z \end{aligned} \tag{7.23}$$

In addition, the equations for the motion of the center of mass are available:

$$F_x = m\ddot{x}_c$$
$$F_y = m\ddot{y}_c$$
$$F_z = 0$$

The rotation of the body about the fixed axis is described by the third moment equation, while the constraining forces which hold the axis stationary may be found from the first two moment equations and the equations of motion of the mass center.

The impulse-momentum equation and the work-energy equation for a rotating body may be derived directly from the third moment equation, $M_z = I_z\dot{\omega}_z$:

$$\int_1^2 M_z\,dt = I_z\omega_z\Big|_1^2 \tag{7.24}$$

The integral of $M_z\,dt$ is the angular impulse or moment of impulse about the axis of rotation, and the term $I_z\omega_z$ is the moment of momentum about the axis of rotation, as may be checked from the general Equation (7.6). Similarly, for the energy equation:

$$\int_1^2 M_z\,d\theta = \int_1^2 I_z\ddot{\theta}\,\frac{d\theta}{dt}\,dt = \frac{1}{2}\,I_z\omega_z^2\Big|_1^2 \tag{7.25}$$

The integral of $M_z\,d\theta$ represents the work done by M_z during the rotation, and the term $(1/2)I_z\omega_z^2$ is the kinetic energy of rotation. This may be checked from the general energy Equation (7.19) since $T_{\text{rot}} = \frac{1}{2}\,\boldsymbol{\omega}\cdot\boldsymbol{H}$ where $\boldsymbol{H} = I_z\omega_z\boldsymbol{k}$, and therefore $T_{\text{rot}} = \frac{1}{2}\,I_z\omega_z^2$.

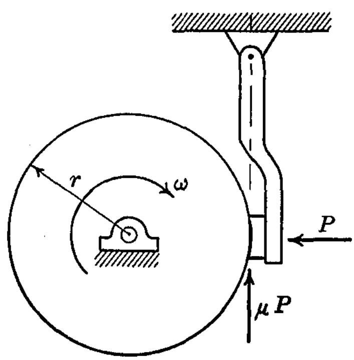


FIG. 7.15

EXAMPLE 1. A flywheel of radius r, having a moment of inertia I about the axis of rotation, has an angular velocity ω rad/sec. At time $t = 0$, a brake is applied with a normal braking force P, as shown in the figure. The brake coefficient of friction is μ. Find the time required to reduce the angular velocity of the flywheel to ω_1 rad/sec, and find the number of revolutions of the flywheel occurring during this time.

Solution. If the normal braking force is P, the tangential frictional force will be μP, and the retarding moment will be $r\mu P$. Applying Equation (7.24) expressing the impulse-momentum relationship:

$$r\mu Pt = I\omega - I\omega_1$$

so:

$$t = \frac{I(\omega - \omega_1)}{r\mu P}$$

The number of revolutions occurring in time t can be found from the work-energy relation of Equation (7.25):

$$r\mu P\theta = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_1{}^2$$

so:

$$\theta = \tfrac{1}{2}I\,\frac{(\omega^2 - \omega_1{}^2)}{r\mu P}$$

EXAMPLE 2. A homogeneous disk of radius R and of uniform thickness is supported by a thin rod or wire as shown in Fig. 7.16. The rod is rigidly attached to the disk and to the support. If the disk is rotated from its equilibrium position through an angle θ, the rod exerts a restoring torque on the disk which is proportional to the displacement and oppositely directed. The disk is rotated through an angle θ_0 and is then released from rest. Describe the subsequent motion of the system.

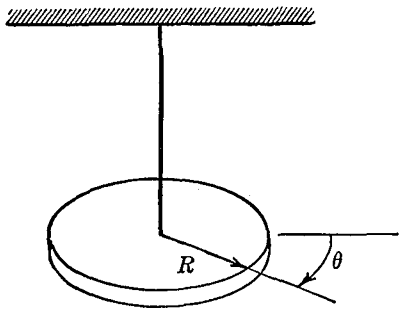


FIG. 7.16

Solution. The disk will rotate about the axis of the rod under the action of a torque $-k\theta$, where k is the torsional spring constant in lb ft/rad. Writing the equation of motion about the fixed axis of rotation, we have, with I as the moment of inertia of the disk about the axis of the rod:

$$I\ddot{\theta} = -k\theta$$

$$\ddot{\theta} + \frac{k}{I}\theta = 0$$

This is the differential equation of simple harmonic motion, the solution of which is:

$$\theta = C_1 \sin \sqrt{\frac{k}{I}}\, t + C_2 \cos \sqrt{\frac{k}{I}}\, t$$

When:

$$t = 0, \quad \dot{\theta} = 0; \quad \text{so} \quad C_1 = 0$$
$$t = 0, \quad \theta = \theta_0; \quad \text{so} \quad C_2 = \theta_0$$

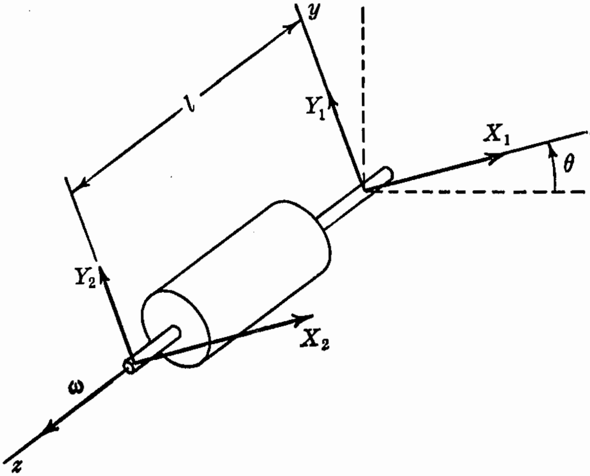


FIG. 7.17

and the complete solution is:

$$\theta = \theta_0 \cos \sqrt{\frac{k}{I}}\, t$$

Thus the disk performs torsional oscillations with an amplitude θ_0, and a frequency $\frac{1}{2\pi}\sqrt{\frac{k}{I}}$ cycles per second. Such an oscillator is called a torsion pendulum. Torsional oscillation problems are in every way similar to the linear oscillation problems treated in the chapter on vibrations and the same methods may be used. Many practical examples of such problems in engineering can be found, such as the torsional oscillations of engine crankshafts and of propeller shafts.

EXAMPLE 3. Investigate the dynamic bearing reactions caused by the rotation of an unbalanced rotor.

Solution. Consider a rotor of total weight W supported horizontally on two bearings a distance l apart as shown in Fig. 7.17.

The z-axis is taken as the fixed axis of rotation, and the xy axes are attached to the rotating body. During the rotation there will be dynamic reactions at the supports of the rotor. We shall determine these dynamic reactions X_1, Y_1, X_2, Y_2 in the rotating xyz system and it is to be understood that, if the total bearing reactions at any instant are required, the dynamic reactions, located in the correct direction at that instant, must be added to the static reactions caused by the weight of the rotor. The equations of motion of the mass center give us directly:

$$\Sigma F_x = m\ddot{x}_c; \quad X_1 + X_2 = -mx_c\omega^2 - my_c\dot{\omega}$$
$$\Sigma F_y = m\ddot{y}_c; \quad Y_1 + Y_2 = mx_c\dot{\omega} - my_c\omega^2$$

The moment of momentum equations give:

$$-Y_2l = I_{yz}\omega^2 - I_{xz}\dot{\omega}$$
$$X_2l = -I_{xz}\omega^2 - I_{yz}\dot{\omega}$$

Knowing m, ω, $\dot{\omega}$, x_c, y_c, I_{xz} and I_{yz}, we can find the four unknown dynamic reaction components from these four equations.

If the products of inertia are zero, and if the center of mass lies on the axis of rotation so that $x_c = y_c = 0$, then it will be seen that there are no dynamic reactions. The rotating body is then said to be dynamically balanced. If the center of mass lies on the axis of rotation so that the system is statically balanced, there is no gravity torque for any position of the body, but there may still be some dynamic unbalance because of the presence of product of inertia terms. Since for static balance the mass center has zero acceleration:

$$X_1 = -X_2$$
$$Y_1 = -Y_2$$

The dynamic reactions hence exert a couple on the rotor.

It is thus seen that static balancing of a rotor is not in general sufficient to remove the dynamic reactions, since a rotating dynamic couple may still be present. Complete dynamic balance is achieved by adding to the system two balance weights, so located that the dynamic reactions set up by the balance weights are equal and opposite to the dynamic reactions resulting from the original unbalanced rotor. This is equivalent to making the products of inertia of the rotating system zero, by the addition of the extra balancing weights.

PROBLEMS

7.43. A rigid body rotates about a fixed axis with an angular velocity ω. Starting with an element of volume dV of the body, show by integration over the volume of the body that the kinetic energy is $\frac{1}{2}I\omega^2$ and that the moment of momentum about the axis of rotation is $I\omega$.

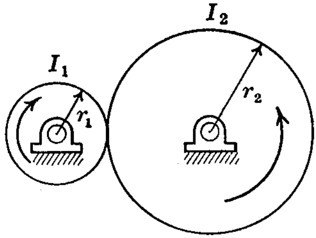
7.44. A rigid body rotates about a fixed axis. When $t = 0$, the angular displacement of the body measured from a fixed position is θ_0, and the angular velocity is $\dot{\theta}_0$. A torque M_z about the axis of rotation is applied when $t = 0$. If $M_z = A + Bt - Ct^2$, find the angular displacement of the body at any time.

7.45. A flywheel with a moment of inertia I starts from rest under the action of a constant torque M. What is the angular velocity of the flywheel after it has rotated through N revolutions? Do this problem in two ways, first using the equation of motion of the flywheel in terms of angular accelerations, and then using the work-energy principle.

7.46. A homogeneous circular disk of radius R and of mass m is fixed on a shaft which coincides with the geometrical axis of the disk. Acting on the shaft is a torque caused by bearing friction which is proportional to the velocity of the disk, $M_F = -k\dot{\theta}$. At the time $t = 0$, the disk has an initial angular velocity ω_0. Find the time required for the disk to come to rest, and find the number of revolutions of the disk occurring in this time.

7.47. A wheel having a moment of inertia I about its axis of rotation is acted upon by a constant torque M. If the motion is resisted by a torque $M_F = -k\dot{\theta}$ due to bearing friction and air resistance, find the maximum speed which will be attained by the wheel.

7.48. A rotor with a moment of inertia I_1 is driven at a constant angular velocity ω_1. It is brought into contact with a second rotor I_2, which is initially at rest. There is a constant normal force of P lb between the rotors, and the coefficient of friction is μ. At first there is slipping between the rotors until the second rotor has attained the angular velocity $\omega_2 = \omega_1 \dfrac{r_1}{r_2}$. How much time is required for the second rotor to reach this velocity? (Assume that the coefficient of friction is independent of velocity.) If the first rotor is free to decelerate after initial contact, how much time is required to overcome slipping? What will be the final angular velocities of the two disks under these conditions?



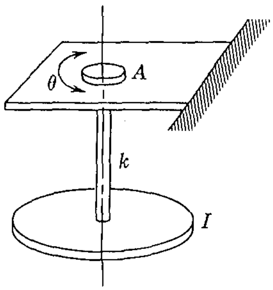
PROB. 7.48

7.49. A torsion pendulum is mounted as shown. The point of suspension A can be rotated by means of a lever. Suppose that the pendulum is initially at rest, and that starting at time $t = 0$ the point of suspension is given a rotation

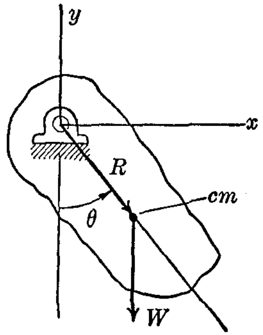
$$\theta = \theta_0 \sin \omega t$$

Find the resulting forced vibration of the pendulum. Neglect damping in the system.

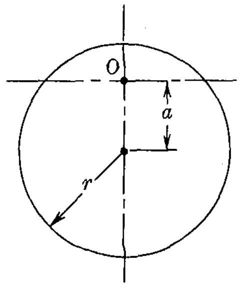
7.50. A flywheel having a moment of inertia about its axis of rotation of 5000 lb ft sec^2 is driven by an electric motor which does work at the rate of 1/4 horsepower. If the wheel starts from rest, what will be its kinetic energy at the end of one hour? If the flywheel is to be brought to rest in 30 sec from the speed attained at the end of one hour, what constant retarding frictional torque is required?



PROB. 7.49



PROB. 7.51

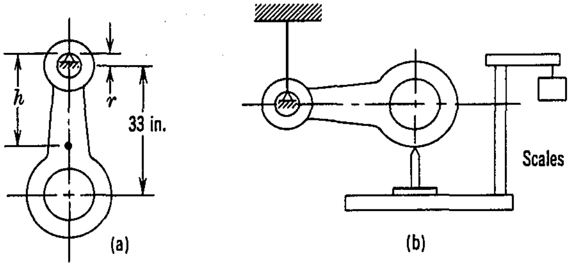


PROB. 7.52

7.51. A body free to oscillate about a fixed axis under the action of gravity is called a compound pendulum. Considering the compound pendulum shown, the total weight of which is W, find the period of small vibrations of the pendulum about the horizontal z-axis. The distance from the point of support to the center of mass is R. Find the length of a simple pendulum with a mass concentrated at one point which would have the same period as the compound pendulum.

7.52. A thin homogeneous circular disk of radius r, mass m, and of uniform thickness, is suspended as a pendulum in a vertical plane from a point a distance a above the center, as shown in the figure. Find a for the maximum frequency of vibration for small oscillations about the vertical equilibrium position in the plane of the disk. Find this maximum frequency.

7.53. For an analysis of the dynamic forces in an engine, the moment of inertia of the connecting rods about an axis through the center of mass must be known. It is proposed that this moment of inertia be experimentally determined in the following way. The connecting rod is supported on a small horizontal knife edge passed through the wrist pin bushing as shown in (a), and the rod is permitted to oscillate as a pendulum through small angles in the vertical plane. It is found that 60 complete oscillations occur in 100 seconds. The distance h between the point of suspension and the center of mass of the rod is found by supporting the wrist pin end as shown in (b), while placing the crank end of the rod on a

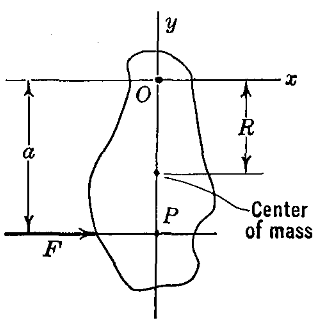


PROB. 7.53

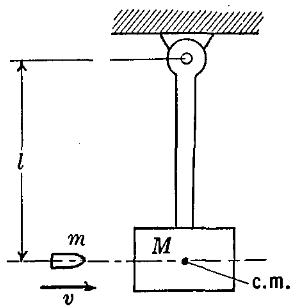
platform scale. The rod is held horizontal and the scale reads 92 lb. The distance between bearing centers is 33 in. and the weight of the rod is 145 lb. The radius of the wrist pin bearing is $1\frac{1}{4}$ in.

Find the moment of inertia I_c of the connecting rod about an axis passing through the center of gravity in a direction parallel to the wrist pin bearing.

7.54. A rigid body can rotate about a fixed point O as shown in the figure. The moment of inertia of the body about the point O is I_0 and the distance between O and the center of mass of the body is R. A force F is applied



PROB. 7.54

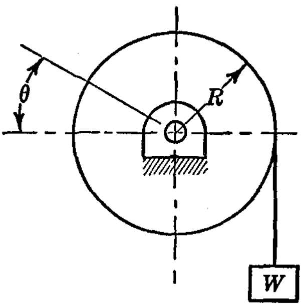


PROB. 7.55

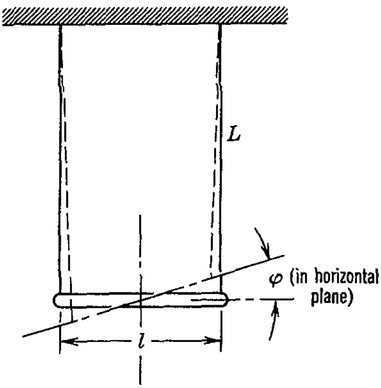
to the body perpendicular to the line joining O and the center of mass, at a distance a from O. Find the distance a for which there will be no reaction at the point O in the direction of F. Neglect gravity forces. The point P, located by a, is called the *center of percussion* of the body. Show that if the body rotates about P, O becomes the center of percussion.

7.55. A ballistic pendulum of mass M has a moment of inertia I about its axis of rotation. A bullet of mass m is fired into the pendulum as shown in the figure. It is observed that the pendulum then undergoes an angular displacement θ_0. Find the velocity of the bullet.

7.56. A wheel of radius R and moment of inertia I about the axis of rotation has a rope wound around it which supports a weight W. Write the equation of conservation of energy for this system, and differentiate to find the equation of motion in terms of acceleration. Check the answer obtained by drawing separate free-body diagrams for the wheel and for the weight, writing the equations of motion for each body, and solving the equations simultaneously. Assume that the mass of the rope is negligible, and that there is no energy loss during the motion.



PROB. 7.56

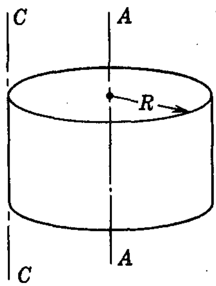


PROB. 7.57

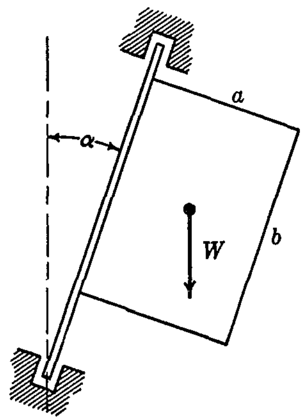
7.57. A horizontal uniform slender bar of length l and mass m is suspended by two massless vertical strings of length L attached at each end as shown in the diagram. The bar is rotated about a vertical axis through its center, and is then released. Find the frequency of small rotational oscillations.

7.58. A solid, homogeneous disc initially rotates with constant angular velocity ω_0 about its own axis (A–A). If this disc is suddenly caught along an element (C–C) parallel to (A–A), find: (a) the angular velocity of the disc about C–C axis immediately afterwards; (b) the impulse exerted by the disc on the axis C–C, letting the mass and radius of the disc be M and R and noting that energy need not be conserved; and (c) the loss of kinetic energy during the impact.

7.59. A thin, homogeneous, rectangular plate of uniform thickness is free to oscillate under the action of gravity about an inclined axis as shown in the figure. Write the equation of conservation of energy for this



PROB. 7.58

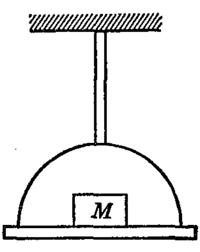


PROB. 7.59

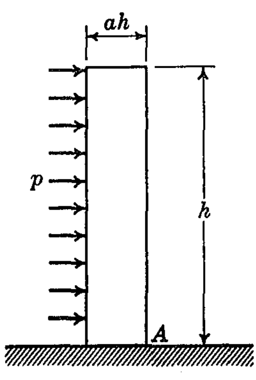
system and differentiate to find the equation of motion in terms of accelerations. Find the period of undamped oscillations of the system, and show how changes in the angle of inclination α change this period.

7.60. A torsional pendulum is arranged as shown in the figure, so that various weights can be placed on the pendulum disk and can be oscillated about the axis of the pendulum. It is observed that with a mass M of known moment of inertia I_1, a torsional frequency of oscillation f_1 is measured. If a body of unknown moment of inertia I_2 is substituted for the known mass, the frequency is observed to be f_2. The frequency of the pendulum alone, with no added weight, is f_0. Find the moment of inertia I_2 in terms of the known quantities.

7.61. A rigid wall of height h and width (ah) rests upon a horizontal surface. It is subjected to a uniform, constant, blast pressure which acts



PROB. 7.60



PROB. 7.61

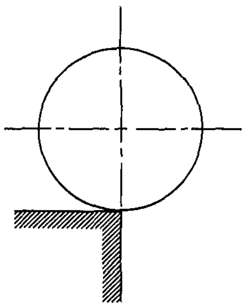
for a short time Δt. With Δt given, what value of p lb/ft^2 will cause the wall to overturn about point A, assuming that there is no sliding? The wall has a weight of W lb/ft of length. The time Δt is so small that it may be assumed that there is no motion of the wall during Δt, the action of p being only to impart an initial angular velocity to the wall.

7.62. A rectangular door of mass m is free to swing on two hinges. It is initially at rest when it is subjected to a uniform blast pressure from a bomb. The blast pressure acts for only a small fraction of a second but reaches a high maximum value of p lb/ft^2. What is the maximum dynamic hinge reaction? A 500-lb bomb detonating at a distance of 100 ft would produce a maximum blast pressure of 8 lb/in.^{2} If the door is 2.5 ft by 7 ft, what is the maximum dynamic hinge reaction?

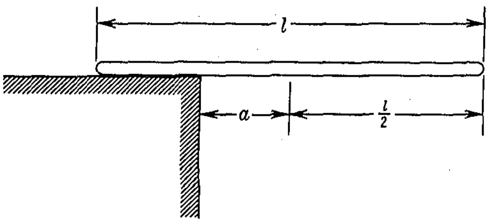
7.63. A homogeneous circular disk of mass M and radius r is mounted on a vertical shaft which is carried on frictionless bearings. The shaft coincides with the geometrical axis of the disk. An insect of mass m lands vertically on the periphery of the disk and crawls around the circumference with a uniform speed v relative to the disk. Find the motion of the disk.

7.64. A homogeneous uniform circular cylinder rests on the edge of a horizontal step as shown in the figure. The cylinder rolls off the step with a negligible initial velocity without sliding. Find the angle through which the cylinder rotates before leaving the step, and the angular velocity of the cylinder after it has rolled off. Assume that any effects of air resistance or of rolling friction may be neglected.

7.65. A straight uniform homogeneous bar of mass m and length l is placed on a horizontal table top with its center of mass a distance a from



PROB. 7.64



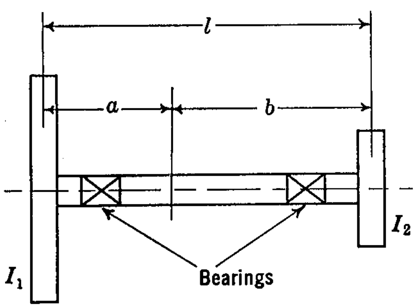
PROB. 7.65

the perpendicular edge as shown in the figure. The bar is released from rest from a horizontal position and begins to rotate about the edge of the table. If the coefficient of static friction for impending motion between the bar and the table edge is μ, find the angle with the horizontal that the bar attains before slipping begins.

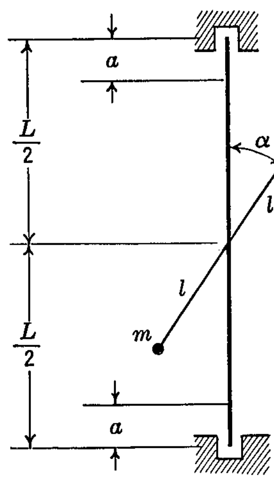
7.66. Two bodies of moment of inertia I_1 and I_2 about the axis of rotation are connected by a shaft as shown. If equal and opposite moments are applied to the bodies, the ends of the shaft will be twisted through a relative angular displacement θ, where $\theta = \frac{M}{k}$. If the moments are suddenly released, the two bodies will then perform torsional vibrations. Apply the principle of the conservation of moment of momentum to the system to show that the two bodies always rotate in opposite directions. From this it may be concluded that there is a certain cross section of the shaft which does not rotate during the oscillatory motion. The location of this cross section may be found by noting that, if the system is divided into two simple torsional pendulums of length a and b, the frequency of oscillation of the two must be equal. In this way show that $a = \frac{I_2 l}{I_1 + I_2}$ and that the frequency of vibration is:

$$f = \frac{1}{2\pi}\sqrt{\frac{k(I_1 + I_2)}{I_1 I_2}}$$

7.67. Two equal particles of mass m are fastened to the ends of a straight rod of length $2l$ and of negligible weight. The rod is attached to the center of a vertical shaft of length L as shown. If the vertical shaft



PROB. 7.66

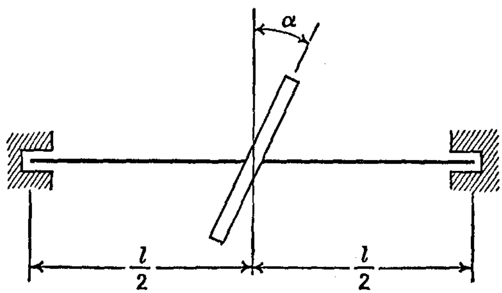


PROB. 7.67

rotates at a constant angular velocity ω, find the dynamic reactions at the bearings. The system is to be dynamically balanced by the addition of two concentrated weights of mass m_1. These weights are to be located in planes at a distance a from the bearings. Show where these weights should be attached and find the radius at which they should be located.

7.68. Show that if a rotating body is in static balance but not dynamic balance, and if the rotational speed is constant, the dynamic bearing reactions have the magnitude:

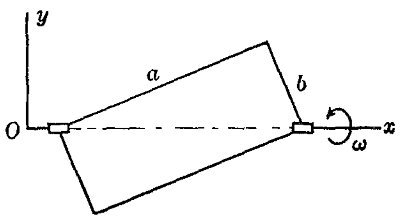
$$R = \frac{\omega^2}{l}\sqrt{I_{yz}^2 + I_{xz}^2}$$



PROB. 7.69

7.69. A thin circular disk of radius r is skewed a small angle α with respect to the axis of rotation, as shown. If the angular velocity of the system is a constant, find the dynamic reactions at the bearings. The total mass of the disk is m and the center of mass of the disk is on the axis of rotation. If the shaft is horizontal, find the total bearing reactions in the position shown in the diagram.

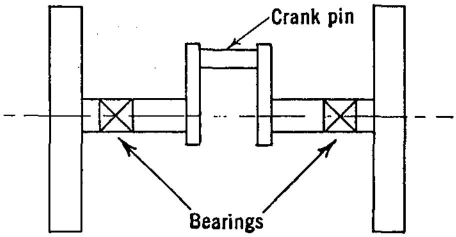
7.70. If in the preceding problem the rotating body is a solid cylinder of radius R and length h, find the dynamic bearing reactions. The center of mass of the cylinder is located on the axis of rotation, and the axis of the cylinder is inclined at an angle α with the axis of rotation. If the center of mass of the cylinder is located at a distance e from the axis of rotation, at the center of the shaft, in addition to the skew of the axis, what are the dynamic bearing reactions?



PROB. 7.71

7.71. A thin rectangular plate of mass m rotates about an axis coinciding with a diagonal of the plate. If the bearings are located at two corners of the plate as shown in the figure, find the dynamic bearing reactions.

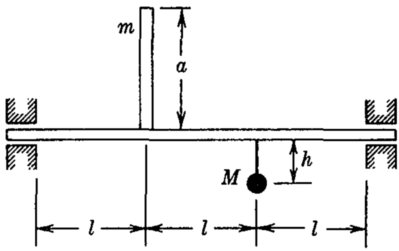
7.72. The diagram represents the two flywheels of a gas engine. The flywheels are 3 ft apart, and the center of the crank pin is located at a distance of 1 ft 4 in. from the left flywheel. The off center crank pin and crank arms are equivalent to a concentrated weight of 80 lb at a distance



PROB. 7.72

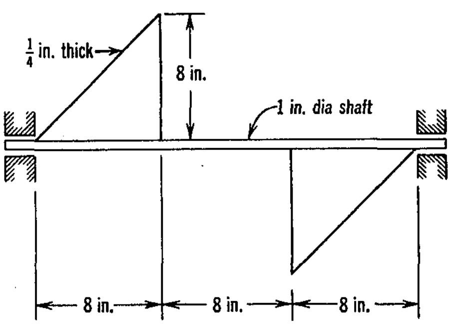
of 8 in. from the bearing centerline. The system is to be balanced by two weights in the planes of the flywheels. If these weights are to be located at a radial distance of 1 ft from the center of the flywheel, find their magnitudes.

7.73. A slender uniform rod of mass m and length a is attached at right angles to a rotating shaft of length $3l$ at a distance l from a bearing as shown in the figure. Lacking a dynamic balancing machine, a machine shop statically balances the shaft by attaching a concentrated mass m at a distance h from the axis, in the plane of the rod and shaft. Find the magnitude of the dynamic bearing reactions for a rotational speed ω and acceleration $\dot{\omega}$.



PROB. 7.73

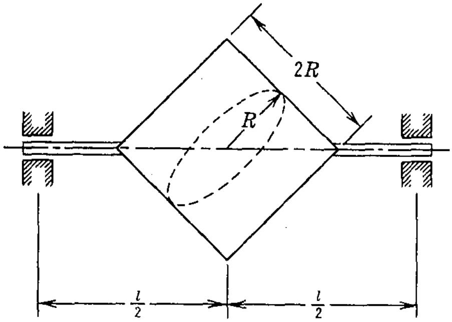
7.74. A certain unbalanced rotor may be idealized as two similar



PROB. 7.74

triangular flat steel plates lying in the same plane attached to a horizontal shaft as shown in the figure. The shaft rotates at a constant speed of 300 rpm.

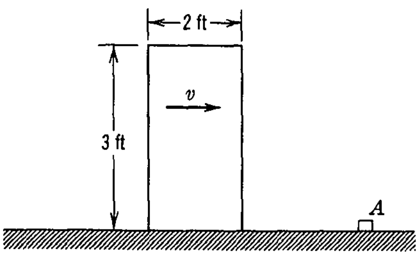
Find the magnitude of the dynamic bearing reactions, and compare with the magnitude of the static reactions.



PROB. 7.75

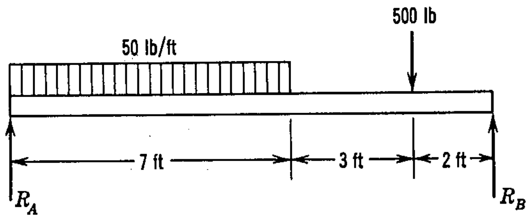
7.75. A homogeneous circular cylinder of mass M, radius R, and height $2R$ is rotated with a constant angular velocity ω about an axis passing through the center of mass and the edge of each circular face. The distance between the bearings is l. Find the magnitudes of the bearing reactions.

7.76. A uniform rectangular block of height 3 ft, width 2 ft and depth 3 ft travels along a frictionless horizontal surface with a velocity v. The block strikes a small projection on the surface which stops the block, but does not prevent the block from rotating freely about the striking edge, which remains in contact with the projection during the subsequent motion. Find the smallest value of approach velocity v for which the block will tip over and fall on the right side of A.



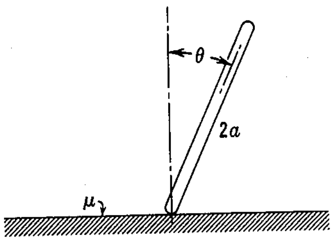
PROB. 7.76

7.77. A 12 ft long uniform horizontal beam weighing 25 lb/ft carries loads as shown in the figure. A uniformly distributed mass weighing 50 lb/ft extends over 7 ft of the length, and a concentrated load of 500 lb acts 2 ft from the other end of the beam. If the support at R_B is suddenly removed, find the reaction R_A, at the instant the beam is still horizontal.

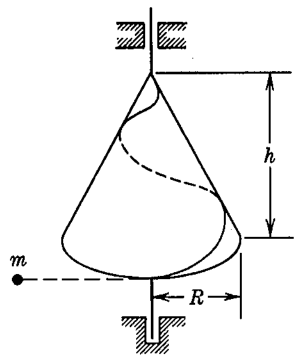


PROB. 7.77

7.78. A slender uniform straight bar of length $2a$ is originally at rest in a vertical position on a rough horizontal plane. The bar falls in a vertical plane under the action of gravity. Find the coefficient of static friction μ between the bar and the plane if the bar starts to slip when it reaches a slope $\theta = 20°$.



PROB. 7.78



PROB. 7.79

7.79. A homogeneous solid right circular cone of mass M, radius of base R, and altitude h, is free to rotate about its vertical geometric axis on frictionless bearings. A particle of mass m starts at the apex of the cone and slides down a frictionless groove in the surface of the cone and emerges horizontally tangent to the base circle as shown. If the cone and the particle are initially at rest, find the angular velocity of the cone after the particle has left the cone.

7.11 Plane Motion of a Rigid Body. If every element of a body moves parallel to a fixed plane, the body is said to have plane motion. If the xy plane is taken as the plane of motion, the angular velocity of the body is $\omega_z\boldsymbol{k}$ and the velocity of the mass center is $\dot{x}_c\boldsymbol{i} + \dot{y}_c\boldsymbol{j}$.

Let the xyz coordinate axes be fixed in the body with the origin at the center of mass. The general equation of motion $\boldsymbol{M}_c = \dot{\boldsymbol{H}}_c$ then has the components:

$$\begin{aligned} M_x &= I_{yz}\omega_z^2 - I_{xz}\dot{\omega}_z \\ M_y &= -I_{xz}\omega_z^2 - I_{yz}\dot{\omega}_z \\ M_z &= I_z\dot{\omega}_z \end{aligned} \tag{7.26}$$

In addition there are the independent equations for the motion of the mass center:

$$\begin{aligned} F_x &= m\ddot{x}_c \\ F_y &= m\ddot{y}_c \\ F_z &= 0 \end{aligned}$$

It will be noted that these six equations are identical with those

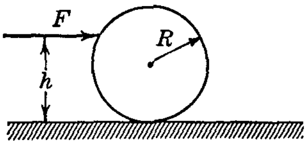


FIG. 7.18

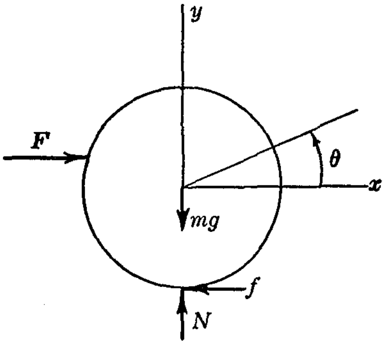


FIG. 7.19

obtained for the rotation of a body about a fixed axis. The origin of the coordinate system, however, is located at the center of mass for plane motion, whereas for rotation it is located on the fixed axis of rotation.

EXAMPLE 1. A circular cylinder of radius R and mass m is pushed along a horizontal plane by a horizontal force F at a distance h above the plane, as shown in Fig. 7.18. The coefficient of kinetic friction between the cylinder and the surface is μ. Describe the motion of the cylinder.

Solution. A free-body diagram of the cylinder is first drawn, as shown in Fig. 7.19. The coordinate x_c describes the linear position of the center of the cylinder, measured from a fixed point, and the coordinate θ describes the angular position. For the motion of the center of mass in the x-direction, we have the equation:

$$\Sigma F_x = m\ddot{x}_c = F - f$$

Taking moments about the z-axis, which passes through the center of mass, we obtain:

$$M_z = I_z\ddot{\theta} = -F(h - R) - fR$$

This gives two equations relating the three unknowns x_c, θ, and f. The additional equation to be used will depend upon whether or not there is slipping between the cylinder and the plane. If there is no slipping, we may write $R\ddot{\theta} = -\ddot{x}_c$, which, together with the first two equations, gives:

$$\ddot{x}_c = \frac{FhR}{I_z + mR^2}$$

$$f = F\left(1 - \frac{hRm}{I_z + mR^2}\right)$$

If the value of the friction force f computed in this way exceeds the value μW, then slipping will occur, and $R\ddot{\theta} \neq \ddot{x}_c$. Then the third equation to be used is $f = \mu W$ and the solution is:

$$\ddot{x}_c = \frac{F - \mu W}{m}$$

$$\ddot{\theta} = \frac{F(h - R) + \mu WR}{I_z}$$

EXAMPLE 2. A cylinder of mass m, radius R, and moment of inertia I about its geometric axis rolls without slipping down a hill under the action of gravity (Fig. 7.20). If the velocity of the center of mass of the cylinder is initially v_0, find the velocity after the cylinder has dropped through a vertical distance h.

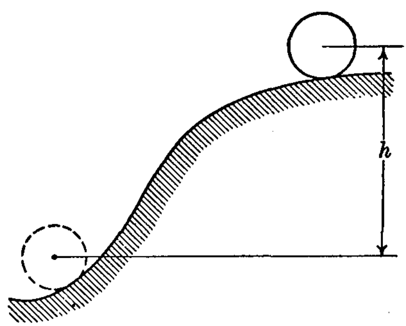


FIG. 7.20

Solution. Since there is no energy loss during the motion of the cylinder, we may write the equation of the conservation of energy, which will lead at once to the desired result. Taking the final position of the cylinder as the point of zero potential energy, we have:

$$\frac{1}{2}mv_0^2 + \frac{1}{2}I\left(\frac{v_0}{R}\right)^2 + mgh = \frac{1}{2}mv_h^2 + \frac{1}{2}I\left(\frac{v_h}{R}\right)^2$$

from which:

$$v_h = \sqrt{v_0^2 + \frac{2mgh}{\left(m + \frac{I}{R^2}\right)}}$$

EXAMPLE 3. Two uniform, homogeneous, circular disks of radius

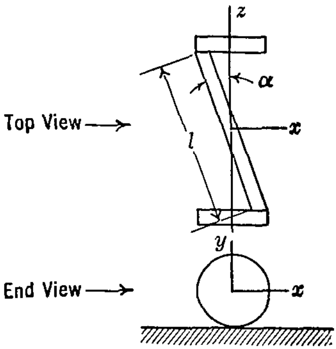


FIG. 7.21

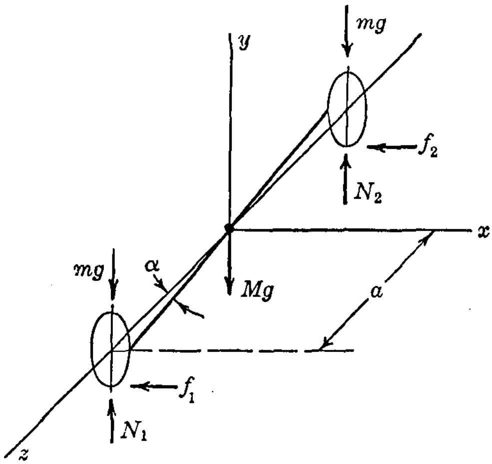


FIG. 7.22

R and mass m are connected by a uniform straight bar of length l (Fig. 7.21). The mass of the straight bar is M. The assembly rolls without slipping along a horizontal plane, the center of the disk having a constant velocity v parallel to the plane. Find the forces exerted on the disks by the plane when the straight bar is parallel to the plane.

Solution. We consider an xyz coordinate system which is attached to the body and whose origin is located at the center of mass. The free-body diagram is shown in Fig. 7.22, where the frictional forces,

f_1 and f_2, and the normal forces, N_1 and N_2 are indicated. The equations of motion for the system become:

$$\Sigma F_x = m\ddot{x}_c = 0 = -f_1 - f_2$$
$$\Sigma F_y = m\ddot{y}_c = 0 = N_1 + N_2 - mg - mg - Mg$$
$$M_x = I_{yz}\omega_z^2 = mga - N_1 a - mga + N_2 a = I_{yz}\left(\frac{v}{R}\right)^2$$
$$M_y = -I_{xz}\omega_z^2 = f_2 a - f_1 a = -I_{xz}\left(\frac{v}{R}\right)^2$$
$$M_z = I_{zz}\dot{\omega}_z = 0 = -(f_1 R + f_2 R)$$

Evaluating the products of inertia gives:

$$I_{yz} = 0$$
$$I_{xz} = \frac{Ml^2}{12}\sin\alpha\cos\alpha$$

From the first or fifth equation, $f_1 + f_2 = 0$, and from the fourth equation:

$$f_1 - f_2 = \frac{Ml^2v^2}{12aR^2}\sin\alpha\cos\alpha$$

Thus:

$$f_1 = -f_2 = \frac{Ml^2v^2}{24aR^2}\sin\alpha\cos\alpha$$

Since:

$$a = \sqrt{\left(\frac{l}{2}\right)^2 - R^2}\,;\ \sin\alpha = 2\frac{R}{l}$$
$$\cos\alpha = \frac{2}{l}\sqrt{\left(\frac{l}{2}\right)^2 - R^2}$$

we have:

$$f_1 = -f_2 = \frac{Mv^2}{6R}$$

From the second equation:

$$N_1 + N_2 = 2mg + Mg$$

From the third equation:

$$N_1 - N_2 = 0$$

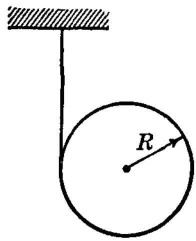
hence:

$$N_1 = N_2 = mg + \frac{Mg}{2}$$

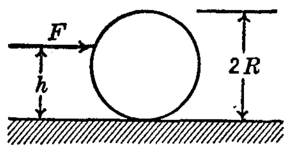
PROBLEMS

7.80. A uniform circular cylinder of weight W and radius R starts from rest and rolls without slipping under the action of gravity down a plane which makes an angle α with the horizontal. Find the acceleration of the cylinder. If the coefficient of friction between the cylinder and the plane is μ, find the maximum angle of inclination of the plane for which the cylinder will roll without slipping.

7.81. A uniform circular cylinder of weight W and radius R has a rope wrapped around it, one end of which is fixed as shown. The system is released from rest with the rope in a vertical position. Describe the subsequent motion of the system and find the force in the rope. The rope is in the plane of the mass center.



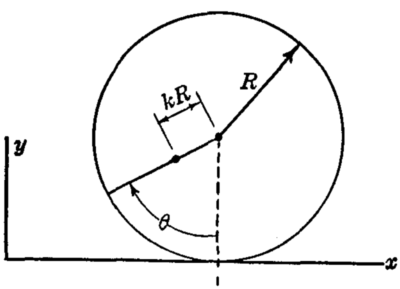
PROB. 7.81



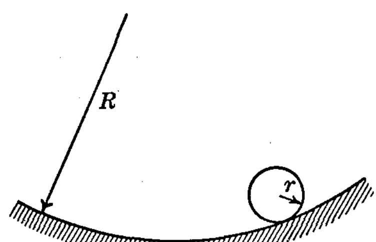
PROB. 7.82

7.82. At what point should a billiard ball be struck with a horizontal impact in order that it will roll without sliding on a frictionless table surface?

7.83. A wheel of weight W is unbalanced so that its center of mass lies at a distance kR from the center of the wheel. The wheel rolls without slipping with a constant velocity v. Determine the normal force exerted by the wheel against the ground.



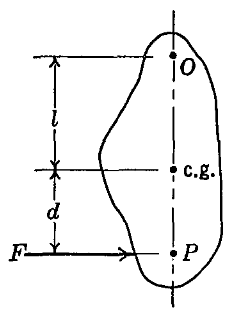
PROB. 7.83



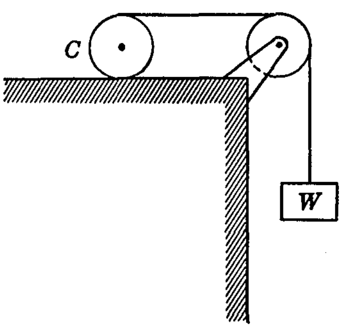
PROB. 7.84

7.84. A sphere of radius r and mass m rolls on a circular surface of radius R under the action of gravity. Find the differential equation describing small oscillations of the systems about the position of equilibrium and show how this problem differs from that of the particle of mass m which slides on the surface.

7.85. A body of mass m and moment of inertia I_c about the center of mass is initially at rest when it is given an impulse $F\Delta t$ as shown. The body moves in a horizontal plane, and the force F is horizontal. Find the distance l from the center of mass to the point O, whose instantaneous acceleration $\ddot{x}_0$ is zero at time $t = 0$. Since the point O has a zero acceleration at $t = 0$, this point could be mounted on an axis of rotation without involving any reaction during the impact. The point P is called the center of percussion corresponding to O. Show that the point O is the center of percussion corresponding to P. Show that the period of vibration of the body as a compound pendulum acted upon by gravity is the same whether O or P is the axis of rotation. Such a compound pendulum is called Kater's Reversible Pendulum.



PROB. 7.85



PROB. 7.86

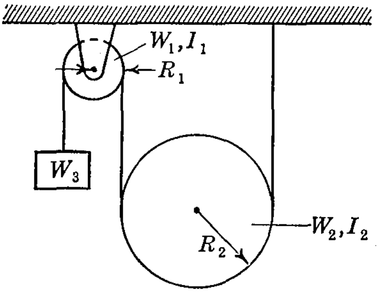
7.86. A circular cylinder having a radius of 1 ft and a weight of 100 lb rolls without slipping along a horizontal surface. A rope wound around the cylinder passes over a frictionless pulley and supports a weight of 200 lb which moves vertically as shown in the figure. If the 200 lb weight is released from rest, find the velocity of the system at the end of 3 sec. Do this first by drawing a separate free-body diagram for each mass, in this way determining the acceleration of the system. Check the answer by applying energy principles to the whole system.

7.87. A fixed pulley, a moving pulley, and a weight which can move vertically are assembled as shown. The sections of rope between the pulleys are vertical, and the frictional forces in the pulleys are assumed to be negligible. Find the equation of motion of W_2 in terms of accelerations by differentiating the energy equation for the system.

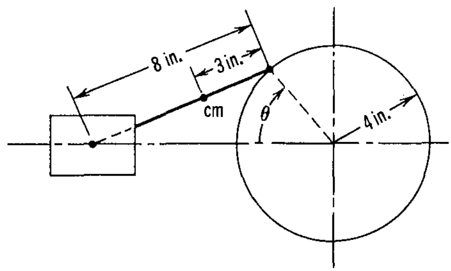
7.88. Two identical solid spheres of mass m and radius r are free to move on a horizontal surface. If one sphere is at rest and the other sphere makes an impact with it at a velocity v, describe the resulting motion of the system. Assume that no slipping occurs between the spheres and the surface, and assume that no energy is lost during the process. The impact is a direct central impact, the direction of rebound being the same as the direction of approach.

7.89. In Example 3, Section 7.11, suppose that the system has rotated through 90°, so that the inclined bar lies in a plane which is perpendicular to the plane on which the disks roll. Solve for the forces on the disks.

7.90. In Example 3, Section 7.11, suppose that the centers of the disks have an acceleration $\ddot{x}$ parallel to the plane as well as a velocity $\dot{x}$. Find the forces on the disks.



PROB. 7.87

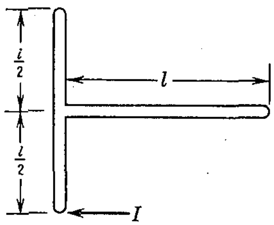


PROB. 7.91

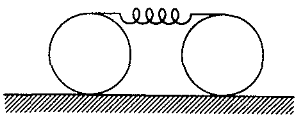
7.91. A crank and connecting rod mechanism has the dimensions shown in the diagram. The piston weighs 20 lb, the connecting rod 15 lb, and the center of mass of the connecting rod is located 3 in. from the crank-pin. The moment of inertia of the connecting rod about its center of mass is 0.2 lb ft sec^2. The crank is rotating at a constant speed of 1200 rpm. Find all the forces acting on the connecting rod at the instant when the crank angle $\theta = 30°$. Neglect friction and gravity.

7.92. Two identical straight uniform slender bars of length l and mass m are rigidly joined as shown in the figure. The system is initially at rest on a smooth horizontal table. An impulse of magnitude I is applied perpendicular to one of the bars as shown. Find the kinetic energy of the system after the impulse has been applied.

7.93. Two homogeneous circular cylinders of equal weight $W = 100$ lb and equal diameter $D = 2$ ft are connected by means of a massless spring of constant $k = 20$ lb/in. which is fastened to two ropes wrapped around the cylinders as shown in the diagram. The cylinders roll without slipping



PROB. 7.92

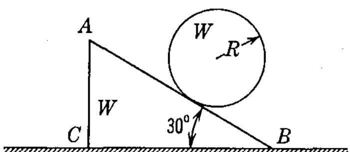


PROB. 7.93

on the horizontal plane, and the ropes do not slip on the cylinders. The two cylinders are rotated in opposite directions until the spring is stretched to a force of 100 lb, and the cylinders are then released from rest. Find the maximum velocity of the center of mass of the cylinders.

7.94. A slender uniform bar of mass m and length l is at rest in a vertical position on a frictionless horizontal plane. The bar falls in a vertical plane under the action of gravity. Find the velocity of the center of mass of the bar at the instant the bar becomes horizontal.

7.95. A 30° triangular prism ABC of weight W is placed on a smooth horizontal plane. A solid homogeneous circular cylinder of equal weight W and of radius R rolls down the face AB without slipping. Find the acceleration of the prism.



PROB. 7.95

7.96. A uniform solid circular cylinder of mass M and radius R spinning about its own horizontal axis with an angular velocity ω_0 is placed on a horizontal table. If the coefficient of sliding friction between the cylinder and the table is μ, determine the subsequent motion of the cylinder. Find the fraction of the original kinetic energy which is dissipated during the motion.

7.12 Rotation About a Fixed Point. The motion of a body rotating about a fixed point is somewhat more complex than the motions hitherto considered. The spinning top and the gyroscope are examples of such motion, in which a rigid body rotates about an axis which is itself rotating. We shall first give an analytical treatment which will be suitable for problems of any degree of generality, and we will show by means of examples how vector methods can be used to simplify certain special types of problems.

We first require a convenient coordinate system for the description of the three-dimensional motions involved. Consider the spinning top of Fig. 7.23, which rotates about its geometric axis Oz at the same time that the geometric axis itself rotates about the vertical axis OZ. Three coordinates will be required to describe the position of the top. Perhaps the most natural coordinates will be the angle ϕ, which describes the rotation of the top about its geometric axis Oz, the angle θ which describes the inclination of this geometric axis from the vertical OZ axis, and the angle ψ, which is the angle in the horizontal plane that locates the vertical plane containing the geometric axis.

In practice, the analysis can be most easily carried out in a

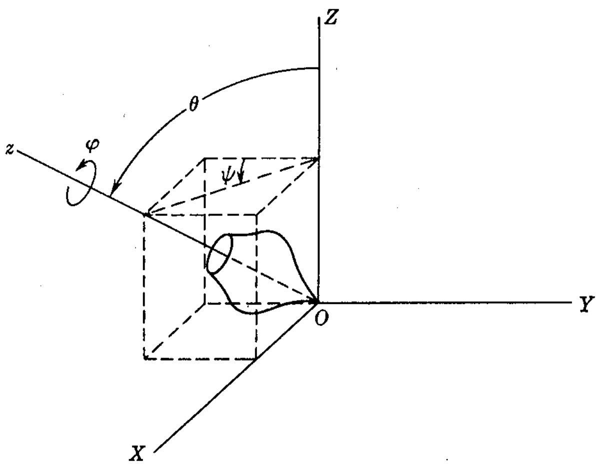


FIG. 7.23

coordinate system which is equivalent to this ϕ, θ, ψ system. This may be explained by the introduction of a new x, y, z coordinate system as shown in Fig. 7.24. XYZ is the same fixed coordinate system as in Fig. 7.23, and the origin O is the fixed point about which motion occurs. The z-axis is the spin axis of the body, and coincides with the geometric axis of the top of Fig. 7.23. The x-axis lies along the intersection of a plane perpendicular to the spin axis Oz and the horizontal XY plane, and hence is always in a horizontal plane. The y-axis is orthogonal to the x, z axes. Note that the xyz system is neither fixed in space nor attached to the body. The body itself can be defined by the $x'y'z'$ system which is attached to the body. z' coincides with z, but the x' and y' are rotated about the z-axis by the angle ϕ. The three angles defining the position of the body can thus be given as θ, ϕ, and ψ, where ψ describes the position of the horizontal x-axis as measured from the fixed OX axis. Comparing Figs. 7.23 and 7.24 it will be noted that the angles θ, ϕ, and ψ are exactly the same in the two coordinate systems. The angles θ, ϕ, and ψ are called *Euler's Angles* and their use, together with the x, y, z coordinate system as defined above, is especially convenient for the discussion of the more complex problems of rotation about a fixed point.

We shall now write the general equations of motion in terms of

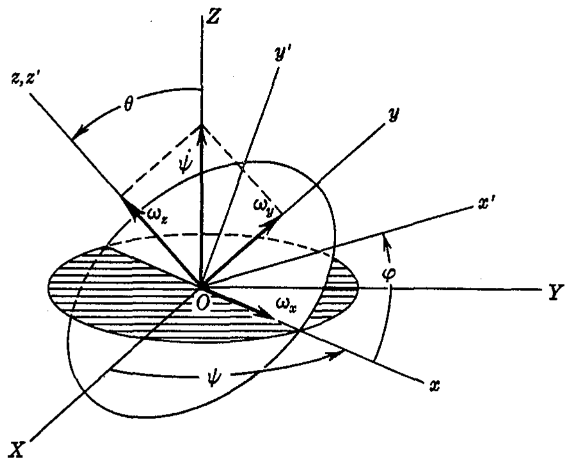


FIG. 7.24

Euler's Angles. Taking $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ as unit vectors in the x, y, z directions, and ω_x, ω_y, and ω_z as the angular velocity components of the x, y, z coordinate system, we have for the angular velocity $\boldsymbol{\omega}$ of the x, y, z coordinate system:

$$\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k} = \dot{\theta}\boldsymbol{i} + \dot{\psi}\sin\theta\boldsymbol{j} + \dot{\psi}\cos\theta\boldsymbol{k}$$

Since the body has an angular velocity of spin $\dot{\phi}$ with respect to the x, y, z system, the total angular velocity of the body is:

$$\boldsymbol{\omega}_b = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + (\omega_z + \dot{\phi})\boldsymbol{k}$$

The equations of motion are obtained in the usual manner by writing the equation of moment of momentum, $\boldsymbol{M} = \dot{\boldsymbol{H}}$, about the fixed point O. As has been shown, this equation, in the rotating x, y, z system is:

$$\begin{aligned} M_x &= \dot{H}_x - H_y\omega_z + H_z\omega_y \\ M_y &= \dot{H}_y - H_z\omega_x + H_x\omega_z \\ M_z &= \dot{H}_z - H_x\omega_y + H_y\omega_x \end{aligned}$$

The ω_x, ω_y, ω_z in these equations are the angular velocity components of the coordinate system, as defined above. H_x, H_y, and H_z must be expressed in terms of the angular velocity components of the body, so:

$$\begin{aligned} H_x &= + I_x\omega_x - I_{xy}\omega_y - I_{xz}(\omega_z + \dot{\phi}) \\ H_y &= - I_{xy}\omega_x + I_y\omega_y - I_{yz}(\omega_z + \dot{\phi}) \\ H_z &= - I_{xz}\omega_x - I_{yz}\omega_y + I_z(\omega_z + \dot{\phi}) \end{aligned} \tag{7.27}$$

The general equations of motion which are obtained by substituting H_x, H_y, H_z into the above moment equations are rather complex. We shall not investigate these general equations further, since the special applications which we shall now consider will permit considerable simplification.

7.13 The Symmetrical Top and the Gyroscope. Consider a body symmetrical about the z-axis and mounted so that it is free to move about a fixed point O as shown in Fig. 7.25. Because of symmetry $I_{xy} = I_{xz} = I_{yz} = 0$, and $I_x = I_y = I$, where I is constant even though the body is rotating with respect to the x, y, z axes.

The components of $\boldsymbol{H}$ are:

$$H_x = I\omega_x; \quad H_y = I\omega_y; \quad H_z = I_z(\omega_z + \dot{\phi})$$

Substituting these components into the general moment equations gives:

$$\begin{aligned} M_x &= I(\dot{\omega}_x - \omega_y\omega_z) + I_z\omega_y(\omega_z + \dot{\phi}) \\ M_y &= I(\dot{\omega}_y + \omega_x\omega_z) - I_z\omega_x(\omega_z + \dot{\phi}) \\ M_z &= I_z(\dot{\omega}_z + \ddot{\phi}) \end{aligned} \tag{7.28}$$

Substituting the expressions for the angular velocity components in terms of Euler's Angles, we obtain:

$$\begin{aligned} M_x &= I(\ddot{\theta} - \dot{\psi}^2 \sin\theta\cos\theta) + I_z\dot{\psi}\sin\theta(\dot{\psi}\cos\theta + \dot{\phi}) \\ M_y &= I(\ddot{\psi}\sin\theta + 2\dot{\psi}\dot{\theta}\cos\theta) - I_z\dot{\theta}(\dot{\psi}\cos\theta + \dot{\phi}) \\ M_z &= I_z(\ddot{\phi} + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta) \end{aligned} \tag{7.29}$$

The solution of this set of differential equations subject to the initial conditions will give a complete description of the motion of the system. Unfortunately, even for the above symmetrical case, general solutions of the non-linear differential equations cannot be obtained, although this is a classical problem which has been intensively studied. It is thus necessary to consider special types of problems which represent still further simplifications. One type of problem that can be solved is the determination of the applied moments that would correspond to certain steady motions of the system, as will be illustrated in the examples to follow. It is also

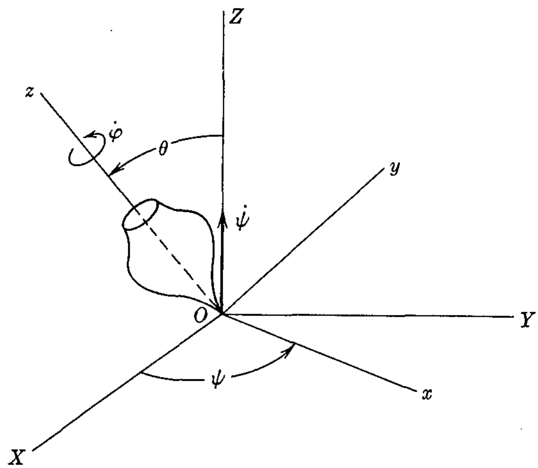


FIG. 7.25

possible to make an analytical study of the stability of such assumed steady motions, and hence to justify the assumptions.

An examination of Equations (7.28) shows first the usual terms involving the angular velocities and the angular accelerations that would be present even if the body were not spinning about its geometric axis. In addition, we find the terms $(I_z\omega_y\dot{\phi})$ and $(-I_z\omega_x\dot{\phi})$ which are consequences of the spin of the body about its axis. The moments associated with these spin terms are called *gyroscopic moments*, and devices in which such terms play a prominent role are called gyroscopes. Note that the gyroscopic moment about the x-axis is associated with an angular velocity about the y-axis, and vice-versa. The fact that an angular velocity is produced at right angles to the external moment which causes it is the characteristic peculiarity of the gyroscope. This motion of the spin axis in a direction perpendicular to the applied moment is called *precession.*

A gyroscope is often mounted in gimbals so that all rotation is

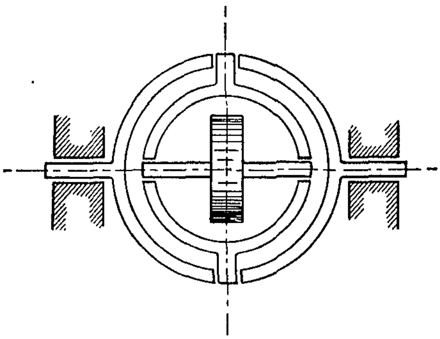


FIG. 7.26

about the center of mass (Fig. 7.26). Since the equation $\dot{\boldsymbol{H}} = \boldsymbol{M}$ may be written with respect to the center of mass, we see that the preceding equations are applicable to a gyroscope mounted in gimbals even though the center of mass is moving.

EXAMPLE 1. A circular disk of moment of inertia I about its geometrical axis rotates about that axis with an angular velocity Ω with respect to the axis. At the same time the axis itself rotates in

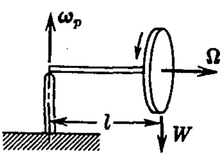


FIG. 7.27

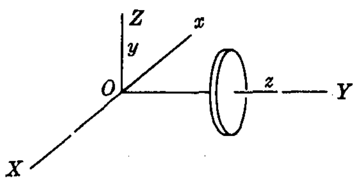


FIG. 7.28

a horizontal plane with a precessional angular velocity $\boldsymbol{\omega}_p$. A gravity moment of magnitude Wl acts on the system as shown in Fig. 7.27. Find the precessional velocity $\boldsymbol{\omega}_p$ compatible with this steady motion of the system.

Solution—Method 1. Using the analytical method we may go directly to Equations (7.29) and the coordinate system of Fig. 7.28. We have, in the notation of Equations (7.29):

$$\theta = 90°; \quad \dot{\theta} = \ddot{\theta} = 0; \quad \dot{\psi} = \omega_p; \quad \dot{\phi} = \Omega$$
$$M_x = Wl; \quad M_y = M_z = 0; \quad I_z = I$$

so, from the first equation:

$$Wl = I\omega_p\Omega$$

or:

$$\omega_p = \frac{Wl}{I\Omega} \quad \text{(counterclockwise viewed from top).}$$

Method 2. The simplicity of this particular problem makes possible a very simple vector solution, as shown in Fig. 7.29. Since

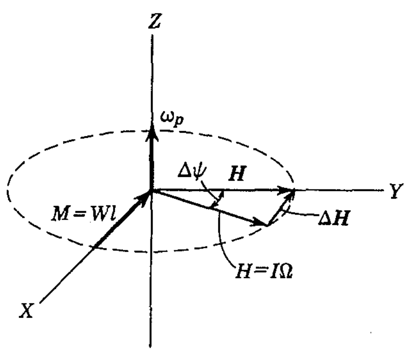


FIG. 7.29

$\boldsymbol{M} = \dot{\boldsymbol{H}}$, there must be a $\Delta \boldsymbol{H}$ in the direction of $\boldsymbol{M}$. If $H = I\Omega =$ constant, this $\Delta \boldsymbol{H}$ can only come about by a rotation of the whole axis as shown, in a direction such that $\Delta \boldsymbol{H}$ has the same direction as $\boldsymbol{M}$. If the axis moves through an angle $\Delta\psi$ in the horizontal XY plane in a time Δt, we have:

$$\Delta H = H\Delta\psi = I\Omega\Delta\psi$$

so:

$$\frac{\Delta H}{\Delta t} = I\Omega\frac{\Delta\psi}{\Delta t}, \quad \text{or} \quad \dot{H} = I\Omega\omega_p$$

Thus, $Wl = I\Omega\omega_p$, and $\omega_p = \dfrac{Wl}{I\Omega}$ as before. Note that in this particular problem the moment of momentum component associated with the precession is always perpendicular to the spin axis and hence does not enter into the moment equation. This would not in general be true, and is only a consequence of the fact that in this particular problem $\theta = 90°$.

EXAMPLE 2. The geometrical axis of a right circular cone of half-angle α and height h makes a constant angle θ with the vertical, and rotates about the vertical axis through the vertex of the cone with a uniform angular velocity ω_p. At the same time, the cone spins about its geometric axis with a constant angular velocity ω, relative to the moving axis, as shown in Fig. 7.30. For given constant values of θ and ω, determine the precessional angular velocity ω_p.

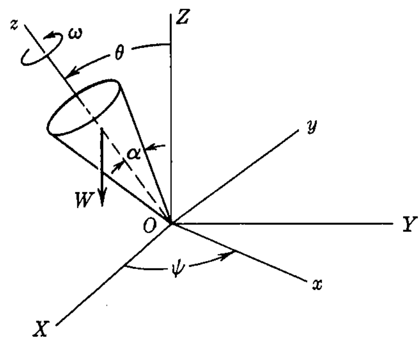


FIG. 7.30

Solution—Method 1. We shall first solve the problem by a direct application of Equations (7.29). Referring to Fig. 7.30 we note that for this particular problem we have:

$$\dot{\theta} = \ddot{\theta} = 0; \quad \dot{\phi} = \omega; \quad \dot{\psi} = \omega_p$$

Thus the M_x equation becomes:

$$M_x = -I\omega_p^2 \sin\theta\cos\theta + I_z\omega_p \sin\theta(\omega_p\cos\theta + \omega)$$

Also, $M_x = (\frac{3}{4}hW\sin\theta)$, so we have:

$$\tfrac{3}{4}Wh\sin\theta = -I\omega_p^2\sin\theta\cos\theta + I_z\omega_p^2\sin\theta\cos\theta + I_z\omega\omega_p\sin\theta$$

$$\omega_p^2 + \frac{I_z\omega}{(I_z - I)\cos\theta}\,\omega_p - \frac{3Wh}{4(I_z - I)\cos\theta} = 0$$

from which:

$$\omega_p = -\frac{I_z\omega}{2(I_z - I)\cos\theta} \pm \sqrt{\left[\frac{I_z\omega}{2(I_z - I)\cos\theta}\right]^2 + \frac{3Wh}{4(I_z - I)\cos\theta}}$$

For the cone $I > I_z$, so we write:

$$\omega_p = \frac{I_z\omega}{2(I - I_z)\cos\theta} \pm \sqrt{\left[\frac{I_z\omega}{2(I - I_z)\cos\theta}\right]^2 - \frac{3Wh}{4(I - I_z)\cos\theta}}$$

Note that this expression gives two positive real values for the precessional angular velocity ω_p. There are thus two precessional speeds that will satisfy the conditions of the problem. These are called the slow and the fast precessional motions. It is the slow precession which is observed in most simple experiments, and it would require an analysis of the stability of the assumed motions to complete the discussion of such problems.

If it is assumed that the angular velocity of precession is small compared to the spin velocity, a simplified solution to the problem can be obtained. The moment equation obtained above can be re-written in the form:

$$\frac{3}{4}\frac{Wh}{\omega^2} = (I_z - I)\left(\frac{\omega_p}{\omega}\right)^2 \cos\theta + I_z\left(\frac{\omega_p}{\omega}\right)$$

if $\left(\frac{\omega_p}{\omega}\right) << 1$, the $\left(\frac{\omega_p}{\omega}\right)^2$ term can be dropped, and we obtain:

$$\frac{3}{4}\frac{Wh}{\omega^2} = I_z\left(\frac{\omega_p}{\omega}\right) \quad \text{or} \quad \omega_p = \frac{3Wh}{4I_z\omega}$$

Note that as the spin velocity increases the precessional velocity decreases. Since most gyroscopes have a very high spin velocity, the assumption that $\omega_p << \omega$ is usually a good one.

To compare the result of the approximate analysis with the exact solution, expand the square root term in the exact answer for ω_p by the bionomial theorem:

$$\omega_p = \frac{I_z\omega}{2(I - I_z)\cos\theta}$$
$$\pm \frac{I_z\omega}{2(I - I_z)\cos\theta}\left\{1 - \frac{1}{2}\left[\frac{3Wh}{4(I - I_z)\cos\theta}\right]\left[\frac{2(I - I_z)\cos\theta}{I_z\omega}\right]^2 + \cdots\right\}$$

taking the negative sign and dropping higher order terms in the expansion, we obtain:

$$\omega_p = \frac{3Wh}{4I_z\omega} \text{ as above.}$$

The approximate solution thus gives the lower precessional speed.

Method 2. The same results can also be obtained by a direct application of the equation $\boldsymbol{M} = \dot{\boldsymbol{H}}$ using the moment of momentum

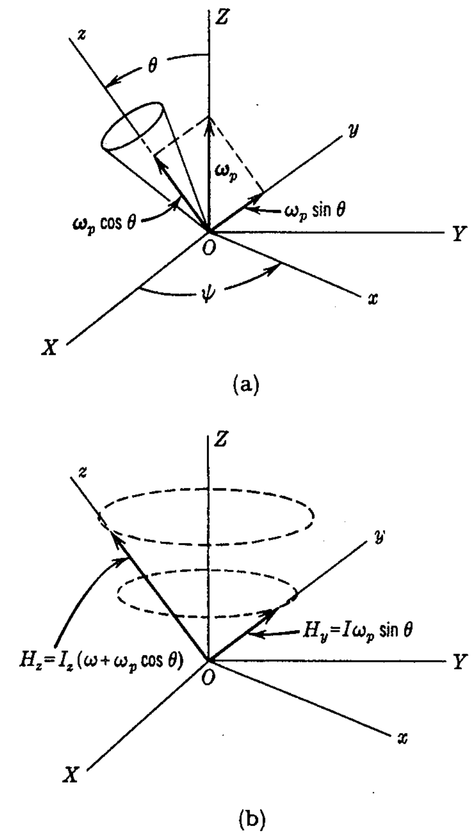


FIG. 7.31

vector, as in Example 1, Method 2 above. In Fig. 7.31 (a) the precessional angular velocity ω_p has been resolved into components along the y and z axes, and in Fig. 7.31 (b) the components of the moment of momentum of the cone along the y and z axes are shown. As the cone precesses, these $\boldsymbol{H}$ components rotate as shown, and have increment changes in the x-direction. Then from $\boldsymbol{M} = \dot{\boldsymbol{H}}$ we obtain:

$$M_x = [I_z(\omega + \omega_p \cos\theta)] \sin\theta\, \omega_p - (I\omega_p \sin\theta) \cos\theta \omega_p = \tfrac{3}{4}Wh \sin\theta$$

or:

$$\omega_p{}^2 + \frac{I_z\omega}{(I_z - I)\cos\theta}\,\omega_p - \frac{3Wh}{4(I_z - I)\cos\theta} = 0$$

which is the same equation that was derived above by Method 1.

7.14 The Gyroscopic Compass. Consider a gyroscope that is mounted at the earth's surface in such a way that it is free to turn in any direction (Fig. 7.26). If no moment is applied, the axis of the gyroscope will maintain a fixed direction in space so that as the earth rotates about its axis the gyroscope axis will rotate relative to the earth. This is illustrated in Fig. 7.32 (a), in which we are looking due south at a gyroscope which is mounted on the equator. The direction of spin of the gyro rotor is indicated by the vector. If now a small weight is attached to the spin axis below the center of mass, a moment is impressed upon the gyroscope by gravity as indicated in Fig. 7.32 (b). This torque, whose vector direction is parallel to

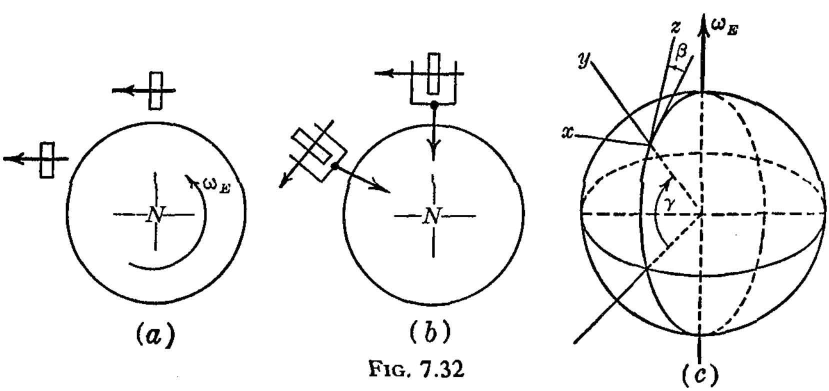


FIG. 7.32

the earth's axis, causes the spin axis to rotate toward the earth's axis. A device of this type will, therefore, point to true north and can be used as a compass. The preceding discussion gives a qualitative indication of the behavior of the gyrocompass, and we shall now show in more detail how the performance can be predicted from the equations of motion.

The effect of the pendulous mass is to constrain the spin axis of the gyroscope to move in a horizontal plane. We therefore take the xz plane, Fig. 7.32 (c), as the horizontal plane containing the spin axis, and the z-axis as the direction of spin of the rotating gyro disk. The location of the spin axis with respect to a meridian is given by the angle β, and γ is the latitude of the gyroscope, measured from the equator. The angular velocity of the earth is ω_E. The angular velocity of the xyz axes is then:

$$\begin{aligned}\omega_x &= -\ \omega_E \cos\gamma \sin\beta \\ \omega_y &= \omega_E \sin\gamma + \dot\beta \\ \omega_z &= \omega_E \cos\gamma \cos\beta\end{aligned}$$

To determine the motion of the spin axis in the horizontal plane with respect to the meridian, we use the general equation for the motion about the y-axis (Equation 7.28):

$$M_y = I\dot\omega_y + I\omega_x\omega_z - I_z\omega_x(\omega_z + \dot\phi)$$

putting $(\omega_z + \dot\phi) = \Omega$, which we may think of as the total spin velocity of the gyroscope, the equation becomes:

$$O = I\ddot\beta - I\omega_E{}^2 \cos^2\gamma \sin\beta\cos\beta + I_z\Omega\omega_E \cos\gamma \sin\beta$$

Since the angular velocity of the earth ω_E is very small compared to the spin velocity Ω, the term containing $\omega_E{}^2$ may be neglected, and the equation of motion may be written:

$$\ddot\beta + \left(\frac{I_z}{I}\Omega\omega_E \cos\gamma\right)\sin\beta = 0$$

For small oscillations of the spin axis about the meridian we may put $\sin\beta \approx \beta$ and the equation becomes:

$$\ddot\beta + \left(\frac{I_z}{I}\Omega\omega_E \cos\gamma\right)\beta = 0$$

This is the equation of simple harmonic motion, from which it may be concluded that the spin axis oscillates about the meridian with a period

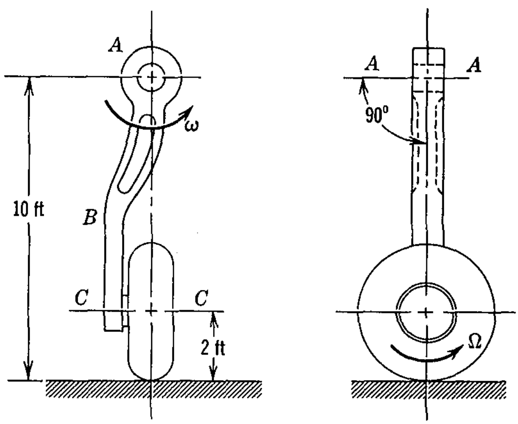
$$\tau = 2\pi\sqrt{\frac{I}{I_z\Omega\omega_E \cos\gamma}}$$

In practical applications of the gyrocompass, sufficient damping is introduced so that the oscillations are damped out with the spin axis finally lined up with the meridian.

The preceding discussion of gyroscopic motion illustrates a common procedure in the solution of dynamics problems. A completely general solution of such problems, which, starting from the general equations would consider all possible motions of the system, often leads to very complex analysis. Since for particular practical problems we are concerned with very special conditions, such as large spin velocities, we make use of these special conditions to simplify the equations of motion at the outset. It must always be realized, however, that such analyses are approximate and are applicable only when their conditions are satisfied.

PROBLEMS

7.97. The figure shows an airplane landing gear assembly. After the airplane takes off, each landing gear is retracted into the wing by rotation about the axis A–A. If the wheel continues to spin as it is being retracted, show that the gyroscopic effect causes a torque in the landing gear strut

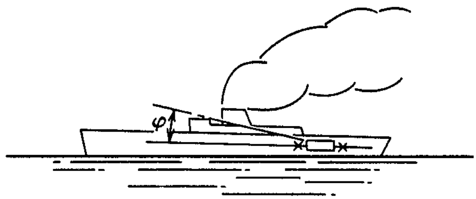


Prob. 7.97

A–B. Indicate clearly by means of a sketch the direction of twist of the landing gear strut caused by this gyroscopic effect. If the weight of the wheel is 100 lb, the radius is 2 ft, the radius of gyration about the axis C–C is 1.5 ft, the take-off speed is 120 mph, and the maximum speed of retraction ω is 3 radians per second, compute the torque in the strut A–B due to the gyroscopic effect of the spinning wheel.

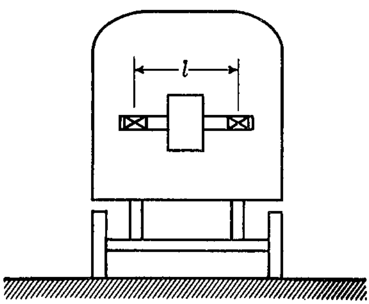
7.98. A "turboprop" plane has a gas turbine and compressor unit with 16 stages. Each turbine or compressor stage can be treated as a solid disk with a 24 in. diameter and weighing 30 lb. The speed of rotation is 12,000 rpm, clockwise viewed from the rear. The propeller has a speed of rotation of 1200 rpm and a moment of inertia of 150 lb ft sec^2. The plane makes a 2g turn to the left at a constant speed of 500 mph. What is the difference in the torque that must be applied to the plane in order to maintain horizontal flight if the propeller rotates in the same direction as the turbine or counter to the turbine?

7.99. A 5000 rpm turbine is mounted in a ship with its axis fore-and-aft as shown in the diagram. The ship pitches about a horizontal axis normal to the fore-and-aft axis with the motion $\phi = \phi_0 \sin \omega t$ where the amplitude of the motion is 10° and the period is 10 sec. The rotor of the turbine weighs 500 lb and may be considered as equivalent to a homogeneous cylinder 2 ft in diameter. The distance between the bearings is 4 ft. Find the maximum magnitude of the bearing reactions due to gyroscopic action.



PROB. 7.99

7.100. The rotor of an electric motor is mounted in an electric locomotive as shown in the diagram. The locomotive travels with a velocity v around a curve of radius R. Find the gyroscopic forces exerted on the bearings of the rotor shaft.



PROB. 7.100

7.101. When an automobile is rounding a curve at high speed, does the gyroscopic effect of the wheels tend to stabilize the car or overturn it?

7.102. The gyroscope in an inertial guidance system has an angular momentum of 10^6 cgs units. In order that the guidance system should hold course within 1 mile in 10 hours of operating time, the drift rate of the gyroscope must be kept below 0.0015 degrees per hour. Find the maximum permissible gimbal bearing friction torque in foot-pounds.

7.15 General Motion in Space. Rolling of a Disk. In the preceding sections we have shown that for certain motions of a restricted type, such as plane motion, or motion about a fixed point, the general equations of motion can be reduced to relatively simple forms. In the present section we shall illustrate, by means of the particular example of a disk rolling on a horizontal plane, the solution of more complex systems by application of the general equations of motion.

A thin uniform circular disk of radius a and mass m rolls without slipping on a horizontal plane. We are to write the equations of motion required to give a complete description of all possible motions of the disk.

We can describe the location and configuration of the system by giving the two coordinates of the point of contact of the disk and the plane, and the angles θ, ψ, and ϕ as shown in Fig. 7.33. Also shown in the diagram are the forces acting on the disk, which consist of the gravity force mg and the three components R_x, R_y, R_z of the reaction of the plane on the disk. We shall select an xyz coordinate system whose origin is at the center of mass of the disk, and whose y-axis passes through the point of contact of the disk and the plane.

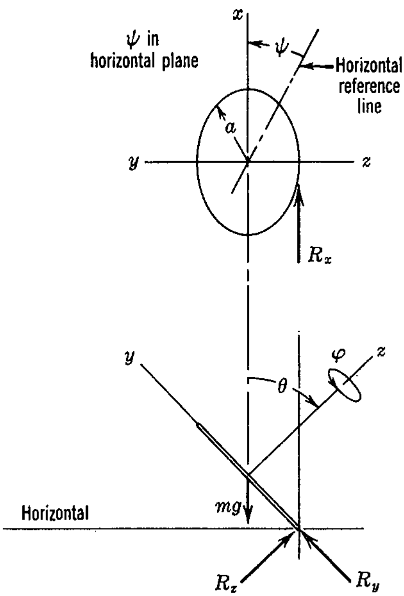


FIG. 7.33

The conditions that the disk roll without slipping on the plane can be expressed analytically as:

$$\dot{x}_c = -a\dot{\phi} - a\dot{\psi}\cos\theta$$
$$\dot{y}_c = 0$$
$$\dot{z}_c = a\dot{\theta}$$

The angular velocity of the disk is:

$$\omega_x = \dot{\theta}$$
$$\omega_y = \dot{\psi}\sin\theta$$
$$\omega_z = \dot{\phi} + \dot{\psi}\cos\theta$$

The angular velocity of the x, y, z coordinate system, however, is not the same as the angular velocity of the disk, since the coordinate system is not attached to the disk. If the angular velocity of the coordinate system is Ω_x, Ω_y, Ω_z, we have:

$$\begin{aligned}
\Omega_x &= \dot\theta \\
\Omega_y &= \dot\psi \sin\theta \\
\Omega_z &= \dot\psi \cos\theta
\end{aligned}$$

To find the acceleration of the center of mass of the body, we write:

$$\boldsymbol{a}_c = \frac{d}{dt}(\dot x_c\boldsymbol{i} + \dot y_c\boldsymbol{j} + \dot z_c\boldsymbol{k}) = \ddot x_c\boldsymbol{i} + \dot x_c\dot{\boldsymbol{i}} + \ddot y_c\boldsymbol{j} + \dot y_c\dot{\boldsymbol{j}} + \ddot z_c\boldsymbol{k} + \dot z_c\dot{\boldsymbol{k}}$$

where:

$$\begin{aligned}
\dot{\boldsymbol{i}} &= \boldsymbol{\Omega} \times \boldsymbol{i} = \Omega_z\boldsymbol{j} - \Omega_y\boldsymbol{k} \\
\dot{\boldsymbol{j}} &= \boldsymbol{\Omega} \times \boldsymbol{j} = \Omega_x\boldsymbol{k} - \Omega_z\boldsymbol{i} \\
\dot{\boldsymbol{k}} &= \boldsymbol{\Omega} \times \boldsymbol{k} = \Omega_y\boldsymbol{i} - \Omega_x\boldsymbol{j}
\end{aligned}$$

Substituting and collecting terms, we find:

$$\begin{aligned}
a_{cx} &= -a\ddot\phi - a\ddot\psi\cos\theta + 2a\dot\psi\dot\theta\sin\theta \\
a_{cy} &= -a\dot\phi\dot\psi\cos\theta - a\dot\psi^2\cos^2\theta - a\dot\theta^2 \\
a_{cz} &= a\ddot\theta + a\dot\phi\dot\psi\sin\theta + a\dot\psi^2\sin\theta\cos\theta
\end{aligned}$$

The three force equations can thus be written:

$$\begin{aligned}
R_x &= -ma(\ddot\phi - 2\dot\theta\dot\psi\sin\theta + \ddot\psi\cos\theta) \\
R_y - mg\sin\theta &= -ma(\dot\theta^2 + \dot\phi\dot\psi\cos\theta + \dot\psi^2\cos^2\theta) \qquad \text{(i)} \\
R_z - mg\cos\theta &= ma(\ddot\theta + \dot\phi\dot\psi\sin\theta + \dot\psi^2\sin\theta\cos\theta)
\end{aligned}$$

For the moment equations we can make use of Equations (7.29), previously derived for the motion of a body of revolution about a fixed point. Since the origin of the coordinate system is the center of mass of the system, the same equations will be valid, and since the angles used to describe the position of the disk are the same Euler's angles used for Equations (7.29) we may substitute directly and obtain:

$$\begin{aligned}
M_x &= -aR_z = I\ddot\theta + (I_z - I)\dot\psi^2\sin\theta\cos\theta + I_z\dot\phi\dot\psi\sin\theta \\
M_y &= 0 = I\ddot\psi\sin\theta + (2I - I_z)\dot\psi\dot\theta\cos\theta - I_z\dot\phi\dot\theta \qquad \text{(ii)} \\
M_z &= aR_x = I_z\ddot\phi + I_z\ddot\psi\cos\theta - I_z\dot\psi\dot\theta\sin\theta
\end{aligned}$$

The six equations of (i) and (ii) constitute the complete set of equations of motion for the problem.

7.16 Stability of Rigid Body Motion. The Rolling Disk. As an example of the use of the equations of the preceding section, we shall find the conditions under which stable rolling of the disk in a straight line is possible. This will also serve as an illustration of one type of stability investigation. In general, the conditions of stability of a given state of motion are determined by introducing a small perturbation of some kind into the motion. If in this perturbed state a restoring force is set up which tends to return the system to its original condition, the motion is said to be stable. In addition to this restoring force it may also be required that there be damping forces so that any oscillatory motions would decay.

Let us suppose that the disk of Fig. 7.33 is rolling along in a straight line with $\psi = \dot\psi = \ddot\psi = 0$ and $\theta = 90^\circ$, $\dot\theta = \ddot\theta = 0$. From the 1st & last of equations (i) & (ii), we obtain

$$R_x = -ma\ddot\varphi$$

$$aR_x = I_z\ddot\varphi$$

so $(I_z + ma^2)\ddot\varphi = 0$; $\ddot\varphi = 0$, and $\dot\varphi = \text{constant} = \dot\varphi_0$.

We investigate the stability of this motion by introducing small perturbations into this reference condition, so that $\dot\varphi = \dot\varphi_0 + \dot\alpha$, $\theta = 90^\circ + \beta$, $\psi = \gamma$, where α, β, γ are small quantities.

Writing equations (i) & (ii) for this perturbed condition, retaining only the 1st order terms, and noting that $\cos(90+\beta) \approx -\beta$, $\sin(90+\beta) \approx 1$, we obtain:

$$R_x = -ma\ddot\alpha \qquad (iii)$$

$$R_y - mg = 0 \qquad (iv)$$

$$R_z + mg\beta = ma\ddot\beta + ma\dot\alpha\dot\gamma \qquad (v)$$

$$-aR_z = I\ddot\beta + I_z\dot\alpha\dot\gamma \qquad (vi)$$

$$0 = I\ddot\gamma - I_z\dot\alpha\dot\beta \qquad (vii)$$

$$aR_x = I_z\ddot\alpha \qquad (viii)$$

From (iii) & (viii), $\ddot\alpha = 0$, so $\dot\alpha = \text{constant} = \dot\alpha_0$

Eliminating R_z between (v) & (vi):

$$-mga\beta + ma^2\ddot\beta + ma^2\dot\alpha_0\dot\gamma + I\ddot\beta + I_z\dot\alpha_0\dot\gamma = 0 \qquad (ix)$$

From (vii), $\ddot\gamma = \frac{I_z}{I}\dot\alpha_0\dot\beta$ which integrate to

Replacing $\dot{r}$ in (ix) by (x), we obtain

$$(I+ma^2)\ddot{\beta} + \left[(I_z+ma^2)\frac{I_z}{I}\dot{\alpha}_0^2 - mga\right]\beta = 0 \qquad (xi)$$

If the square bracket in (xi) is negative, β will grow exponentially, i.e., the motion is unstable. The condition for stability is thus:

$$(I_z+ma^2)\frac{I_z}{I}\dot{\alpha}_0^2 > mga$$

The angular velocity of the disk is $\dot{\varphi} = \dot{\varphi}_0 + \dot{\alpha}_0 = \omega_d$ so finally the condition for stability becomes:

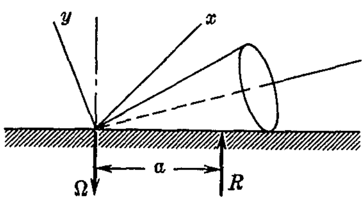
$$\omega_d^2 > \frac{Imga}{I_z(I_z + ma^2)}$$

For a uniform disk, $I_z = \frac{1}{2}ma^2$, $I = \frac{1}{4}ma^2$, and $v = a\omega_d$, where v is the velocity of the center of the disk. The stable velocity of rolling in a straight line is thus given by:

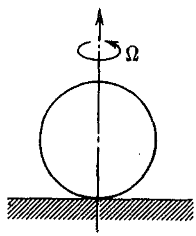
$$v > \sqrt{\frac{ga}{3}}$$

PROBLEMS

7.103. A homogeneous right circular cone of half-angle α and height h rolls without slipping on a rough horizontal plane. The angular velocity of the axis of the cone about the vertical line through the vertex is Ω.



PROB. 7.103



PROB. 7.104

The resultant of the reaction forces of the plane on the cone is R, located at a distance a from the vertex. Find a as a function of Ω, and determine the limiting value of Ω for which such a rolling motion is possible.

7.104. A thin homogeneous circular disk of radius a rests on a rough horizontal plane and spins about a vertical axis with its plane always vertical. Using the general theory developed for the rolling disk, find the value of Ω for which such a stable spinning motion will be possible.

7.17 D'Alembert's Principle. It was pointed out by D'Alembert that Newton's Second Law of Motion could be considered from a slightly different viewpoint by writing it in the form:

$$\boldsymbol{F} + (-m\ddot{\boldsymbol{r}}) = 0$$

and treating the term $(-m\ddot{\boldsymbol{r}})$ as if it were a force. When this is done the terms $(-m\ddot{\boldsymbol{r}})$ are called *inertia forces*, and it should be particularly noted that inertia forces are not actual forces in the sense that the word "force" has been used in the preceding portion of this book. The concept of the inertia force makes it possible to apply the general methods of statics to the solution of dynamics problems, since Newton's Law may be written as $\Sigma \boldsymbol{F} = 0$ if it is understood that inertia forces are to be included in the summation.

This viewpoint can be extended to systems of particles and to rigid bodies. For any system of particles we have

$$\Sigma \boldsymbol{F}_i + \Sigma(-m_i\ddot{\boldsymbol{r}}_i) = 0$$

For a rigid body performing plane motion, the equations of motion are:

$$\Sigma \boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$$
$$\Sigma \boldsymbol{M}_c = I_c\ddot{\boldsymbol{\theta}}$$

These may be written:

$$\Sigma \boldsymbol{F} + (-m\ddot{\boldsymbol{r}}_c) = 0$$
$$\Sigma \boldsymbol{M}_c + (-I_c\ddot{\boldsymbol{\theta}}) = 0$$

where $\ddot{\boldsymbol{r}}_c$ is the acceleration of the center of mass of the body, and I_c is the moment of inertia of the body about the center of mass. If we imply that the inertia force $(-m\ddot{\boldsymbol{r}}_c)$ and the inertia torque $(-I_c\ddot{\boldsymbol{\theta}})$ are included in the summation, we have the equations in the same form as the equations of statics:

$$\Sigma \boldsymbol{F} = 0$$
$$\Sigma \boldsymbol{M} = 0$$

It should be noted that in this moment equation, the moments can be taken about any axis, as in statics, since the inertia torque has already been included with the appropriate moment of inertia about the mass center. It is this feature which may sometimes lead to a simplification in writing the equations of motion for a system. Since any point can be taken as a moment center it is sometimes possible to select a point through which several unknown forces pass, thus eliminating such unknown forces from the equations. The equations of motion as formulated in the previous sections of the present chapter, on the other hand, usually require that some particular point such as the mass center or a fixed point should be used as the moment center.

The introduction of the concept of an inertia force and an inertia torque does not, of course, represent any new information. For some problems, however, this method of writing the equation of motion leads to a convenient way of visualizing the dynamics of a situation, as will be illustrated in the following examples. D'Alembert's principle is also often used for non-rigid systems, as in Prob. 7.107. By applying the appropriate inertia force to each element of the system in such cases, a complete description of the dynamics of the system is obtained. It is important to note that D'Alembert's method should be considered as an alternative to the method employed previously in this chapter, and that mixing the two methods in one solution can lead to considerable confusion. In such a case it is quite possible that the same term will be included twice—once in the guise of an inertia force, and once as an acceleration term.

The concept of an inertia force can be combined with the principle of virtual displacements to give Lagrange's form of D'Alembert's principle. For example, the equation of motion for a particle is:

$$m\ddot{\boldsymbol{r}} - \boldsymbol{F} = 0$$

If, according to D'Alembert's notion, we consider this to be an equation of static equilibrium, the principle of virtual displacements states that:

$$(m\ddot{\boldsymbol{r}} - \boldsymbol{F})\cdot\delta\boldsymbol{r} = 0$$

where $\delta\boldsymbol{r}$ is a virtual displacement. The general statement of D'Alembert's principle for a system of particles is:

$$\Sigma[(m\ddot{x}_i - F_{xi})\delta x_i + (m\ddot{y}_i - F_{yi})\delta y_i + (m\ddot{z}_i - F_{zi})\delta z_i] = 0 \tag{7.30}$$

This is the equation from which Lagrange developed analytical mechanics. It introduces into dynamics the same advantage that the principle of virtual displacements introduces into statics—the conditions of equilibrium or of motion may be studied without introducing the constraining forces which may be acting. A further development of this method of formulation of the dynamics problem will be discussed in Chapter 9.

EXAMPLE 1. Solve the problem of the example of Section 7.9 using the concept of the inertia force.

Solution. We draw a free-body diagram indicating the forces as solid vectors and the inertia forces as dotted vectors (Fig. 7.34). The

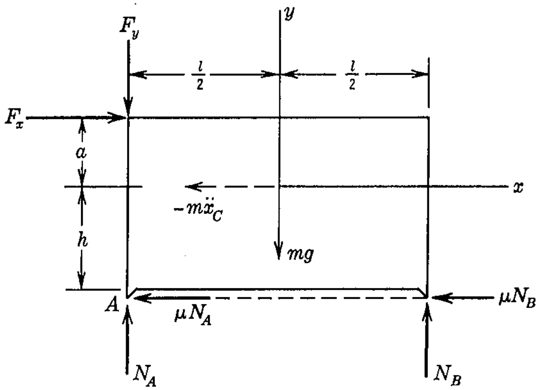


FIG. 7.34

inertia force $(-m\ddot{x}_c)$ is shown at the center of mass of the system. Since the body is translating, there is no inertia torque acting on the system. The problem has now been reduced to a problem in statics, and the equations may be written:

$$\Sigma F_x = 0 = F_x - \mu N_A - \mu N_B - m\ddot{x}_c$$
$$\Sigma F_y = 0 = -F_y + N_A + N_B - mg$$
$$\Sigma M_A = 0 = -F_x(a + h) - mg\left(\frac{l}{2}\right) + lN_B + hm\ddot{x}_c$$

These equations lead directly to the solution given in the example of Section 7.9. Note that we have taken advantage of the fact that any point can be selected as the moment center, and have picked the point A, which eliminates several unknown forces from the equations.

EXAMPLE 2. A pulley of radius R and moment of inertia I supports two masses m_1 and m_2 fastened together by a rope as shown in Fig. 7.35. Find the equation of motion of the system.

Solution. We shall describe the motion of this single-degree of freedom system by the coordinate x, as shown in the free-body diagram. The inertia forces $(-m_1\ddot{x})$ and $(-m_2\ddot{x})$ are shown as dotted vectors, and the inertia torque $(-I\ddot{\theta})$ is also indicated. The equation of motion can now be written:

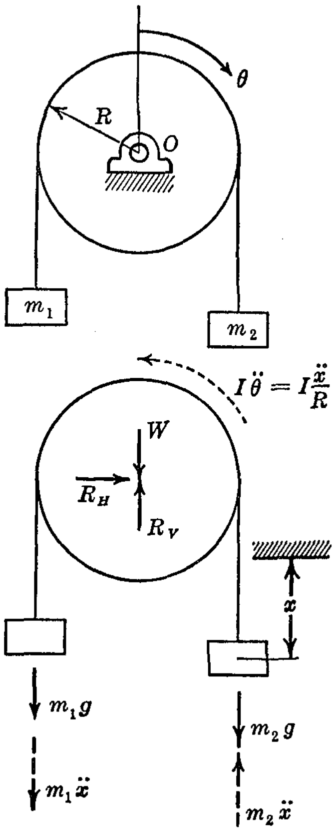


FIG. 7.35

$$\Sigma M_0 = m_1\ddot{x}R + \frac{I\ddot{x}}{R} + m_2\ddot{x}R + m_1gR - m_2gR = 0$$

From which:

$$\ddot{x} = \frac{(m_2 - m_1)g}{m_1 + m_2 + \dfrac{I}{R^2}}$$

EXAMPLE 3. A thin hoop of radius R rotates about an axis through the center perpendicular to the plane of the hoop with a constant angular velocity ω. Find the circumferential tension force in the hoop.

Solution. We first draw a free-body diagram of one-half of the hoop (Fig. 7.36). Each element of mass of the hoop is acted upon by an inertia force directed as shown in the diagram. Consider an element of mass included by the angle $d\phi$ and let ρ be the mass per unit length of the hoop; then the inertia force is $\rho(R\omega^2)Rd\phi$ directed radially outward. We may now write:

$$\Sigma F_y = 0 = \int_0^\pi R^2\omega^2\rho \sin \phi d\phi - 2F$$

From which

$$F = R^2\omega^2\rho$$

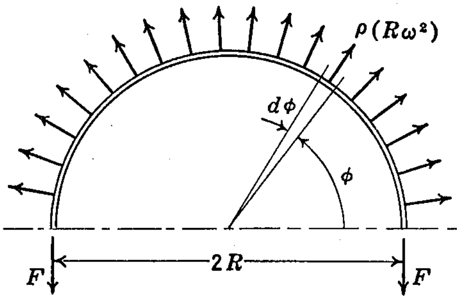


FIG. 7.36

EXAMPLE 4. A type of governor mechanism used for the control of the speed of rotating shafts is shown in Fig. 7.37. A simple pendulum having a concentrated mass m and a length l is mounted on the rim of a wheel of radius R which rotates with an angular velocity ω. The pendulum is constrained by two springs which are also attached to the rim of the wheel. For small displacements x, of the pendulum, the restoring force of the springs can be taken as $F_x = -kx$. When the wheel is rotating with a constant angular velocity, the pendulum remains in a radial position, but if the wheel accelerates or decelerates the pendulum moves to one side or the other of its neutral position. By allowing the pendulum displacement to control the power input to the system, the angular velocity can be regulated. Find the differential equation of motion of the pendulum with respect to the flywheel, in terms of angular velocity and acceleration of the flywheel.

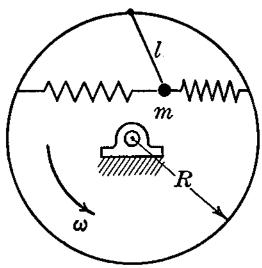


FIG. 7.37

Solution. Since we wish to express the absolute acceleration of m in terms of the relative motion, we use Equation (2.16):

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

By D'Alembert's method the equation of equilibrium is:

$$\mathbf{F} - m\ddot{\mathbf{R}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) - m\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} - m\ddot{\boldsymbol{\rho}}_r - 2m\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r = 0$$

Letting the xy axes be fixed on the wheel (Fig. 7.38), each term of the equation is as follows:

$$\begin{aligned}
\boldsymbol{F} &= -kl\sin\phi\boldsymbol{i} \\
-m\ddot{\boldsymbol{R}} &= -mR\dot{\omega}\boldsymbol{i} - mR\omega^2\boldsymbol{j} \\
-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) &= ml\omega^2\boldsymbol{e}_\rho \\
-m\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} &= m\dot{\omega}l\boldsymbol{e}_\phi \\
-m\ddot{\boldsymbol{\rho}}_r &= ml\dot{\phi}^2\boldsymbol{e}_\rho - ml\ddot{\phi}\boldsymbol{e}_\phi \\
-2m\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r &= -2m\omega l\dot{\phi}\boldsymbol{e}_\rho
\end{aligned}$$

A free-body diagram with all of the inertia forces is shown in Fig. 7.39. The force in the pendulum rod now can be determined by statics and

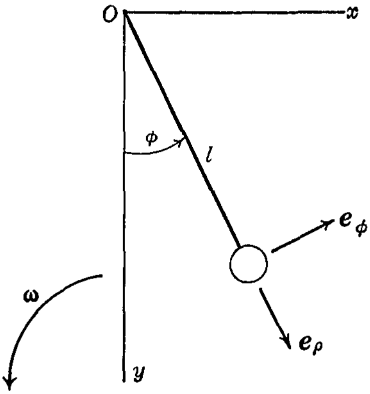


FIG. 7.38

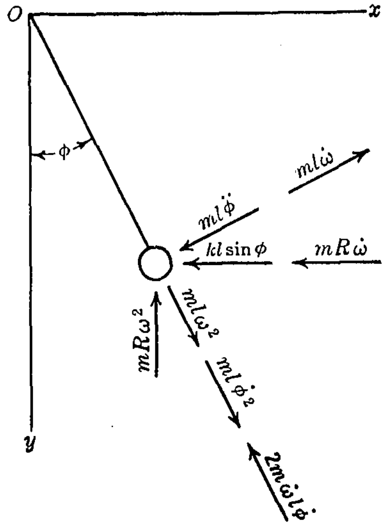


FIG. 7.39

the equation of motion can be derived by writing the moment equation $\Sigma M_0 = 0$.

$$-l(-ml\dot{\omega} + ml\ddot{\phi}) - l\cos\phi(kl\sin\phi + mR\dot{\omega}) + l(\sin\phi)mR\omega^2 = 0$$

For small oscillations we may set $\cos\phi \approx 1$, $\sin\phi \approx \phi$ and obtain:

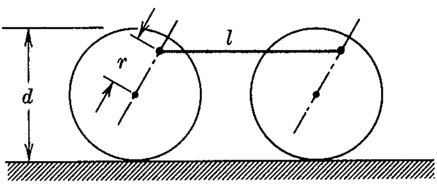
$$\ddot{\phi} + \left(\frac{k}{m} - \frac{R\omega^2}{l}\right)\phi = \left(\frac{l-R}{l}\right)\dot{\omega}$$

This equation has the same form as the equation describing the vibration of a pendulum about a fixed point. If the spring constant is large so that $\frac{k}{m} > \frac{R\omega^2}{l}$, the pendulum will oscillate about its equilibrium position under the action of the term $\left(\frac{l-R}{l}\right)\dot{\omega}$, which acts like an exciting force.

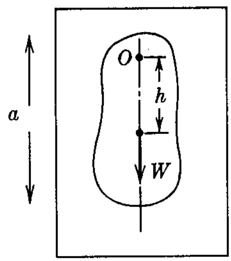
For the complete analysis of a control system of the above type, one would have to take account of the fact that the mechanisms connected to the pendulum and operated by it have dynamic characteristics which affect the behavior of the system. Considerable work has been done in recent years on control systems of all types, and much of this work is summarized in books on the theory of servomechanisms.

PROBLEMS

7.105. The side-crank connecting rod of a locomotive drive system has a length l and is connected to the wheels at a point a distance r from the center. The wheels have a diameter d, and the velocity of the locomotive horizontally is v. Assuming that the connecting rod is a uniform straight bar of weight W, find the maximum bending moment in the bar.



PROB. 7.105

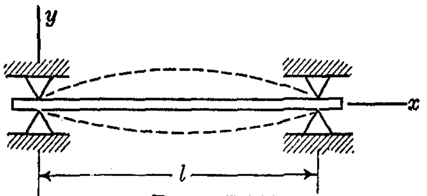


PROB. 7.106

7.106. A compound pendulum of moment of inertia I_0 about the point of support is attached to an elevator which moves either up or down with an acceleration a. The distance from the point of support of the pendulum to the center of gravity is h. Find the way in which the frequency of vibration of the pendulum is influenced by the elevator acceleration.

7.107. A slender steel rod of length l and radius r rotates about an axis through one end perpendicular to the bar with a constant angular velocity ω. Find the maximum tension force in the rod using the inertia force method. Find the numerical value of this force if $l = 3$ ft, $r = \frac{1}{4}$ in., and $\omega = 500$ rpm.

7.108. A steel beam of length l and weight w lb/ft is simply supported at each end. It is observed that the beam vibrates with a motion $y = A \sin\left(\frac{\pi x}{l}\right) \sin pt$. Find the expression for the maximum dynamic reaction which occurs at a support, using the concept of an inertia force. Find the numerical value of this force when $l = 20$ ft, $w = 20$ lb/ft, $A = \frac{1}{8}$ in., and the frequency of the vibration is 10 cycles per second.



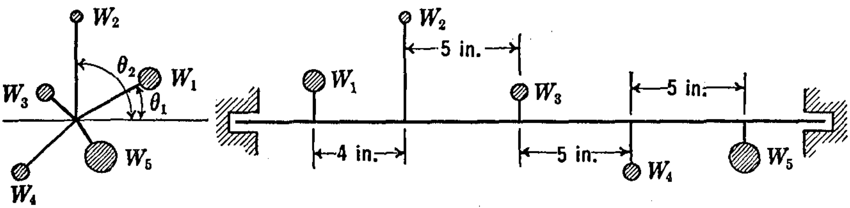
PROB. 7.108

7.109. Solve Prob. 7.35 using the concept of inertia force.

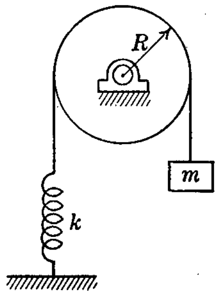
7.110. Solve Prob. 7.37 using the concept of inertia force.

7.111. Five weights are attached to a rigid horizontal shaft as shown in the figure. The weights, radii, and angles locating the weights are given in the accompanying table. The system is to be balanced by the addition of two weights, one in the plane of W_1 and the other in the plane of W_5. Each balance weight has a radius of 2 in. Find the magnitudes and the angular positions of the balance weights. Show that the products of inertia are zero for the balanced system.

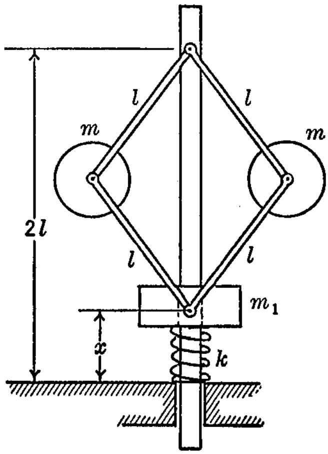
No.	*W (lb)*	*r* (in.)	θ
W_1	10	2	30°
W_2	5	3	90°
W_3	8	1	135°
W_4	8	2	225°
W_5	15	1	300°



PROB. 7.111



PROB. 7.112

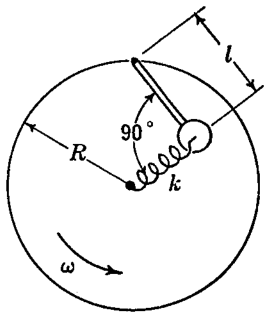


PROB. 7.113

7.112. A pulley having a moment of inertia I about its axis of rotation supports a rope which carries a mass m at one end, while the other end is connected to a spring of spring constant k as shown in the diagram. Find the period of oscillation of the system. Assume that the rope does not slip on the pulley.

7.113. A governor is constructed as is shown in the diagram. The assembly of four linked bars of equal length l rotates with an angular velocity ω about a vertical axis. The mass m_1 slides on the vertical axis and is restrained by a spring force $(-kx)$. Find the displacement x of the mass m_1 in terms of the constant angular velocity ω of the system.

7.114. A governor is mounted on a rotating wheel as is shown in the figure. When operating in a steady-state condition (ω = constant) the angle between the spring and the pendulum is 90°. The total mass of the pendulum is m, and the moment of inertia of the pendulum about its suspension axis is I. Find the equation of motion of the pendulum for small displacements from the steady-state position.



PROB. 7.114

Chapter 8

NON-RIGID SYSTEMS OF PARTICLES

But I consider philosophy rather than the arts and write not concerning manual but natural powers, and consider chiefly those things which relate to gravity, levity, elastic forces, the resistance of fluids, and the like forces, whether attractive or impulsive.—I. Newton, *Principia Philosophiae* (1686).

The analysis of the dynamics of systems of particles is greatly influenced by the characteristics of the particular system being studied. For example, solid bodies, fluids, and gases are all systems of particles and as such can be treated by the general methods of dynamics which have already been discussed. The physical characteristics of these various systems differ so greatly, however, that the analysis must be handled in a distinctive fashion for the various materials. The analysis is, of course, always based upon the equations of motion, but it is developed in different ways in order to take advantage of the particular characteristics of a given system.

In the following sections some of the methods which can be applied to non-rigid systems of particles, such as elastic bodies, fluids, and gases will be briefly discussed. The non-rigid system which is most closely related to the rigid body of the preceding chapter is the elastic body, in which small motions of the particles with respect to each other may occur. These small motions may be the only motions involved in the problem, or they may be superimposed on a "rigid-body" motion of the whole system.

8.1 Longitudinal Waves in an Elastic Bar. As one example of a type of dynamics problem in which the motions are associated with the non-rigidity of the system, we shall consider the propagation of elastic strain waves in a long, slender bar.

Fig. 8.1 (a) shows a portion of a long, slender bar of uniform cross section area A, and of density ρ. It is supposed that loads are applied in such a way that only longitudinal strains are set up, and translational motions of the bar as a rigid body will not be considered. It will be assumed that plane sections normal to the axis of the bar remain plane, i.e., the axial stresses and strains are uniformly distributed across the area of the bar.

Referring to Fig. 8.1 (a), consider a section of the bar located at a distance x from a reference point. During the strain of the bar, this section is displaced a distance u from its original position, while another section, originally at a distance $x + dx$ is displaced by $u + du$. The total elongation of the element of length dx is thus du, and the

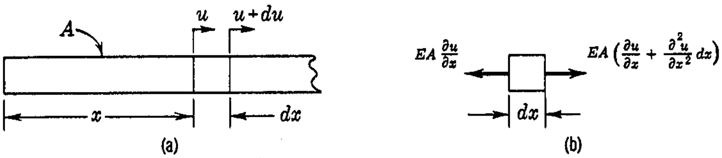


FIG. 8.1

strain is $\frac{\partial u}{\partial x}$, which we write as a partial derivative since the displacements u will be functions of both x and t.

If it is assumed that the material of the bar obeys Hooke's law, the uniformly distributed stress across the bar will be $\sigma = E\,\frac{\partial u}{\partial x}$, and the total force on the section will be $EA\,\frac{\partial u}{\partial x}$. Writing the equation of motion $F = ma$ for the free-body diagram of Fig. 8.1 (b)

$$-EA\,\frac{\partial u}{\partial x} + EA\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\,dx\right) = (\rho A\,dx)\,\frac{\partial^2 u}{\partial t^2}$$

we obtain the basic partial differential equation describing the motions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\,\frac{\partial^2 u}{\partial x^2} \tag{8.1}$$

The solution of this partial differential equation, subject to prescribed initial and boundary conditions, leads to a complete description of all motions permitted by the basic assumptions. It will next be shown that the solution to Equation (8.1) can be obtained in two different forms, which must, of course, be physically equivalent, but which lead to two different points of view for problems of this kind.

8.2 The Traveling Wave Solution. Consider a function of the form

$$u = f(x - ct) \tag{8.2}$$

where f indicates any function and $c = \sqrt{\frac{E}{\rho}}$. It will be seen that this is a solution of Equation (8.1), for*:

$$u = f(x - ct); \quad \frac{\partial u}{\partial x} = f'(x - ct); \quad \frac{\partial^2 u}{\partial x^2} = f''(x - ct)$$

$$\frac{\partial u}{\partial t} = -cf'(x - ct); \quad \frac{\partial^2 u}{\partial t^2} = c^2 f''(x - ct)$$

Substituting into Equation (8.1):

$$c^2 f''(x - ct) = \frac{E}{\rho} f''(x - ct)$$

which is always satisfied if $c^2 = \frac{E}{\rho}$.

We now investigate the physical interpretation of a solution in the form of Equation (8.2). Fig. 8.2 (a) shows a representation of the function $u = f(x - ct)$ when $t = 0$. In Fig. 8.2 (b) the more general case $u = f(x - ct)$, $t > 0$ is plotted in an x_0, u_0 coordinate system which has been transformed by the translation along the x-axis $x = x_0 + ct$. It will be noted that this transformation has the effect of moving the u-axis to the right by an amount ct. In this new

* Note that writing $(x - ct) = z$; $\frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{\partial z}\frac{dz}{dx} = f'$; and similarly $\frac{\partial f(z)}{\partial t} = \frac{\partial f}{\partial z}\frac{dz}{dt} = -cf'$.

coordinate system we have $u_0 = f(x_0 + ct - ct) = f(x_0)$ and hence in the translated coordinate system the same shape of function $f(x_0)$ is obtained that was originally plotted for $t = 0$. It is thus evident that as time increases, and hence the distance ct increases, the original curve $f(x)$ can be thought of as moving along the x-axis to the right with a constant velocity c.

It will thus be seen that a solution of the form $f(x - ct)$ represents physically a wave traveling along the bar with a velocity $c = \sqrt{\dfrac{E}{\rho}}$. For this reason Equation (8.1) is often called the one-dimensional wave equation. The wave equation in more general terms is:

$$\frac{\partial^2\Phi}{\partial t^2} = a^2\nabla^2\Phi = a^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\right)$$

the solutions of which describe waves that can be propagated in

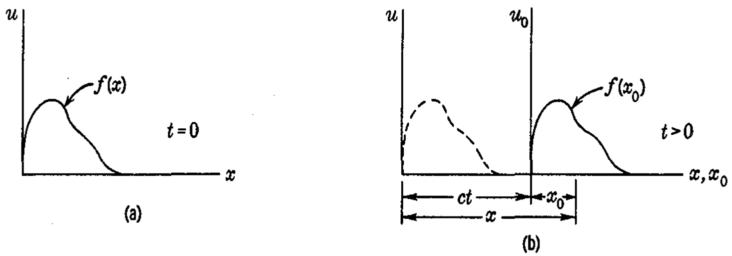


FIG. 8.2

certain three-dimensional systems, as, for example, sound waves in air.

By repeating the steps which lead to the solution $u = f(x - ct)$ it can be shown that the solution $u = f(x + ct)$ also satisfies the wave equation, and that it represents a wave traveling to the left with the velocity c. The solution $u = f(x \pm ct)$ thus represents two waves traveling in opposite directions with equal constant velocities c. Since Equation (8.1) is linear, a general solution can be built up from the two wave solutions by superposition, thus giving:

$$u(x, t) = f_1(x - ct) + f_2(x + ct)$$

The forms of the functions f_1 and f_2 can be determined from the prescribed initial conditions. If, for example, the initial displacements are $u_0(x)$ and the initial velocities are $\left(\frac{\partial u}{\partial t}\right)_0$, we would have:

$$u_0(x) = f_1(x) + f_2(x)$$

$$\left(\frac{\partial u}{\partial t}\right)_0 = c[f_2'(x) - f_1'(x)]$$

where $f_1(x)$ and $f_2(x)$ are the shapes of the two oppositely moving waves.

A superposition of waves may also be used to satisfy the boundary conditions at the ends of the bar. Consider two identical waves as in Fig. 8.3 (a). In this figure we will plot $\sigma = E\left(\frac{\partial u}{\partial x}\right)$ instead of u, so that the waves are stress waves, and we show the positions of the waves at three successive intervals of time.

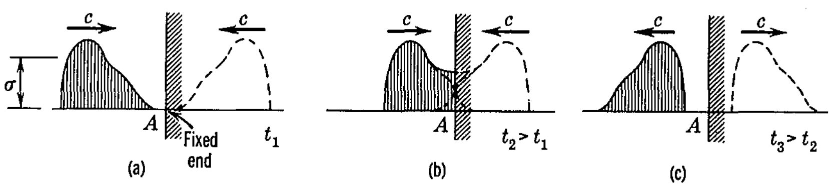


FIG. 8.3

If the two waves are of equal sign and are mirror-images, then point A midway between the waves will at all times be in equilibrium, and will hence remain at rest. The point A, therefore, can represent a fixed point of the bar, as at a built-in end. We thus conclude that a wave is "reflected" from a fixed end of the bar as a mirror image of the same sign as the original wave, i.e., a tension wave reflects from a fixed end as a similar tension wave.

If we now imagine that the two approaching waves have the same form but are of opposite sign, as in Fig. 8.4 (a), we note that at the midpoint A the stresses cancel but the displacements add. Point A thus in this case corresponds to a free end of a bar, and we conclude that a wave is reflected from a free end with a change in sign, i.e., a tension wave reflects from a free end as a similar compression wave.

In connection with the above discussion of wave propagation, it will be useful to refer back to Prob. 3.38, where it is indicated how the velocity of propagation of elastic waves can be calculated from impulse-momentum principles.

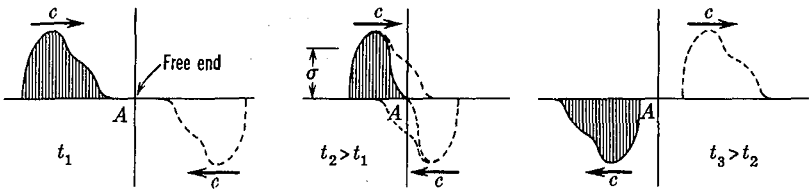


FIG. 8.4

EXAMPLE. A longitudinal stress wave of the shape shown in Fig. 8.5 starts at the free end of a bar of length l and travels down the bar toward a built-in end. What time will elapse before the bar will again be in its original configuration?

Solution. Since the wave reflects from a fixed end with the same sign, and from the free end with a reversed sign, it will require four complete traverses of the bar to get the wave back to its original configuration, as shown in Fig. 8.6 which indicates the sequence of

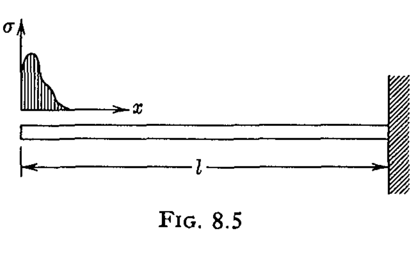


FIG. 8.5

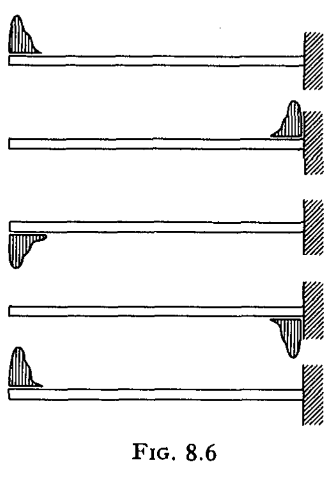


FIG. 8.6

events. Each traverse takes a time $t = \frac{l}{c}$, so the total time is $\frac{4l}{c} = 4l\sqrt{\frac{\rho}{E}}$. The resultant motion of the bar, which involves a longitudinal to-and-fro motion of the individual particles, may be thought of as a vibration of the bar. The time calculated above may thus be regarded as the fundamental period of longitudinal vibration of the bar.

In the next section this vibration problem will be studied from another point of view.

8.3 The Longitudinal Vibrations of a Bar. We shall now return to Equation (8.1) and show that a different form of solution can be obtained.

The differential equation is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Try as a solution a function of the form:

$$u(x, t) = u_1(t) \cdot u_2(x)$$

where $u_1(t)$ is a function of t alone, and $u_2(x)$ is a function of x alone. Substituting this trial solution into the differential equation gives:

$$u_2 \frac{d^2 u_1}{dt^2} = c^2 u_1 \frac{d^2 u_2}{dx^2}$$

or:

$$\frac{1}{u_1} \frac{d^2 u_1}{dt^2} = c^2 \frac{1}{u_2} \frac{d^2 u_2}{dx^2}$$

The left side of this equation is a function of t alone, and the right is a function of x alone; these two sides can be equal only if they equal the same constant. If we call this constant $(-p^2)$, we have:

$$\frac{1}{u_1} \frac{d^2 u_1}{dt^2} = c^2 \frac{1}{u_2} \frac{d^2 u_2}{dx^2} = -p^2$$

which leads to the two equations:

$$\frac{d^2 u_1}{dt^2} + p^2 u_1 = 0 \tag{8.3}$$

$$\frac{d^2 u_2}{dx^2} + \frac{p^2}{c^2} u_2 = 0 \tag{8.4}$$

By the use of the product solution technique the single partial differential equation has been replaced by two ordinary differential equations. This method is often useful in the solution of partial differential equations.

Equation (8.3) will be seen to be the simple vibration equation as discussed in Chapter 5, the solution of which is:

$$u_1(t) = A \sin pt + B \cos pt$$

where p is the frequency of the vibrations.

Equation (8.4) is an equation of similar form, and will have the solution:

$$u_2(x) = C \sin \frac{p}{c} x + D \cos \frac{p}{c} x.$$

And thus a solution of the differential equation is:

$$u(x, t) = (A \sin pt + B \cos pt)\left(C \sin \frac{p}{c} x + D \cos \frac{p}{c} x\right) \quad (8.5)$$

We shall show how a complete solution of the differential equation can be formed from solutions of the above type by considering a specific example. Suppose that we are to determine the natural frequencies of longitudinal vibrations of a uniform slender bar of length l, area A, material of density ρ, that is free at one end and built in at the other end. The boundary conditions are:

(1) at the free end ($x = 0$) the strain is zero, therefore $\left(\frac{\partial u_2}{\partial x}\right)_{x=0} = 0$

(2) at the fixed end ($x = l$) the displacement is zero.

Substituting in the equation for u_2, we obtain:

$$(1) \quad \frac{\partial u_2}{\partial x} = C\frac{p}{c} \cos \frac{p}{c} x - D\frac{p}{c} \sin \frac{p}{c} x$$

$$0 = C\frac{p}{c}; \quad \text{so} \quad C = 0$$

and:

$$(2) \quad 0 = D \cos \frac{pl}{c}$$

For a non-trivial solution $D \neq 0$, so we have as a requirement for satisfying the boundary conditions:

$$\cos \frac{p_n l}{c} = 0 \quad (8.6)$$

There are an infinite number of values p_n which satisfy this equation, corresponding to the fundamental frequency and the higher harmonics.

The situation represented by Equation (8.6), in which a differential equation and its boundary conditions are satisfied only for a particular set of values of some parameter, often occurs in problems in applied mathematics. These are called the "characteristic values" or "eigenvalues" for the problem.

In this particular case, the natural frequencies of vibration of the bar are given by the solution of Equation (8.6):

$$p_n = \frac{\pi}{2}\frac{c}{l}, \quad \frac{3\pi}{2}\frac{c}{l}, \quad \frac{5\pi}{3}\frac{c}{l}, \ldots$$

It will be recalled that in Chapter 5 vibrating systems having one degree of freedom and hence one natural frequency of vibration were discussed. The elastic body is a system having an infinite number of degrees of freedom and hence, as we would expect, there are an infinite number of natural frequencies.

The fundamental frequency of longitudinal vibrations is the lowest of the above values:

$$p_1 = \frac{\pi}{2l}\sqrt{\frac{E}{\rho}}$$

The fundamental period of the vibration is:

$$\frac{2\pi}{p_1} = \frac{2\pi}{\frac{\pi}{2}\frac{c}{l}} = \frac{4l}{c} = 4l\sqrt{\frac{\rho}{E}}$$

Note that this agrees with the answer obtained by wave propagation techniques in the example of the preceding section.

The fact that there are an infinite number of frequency parameters p_n that will satisfy an equation of the type of Equation (8.5) for a given set of boundary conditions enables us to form a complete solution to the problem. Since the original differential equation is linear, solutions of the type of Equation (8.5) can be superimposed to give a complete solution which will satisfy the initial conditions of displacement and velocity as well as the boundary conditions. This general solution will thus be of the form:

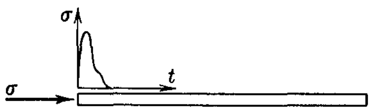
$$u(x, t) = \sum_{n=1}^{\infty} (A_n \sin p_n t + B_n \cos p_n t)\left(C_n \sin \frac{p_n}{c} x + D_n \cos \frac{p_n}{c} x\right)$$

It will often be found that many problems can be analyzed either as a wave propagation or as the superposition of harmonic solutions. Depending on the type of loading, the initial and boundary conditions, and on the information desired, one or the other of the methods may be preferable.

PROBLEMS

8.1. A straight uniform bar of length l, cross-section area A, and material of density ρ is free at both ends. A longitudinal stress pulse is applied to one end of the bar as shown in the diagram.

(a) By wave propagation techniques find the time that would elapse before the bar has again its original configuration.



PROB. 8.1

(b) Compare the time of part (a) with the fundamental natural period of vibration as calculated from the differential equation of motion of the bar.

(c) Repeat (a) and (b) for the case in which the bar is built in at both ends.

8.2. A straight shaft has a uniform circular cross section having a polar moment of inertia I_p, and an area A. The material of the shaft has a density ρ. The angle of twist of the shaft is ϕ, where ϕ is a function of the distance x along the shaft and of the time. Derive the partial differential equation describing the twist of the shaft. Show that torsional waves can be transmitted along the shaft and determine the velocity of these waves.

8.3. The partial differential equation describing the transverse bending vibrations of a uniform beam is:

$$\frac{\partial^4 y}{\partial x^4} + \frac{\mu}{EI}\frac{\partial^2 y}{\partial t^2} = 0$$

where μ is the mass per unit length, E is the modulus of elasticity, and I is the moment of inertia of the cross-sectional area about the neutral axis.

(a) Show that a simple traveling wave type solution $y(x, t) = f(x - ct)$ does not apply to the beam equation.

(b) Show that a solution of the type:

$$y(x, t) = A \sin \omega\left(t - \frac{x}{V}\right)$$

satisfies the equation, and that this solution represents the propagation of a sinusoidal bending wave of frequency ω with a velocity V. Find the relationship between V and ω. Systems for which wave propagation velocities are not constant, but depend upon the frequencies of the disturbances, are called "dispersive" systems.

8.4. A uniform flexible string of length l and mass per unit length μ is fastened at each end to rigid supports. The string is stretched by a large tension force F, which may be assumed to be constant for small transverse motions of the string.

(a) By writing the partial differential equation describing small transverse motions of the string, find the velocity of propagation of transverse waves along the string.

(b) Find the lowest natural frequency of vibration for small transverse motions of the string.

8.4 The Equations of Motion of a Non-viscous Fluid. In the analysis of fluid motion it is possible to introduce certain simplifications because of the special properties of fluids. First, it is not necessary to treat the fluid as being composed of discrete particles. Instead, we consider it to be composed of elements of volume dV, an element having a mass ρdV, where ρ is the density of the fluid. Second, we make use of the fact that the pressure at a point in a fluid is the same in all directions:

$$p_x = p_y = p_z = p$$

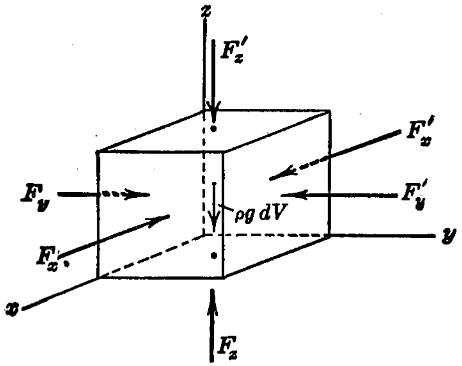
The equation of motion for an element of fluid is:

$$(\rho dV)\frac{d\boldsymbol{v}}{dt} = \Sigma \boldsymbol{F} \tag{8.7}$$

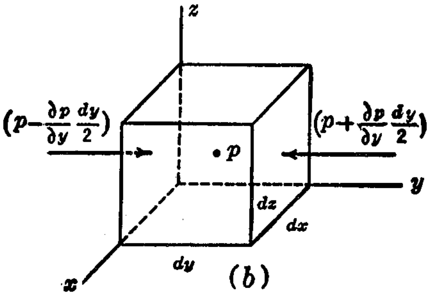
where $\boldsymbol{v}$ is the velocity of the element and $\Sigma \boldsymbol{F}$ is the resultant force acting upon the element. In rectangular coordinates the equations of motion are:

$$(\rho\, dx\, dy\, dz)\ddot{x} = \Sigma F_x \quad \text{etc.}$$

Fig. 8.7 (a) shows a free-body diagram of a fluid element. If we assume a perfect or non-viscous fluid, there are no viscous shearing forces on the sides of the element, so that the only forces are the normal forces acting on the faces, and the gravity force acting at the center of the element. In Fig. 8.7 (b) are shown the pressures exerted on two faces of the element. The pressure at the center of the element is taken to be p. Then on the right face of the element, at a distance $\frac{dy}{2}$ from the center, the pressure is $\left(p + \frac{\partial p}{\partial y}\frac{dy}{2}\right)$,



(a)



(b)

FIG. 8.7

whereas on the left face the pressure is $\left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)$. Since the force in the y-direction is given by the pressure multiplied by the area on which it acts:

$$\Sigma F_y = \left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)dx\,dz - \left(p + \frac{\partial p}{\partial y}\frac{dy}{2}\right)dx\,dz$$

$$\Sigma F_y = -\frac{\partial p}{\partial y}\,dx\,dy\,dz$$

The minus sign occurs because for $\frac{\partial p}{\partial y}$ positive the resultant force acts in the negative y-direction. The y-component of the equation of motion now may be written:

$$(\rho\,dx\,dy\,dz)\,\ddot{y} = -\frac{\partial p}{\partial y}\,dx\,dy\,dz$$

or:

$$\rho \ddot{y} = -\frac{\partial p}{\partial y}$$

In the x and z directions the equations are found in the same way to be:

$$\rho \ddot{x} = -\frac{\partial p}{\partial x}; \quad \rho \ddot{z} = -\left(\frac{\partial p}{\partial z} + \rho g\right)$$

or, in vector notation:

$$\rho \dot{\boldsymbol{v}} = -\left[\frac{\partial p}{\partial x}\boldsymbol{i} + \frac{\partial p}{\partial y}\boldsymbol{j} + \left(\frac{\partial p}{\partial z} + \rho g\right)\boldsymbol{k}\right] \tag{8.8}$$

With this equation it is possible to determine the motion when the pressure distribution is known, or to determine the pressure distribution when the motion is known.

There is an obvious difficulty in the practical application of this equation, since, in general, neither the motion nor the pressure in the fluid will be known. The only facts that are usually known are certain boundary values, such as the pressures at free surfaces or the directions of the velocity at a boundary, etc. The problem is to find the motion that will satisfy both the differential equation of motion and the particular boundary conditions. In general, this is a difficult problem, and many special techniques have been developed to analyze such fluid dynamics problems. These methods are treated in texts on hydrodynamics and aerodynamics. In the following paragraphs, we shall discuss only two of such special methods—that of the energy equation, and that of the momentum equation.

8.5 The Energy Equation. The equation of motion for an element of the fluid can be integrated directly to obtain the work-energy equation. Assuming an incompressible fluid, that is $\rho =$ constant, forming the dot product with $d\boldsymbol{r}$ and integrating, we obtain:

$$\int_1^2 \rho \dot{\boldsymbol{v}} \cdot d\boldsymbol{r} = -\int_1^2 \left[\frac{\partial p}{\partial x}\boldsymbol{i} + \frac{\partial p}{\partial y}\boldsymbol{j} + \left(\frac{\partial p}{\partial z} + \rho g\right)\boldsymbol{k}\right] \cdot d\boldsymbol{r}$$

$$\int_1^2 \rho \boldsymbol{v} \cdot d\boldsymbol{v} = -\int_1^2 \left[\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz + \rho g\,dz\right]$$

$$\tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2 = -\int_1^2 dp - \int_1^2 \rho g\,dz = -(p_2 - p_1) - \rho g(z_2 - z_1)$$

Collecting terms, this equation may be written:

$$\tfrac{1}{2}\rho v_1^2 + p_1 + \rho g z_1 = \tfrac{1}{2}\rho v_2^2 + p_2 + \rho g z_2$$

This states that the sum of the three terms is the same at all points, or:

$$\tfrac{1}{2}\rho v^2 + p + \rho g z = \text{constant} \tag{8.9}$$

This equation applies to an element of fluid no matter what type of fluid motion is involved. In fluid-mechanics problems, however, it is not in general possible to follow the motion of one particular element, and the usefulness of the above equation lies in the fact that, subject to certain restrictions, the equation can be applied from point to point in a fluid. The nature of these restrictions may be shown in the following way. A streamline in a fluid is defined as a line which has at every point the direction of the velocity of the fluid at that point. If we assume steady flow of the fluid, that is, no change with time, then at any point along a stream-line, successive fluid elements will have identical characteristics. The above equation can thus be applied between any two points on a streamline in a steady flow. Equation 8.9 is the well-known Bernoulli's Equation.

8.6 Bernoulli's Equation by Euler's Method. In the preceding discussion of fluid motion we selected a particular element of fluid and studied the forces on and the motion of that element. This is, of course, the procedure that we have used for all of the problems hitherto taken up in this book. This was the general method used by Lagrange in studying fluid mechanics, and hence this procedure in such problems is usually called the Lagrangian method. It is also possible to adopt a somewhat different point of view, by considering a fixed point in space and observing the motion of the fluid as it passes that point. The aim of the equations will then be to describe what happens to a particle at the instant that it arrives at a particular fixed point, rather than to describe the path of the particle throughout its motion. This was the method used by Euler, and for many fluid mechanics problems it has advantages over the Lagrangian formulation. We shall now derive Bernoulli's equation from this Eulerian point of view as a comparison with the derivation of the preceding section. An advantage of the Eulerian method for this particular problem will be the fact that the non-steady problem can also be considered in the derivation if desired.

Consider a volume element in the form of a section of a tube along a stream-line, as in Fig. 8.8. The cross-section area of the stream-tube is da and the length of the element is ds. We shall suppose that this elementary volume is fixed in space, and we shall consider the fluid flowing through it. The equation of motion for the mass which occupies the volume at a given instant will now be written. We observe that the acceleration of this mass will consist of two parts; first, the velocity of the fluid in the element may vary along the length of the element (the velocity is assumed to be constant across the area), and second, at a given point the velocity may vary with time. Thus, we have:

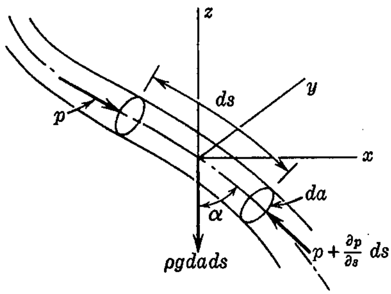


FIG. 8.8

$$v = f(s, t), \quad dv = \frac{\partial v}{\partial s} ds + \frac{\partial v}{\partial t} dt$$

so:

$$\frac{dv}{dt} = v \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}$$

The term $v \dfrac{\partial v}{\partial s}$ may be thought of as the acceleration associated with the particle moving from one point to another point where the velocity is different. The term $\dfrac{\partial v}{\partial t}$ is the acceleration associated with the change of flow with time at any one point.

The equation $F = ma$ can now be written, summing the force components along a stream-line:

$$p\, da - \left(p + \frac{\partial p}{\partial s} ds\right) da + \rho g\, da\, ds \cos \alpha = \rho\, da\, ds \left(v \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right)$$

Noting that $\cos \alpha = -\dfrac{\partial z}{\partial s}$, this becomes:

$$-\frac{\partial p}{\partial s} - \rho g \frac{\partial z}{\partial s} = \rho\left(v \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right)$$

Writing $v \frac{\partial v}{\partial s} = \frac{\partial}{\partial s}(\frac{1}{2}v^2)$, and limiting the problem, as in the previous derivation to steady flow, for which $\frac{\partial v}{\partial t} = 0$, we obtain:

$$-\frac{\partial p}{\partial s} - \rho g \frac{\partial z}{\partial s} = \rho \frac{\partial}{\partial s}(\tfrac{1}{2}v^2)$$

Integrating this equation along the stream-line we obtain Bernoulli's equation:

$$\tfrac{1}{2}\rho v^2 + p + \rho g z = \text{constant.}$$

EXAMPLE. There is a steady flow of fluid from a reservoir through a pipe as shown in Fig. 8.9. Assuming no energy loss in the system, find the velocity v_B with which the fluid issues from the pipe.

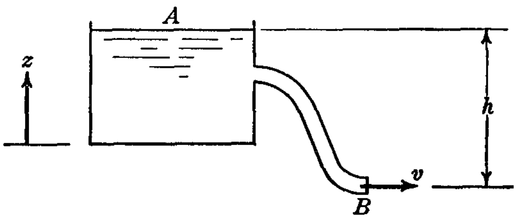


FIG. 8.9

Solution. Applying the energy equation to the two points A and B gives:

$$\tfrac{1}{2}\rho v_A^2 + p_A + \rho g z_A = \tfrac{1}{2}\rho v_B^2 + p_B + \rho g z_B$$

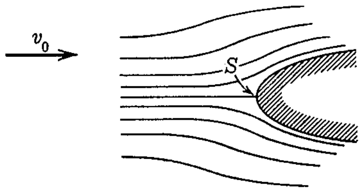
At point A the velocity of the fluid is very small so that to a good approximation $v_A = 0$. The pressure at A is atmospheric pressure, and we can assume with good accuracy that the pressure at B is also atmospheric. Thus:

$$\tfrac{1}{2}\rho v_B^2 = \rho g(z_A - z_B) = \rho g h$$

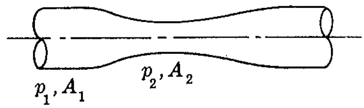
$$v_B = \sqrt{2gh}$$

PROBLEMS

8.5. A solid body is placed in a fluid which flows uniformly with an undisturbed velocity v_0. The fluid divides to flow around the body, and at one point S the fluid comes to rest. If p_s is the "stagnation pressure" at the stagnation point S, and p_0 is the pressure at the same depth in the undisturbed fluid, find p_s in terms of p_0, v_0, and the density of the fluid ρ.



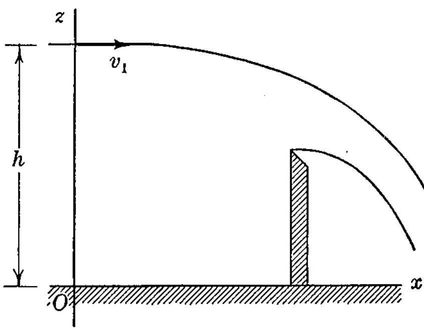
PROB. 8.5



PROB. 8.6

8.6. The velocity of fluid flow is often measured by a device called a Venturi meter, which consists of a section of pipe of a gradually reduced area, as shown in the figure. Find the velocity of flow v in terms of the known areas A_1 and A_2, and the measured pressures p_1 and p_2. In the actual instrument a correction coefficient must be applied to account for non-uniform flow conditions and for friction losses.

8.7. Fluid flows over a weir as shown in the diagram. By applying the energy equation to elements in the top surface of the fluid, express the velocity at any point in the surface as a function of z.

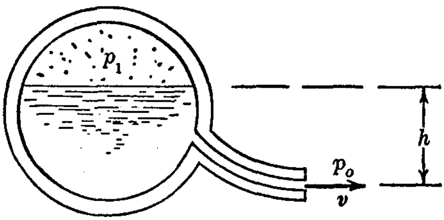


PROB. 8.7

8.8. A pressure vessel is partly filled with liquid and partly with gas under a pressure p_1. Find the discharge velocity v.

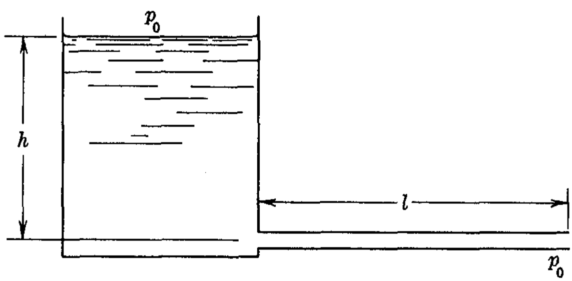
8.9. Using the Eulerian method, carry out the steps for the derivation of Bernoulli's Equation for non-steady flow, and show that the equation becomes:

$$\frac{1}{2}pv^2 + p + \rho gz + \rho \int_0^s \frac{\partial v}{\partial t}\, ds = \text{constant.}$$



PROB. 8.8

8.10. A horizontal pipe of length l and cross-section area A is connected to the bottom of a large tank as shown in the figure. The free surface of the fluid in the tank is a distance h above the pipe. A valve at



PROB. 8.10

the end of the pipe is suddenly opened to the atmosphere, and fluid begins to flow from the pipe. Find the way in which the outflow velocity increases with time, and find the time required to reach the steady value. Assume that the velocities of all the fluid particles in the pipe are equal, and that the velocity of all of the fluid in the tank is zero.

8.7 The Momentum Equation. For some types of fluid problems, the equations of motion are found to be in their most convenient form when expressed in terms of a momentum principle. To derive a general momentum equation in a form suitable for fluid applications, we shall use the Eulerian approach, as described in the preceding section.

As in Fig. 8.8 above, let us consider the fluid flowing through a particular volume $dV = da\, ds$ which is fixed in space. If $\boldsymbol{F}$ is the resultant force acting on the fluid within the volume dV, then we have:

$$\boldsymbol{F} = (\rho\, dV)\frac{d\boldsymbol{v}}{dt}$$

Also, as shown above:

$$\frac{d\boldsymbol{v}}{dt} = v\frac{\partial \boldsymbol{v}}{\partial s} + \frac{\partial \boldsymbol{v}}{\partial t}$$

so we obtain:

$$\boldsymbol{F} = \rho\, dV\left(\frac{\partial \boldsymbol{v}}{\partial t} + v\frac{\partial \boldsymbol{v}}{\partial s}\right)$$

We next refer to Fig. 8.10, in which a total volume V fixed in space

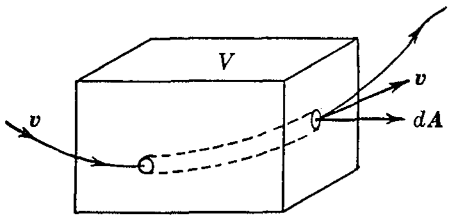


FIG. 8.10

is considered. We imagine that this volume is made up of stream-tubes, of which a typical one is shown dotted. The preceding equation, which applies to the typical dotted stream-tube, can now be integrated over the total volume V to give:

$$\Sigma\boldsymbol{F} = \int_V \rho\frac{\partial \boldsymbol{v}}{\partial t}\, dV + \int_V \rho v\frac{\partial \boldsymbol{v}}{\partial s}\, da\, ds$$

where $\Sigma\boldsymbol{F}$ is the resultant force acting on all of the fluid within the volume V. We shall now transform these integrals into forms which are readily evaluated.

Using the formula for the derivative of a product, we have:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = \frac{\partial}{\partial t}(\rho\boldsymbol{v}) - \boldsymbol{v}\frac{\partial \rho}{\partial t}$$

and

$$\rho v\frac{\partial \boldsymbol{v}}{\partial s}\, da = \frac{\partial}{\partial s}(\rho v\boldsymbol{v}\, da) - \boldsymbol{v}\frac{\partial}{\partial s}(\rho v\, da)$$

Substituting these expressions into the above integrals gives:

$$\Sigma \boldsymbol{F} = \int \left\{ \frac{\partial}{\partial t} (\rho \boldsymbol{v}) dV + \frac{\partial}{\partial s} (\rho v \boldsymbol{v} \, da) ds - \boldsymbol{v} \left[\frac{\partial \rho}{\partial t} dV + \frac{\partial}{\partial s} (\rho v \, da) ds \right] \right\}$$

It can be shown that the term in square brackets vanishes by virtue of the conservation of mass. Consider an element of volume $dV = da\, ds$ as shown in Fig. 8.11. The rate at which mass is accumulating in this element, because of changing ρ, is $\frac{\partial \rho}{\partial t} dV$. This must be equal

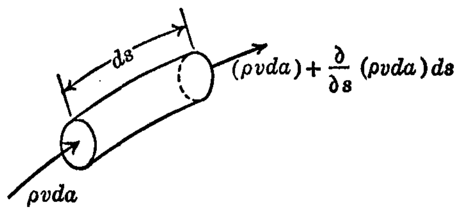


FIG. 8.11

to the difference between the rate of inflow of mass and the rate of outflow:

$$\frac{\partial \rho}{\partial t} dV = \rho v da - \left[(\rho v \, da) + \frac{\partial}{\partial s} (\rho v \, da) ds \right]$$

which gives

$$\frac{\partial \rho}{\partial t} dV + \frac{\partial}{\partial s} (\rho v \, da) ds = 0.$$

This expression is usually called the continuity equation, which is often written in a vector form as

$$\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{v}) = 0.$$

The expression for the resultant force acting on the fluid within V thus reduces to:

$$\Sigma \boldsymbol{F} = \int_V \frac{\partial}{\partial t} (\rho \boldsymbol{v}) dV + \int_V \frac{\partial}{\partial s} (\rho v \boldsymbol{v} \, da) ds$$

The first integral may be written as:

$$\int_V \frac{\partial}{\partial t} (\rho \boldsymbol{v}) dV = \frac{\partial}{\partial t} \int \rho \boldsymbol{v} \, dV = \frac{\partial \mathfrak{M}}{\partial t}$$

where $\mathfrak{M}$ is the resultant momentum of the fluid within V.

The second integral may be integrated along the length s of the stream-tube to give:

$$\int_V \frac{\partial}{\partial s}(\rho v \boldsymbol{v}\, da)ds = \int_A (\rho v \boldsymbol{v}\, da)\Big|_{s_1}^{s_2}$$

The notation $(\rho v \boldsymbol{v}\, da)\Big|_{s_1}^{s_2}$ indicates that this term is to be evaluated at the two end points of the stream-tube; that is, at the points where the stream tube intersects the boundary of the volume. The resulting area integral is to be evaluated, therefore, over the areas of the ends of the stream-tubes lying on the surface of the volume. We note that the direction of a stream-tube as it intersects the boundary is in general inclined at some angle to the normal to the boundary surface. If the vector $d\boldsymbol{A}$, normal to the surface and directed out of the volume, describes the increment of surface area, then on the surface $v\, da = \boldsymbol{v}\cdot d\boldsymbol{A}$, and the area integral may be written as:

$$\int_A \rho\boldsymbol{v}(\boldsymbol{v}\cdot d\boldsymbol{A})$$

Since $\rho\boldsymbol{v}\cdot d\boldsymbol{A}$ represents the mass flow per unit time out through $d\boldsymbol{A}$, the term $\rho\boldsymbol{v}(\boldsymbol{v}\cdot d\boldsymbol{A})$ represents the rate at which momentum is flowing out through $d\boldsymbol{A}$. *The integral thus represents the net rate of outflow of momentum through the surface of V.*

We thus obtain finally the following expression for the resultant force acting upon the fluid within the volume V:

$$\Sigma\boldsymbol{F} = \frac{\partial\mathfrak{M}}{\partial t} + \int_A \rho\boldsymbol{v}(\boldsymbol{v}\cdot d\boldsymbol{A}) \qquad (8.10)$$

The first term represents the time rate of change of the total momentum of the fluid within the element at a particular time, and the second term represents the total momentum per unit time passing through the area of the element. Note particularly that this form of Newton's law does not apply to a particular mass element, but rather considers a fixed volume element within which the mass may be changing. The above analysis places no restrictions upon the compressibility or viscosity of the fluid.

The foregoing analysis can be repeated, beginning with the equation of moment of momentum of the fluid within the fixed element of volume dV:

$$\boldsymbol{r} \times \left(\rho\, dV \frac{d\boldsymbol{v}}{dt}\right) = \boldsymbol{r} \times \boldsymbol{F}$$

and the following result will be obtained:

$$\Sigma \boldsymbol{M}_t = \frac{\partial \boldsymbol{H}}{\partial t} + \int_A \boldsymbol{r} \times \rho \boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A}) \tag{8.11}$$

where $\Sigma \boldsymbol{M}_t$ is the resultant moment acting on the fluid within the volume V about some fixed point O, and $\boldsymbol{H}$ is the total moment of momentum of the fluid within the volume about the point O.

The essence of the momentum flow method is that instead of observing a particular mass element of fluid during its motion, attention is focused on a region of space. By noting the inflow and outflow of momentum and the change of momentum within the region it is possible to deduce the resultant force acting on the fluid within the region.

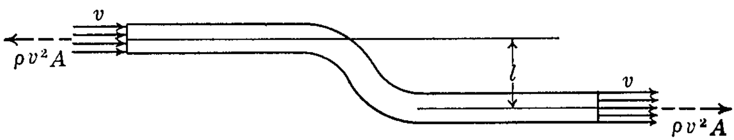


FIG. 8.12 (a)

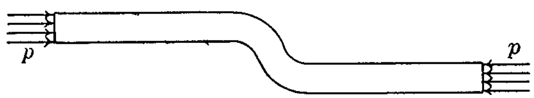


FIG. 8.12 (b)

EXAMPLE 1. A pipe of uniform cross-sectional area A has a horizontal offset l as shown in Fig. 8.12 (a). Fluid flows through the pipe with a uniform velocity v. Find the moment exerted on the pipe by the fluid.

Solution. Considering the flow of momentum through the volume of the pipe indicated in the figures, and evaluating the integral $\int \rho \boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A})$ we obtain the two vectors of magnitude $\rho v^2 A$ shown in Fig. 8.12 (a). Note that the directions are such that both these vectors are pointing out of the volume. There is thus a counter-clockwise moment of magnitude $\rho v^2 A l$ exerted upon the fluid within the volume. This moment is the resultant of the moment due to the pressure forces, and the moment M_p which is applied to the fluid by the pipe (Fig. 8.12 b). Thus:

$$\rho v^2 A l = -pAl + M_p$$

The moment exerted on the pipe by the fluid is equal and opposite to M_p, and is hence a clockwise moment of magnitude $(\rho v^2 + p)Al$.

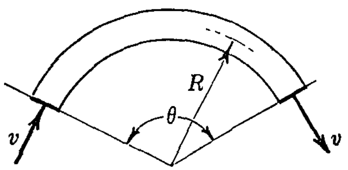


FIG. 8.13

EXAMPLE 2. Fluid flows with a uniform velocity v through a pipe of uniform cross-sectional area in a horizontal plane as shown in Fig. 8.13. Find the external force which must be applied to the pipe to maintain the system in equilibrium.

Solution. Fig. 8.14 shows the momentum flow vectors $\int \rho \boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A})$ and the pressure forces acting upon the volume of fluid. The resultant momentum-flow vector is equal to the force acting upon the fluid within the pipe as shown in (b). This resultant force is the sum of the two pressure forces, and the force F_p exerted by the pipe on the fluid. This force F_p can

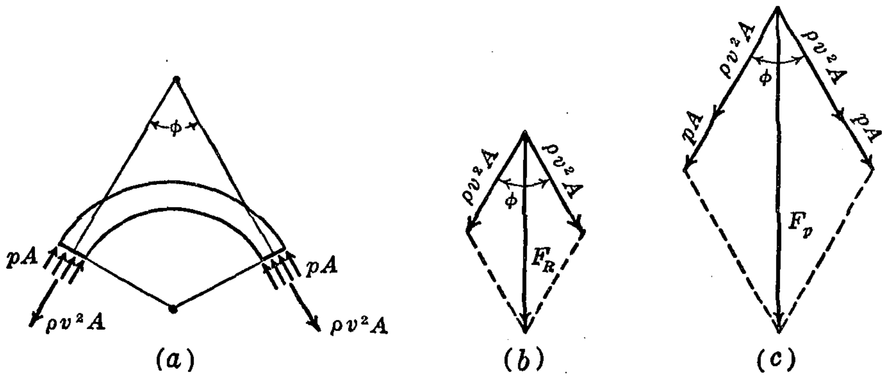
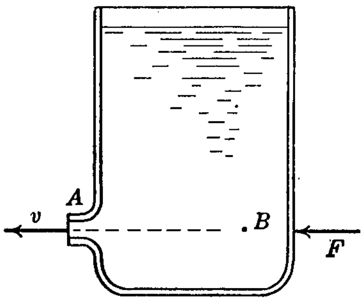


FIG. 8.14

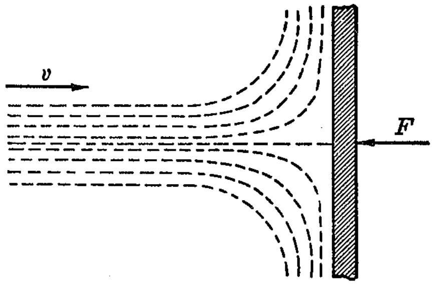
thus be found as shown in (c). The fluid exerts a force on the pipe which is equal and opposite to F_p, and hence the external force which must be applied to the pipe in order to maintain it in equilibrium is F_p, directed as shown in (c).

PROBLEMS

8.11. Fluid is discharged from a tank with velocity v through an outlet of area A. Show that in order to maintain equilibrium of the system an external force $F = 2(p_B - p_A)A$ must act as shown in the figure, where p_A is the atmospheric pressure, and p_B is the pressure within the tank at the same elevation as the outlet, at a point where the fluid velocity is zero.



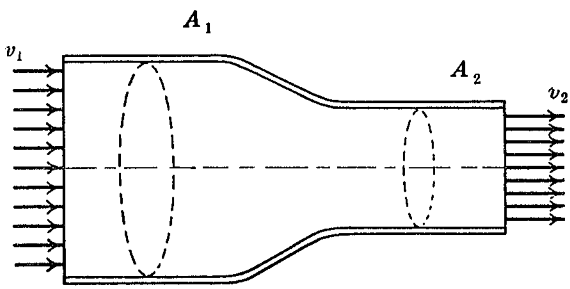
PROB. 8.11



PROB. 8.12

8.12. A stream of fluid impinges on a stationary surface as shown in the diagram. Find the force exerted by the fluid on the surface, by the method of the preceding section.

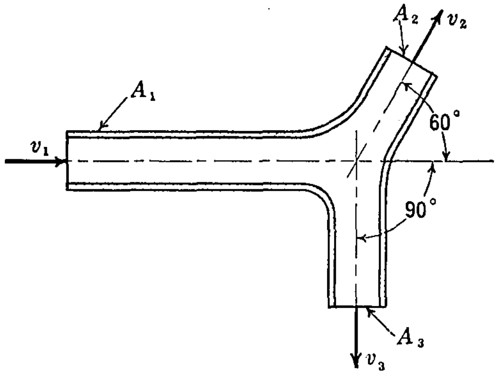
8.13. An incompressible fluid flows through a pipe which has a change in cross-section as shown. Both ends of the pipe are at the same elevation.



PROB. 8.13

Assuming that the velocity is uniform across the area of both sections, find the force exerted on the pipe by the fluid.

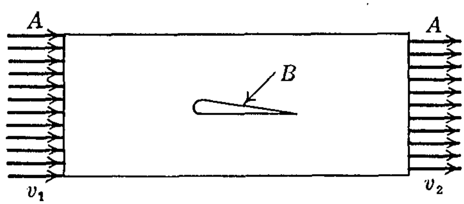
8.14. Three pipes each of uniform cross-section lie in a horizontal plane and converge at a point as shown in the figure. Fluid flows through A_1 with a velocity v_1, and out of A_2 and A_3 with velocities v_2 and v_3. The dimensions of the system are such that $A_2v_2 = \frac{1}{2}A_1v_1$. Find the resultant force required to hold the joint in equilibrium, assuming that the velocities are uniform across the sections of the pipe, and that the pressure in the fluid is negligible.



PROB. 8.14

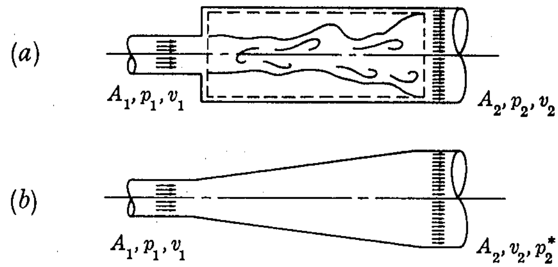
8.15. Carry through the steps indicated in the text leading to Equation (8.11) above, and thus derive the expression for the moment of momentum relationship in fluid flow.

8.16. A frictionless compressible fluid flows through a straight pipe. An object B is held in a fixed position in the stream. Assuming that there is steady flow, find the relation between the force exerted by the fluid on B, and the velocities, densities, and pressures of the fluid at the two ends of the pipe.



PROB. 8.16

8.17. When the area of a pipe in which fluid is flowing increases, the velocity of the fluid decreases and hence the pressure increases. Compare this increase of pressure in the two cases shown in the diagram. In (a) the pipe increases suddenly in area from A_1 to A_2, while in (b) the transition is made gradually. Note that in (a) there will be a mixing region of irregular flow, but that after a certain distance the flow will be again approximately uniform with an averge velocity v_2. The pressure change in case (a) may be determined, without a consideration of the details in



PROB. 8.17

the mixing region, by applying the momentum equation to the dashed volume shown in (a).

8.8 The Momentum Equation for an Accelerating Volume. In the preceding section we considered the problem of flow of a fluid through a stationary volume. We shall now consider the more general problem in which the volume itself is accelerating. This is, for example, the type of problem involved in an analysis of the dynamics of rocket flight.

Let the absolute velocity of a point in the fluid be $\boldsymbol{v}$ and consider a coordinate system fixed in the moving volume. If $\boldsymbol{v}_r$ is the velocity of a point in the fluid relative to the moving coordinate system and $\boldsymbol{u}$ is the absolute velocity of the corresponding point in the coordinate system, then $\boldsymbol{v} = \boldsymbol{v}_r + \boldsymbol{u}$. Let s be measured along a stream-line in the moving coordinate system. By a stream-line in the moving coordinate system we mean a line which has everywhere the direction of $\boldsymbol{v}_r$. We may now write:

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial s}\frac{ds}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + v_r \frac{\partial \boldsymbol{v}}{\partial s}$$

The resultant external force acting upon the mass contained within the volume is:

$$\Sigma \boldsymbol{F} = \int \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + v_r \frac{\partial \boldsymbol{v}}{\partial s} \right) dV$$

This integral may be transformed in the same way as the corresponding integral obtained for the fixed volume of the preceding section, with the result:

$$\Sigma \boldsymbol{F} = \frac{\partial \mathfrak{M}}{\partial t} + \int_A \rho \boldsymbol{v}(\boldsymbol{v}_r \cdot d\boldsymbol{A}) \tag{8.12}$$

where $\mathfrak{M}$ is the total momentum of the mass within the volume at any instant, and the integral represents the net rate of outflow of momentum from the volume. Note that the term $\rho(\boldsymbol{v}_r \cdot d\boldsymbol{A})$ is the mass per unit time which passes through the surface $d\boldsymbol{A}$, and hence physically is the same thing as the $\rho(\boldsymbol{v} \cdot d\boldsymbol{A})$ term obtained for the fixed volume.

From the moment of momentum relationship the following equation can be derived:

$$\Sigma \boldsymbol{M}_t = \frac{\partial \boldsymbol{H}}{\partial t} + \int_A \boldsymbol{r} \times \rho \boldsymbol{v}(\boldsymbol{v}_r \cdot d\boldsymbol{A}) \tag{8.13}$$

$\boldsymbol{H}$ represents the total moment of momentum of the mass within the volume under consideration, and $\Sigma \boldsymbol{M}_t$ is the resultant external torque acting upon the material within the volume. $\boldsymbol{H}$ and $\Sigma \boldsymbol{M}_t$ may be measured either with respect to a fixed point or with respect to the moving center of mass of the system. If $\boldsymbol{H}$ and $\Sigma \boldsymbol{M}_t$ are measured with respect to the moving center of mass, however, the velocity $\boldsymbol{v}$ appearing in Equation (8.13) is no longer the absolute velocity as used in the preceding analysis, but is the velocity of the element with respect to the center of mass.

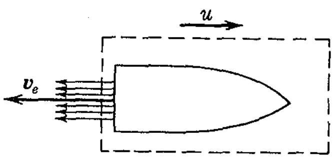


FIG. 8.15

EXAMPLE 1. A rocket travels in straight horizontal flight. The exhaust velocity of the jet relative to the rocket is v_e and the gage pressure in the jet of area A is p. The mass of fuel burned per unit time is $\rho v_e A$ and the total mass of the rocket at any time is $m = m_0 - \rho v_e A t$. Assuming that gravity forces and drag forces are negligible, derive the differential equation of motion of the rocket, and find the velocity of the rocket as a function of time.

Solution. As the volume to which momentum flow considerations are to be applied let us take the complete rocket, including shell and propellant, as in Fig. 8.15. This volume is attached to the rocket, and hence moves with the same velocity $\boldsymbol{u}$ as the rocket. The only external force acting on the material within the volume is pA, since gravity and drag forces are assumed negligible, and hence Equation (8.12) gives directly:

$$\Sigma \boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}_c) + \int_A \rho(\boldsymbol{u} - \boldsymbol{v}_e)(\boldsymbol{v}_e \cdot d\boldsymbol{A}) = pA$$

$\boldsymbol{v}_c$ is the velocity of the center of mass of the material within the volume, which to a very good approximation may be taken as $\boldsymbol{u}$, the velocity of the rocket. The equation thus becomes:

$$m\dot{u} + u\frac{dm}{dt} + \rho(u - v_e)v_eA = pA$$

but $\dfrac{dm}{dt} = -\rho v_e A$, so we obtain:

$$m\dot{u} = \rho v_e^2 A + pA$$

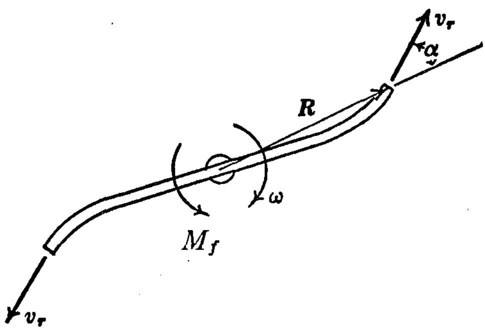


FIG. 8.16

This equation may be integrated by putting it into the form:

$$du = (\rho v_e^2 A + pA)\frac{dt}{(m_0 - \rho v_e A t)}$$

from which, assuming that the rocket starts from rest, we obtain:

$$u = v_e\left(1 + \frac{p}{\rho v_e^2}\right)\log_e\left(\frac{m_0}{m_0 - \rho v_e A t}\right)$$

This general treatment of the rocket problem by momentum flow principles should be compared with the special rocket problems discussed previously in Chapter 3.

EXAMPLE 2. A lawn sprinkler consists of two sections of curved pipe rotating about a vertical axis as shown in Fig. 8.16. The sprinkler rotates with an angular velocity $\boldsymbol{\omega}$, and the effective discharge area is A, so that the water is discharged at a rate $Q = 2v_rA$, where v_r is the velocity of the water relative to the rotating pipe. A constant friction torque M_f resists the motion of the sprinkler. Find an expression for the speed of the sprinkler in terms of the significant variables.

Solution. We apply the momentum flow principle in the form of the moment of momentum relationship for a moving volume, taking a volume which coincides with the sprinkler pipe:

$$\Sigma \boldsymbol{M}_t = \frac{\partial \boldsymbol{H}}{\partial t} + {}_{A}\ \boldsymbol{r} \times \rho \boldsymbol{v}(\boldsymbol{v}_r \cdot d\boldsymbol{A})$$

If we consider only steady motion $\frac{\partial \boldsymbol{H}}{\partial t} = 0$, the only external force acting is the friction torque $\boldsymbol{M}_f$, and we obtain, taking vertical-upward as the negative direction:

$$-\boldsymbol{M}_f = 2\boldsymbol{R} \times \rho \boldsymbol{v}(v_r A)$$

The absolute velocity is given by:

$$\boldsymbol{v} = \boldsymbol{v}_r + \boldsymbol{\omega} \times \boldsymbol{R}$$

$$-\boldsymbol{M}_f = 2\boldsymbol{R} \times \rho(\boldsymbol{v}_r + \boldsymbol{\omega} \times \boldsymbol{R})v_r A$$

$$-M_f = -2R\rho A v_r^2 \sin\alpha + 2R^2 \rho A v_r \omega$$

$$\omega = \frac{1}{2R^2 \rho A v_r}(2R\rho A v_r^2 \sin\alpha - M_f)$$

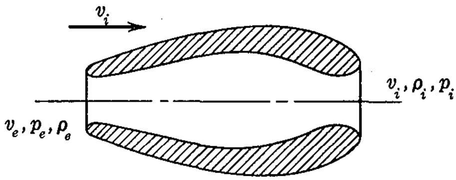
from which

$$\omega = \frac{v_r \sin\alpha}{R} - \frac{M_f}{Q\rho R^2}$$

PROBLEMS

8.18. Show that the jet thrust force acting upon an accelerating rocket is equal to the thrust force acting on the same rocket when it is held stationary in a test stand. The exhaust velocities and exhaust pressures are assumed to be the same in each case.

8.19. A ramjet missile is traveling in a straight horizontal path with a velocity v_i. Air is taken in at the inlet with a relative velocity v_i, a pressure p_i, and a density ρ_i. After an internal combustion process the exhaust gases are discharged with a relative velocity v_e, at a pressure p_e, and density ρ_e. The intake and exhaust areas are A_i and A_e. Using momentum flow principles, find the propulsive force acting on the ramjet.

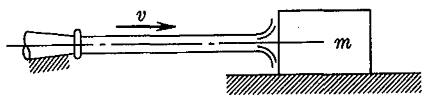


PROB. 8.19

8.20. A fluid jet of area A and velocity v from a fixed nozzle strikes the vertical face of a block of mass m. After striking the block the fluid leaves with a velocity parallel to the face of the block.

(a) If the block slides on a frictionless horizontal plane, find the velocity u of the block as a function of time, if $u = 0$ when $t = 0$.

(b) If the coefficient of sliding friction between the block and the horizontal plane is μ, independent of velocity, find the terminal velocity of the block.



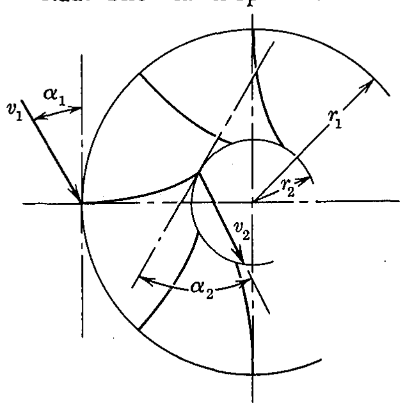
PROB. 8.20

8.21. Water enters a turbine wheel with an absolute velocity v_1 at an angle α_1 with the tangent to the wheel at a radius r_1 as shown in the diagram. The water leaves the wheel with an absolute velocity v_2 at an angle α_2 with the tangent at a radius r_2. The mass of water flowing through the turbine per unit time is m'. Show that the torque M_t exerted on the wheel by the water is given by:

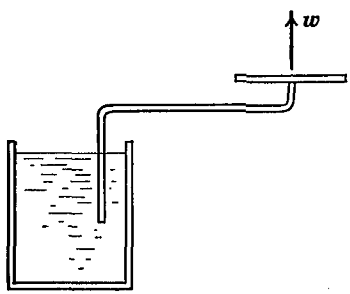
$$M_t = m'(v_1 r_1 \cos \alpha_1 - v_2 r_2 \cos \alpha_2)$$

This expression is often called Euler's turbine relation.

8.22. The "lawn sprinkler" turbine of Example 2 above is to be used

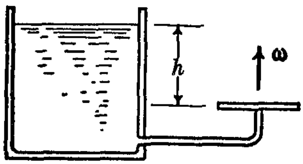


PROB. 8.21



PROB. 8.22

as a pump, as indicated in the diagram. A pumping torque M_p applied to the turbine gives an angular velocity to the device. Find the quantity of water discharged per unit time. Note that, if the torque and speed are specified, the pressure-elevation relations would be determined by energy considerations (Bernoulli's Equation).



PROB. 8.23

8.23. The turbine of Prob. 8.22 discharges water under a head h. If the turbine is held stationary, what quantity of water is discharged per unit time? If the turbine is free to rotate with no friction, what is the discharge rate? Assume steady flow with the turbine running at constant speed. Determine the speed of the turbine and discuss the physical meaning of the relationships found for flow rate and speed.

Chapter 9

ADVANCED METHODS IN DYNAMICS

"When one obtains a simple result by means of complicated calculations, there must exist a more direct method of obtaining the result; the simplifications which occur and the terms which disappear during the course of the calculations are certain indications that a method exists for which these simplifications have already been made and in which these terms do not appear."—M. Lamé, *Théorie de l'Élasticité* (1866).

As the problems in dynamics become more complex it naturally becomes increasingly difficult to work out the solutions. This difficulty is associated not only with the solution of the equations of motion, but with their formulation as well. In fact, the derivation of the basic differential equations of motion in a form suitable for a particular complicated problem may well be the most difficult part of the investigation. A number of methods, more powerful than those hitherto considered in this book, have been developed for deriving the equations for these more involved situations. Perhaps the most generally useful of these more advanced methods for engineering problems is that of Lagrange, who has put the basic equations of motion in such a form that the simplifying features of a particular problem can be utilized most advantageously. In the present chapter we shall derive Lagrange's equations, and we shall indicate their application by a number of examples.

9.1 Generalized Coordinates. One of the principal advantages of Lagrange's method is that one uses for each problem that coordinate system which most conveniently describes the motion. We have already seen that the position of a particle can be described in a large number of different ways, and we have found in the problems already discussed that the choice of a proper coordinate system may introduce a considerable simplification into the solution of a problem. In general, the requirement for a system of coordinates is that the specification of the coordinates must locate completely the position of each part of the system. This means that there must be one coordinate associated with each degree of freedom of the system.* We shall restrict the following treatment to systems whose coördinates are independent, in the sense that a change can be given to any one of the coordinates without changing any of the other coordinates.† By the *generalized coordinates* $(q_1, q_2, \ldots q_n)$ we shall mean a set of independent coordinates, equal in number to the degrees of freedom of the system. We use the word "generalized" to emphasize the fact that such coordinates are not necessarily of the type of the simple (x, y, z) or (r, θ, ϕ) systems which we have already used, and to indicate that they are not necessarily lengths or angles, but may be any quantity appropriate to the description of the position of the system.

The (x, y, z) coordinates of a point are expressible in terms of the generalized coordinates (q_1, q_2, q_3) by functional relations‡:

$$\begin{aligned} x &= \phi_1(q_1, q_2, q_3) \\ y &= \phi_2(q_1, q_2, q_3) \\ z &= \phi_3(q_1, q_2, q_3) \end{aligned} \tag{9.1}$$

For example, if (q_1, q_2, q_3) are the cylindrical coordinates (r, θ, z) the foregoing equations become:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

The equations of motion in generalized coordinates for any particular system could always be obtained by writing the equations

* More exactly, there must be at least one coordinate associated with each degree of freedom. So called *non-holonomic* systems exist, for which, because of the particular geometrical constraints involved, more coordinates are required than there are degrees of freedom. Such systems are not often encountered and will not be considered here. See Appendix I, references 32 and 33.

† In some problems it may be more convenient to use coordinates which are not all independent. See Appendix I, references 2, 7, 32, and 33.

‡ We have supposed that the relation between the coordinate systems does not involve time. In the more general treatment in which $x = \phi(q_1, q_2, \ldots q_n, t)$ the analysis can proceed along essentially the same lines. See Appendix I, reference 2.

first in an (x, y, z) system, and then transforming to the q's by Equations (9.1). This procedure usually leads to involved algebraic manipulations, and it is better to make the transformation in general terms and to write the equations of motion directly in generalized coordinates.

9.2 Lagrange's Equations for a Particle. We shall first show how the transformation to generalized coordinates may be made for one particle. The extension to systems of particles will be a direct one involving only a summation of the result over all of the particles of the system.

The equations of motion of the particle will first be written in the form:

$$F_x = m\ddot{x} \qquad F_y = m\ddot{y} \qquad F_z = m\ddot{z}$$

Multiply through the first equation by δx, the second equation by δy, and the third equation by δz, and add the resulting equations:

$$F_x\delta x + F_y\delta y + F_z\delta z = m\ddot{x}\delta x + m\ddot{y}\delta y + m\ddot{z}\delta z \tag{9.2}$$

The quantities δx, δy and δz are small displacements, and they have been written as deltas to emphasize that they may have arbitrary values, consistent with the physical restraints, and hence are not differentials.* The quantities δx, δy and δz are virtual displacements, in the sense that this term is used in statics, and Equation (9.2) is equivalent to Equation (7.30) previously obtained by a combination of D'Alembert's Principle and the Principle of Virtual Displacements.

We now transform the coordinates in Equation (9.2) from the (x, y, z) coordinates to the (q_1, q_2, q_3) system, using the general transformation Equations (9.1) above. From Equation (9.1) the displacements δx, δy, and δz can be written in terms of the q's as:

$$\begin{aligned}
\delta x &= \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \frac{\partial x}{\partial q_3}\delta q_3 \\
\delta y &= \frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2 + \frac{\partial y}{\partial q_3}\delta q_3 \\
\delta z &= \frac{\partial z}{\partial q_1}\delta q_1 + \frac{\partial z}{\partial q_2}\delta q_2 + \frac{\partial z}{\partial q_3}\delta q_3
\end{aligned} \tag{9.3}$$

* If, for example, in a particular situation $y = ax^2$, then $dy = 2ax\,dx$, and dy and dx are related. The symbol d rather than δ would be used in such a case to indicate that dy is not arbitrary but depends upon the value of dx.

In order to introduce a simplification into the algebra of the derivation, we shall now suppose that $\delta q_1 \neq 0$ while $\delta q_2 = \delta q_3 = 0$. This is permissible since we have defined the generalized coordinates above as independent coordinates. Equations (9.3) thus become:

$$\delta x = \frac{\partial x}{\partial q_1}\,\delta q_1; \quad \delta y = \frac{\partial y}{\partial q_1}\,\delta q_1; \quad \delta z = \frac{\partial z}{\partial q_1}\,\delta q_1$$

Substituting these values into Equation (9.2), we obtain:

$$\left(F_x \frac{\partial x}{\partial q_1} + F_y \frac{\partial y}{\partial q_1} + F_z \frac{\partial z}{\partial q_1}\right)\delta q_1 = \left(m\ddot{x}\,\frac{\partial x}{\partial q_1} + m\ddot{y}\,\frac{\partial y}{\partial q_1} + m\ddot{z}\,\frac{\partial z}{\partial q_1}\right)\delta q_1 \tag{9.4}$$

The left side of Equation (9.4) has a simple physical meaning: it is the work done by the external forces of the system during the displacement δq_1. We shall equate this work to the expression $(Q_1 \delta q_1)$ and thus define the *generalized force* Q_1 as:

$$Q_1 = F_x \frac{\partial x}{\partial q_1} + F_y \frac{\partial y}{\partial q_1} + F_z \frac{\partial z}{\partial q_1} \tag{9.5}$$

This definition of the generalized force Q indicates the way in which Q can be determined in specific problems. To find Q_i, the total work done by all of the external forces during a small displacement δq_i of one of the coordinates is calculated, and Q_i is then obtained by dividing this total work by δq_i. It should be noted that the generalized force does not necessarily have the dimensions of a force; for example, if q_i is an angle, δq_i is dimensionless, and Q_i would have the dimensions $[F^1L^1T^0]$. In this particular case, Q_i would be a moment. Similarly if the generalized coordinate is a volume, then the dimensions of the generalized force would be $[F^1L^{-2}T^0]$.

We next transform the right side of Equation (9.4). From the formula for the differentiation of a product we have:

$$\ddot{x}\,\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\,\frac{\partial x}{\partial q_1}\right) - \dot{x}\,\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) \tag{9.6}$$

From Equations (9.1) we may write:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}\,\dot{q}_1 + \frac{\partial x}{\partial q_2}\,\dot{q}_2 + \frac{\partial x}{\partial q_3}\,\dot{q}_3$$

so that: $\dfrac{\partial \dot{x}}{\partial \dot{q}_1} = \dfrac{\partial x}{\partial q_1}$; etc.

We may also put:

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial}{\partial q_1}\left(\frac{dx}{dt}\right) = \frac{\partial \dot{x}}{\partial q_1}; \text{ etc.}$$

With these substitutions, Equation (9.6) becomes:

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2}\right)\right] - \frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2}\right)$$

Substituting this into the right hand side of Equation (9.4) along with similar expressions for y and z, we obtain:

$$Q_1\delta q_1 = m\left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2}\right)\right] - \frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2}\right)\right\}\delta q_1$$

But the kinetic energy T of the particle is:

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

So we have:

$$Q_1\delta q_1 = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1}\right]\delta q_1$$

from which:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1 \qquad (9.7)$$

This is Lagrange's equation in the form in which it is most useful for engineering problems.

If in the above derivation we had put $\delta q_2 \neq 0$ while $\delta q_1 = \delta q_3 = 0$, we would have obtained the same Equation (9.7) in terms of q_2 instead of q_1, and in a similar way an equation in terms of q_3 could be obtained. The above procedure thus leads to three independent Lagrangian equations of motion corresponding to the three degrees of freedom of the single particle, which we can write as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, 2, 3) \qquad (9.8)$$

If the forces acting on the system are conservative forces, so that they can be derived from a potential energy V we may say by definition that:

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad (9.9)$$

Thus for conservative systems Lagrange's equations can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \qquad (9.10)$$

The potential energy V is a function of the generalized coordinates q_i but does not involve the velocities $\dot{q}_i$, thus $\frac{\partial V}{\partial \dot{q}_i} = 0$, and Equation (9.10) may be written in the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (9.11)$$

where $L = (T - V)$. The quantity L, the difference between the kinetic and the potential energy, is called the *Lagrangian Function*, or the *Kinetic Potential* of the system.

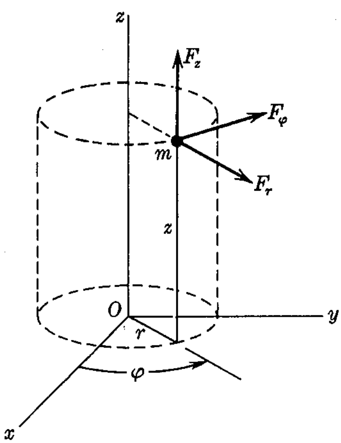


FIG. 9.1

It should be noted that Lagrange's equations in the form of Equations (9.10) or (9.11) apply to conservative systems only, whereas Equation (9.8) applies to non-conservative systems as well.

EXAMPLE 1. Derive the equations of motion for a particle in cylindrical coordinates, using Lagrange's equations.

Solution. In this problem we have:

$$q_1 = r$$
$$q_2 = \phi$$
$$q_3 = z$$

The kinetic energy T of the particle is:

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)$$

Thus, Lagrange's equation for the r-coordinate becomes:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r}; \quad \frac{\partial T}{\partial r} = mr\dot{\phi}^2$$

$$Q_r\delta r = F_r\delta r; \quad Q_r = F_r$$

and the differential equation of motion is:

$$m(\ddot{r} - r\dot{\phi}^2) = F_r$$

For the ϕ-coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = Q_\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \frac{d}{dt}(mr^2\dot{\phi}) = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi}$$

$$\frac{\partial T}{\partial \phi} = 0; \quad Q_\phi \delta\phi = F_\phi r \delta\phi, \quad Q_\phi = F_\phi r$$

and the differential equation of motion is:

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = F_\phi$$

For the z-coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = Q_z$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = m\ddot{z}; \quad \frac{\partial T}{\partial z} = 0$$

$$Q_z \delta z = F_z \delta z, \quad Q_z = F_z$$

and the differential equation of motion is:

$$m\ddot{z} = F_z$$

It will be noted that these are the same expressions obtained in Chapter 2, Equations (2.8).

EXAMPLE 2. A simple pendulum consisting of a concentrated mass m and a weightless string of length l is mounted on a massless support which is elastically restrained horizontally by means of a spring having a spring constant k, as shown in Fig. 9.2. Write the equations of motion for the system, using Lagrange's equations, and find the frequency of small oscillations of the pendulum.

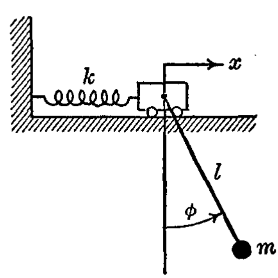


FIG. 9.2

Solution. We shall use as generalized coordinates for this problem the angle ϕ of the pendulum with the vertical, and the horizontal displacement x of the point of support. The velocity of the mass m in terms of these coordinates can be found by combining the two velocity components $\dot{x}$ and $l\dot{\phi}$ by the cosine law, the angle between these velocity component vectors being ϕ. The kinetic energy of the mass is thus given by:

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi)$$

Since this is a conservative system, the forces can be determined from the potential energy, or they can be determined directly from the basic definition of the generalized force. Lagrange's equations can thus be used either in the form of Equation (9.8) or of Equation (9.10). We shall use in this example both methods for purposes of comparison.

The potential energy of the system may be computed, from its basic definition, as the negative of the work done by the forces of the system as the coordinates are increased from zero to general positive values.

For the spring:

$$V_s = -\int_0^x (-kx'dx') = \tfrac{1}{2}kx^2$$

For the pendulum, taking h as a vertical displacement:

$$V_p = -\int_0^{l(1-\cos\phi)} (-mg)\,dh = mgl(1 - \cos\phi)$$

thus the total potential energy is:

$$V = \tfrac{1}{2}kx^2 + mgl(1 - \cos\phi)$$

To find the generalized force Q_x, imagine that the spring is stretched a positive amount x, and consider the work done by all of the forces of the system during an additional displacement δx. The gravity force on the pendulum, mg, does no work, and the spring force does the work $(-kx)\delta x$. Equating the total work to $Q_x\delta x$ we obtain:

$$Q_x\delta x = -kx\delta x, \quad Q_x = -kx$$

Similarly, to find Q_ϕ, start the pendulum at some positive angle ϕ and increase this angle by an amount $\delta\phi$. During this increase the spring force does no work, but the gravity force mg is raised by an amount Δh, as shown in Fig. 9.3.

Therefore:

$$Q_\phi \delta\phi = -mg\Delta h$$

but:

$$\Delta h = l(\delta\phi) \sin \phi$$

so:

$$Q_\phi \delta\phi = -mgl \sin \phi \delta\phi$$

and:

$$Q_\phi = -mgl \sin \phi$$

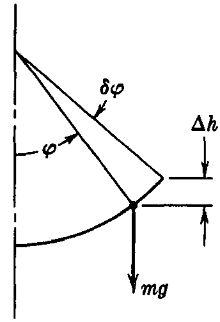


FIG. 9.3

We may check these expressions for the generalized forces by deriving them from the potential energy:

$$Q_x = -\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}[\tfrac{1}{2}kx^2 + mgl(1 - \cos \phi)] = -kx$$

$$Q_\phi = -\frac{\partial V}{\partial \phi} = -\frac{\partial}{\partial \phi}[\tfrac{1}{2}kx^2 + mgl(1 - \cos \phi)] = -mgl \sin \phi$$

Now, using Lagrange's equations (Equations (9.8) or (9.10)) we obtain, for the x-coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x} + ml\dot{\phi} \cos \phi) = m\ddot{x} + ml\,\ddot{\phi} \cos \phi - ml\dot{\phi}^2 \sin \phi$$

$$\frac{\partial T}{\partial x} = 0; \quad Q_x = -\frac{\partial V}{\partial x} = -kx$$

so the differential equation becomes:

$$m\ddot{x} + ml\ddot{\phi} \cos \phi - ml\,\dot{\phi}^2 \sin \phi + kx = 0$$

For the ϕ-coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \frac{d}{dt}(ml^2\dot{\phi} + ml\dot{x} \cos \phi) = ml^2\ddot{\phi} + ml\ddot{x} \cos \phi - ml\dot{x}\dot{\phi} \sin \phi$$

$$\frac{\partial T}{\partial \phi} = -ml\dot{x}\dot{\phi} \sin \phi; \quad Q_\phi = -\frac{\partial V}{\partial \phi} = -mgl \sin \phi$$

so the differential equation becomes:

$$ml^2\ddot{\phi} + ml\ddot{x} \cos \phi + mgl \sin \phi = 0$$

A general solution of these two simultaneous non-linear differential equations would be difficult, if not impossible, to obtain. In practice, solutions would be calculated by numerical techniques. If we limit ourselves to small oscillations of the system, however, the equations become linear, and a solution can easily be obtained. Setting $\sin\phi \approx \phi$, $\cos\phi \approx 1$ and neglecting terms of higher than second order in displacements or velocities, the two differential equations become:

$$m\ddot{x} + ml\ddot{\phi} + kx = 0$$
$$m\ddot{x} + ml\ddot{\phi} + mg\phi = 0$$

Subtracting these two equations, we obtain:

$$kx = mg\phi,$$

Eliminating the $\ddot{x}$ from the second equation by means of this expression gives:

$$\ddot{\phi} + \frac{g}{\left(l + \frac{mg}{k}\right)}\phi = 0$$

Thus the pendulum executes small sinusoidal oscillations of frequency:

$$\omega = \sqrt{\frac{g}{l + \frac{mg}{k}}}$$

If the point of support of the pendulum is fixed, $k = \infty$ and the frequency reduces to the known value for a simple pendulum.

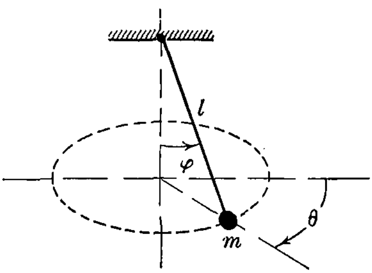
PROBLEMS

9.1. By means of Lagrange's equations, derive the differential equations of motion for a particle in spherical coordinates r, θ, ϕ.

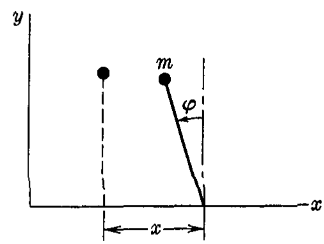
9.2 A spherical pendulum consists of a particle of mass m supported by a massless string of length l. Using the angles θ and ϕ as generalized coordinates, derive the two differential equations of motion by means of Lagrange's equations. Show that these equations reduce to previously known results when θ and ϕ are successively held constant.

9.3. A particle moves in a plane and is attracted toward the origin of a coordinate system by a force which is inversely proportional to the square of the distance from the origin. Find, using Lagrange's equations, the differential equations of motion of the system in plane polar coordinates.

9.4. A particle of mass m rests on a smooth horizontal table whose surface lies in the x, y plane. The particle is connected by a massless string of length l to a point which moves along the x-axis according to the law $x = f(t)$, starting from a position in which the string is parallel to the y-axis as shown in the figure. (a) Using the coordinates x, ϕ as shown, derive by the use of Lagrange's equations the differential equations of motion of the particle. (b) If $x = f(t) = vt$, solve the equations of part (a), determining the motion of the particle and the tension in the string.



PROB. 9.2



PROB. 9.4

9.5. A particle of mass m is supported by a frictionless horizontal disk which rotates about a vertical axis through its center with a constant angular velocity ω. The particle is connected by a massless string of length l to a point located a distance a from the center of the disk. Show that the motion of the particle with respect to the disk is similar to that of a simple pendulum, and find the frequency of small oscillations of the system. Set up the equations in this problem by means of Lagrange's equations.

9.3 Lagrange's Equations for a System of Particles. The methods of the preceding section, which lead to Lagrange's equations for a single particle, can be extended directly to a system of particles. There will be $(q_1, q_2, \ldots q_n)$ independent coordinates required, where n is the number of degrees of freedom of the system. The derivation follows the same lines as for the single particle, except that the equations must now be summed over all of the particles of the system. Thus, equation (9.4) becomes:

$$\sum_{i=1}^{n}\left(F_{x_i}\frac{\partial x_i}{\partial q_1} + F_{y_i}\frac{\partial y_i}{\partial q_1} + F_{z_i}\frac{\partial z_i}{\partial q_1}\right)\delta q_1$$
$$= \sum_{i=1}^{n}\left(m_i\ddot{x}_i\frac{\partial x_i}{\partial q_1} + m_i\ddot{y}_i\frac{\partial y_i}{\partial q_1} + m_i\ddot{z}_i\frac{\partial z_i}{\partial q_1}\right)\delta q_1 \qquad (9.12)$$

the total kinetic energy of the system becomes:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \tag{9.13}$$

and as in the preceding section, we obtain the result:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i,\ i = 1, 2, 3, \ldots n \tag{9.14}$$

and the number of equations obtained will just be equal to the number of degrees of freedom of the system.

The total kinetic energy T can be computed by any means appropriate to the type of system involved. For rigid bodies, Equation (9.13) has already been evaluated in terms of the moments and products of inertia of the body, as in Equations (7.19) and (7.20) of Chapter 7, and can be introduced into Lagrange's equations in that form.

The following advantages of Lagrange's equations as compared with some of the methods used in preceding chapters may be noted:

(1) The equations of motion are derived in the same way for any set of coordinates. For each problem the most suitable set of coordinates can thus be selected, without altering the basic methods used.

(2) Since the kinetic energy involves velocities only, accelerations do not have to be determined. This considerably simplifies the kinematics part of many problems. Since the velocities occur in squared terms, some possible difficulties with algebraic signs are avoided.

(3) The required number of equations of motion are automatically obtained.

Some of these advantages are simply the basic advantages of the energy method, and Lagrange's equations may be thought of as perhaps the most general and the most useful expression of the energy principle.

EXAMPLE 1. A rigid body rotates about a fixed axis under the action of a torque M_t. Find the equation of motion of the body by means of Lagrange's equations.

Solution. The position of the rotating body can be completely specified by one angle ϕ, which we shall take as the coordinate for this one degree of freedom system. The kinetic energy of the body is $T = \frac{1}{2}I\dot{\phi}^2$ where I is the moment of inertia of the body about the axis of rotation. The work done by the torque M_t as the coordinate is changed by an amount $\delta\phi$ is $M_t\delta\phi$, therefore the generalized force $Q_\phi = M_t$. Then:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = M_t; \quad \frac{d}{dt}(I\dot{\phi}) - 0 = M_t$$

or

$$I\ddot{\phi} = M_t$$

which checks the results of the analysis made in Chapter 7, Equation (7.23).

EXAMPLE 2. A pulley of moment of inertia I about its horizontal axis of rotation is restrained by a spring of spring constant k_1 as shown in Fig. 9.4. From the other side of the pulley a spring of spring constant k_2 and a concentrated mass m are suspended, as shown. Find by the use of Lagrange's equations the equations of motion of the system.

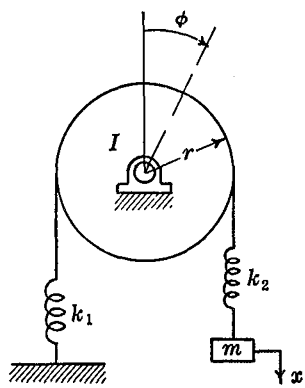


FIG. 9.4

Solution. We choose as the two coordinates for this two degree of freedom system the clockwise angle of rotation of the pulley ϕ, and the downward displacement x of the mass, measured from the position of static equilibrium. There is no energy loss in the system, and the potential energy of the system may be written as:

$$V = \tfrac{1}{2}k_1(r\phi)^2 + \tfrac{1}{2}k_2(x - r\phi)^2$$

Note that since we are measuring the coordinates from the position of static equilibrium there are initial forces mg in the springs, which lead to terms in the potential energy expressions for the springs which cancel the potential energy change of the mass due to the gravity force (See Example 1, Section 3.6.) The kinetic energy of the system is:

$$T = \tfrac{1}{2}I\dot{\phi}^2 + \tfrac{1}{2}m\dot{x}^2$$

Substituting these expressions directly into Lagrange's equations, we obtain:

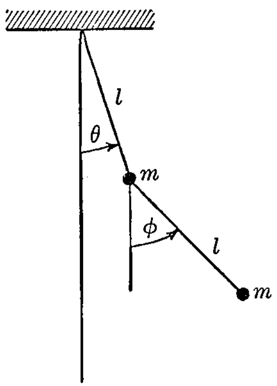
$$m\ddot{x} + k_2 x = k_2 r\phi$$
$$I\ddot{\phi} + (k_1 + k_2)r^2\phi = k_2 r x$$

A method of finding the solution of such simultaneous differential equations will be discussed in the next section.

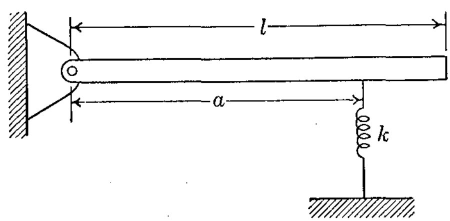
PROBLEMS

9.6. A rigid body oscillates about a horizontal axis as a compound pendulum. The moment of inertia of the body about its axis of rotation is I, and the distance from the axis to the center of mass of the body is a. Derive the differential equation of motion of the system by Lagrange's equations, and find the period of small oscillations.

9.7. A double pendulum consists of two equal masses and two strings of equal length and of negligible mass. Using as coordinates the angles between the strings and the vertical, as shown in the figure, find the differential equations of motion for the system, by means of Lagrange's equations. Show also the simplified form assumed by these equations for small oscillations.



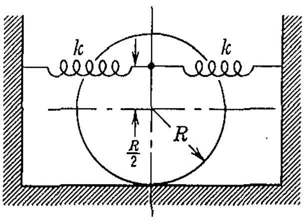
PROB. 9.7



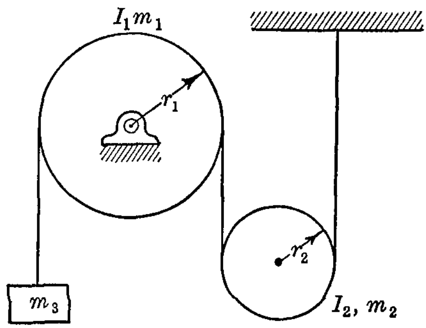
PROB. 9.8

9.8. A rigid straight uniform bar of length l and mass m is pinned at one end and is supported at distance a from the pinned end by a spring having a spring constant k. Find by the use of Lagrange's equations the differential equation of motion describing small oscillations of the bar about the position of static equilibrium, and find the frequency of the motion.

9.9. A solid homogeneous disk of radius R and mass m is constrained to roll, without slipping, in its own plane. Two massless springs, each of spring constant k, are attached to the disk in such a way that the position of static equilibrium, with the springs in an unstretched position, is as



PROB. 9.9

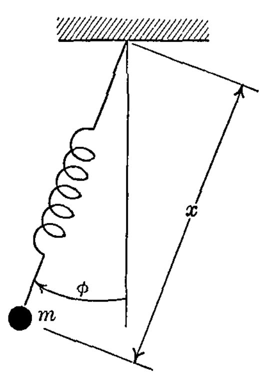


PROB. 9.10

shown in the figure. Derive the equations of motion of the system, using Lagrange's equations, and find the natural period of the motion.

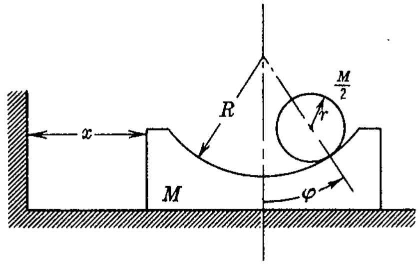
9.10. A rope of negligible mass passes over a fixed pulley of moment of inertia I_1, mass m_1, and radius r_1, and supports a movable pulley of moment of inertia I_2, mass m_2, and radius r_2 as shown in the diagram. A concentrated mass is attached to one end of the rope, and the other end is fixed, with the rope sections vertical as shown. Assuming no slipping of the rope on the pulleys, find by means of Lagrange's equations the differential equation describing the motion of the mass m_3, as the system moves under the action of gravity.

9.11. The string of a simple pendulum is assumed to be elastic with a spring constant k, as indicated in the diagram. Taking as generalized coordinates of the mass a displacement x in the direction of the spring, and the angle ϕ between the spring and the vertical, find the differential equations describing small oscillations of the system.

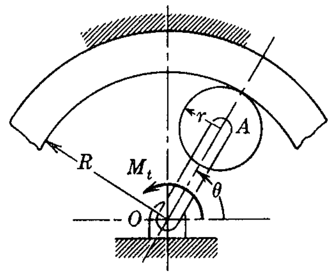


PROB. 9.11

9.12. A block of mass M has a cylindrical groove of radius R, as shown in the diagram. A small cylinder of radius r and mass $\frac{M}{2}$ rolls without slipping in the groove under the action of gravity. The contact between M and the supporting horizontal surface is frictionless. Find, using Lagrange's equations, the two differential equations of motion of the system.

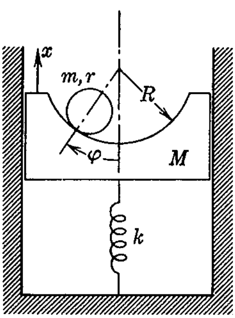


PROB. 9.12



PROB. 9.13

9.13. A circular gear of radius r, mass m, and moment of inertia about the geometric axis I, rolls around the inside of a fixed circular gear of radius R. The plane of the gear is horizontal. A straight uniform bar of mass M connects the axis of small gear with a fixed point O. A torque M_t is applied to the system as shown in the diagram. Find, by the use of Lagrange's equations, the equation of motion of the system in terms of the angle θ. The connecting bar OA may be assumed to be of length $(R - r)$.

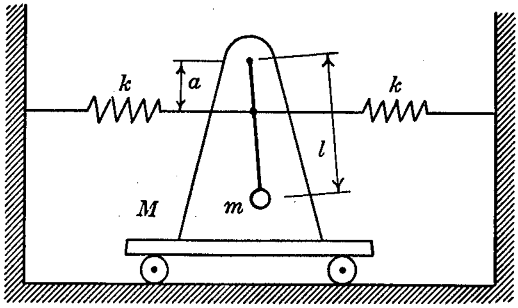


PROB. 9.14

9.14. A circular cylinder of radius r and mass m rolls without slipping in a semi-circular groove of radius R cut in a block of mass M which is constrained to move without friction in a vertical guide. The block is supported by a spring of spring constant k, as shown in the figure. Taking as coordinates the vertical displacement of the block x, and the angular position of the cylinder ϕ, both measured from the position of static equilibrium, find, by means of Lagrange's equations, the equations of motion of the system. Write also the simplified equations of motion for small oscillations of the system.

9.15. A car of mass M moves along a frictionless horizontal plane. The car carries a simple pendulum of length l and concentrated mass m as shown in the diagram. Two equal springs of spring constant k attach the pendulum, at a distance a from the axis of rotation O, to fixed walls. Find, using Lagrange's equations, the differential equations of motion describing small oscillations of the system.

9.4 Oscillations of Two Degree of Freedom Systems. Consider a conservative system consisting of two equal masses m and



PROB. 9.15

three equal springs of spring constant k connected as shown in Fig. 9.5. This is the simplest type of system which illustrates the new features which appear in a vibration problem when more than one degree of freedom is present. Taking x_1 and x_2, the displacements of the two masses from the equilibrium position, as the coordinates, we have:

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) \\ V &= \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_1 - x_2)^2 + \tfrac{1}{2}kx_2^2 \\ &= kx_1^2 + kx_2^2 - kx_1x_2 \end{aligned}$$

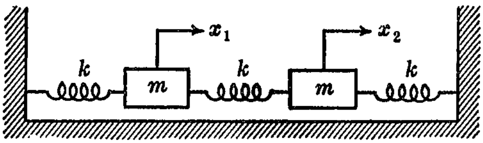


FIG. 9.5

Substituting these expressions into Lagrange's equations, we obtain the two simultaneous differential equations of motion:

$$\ddot{x}_1 + 2\frac{k}{m}x_1 - \frac{k}{m}x_2 = 0$$

$$\ddot{x}_2 + 2\frac{k}{m}x_2 - \frac{k}{m}x_1 = 0$$

Each of these equations involves both x_1 and x_2. Since a displacement of x_1 changes the force applied to the x_2 mass, such coordinates are said to be statically coupled.

From the form of the equations, and from the nature of the physical problem, we expect the oscillations of the system to be harmonic, i.e., a solution of the form $\sin \omega t$ or $\cos \omega t$ should be suitable. We shall thus take as trial solution:

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin \omega t$$

and we shall see, by direct substitution in the equations of motion, whether we can make this form, involving two constants and a common frequency, satisfy all of the conditions of the problem. Substituting these assumed solutions into the differential equation and cancelling common factors, we obtain:

$$\left(2\frac{k}{m} - \omega^2\right)A_1 - \frac{k}{m}A_2 = 0$$

$$-\frac{k}{m}A_1 + \left(2\frac{k}{m} - \omega^2\right)A_2 = 0$$

Each of these equations gives a value for the ratio A_1/A_2; equating these ratios gives:

$$\frac{\dfrac{k}{m}}{\left(2\dfrac{k}{m} - \omega^2\right)} = \frac{\left(2\dfrac{k}{m} - \omega^2\right)}{\dfrac{k}{m}}$$

$$\left(\omega^2 - 2\frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

This is the frequency equation which determines the proper values of ω. The equation is quadratic in ω^2 and has two roots giving the frequencies:

$$\omega_1^2 = \frac{k}{m}$$

$$\omega_2^2 = 3\frac{k}{m}$$

The frequency equation can also be obtained by noting that since the two simultaneous algebraic equations in A_1 and A_2 are homogeneous, they can have a solution other than zero only if the determinant formed from the coefficients of the A's disappears:

$$\begin{vmatrix} \left(2\dfrac{k}{m} - \omega^2\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \left(2\dfrac{k}{m} - \omega^2\right) \end{vmatrix} = 0$$

This, of course, gives the same frequency equation.

To find the configuration of the system corresponding to the two natural frequencies of vibration found above, we solve for the ratio $\dfrac{A_1}{A_2}$ from either of the algebraic equations; for example:

$$\frac{A_1}{A_2} = \frac{\dfrac{k}{m}}{2\dfrac{k}{m} - \omega^2}$$

Substituting $\omega_1^2 = \dfrac{k}{m}$, we obtain:

$$\left(\frac{A_1}{A_2}\right)_1 = +1$$

hence, the two masses move in phase with equal amplitudes. Substituting $\omega_2^2 = 3\dfrac{k}{m}$ we obtain:

$$\left(\frac{A_1}{A_2}\right)_2 = -1$$

and the two masses move with equal amplitudes but in opposite directions, that is, with a 180° phase difference.

The physical significance of these two motions can easily be seen. If $A_1 = A_2$ the two masses move together with equal amplitudes and the center spring connecting them is neither extended nor compressed; hence it cannot affect the motion. The frequency of vibration should thus be the same as for one of the masses restrained by only one of the springs, that is, $\omega^2 = \dfrac{k}{m}$. For the motion in which the two masses always move in opposite directions, we note that the displacements are symmetrical and that the center point of the center spring can be considered as fixed. We can thus represent the system as shown in Fig. 9.6, which is a single degree of freedom system having a spring constant of $k + 2k = 3k$ and hence a frequency

$$\omega^2 = 3\frac{k}{m}$$

The above analysis started with the assumption that the solution was of the form $x = A \sin \omega t$. We could just as well have started

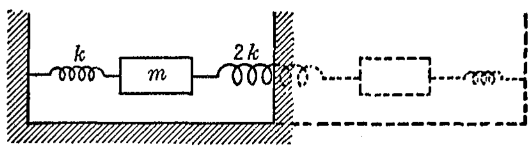


FIG. 9.6

with a solution of the form $x = A \cos \omega t$, in which case the same algebraic equations for the A's would have been obtained. Since both sine and cosine terms will satisfy the original equations, and since superposition of solutions will be valid for the linear differential equations involved, the general solution can be made up of a sum of sines and cosines as:

$$x_1 = A_1^{(1)} \sin \omega_1 t + A_1^{(2)} \sin \omega_2 t + B_1^{(1)} \cos \omega_1 t + B_1^{(2)} \cos \omega_2 t$$

$$x_2 = A_2^{(1)} \sin \omega_1 t + A_2^{(2)} \sin \omega_2 t + B_2^{(1)} \cos \omega_1 t + B_2^{(2)} \cos \omega_2 t$$

where the subscripts on the A's and B's refer to the coordinates, and the superscripts refer to the frequencies. We have already seen that the A's and hence the B's, which must satisfy the same equations as the A's, are not arbitrary, but must satisfy the conditions:

$$\frac{A_1^{(1)}}{A_2^{(1)}} = \frac{B_1^{(1)}}{B_2^{(1)}} = +1$$

$$\frac{A_1^{(2)}}{A_2^{(2)}} = \frac{B_1^{(2)}}{B_2^{(2)}} = -1$$

so the general solution becomes:

$$x_1 = A_1^{(1)} \sin \omega_1 t + A_1^{(2)} \sin \omega_2 t + B_1^{(1)} \cos \omega_1 t + B_1^{(2)} \cos \omega_2 t$$

$$x_2 = A_1^{(1)} \sin \omega_1 t - A_1^{(2)} \sin \omega_2 t + B_1^{(1)} \cos \omega_1 t - B_1^{(2)} \cos \omega_2 t$$

We thus have four arbitrary constants $A_1^{(1)}$, $A_1^{(2)}$, $B_1^{(1)}$ and $B_1^{(2)}$ which will suffice to specify the two initial displacements and the two initial velocities for the two masses. The most general motion of the system can thus be thought of as a superposition of the motions corresponding to the two natural frequencies.

9.5 Principal Modes of Vibration. The two configurations in the preceding section which correspond to motions having the two natural frequencies of vibration of the system, are called the two *natural modes* of vibration. In general, by the word "mode" we refer to a motion of the system that can be described by a single frequency. We have seen that the solution of the preceding problem is a superposition of two modes of vibration, each having its characteristic frequency. There is often a decided advantage in choosing for the system that set of coordinates for which each coordinate has only one frequency. In the present problem this can be done by introducing the new coordinates ξ_1 and ξ_2, defined as:

$$\xi_1 = \frac{x_1 + x_2}{2}$$

$$\xi_2 = \frac{x_1 - x_2}{2}$$

If we write the general solution for x_1 and x_2, as obtained in the preceding section, in terms of these new coordinates, we obtain:

$$\xi_1 = A_1^{(1)} \sin \omega_1 t + B_1^{(1)} \cos \omega_1 t$$
$$\xi_2 = A_1^{(2)} \sin \omega_2 t + B_1^{(2)} \cos \omega_2 t$$

and each coordinate involves only one frequency and one mode of vibration. Coordinates which have this property are called *normal coordinates*, and the modes of vibration corresponding to them are called the *principal modes*, or the *normal modes* of the system.

Once the normal coordinates of a system have been determined, the problem has been essentially reduced to a study of a set of independent single degree of freedom systems. This can be illustrated by the system of Fig. 9.5 with a force $F = F_0 \sin \omega t$ applied to the left mass, as shown in Fig. 9.7. Since we have already seen that the coordinates ξ_1, ξ_2 describe the principal modes of the system, we shall use these normal coordinates as the generalized coordinates for the forced oscillation problem. Substituting $x_1 = \xi_1 + \xi_2$; $x_2 = \xi_1 - \xi_2$

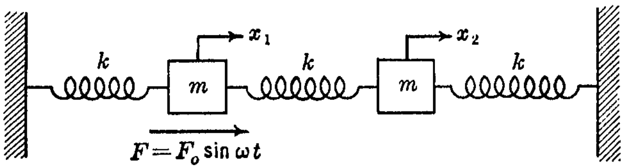


FIG. 9.7

in the equation for kinetic and potential energy of the preceding section, we obtain:

$$T = m(\dot{\xi}_1^2 + \dot{\xi}_2^2)$$
$$V = k(\xi_1^2 + 3\xi_2^2)$$

To find the generalized force Q associated with the applied external force F, we note that the work done by F during the displacement $\delta x_1 = \delta\xi_1 + \delta\xi_2$ is:

$$F\delta x_1 = F\delta\xi_1 + F\delta\xi_2 = Q_1\delta\xi_1 + Q_2\delta\xi_2$$

from which we see that:

$$Q_1 = F, \; Q_2 = F$$

Lagrange's equations are now used in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

Note that some of the forces acting on the masses have been included in the potential energy, while some have been left as generalized forces. This is often a convenient procedure in problems in which the forces fall naturally into two different categories as in the above example. It is necessary, however, to be careful that all of the forces are included, and that no forces are included twice.

Substituting the expressions for T, V and Q into Lagrange's equations, we obtain:

$$m\ddot{\xi}_1 + k\xi_1 = \frac{F_0}{2} \sin \omega t$$

$$m\ddot{\xi}_2 + 3k\xi_2 = \frac{F_0}{2} \sin \omega t$$

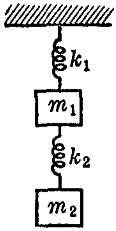
These are two independent equations, each involving only one unknown, which describe the forced oscillations of the system. They are equations of the same type as those discussed in Chapter 5, so that the conclusions reached in that chapter as to the behavior of a single degree of freedom system can be applied directly to the motion of the present two degree of freedom system. It should be noted that any coordinate system for which the expressions for the kinetic and potential energies are sums of squares and do not involve cross-product terms will lead directly to a set of independent equations of motion. One of the ways in which the normal coordinates of a system can be found is to determine the transformation of coordinates required to make the cross-product terms in the kinetic and potential energies disappear simultaneously. In the above example we were able to determine the normal coordinates by inspection, because of the simplicity of the system and the symmetry involved. In most problems this is not possible, and the normal coordinates must be determined from the natural mode shapes, as calculated for the free vibration problem.

In the next section we shall show how the theory of the above section can be extended to include a wide class of vibration problems of any number of degrees of freedom. The fact that linear vibration problems of any complexity can be considered as a number of superimposed single degree of freedom systems makes it possible to apply the theory of Chapter 5 directly to the more complex systems, and provides an additional justification for a detailed study of the single degree of freedom system.

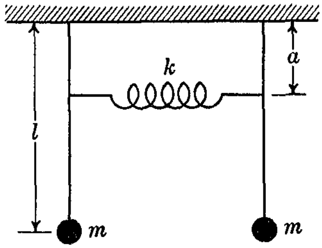
PROBLEMS

9.16. Derive the differential equations of motion for the system of Fig. 9.5 by a direct application of Newton's second law in the form $F = ma$.

9.17. Find the natural frequencies of vibration of the two-mass system shown in the figure.

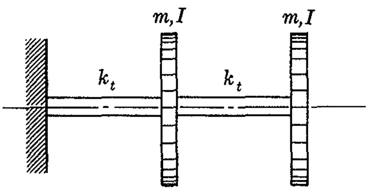


PROB. 9.17



PROB. 9.18

9.18. Two simple pendulums having equal lengths l and equal concentrated masses m are connected by a spring of spring constant k as shown in the figure. Find the frequencies of vibration for small oscillations, and the principal modes of vibration of the system. What initial conditions would be necessary to obtain free oscillations of the system in the first mode without exciting the second mode, and vice versa?



PROB. 9.19

9.19. For the purpose of studying torsional oscillations, a particular engine-pump system is idealized as two equal disks of mass m and moment of inertia I about the geometric axis, equally spaced on a shaft one end of which is built in. The torsional spring constant k_t lb ft/rad of the two portions of the shaft are equal. Find the natural frequencies and mode shapes of torsional oscillations of the system.

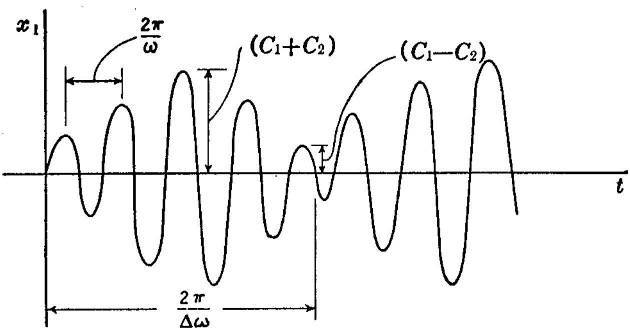
9.20. Find the natural frequencies of small oscillations of the double pendulum of Prob. 9.7, and determine the shapes of the corresponding modes of vibration.

9.21. In the expressions given above for the general solution x_1 and x_2 for the two degree of freedom system, determine the four constants $A_1^{(1)}$, $A_1^{(2)}$, $B_1^{(1)}$ and $B_1^{(2)}$ in terms of the initial displacements x_{10}, x_{20} and the initial velocities $\dot{x}_{10}$, $\dot{x}_{20}$.

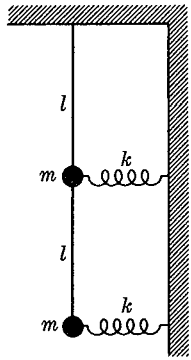
9.22. (a) Starting with the expressions given above for the general solution x_1 and x_2 for the two degree of freedom system in terms of the four constants $A_1^{(1)}$, $A_1^{(2)}$, $B_1^{(1)}$ and $B_1^{(2)}$ show that if when $t = 0$, $x_1 = x_2 = 0$ and $\dot{x}_1 = \dot{x}_{10}$, $\dot{x}_2 = \dot{x}_{20}$, the displacement x_1 is given by:

$$x_1 = C_1 \sin \omega_1 t + C_2 \sin \omega_2 t$$

(b) Using the result of part (a), show that if ω_1 is nearly the same as ω_2 so that $(\omega_1 - \omega_2) = \Delta\omega$, x_1 varies between the values $(C_1 + C_2)$ and $(C_1 - C_2)$ with a frequency $\Delta\omega$, as shown in the figure. This modulation



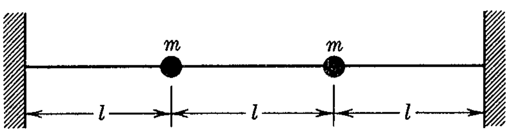
PROB. 9.22



PROB. 9.23

of the amplitude of a sinusoidal motion resulting from the superposition of two sinusoidal motions of different frequencies is called the phenomenon of *beats*, and the frequency $\Delta\omega = (\omega_1 - \omega_2)$ is called the beat frequency. Note that if $C_1 = C_2$, x_1 passes through zero at periodic times corresponding to the beat frequency.

9.23. A double pendulum consists of two equal masses suspended from massless strings of equal length l. The masses are connected to a wall by equal springs of spring constant k as shown in the figure. Initially the



PROB. 9.24

system has a vertical position with the springs in an unstretched condition. Find the natural frequencies of small oscillation about the position of equilibrium, and determine the corresponding mode shapes.

9.24. Two equal masses are attached to a stretched string of length $3l$ as shown in the diagram. The tensile force in the string is F and can be assumed to be constant during small transverse oscillations of the masses.

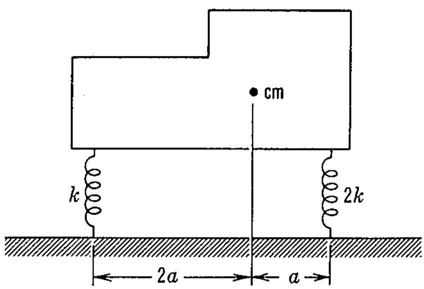
Neglecting gravity forces and the mass of the string, find the principal modes for small transverse vibrations of the system and their frequencies.

9.25. Show that if the coordinates of a system are selected so that the kinetic and potential energies can be written as the sums of squares with no cross-product terms in the form:

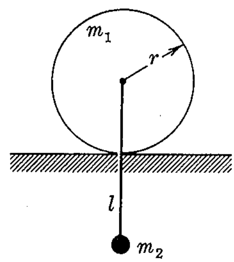
$$T = \tfrac{1}{2}(a_1\dot{q}_1^2 + a_2\dot{q}_2^2 + \cdots)$$
$$V = \tfrac{1}{2}(b_1q_1^2 + b_2q_2^2 + \cdots)$$

the equations of motion will each involve only one unknown and hence can be solved directly. Note that $q_1, q_2, \ldots$ in this case would be the normal coordinates for the system.

9.26. A rigid machine of mass m, and moment of inertia I about an axis through the center of mass normal to the plane of the figure, is supported



PROB. 9.26



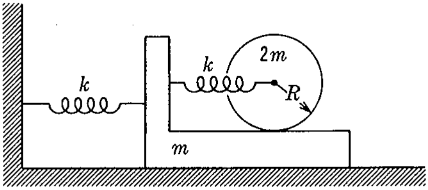
PROB. 9.27

on springs as shown in the diagram. The supports are equivalent to a spring of spring constant $2k$ located at a distance of a from the center of mass, and a spring having a spring constant of k located a distance of $2a$ from the center of mass and $3a$ from the first spring as shown. Considering only plane oscillations of the system it will be seen that the machine can vibrate vertically and can also perform rotational oscillations about an axis through the center of mass. Find the natural frequencies of small oscillations of the machine about the position of static equilibrium.

9.27. A uniform disk of mass m_1 and radius r is free to roll without slipping on a horizontal surface. At the center of the disk is attached a weightless string of length l which carries a concentrated mass m_2 at its lower end. (a) Find the natural frequencies of vibration of the system for motion in the plane of the disk for small oscillations of the pendulum.

(b) Describe the motions of the system corresponding to each of the frequencies found in part (a).

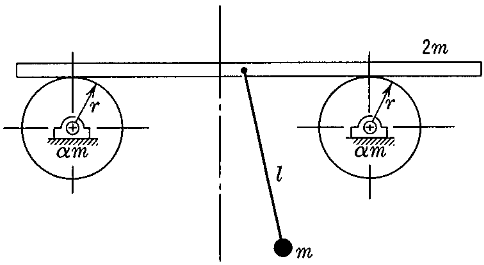
9.28. A solid homogeneous cylinder of mass $2m$ and radius R rolls without slipping on a car of mass m which slides without friction on a smooth horizontal plane. The car is attached to a wall by a horizontal spring of spring constant k, and an identical spring attaches the center of the cylinder to the car as shown in the figure. Taking as coordinates the absolute displacements of the car x_1, and of the center of the cylinder x_2,



PROB. 9.28

find the frequencies of vibration of the system and the corresponding mode shapes.

9.29. A horizontal uniform beam rests on two similar circular cylinders which can rotate freely without friction about their geometric axes. The beam rolls freely on the cylinders without slipping. The mass of each cylinder is αm and the mass of the beam is $2m$. A pendulum having a



PROB. 9.29

length l and a concentrated mass m is attached to the center of the beam which is initially midway between the cylinders. (a) Write the differential equations of motion for the system. Do not assume small motions. (b) Assuming small displacements and velocities, determine the natural frequencies of the system.

9.6 Small Oscillations of a Conservative System.

In the preceding sections and in Chapter 5 a number of vibration problems of various kinds have been discussed, and many common features have been noted in the behavior of systems that were physically quite dissimilar. We shall now show that a more general approach to the vibration problem can be made, and that many of the characteristics of vibrating systems can be derived in general terms that can be applied to a large variety of physical situations.

We shall limit the present discussion to systems in which there are no energy dissipations, and for which the forces can be derived from a potential energy function. It will further be supposed that the displacements and velocities of the system are small, so that higher order terms in these quantities can be dropped. This assumption of small oscillations will usually result in a set of linear differential equations, for which solutions can readily be obtained. We will consider an n degree of freedom system having the generalized coordinates $(q_1, q_2, \ldots q_n)$, which are defined in such a way that they all have zero values at a position of stable equilibrium of the system.

9.7 The Potential Energy Function. The potential energy V will be a function of the generalized coordinates $(q_1, q_2, \ldots q_n)$ Expanding this function in a Taylor series about the position of static equilibrium we obtain:

$$\begin{aligned} V(q_1, q_2, \ldots q_n) = V_0 &+ \left(\frac{\partial V}{\partial q_1}\right)_0 q_1 + \left(\frac{\partial V}{\partial q_2}\right)_0 q_2 + \cdots \\ &+ \tfrac{1}{2}\left(\frac{\partial^2 V}{\partial q_1^2}\right)_0 q_1^2 + \tfrac{1}{2}\left(\frac{\partial^2 V}{\partial q_2^2}\right)_0 q_2^2 + \cdots \\ &+ \left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 q_1 q_2 + \left(\frac{\partial^2 V}{\partial q_2 \partial q_3}\right)_0 q_2 q_3 + \cdots \\ &+ \text{higher order terms} \end{aligned}$$

where the derivatives are all evaluated at the origin. If we define the potential energy of the system as zero at the origin, $V_0 = 0$. Since the system is in static equilibrium at the origin, the sum of the forces on the system vanishes at the origin, thus:

$$\left(\frac{\partial V}{\partial q_1}\right)_0 = \left(\frac{\partial V}{\partial q_2}\right)_0 = \cdots = \left(\frac{\partial V}{\partial q_n}\right)_0 = 0$$

Also, by virtue of the assumption of small oscillations, terms of higher order than the second can be dropped, as being small compared to the second order terms. The expression for V thus becomes:

$$V = \tfrac{1}{2}\left[\left(\frac{\partial^2 V}{\partial q_1^2}\right)_0 q_1^2 + \left(\frac{\partial^2 V}{\partial q_2^2}\right)_0 q_2^2 + \cdots + 2\left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 q_1 q_2 + \cdots\right]$$

Since $\dfrac{\partial^2 V}{\partial q_i \partial q_j} = \dfrac{\partial^2 V}{\partial q_j \partial q_i}$, the term involving the mixed derivative can be written:

$$2\left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 q_1 q_2 = \left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 q_1 q_2 + \left(\frac{\partial^2 V}{\partial q_2 \partial q_1}\right)_0 q_2 q_1$$

All the terms in the expression for V will thus have the same form, and a summation convention can be used as follows:

$$V = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i q_j$$

The derivatives in this expression, since they are all evaluated at the origin, are constants, which we may write in the form:

$$k_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0; \quad k_{ij} = k_{ji} = \text{constant}$$

and we have finally for the potential energy:

$$V = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j \tag{9.15}$$

The terms k_{ij} may be called the generalized spring constants for the system. In some simple systems these k's will be the same as the familiar spring constants of a spring element, but more generally they are combinations of such spring constants, and may relate to torsional restoring forces, gravitational restoring forces, etc.

9.8 The Kinetic Energy Function. The total kinetic energy of a system can be expressed in terms of the orthogonal coordinates (x_i, y_i, z_i) of each particle as:

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

We shall now express T in terms of the generalized coordinates. We have

$$x_i = \phi_1(q_1, q_2, \cdots q_n)$$
$$y_i = \phi_2(q_1, q_2, \cdots q_n)$$
$$z_i = \phi_3(q_1, q_2, \cdots q_n)$$

and thus:

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \frac{\partial x_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial x_i}{\partial q_n}\dot{q}_n$$

$$\dot{y}_i = \frac{dy_i}{dt} = \frac{\partial y_i}{\partial q_1}\dot{q}_1 + \frac{\partial y_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial y_i}{\partial q_n}\dot{q}_n$$

$$\dot{z}_i = \frac{dz_i}{dt} = \frac{\partial z_i}{\partial q_1}\dot{q}_1 + \frac{\partial z_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial z_i}{\partial q_n}\dot{q}_n$$

The expression for T thus becomes:

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i\left[\left(\frac{\partial x_i}{\partial q_1}\dot{q}_1 + \frac{\partial x_i}{\partial q_2}\dot{q}_2 + \cdots\right)^2 + \left(\frac{\partial y_i}{\partial q_1}\dot{q}_1 + \frac{\partial y_i}{\partial y_2}\dot{q}_2 + \cdots\right)^2 + \left(\frac{\partial z_i}{\partial q_1}\dot{q}_1 + \frac{\partial z_i}{\partial q_2}\dot{q}_2 + \cdots\right)^2\right]$$

Collecting terms, this becomes:

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i\left\{\left[\left(\frac{\partial x_i}{\partial q_1}\right)^2 + \left(\frac{\partial y_i}{\partial q_1}\right)^2 + \left(\frac{\partial z_i}{\partial q_1}\right)^2\right]\dot{q}_1^2 + \left[\left(\frac{\partial x_i}{\partial q_2}\right)^2 + \left(\frac{\partial y_i}{\partial q_2}\right)^2 + \left(\frac{\partial z_i}{\partial q_2}\right)^2\right]\dot{q}_2^2 + \cdots + \left[2\left(\frac{\partial x_i}{\partial q_1}\right)\left(\frac{\partial x_i}{\partial q_2}\right) + 2\left(\frac{\partial y_i}{\partial q_1}\right)\left(\frac{\partial y_i}{\partial q_2}\right) + 2\left(\frac{\partial z_i}{\partial q_1}\right)\left(\frac{\partial z_i}{\partial q_2}\right)\right]\dot{q}_1\dot{q}_2 + \cdots\right.$$

using the summation convention as above:

$$T = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j$$

where, for example:

$$m_{11} = m_1\left[\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial y_1}{\partial q_1}\right)^2 + \left(\frac{\partial z_1}{\partial q_1}\right)^2\right] + m_2\left[\left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial y_2}{\partial q_1}\right)^2 + \left(\frac{\partial z_2}{\partial q_1}\right)^2\right] + \cdots$$

In general, the coefficients m_{ij} are not constants, but are functions of the coordinates. If a typical m_{ij} be expanded in a Taylor series about the origin, we obtain:

$$m_{ij}(q_1, q_2, \ldots q_n) = (m_{ij})_0 + \left(\frac{\partial m_{ij}}{\partial q_1}\right)_0 q_1 + \left(\frac{\partial m_{ij}}{\partial q_2}\right)_0 q_2 + \cdots$$
$$+ \tfrac{1}{2}\left(\frac{\partial^2 m_{ij}}{\partial q_1^2}\right)_0 q_1^2 + \cdots$$

We shall now make use of one of our basic assumptions that the velocities $(\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n)$ remain small. If any of the terms in the above series expansion for m_{ij} beyond the constant term $(m_{ij})_0$ were to be retained, the expression for T would have third order terms in it, since the m_{ij} is multiplied by $(\dot{q}_i\dot{q}_j)$. We thus see that for small oscillation problems the m_{ij} terms are constant, and we have finally for the kinetic energy expression:

$$T = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j; \quad m_{ij} = m_{ji} = \text{constant} \tag{9.16}$$

The constants m_{ij} are called the inertia constants of the system. For simple systems they may be simple masses or moments of inertia, but more commonly they are combinations of such quantities depending on the coordinate system used.

It will be noted that both the potential energy and the kinetic energy are similar quadratic forms. This fact has made it possible to apply directly to small oscillation problems many of the mathematical techniques which have been developed for the study of quadratic forms.

9.9 The General Equations of Free Oscillations. Since the kinetic energy expression, Equation (9.16), does not involve the q's directly, but only the $\dot{q}$'s, the term $\dfrac{\partial T}{\partial q_i}$ in Lagrange's equations will vanish, and we have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = 0 \tag{9.17}$$

where:

$$T = \tfrac{1}{2}m_{11}\dot{q}_1^2 + \tfrac{1}{2}m_{22}\dot{q}_2^2 + \cdots + m_{12}\dot{q}_1\dot{q}_2 + \cdots$$
$$V = \tfrac{1}{2}k_{11}q_1^2 + \tfrac{1}{2}k_{22}q_2^2 + \cdots + k_{12}q_1q_2 + \cdots$$

The terms involving the k_{ij} cross-products are referred to as the *static coupling* terms, and those containing the m_{ij} cross-products as *dynamic coupling* terms. Note that the word "coupling" refers to the coordinates and not to the systems, so that the kinds of coupling present depend upon the coordinates chosen rather than upon the characteristics of the system itself. It can be shown in general that it is always possible to choose the generalized coordinates in such a way that all the static coupling terms, or all the dynamic coupling terms, or all the static and dynamic coupling terms together, will be zero.

We shall suppose, in the following sections, that the coordinates have been selected in such a way that all of the dynamic coupling terms are zero. This simplification is for convenience only, and does not represent any limitation of the basic theory.

The expression for the kinetic and potential energies for this statically coupled system are:

$$\begin{aligned} T &= \tfrac{1}{2}m_1\dot{q}_1{}^2 + \tfrac{1}{2}m_2\dot{q}_2{}^2 + \cdots \\ V &= \tfrac{1}{2}k_{11}q_1{}^2 + \tfrac{1}{2}k_{22}q_2{}^2 + \cdots + k_{12}q_1q_2 + \cdots \end{aligned}$$

where we no longer need a double subscript for the m_{ij} terms, since there are no cross-product terms. Substituting these expressions for T and V into Lagrange's equations, Equation (9.17) above, we obtain:

$$\begin{aligned} m_1\ddot{q}_1 + k_{11}q_1 + k_{12}q_2 + \cdots + k_{1n}q_n &= 0 \\ m_2\ddot{q}_2 + k_{21}q_1 + k_{22}q_2 + \cdots + k_{2n}q_n &= 0 \\ &\vdots \\ m_n\ddot{q}_n + k_{n1}q_1 + k_{n2}q_2 + \cdots + k_{nn}q_n &= 0 \end{aligned} \tag{9.18}$$

Since there is no damping, the solution of this set of equations will be of the form:

$$q_i = A_i \sin \omega t + B_i \cos \omega t$$

since the sine and cosine terms will lead to the same equations, we shall take as the solution, following the same method as in the two degree of freedom example of Section 9.4,

$$q_i = A_i \sin \omega t \tag{9.19}$$

where A_i is the amplitude of the harmonic solution of frequency ω.

Substituting Equation (9.19) into Equations (9.18), we obtain the following set of algebraic equations:

$$\begin{aligned}
(k_{11} - m_1\omega^2)A_1 + k_{12}A_2 + \cdots + k_{1n}A_n &= 0 \\
k_{21}A_1 + (k_{22} - m_2\omega^2)A_2 + \cdots + k_{2n}A_n &= 0 \\
&\vdots \\
k_{n1}A_1 + k_{n2}A_2 + \cdots + (k_{nn} - m_n\omega^2)A_n &= 0
\end{aligned} \tag{9.20}$$

This set of homogeneous simultaneous equations will have a nontrivial solution only if the determinant of the coefficients vanishes:

$$\begin{vmatrix}
(k_{11} - m_1\omega^2) & k_{12} & \cdots k_{1n} \\
k_{21} & (k_{22} - m_2\omega^2) & \cdots k_{2n} \\
\vdots & & \\
k_{n1}\cdots & & (k_{nn} - m_n\omega^2)
\end{vmatrix} = 0 \tag{9.21}$$

This determinant is the frequency equation, from which the natural frequencies of free vibration, ω_r, can be determined if the k's and m's for the system are known. This frequency equation will give n values of ω^2 from which the n positive real values, $(\omega_1, \omega_2, \ldots \omega_r, \ldots \omega_n)$ can be determined.

Corresponding to each natural frequency ω_r will be a set of A values $(A_1^{(r)}, A_2^{(r)}, A_3^{(r)}, \ldots A_n^{(r)})$ which will describe the configuration of the system as it vibrates harmonically with the frequency ω_r. Such a set of $A^{(r)}$'s defines the shape of a *principal mode* of oscillation of the system. Note that the values of the A's are not determined; it is only the ratios between the A's, that is, the "shape" of the mode, that can be determined from the set of homogeneous Equations (9.20). Physically this means that while the shape of the harmonic mode is fixed, its absolute amplitude can have any value, so long as it remains small. The notation that is used for the A's should be carefully considered; note that the subscript refers to the coordinate, while the superscript in parentheses refers to the mode of vibration. On the other hand, for the frequency ω, which involves the mode only, and not the individual coordinates, it is more convenient to use the single subscript referring to the mode, as ω_r.

The general solution of the set of simultaneous differential equations of motion Equation (9.18) can be obtained by a superposition of the solutions corresponding to the principal modes:

$$q_i = \sum_{r=1}^{n} A_i^{(r)} \sin \omega_r t \tag{9.22}$$

A similar expression involving a cos $\omega_r t$ instead of the sin $\omega_r t$ could be added if necessary to satisfy the initial conditions of a particular problem.

9.10 Orthogonality of the Principal Modes. In the present section a basic property of the principal modes of vibration will be developed which will be of considerable use for the calculation of natural frequencies and mode shapes, as well as for further extensions of the theory.

Consider two particular principal oscillations having different frequencies ω_r and ω_s. From Equations (9.20) we may write:

$$\begin{aligned} k_{i1}A_1^{(r)} + k_{i2}A_2^{(r)} + \cdots + k_{in}A_n^{(r)} &= m_i\omega_r^2 A_i^{(r)} \\ k_{i1}A_1^{(s)} + k_{i2}A_2^{(s)} + \cdots + k_{in}A_n^{(s)} &= m_i\omega_s^2 A_i^{(s)} \end{aligned}$$

where each equation is of the form:

$$\sum_{j=1}^{n} k_{ij}A_j^{(r)} = m_i\omega_r^2 A_i^{(r)} \tag{9.23}$$

Multiply the first equation by $A_i^{(s)}$, and the second equation by $A_i^{(r)}$, obtaining:

$$\begin{aligned} k_{i1}A_1^{(r)}A_i^{(s)} + k_{i2}A_2^{(r)}A_i^{(s)} + \cdots + k_{in}A_n^{(r)}A_i^{(s)} &= m_i\omega_r^2 A_i^{(r)}A_i^{(s)} \\ k_{i1}A_1^{(s)}A_i^{(r)} + k_{i2}A_2^{(s)}A_i^{(r)} + \cdots + k_{in}A_n^{(s)}A_i^{(r)} &= m_i\omega_s^2 A_i^{(s)}A_i^{(r)} \end{aligned}$$

each of these equations is a set of $(i = 1, 2, 3, \ldots n)$ equations; adding this system of equations term by term on each side, we obtain:

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}A_j^{(r)}A_i^{(s)} &= \omega_r^2 \sum_{i=1}^{n} m_i A_i^{(r)}A_i^{(s)} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}A_i^{(r)}A_j^{(s)} &= \omega_s^2 \sum_{i=1}^{n} m_i A_i^{(r)}A_i^{(s)} \end{aligned}$$

In the left side of the second equation the subscript i may be written as j and the subscript j may be written as i without affecting the value of the term, since $k_{ij} = k_{ji}$ and in each case we are only summing from 1 to n on the subscripts. We thus see that the left sides of the two equations are identical and hence if the second is subtracted from the first there results:

$$(\omega_r^2 - \omega_s^2) \sum_{i=1}^{n} m_i A_i^{(r)} A_i^{(s)} = 0$$

Since we have picked $\omega_r \neq \omega_s$, we have, finally:

$$\sum_{i=1}^{n} m_i A_i^{(r)} A_i^{(s)} = 0 \tag{9.24}$$

This equation is called the *orthogonality relation*. The reason for this name may be seen by considering two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ which are perpendicular, or orthogonal, to each other. The analytical condition for this orthogonality is $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, or:

$$a_x b_x + a_y b_y + a_z b_z = 0$$

$$\sum_{i=1}^{3} a_i b_i = 0$$

Thus, Equation (9.24) may be thought of as the condition that two n-dimensional vectors $m^{\frac{1}{2}}A^{(s)}$ and $m^{\frac{1}{2}}A^{(r)}$ are orthogonal.

9.11 Example: The Calculation of Natural Frequencies and Mode Shapes. As an illustration of the above general theory, we shall work out a particular example. Three equal masses slide

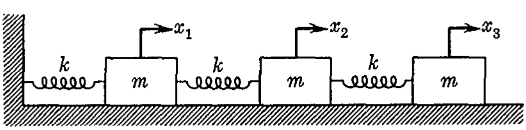


FIG. 9.8

without friction on a horizontal plane. They are connected together by three equal springs of spring constant k as shown in Fig. 9.8. Find the natural frequencies and mode shapes for this system.

Solution. Take as coordinates the three absolute displacements of the masses, x_1, x_2, and x_3. Then, in terms of these coordinates:

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \\ V &= \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_1 - x_2)^2 + \tfrac{1}{2}k(x_2 - x_3)^2 \\ &= \tfrac{1}{2}(2k)x_1^2 + \tfrac{1}{2}(2k)x_2^2 + \tfrac{1}{2}kx_3^2 - kx_1x_2 - kx_2x_3 \end{aligned}$$

Thus, in this particular problem, we have a statically coupled set of coordinates, with:

$$\begin{aligned} m_1 &= m_2 = m_3 = m \\ k_{11} &= k_{22} = 2k \\ k_{33} &= k \\ k_{12} &= k_{21} = k_{23} = k_{32} = -k \\ k_{13} &= k_{31} = 0 \end{aligned}$$

Substituting these values into Equation (9.20) we obtain:

$$\begin{aligned} (2k - m\omega^2)A_1 - kA_2 &= 0 \\ -kA_1 + (2k - m\omega^2)A_2 - kA_3 &= 0 \\ -kA_2 + (k - m\omega^2)A_3 &= 0 \end{aligned}$$

or, putting
$$\frac{m\omega^2}{k} = f$$

$$\begin{aligned} (2 - f)A_1 - A_2 &= 0 \\ -A_1 + (2 - f)A_2 - A_3 &= 0 \\ -A_2 + (1 - f)A_3 &= 0 \end{aligned}$$

The frequency determinant, Equation (9.21), becomes:

$$\begin{vmatrix} (2-f) & -1 & 0 \\ -1 & (2-f) & -1 \\ 0 & -1 & (1-f) \end{vmatrix} = 0$$

Expanding the determinant, we obtain the following polynomial equation for f:

$$f^3 - 5f^2 + 6f - 1 = 0$$

from which:

$$f_1 = 0.198; \quad f_2 = 1.555; \quad f_3 = 3.247$$

Thus, the three natural frequencies of vibration are:

$$\omega_1 = 0.445\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.25\sqrt{\frac{k}{m}}$$

$$\omega_3 = 1.80\sqrt{\frac{k}{m}}$$

We find the mode shapes from the three algebraic equations for A_1, A_2, A_3 in terms of the f. Since it is only the ratios between the A's that can be determined, we can arbitrarily set $A_1 = 1$ for each of the modes, so that we can work with numbers; then:

$$A_1^{(r)} = 1; \quad A_2^{(r)} = (2 - f); \quad A_3^{(r)} = \frac{(2 - f)}{(1 - f)}$$

For the first mode, for which $f_1 = 0.198$

$$A_1^{(1)} = 1; \quad A_2^{(1)} = 2 - 0.198 = 1.802; \quad A_3^{(1)} = \frac{1.802}{0.802} = 2.245$$

For the second mode, $f_2 = 1.555$

$$A_1^{(2)} = 1; \; A_2^{(2)} = 2 - 1.555 = 0.445; \; A_3^{(2)} = \frac{0.445}{-0.555} = -0.802$$

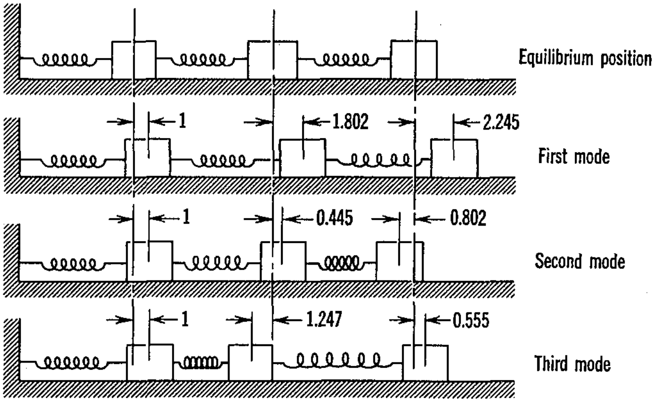


FIG. 9.9

For the third mode, $f_3 = 3.247$

$$A_1^{(3)} = 1; A_2^{(3)} = 2 - 3.247 = -1.247; A_3^{(3)} = \frac{-1.247}{-2.247} = 0.555$$

Thus, in the lowest mode of vibration the three masses always move in the same direction, with m_3 having 2.245 times the amplitude of m_1, and m_2 having 1.802 times the amplitude of m_1. In the second mode, m_1 and m_2 move in the same direction, while m_3 moves in the opposite direction, and in the third mode m_1 and m_3 move in the same direction while m_2 moves in the opposite direction. The configuration of the system as it vibrates in each of the principal modes of vibration can be indicated diagrammatically as in Figure 9.9.

We may use the orthogonality relationships of Equation (9.24) to check the values of the A's:

$$\sum_{i=1}^{n} m_i A_i^{(r)} A_i^{(s)} = 0$$

So:

$$A_1^{(1)}A_1^{(2)} + A_2^{(1)}A_2^{(2)} + A_3^{(1)}A_3^{(2)} = 0$$
$$(1)(1) + (1.802)(0.445) + (2.245)(-0.802) = 0$$
$$1 + 0.802 - 1.802 = 0$$

similarly:

$$A_1^{(1)}A_1^{(3)} + A_2^{(1)}A_2^{(3)} + A_3^{(1)}A_3^{(3)} = 0$$
$$A_1^{(2)}A_1^{(3)} + A_2^{(2)}A_2^{(3)} + A_3^{(2)}A_3^{(3)} = 0$$

Note that in this particular problem the m's drop out of the orthogonality relation, since all of the masses are equal.

PROBLEMS

9.30. In the expression for the kinetic energy T, Equation (9.16), write the general term m_{ij} in the form of a summation, and show that $m_{ij} = m_{ji}$.

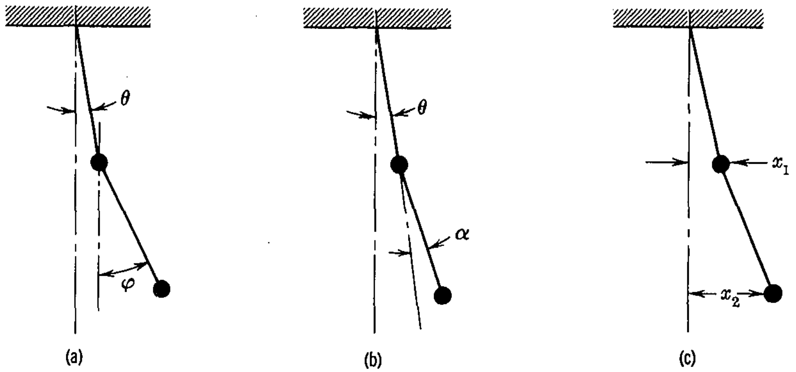
9.31. Refer to Example 2, Section 9.2 above, and show that for small oscillations the equations for kinetic and potential energy derived there in terms of the coordinates x and ϕ reduce to the form of Equations (9.15) and (9.16). Find the m_{ij}'s and the k_{ij}'s for this particular example. What kind of coupling exists between the (x, ϕ) coordinates?

9.32. Referring to the two masses on a stretched string of Prob. 9.24 above, write the kinetic and potential energies in terms of suitable coordinates, and determine the k_{ij}'s and the m_{ij}'s for the problem. Calculate the natural frequencies of vibration of the system by a direct substitution of the k's and m's into the frequency determinant Equation (9.21).

9.33. The figure shows three different coordinate systems suitable for the description of small oscillations of a double pendulum consisting of two equal masses and two massless strings of equal length. Write expressions for the kinetic and potential energy of the system in each of the three coordinate systems, and state whether static coupling or dynamic coupling is present in each.

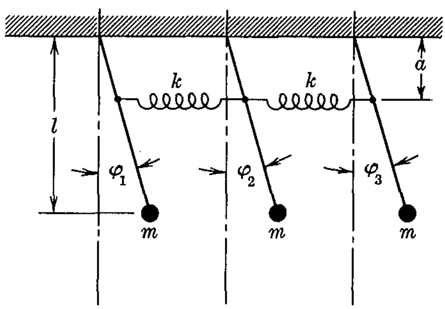
9.34. Three equal simple pendulums of length l and concentrated mass m are coupled by two equal springs of spring constant k located a distance a from the point of support as shown in the figure. (a) Using as coordinates the three angles ϕ_1, ϕ_2, ϕ_3 and considering small oscillations only, write the expressions for the potential and kinetic energy of the system, and find the values of the k_{ij}'s and the m_{ij}'s for this problem. (b) Find the natural frequencies of vibration of the system. (c) Find the mode shapes for the system.

9.35. An approximate dynamic model of a three story steel-frame

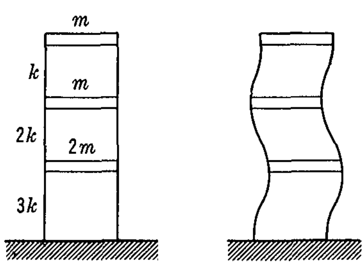


PROB. 9.33

building can be formed from three concentrated masses and three massless springs as shown in the figure. During lateral vibrations of the building, as for example, during an earthquake, it is assumed that the floors move parallel to each other, so that the action is primarily that of shear. The



PROB. 9.34



PROB. 9.35

masses of the three floors and the shearing spring constants are as shown in the figure. Show that the frequency equation for small lateral oscillations is:

$$2f^3 - 13f^2 + 20f - 6 = 0; \quad f = \frac{m\omega^2}{k}$$

9.12 Forced Oscillations. We shall now suppose that the statically coupled conservative system which has been considered above is acted upon by a system of periodic exciting forces $F_i \sin \omega t$ so that the basic equations of motion become:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = F_i \sin \omega t \tag{9.25}$$

Following the same procedure that was successful for the single degree of freedom system of Chapter 5, we shall assume that the steady state solution will be harmonic and of the same frequency as the exciting force. In what follows we shall consider steady state motion only. We suppose that there is a small amount of damping present which after a time will eliminate the transient terms, but which is not large enough to change appreciably the steady state forced amplitudes.

It cannot be expected that the solution will involve only a single mode of vibration, since all of the modes may be excited simultaneously. We accordingly write a trial solution in the form:

$$q_i = \sum_{r=1}^{n} c_i^{(r)} \sin \omega t \tag{9.26}$$

where the coefficients $c_i^{(r)}$ are the steady state amplitudes of the forced oscillations of the various modes. We do not need to include a phase angle in this expression, since for the undamped system the phase shift will be either zero or 180°.

Substituting the general expressions for kinetic and potential energies into Equation (9.25) we obtain:

$$m_i \ddot{q}_i + k_{i1} q_1 + k_{i2} q_2 + \cdots + k_{in} q_n = F_i \sin \omega t$$

Putting Equation (9.26) into this equation, and cancelling the common sin ωt term leads to:

$$-\, m_i \omega^2 \sum_{r=1}^{n} c_i^{(r)} + k_{i1} \sum_{r=1}^{n} c_1^{(r)} + k_{i2} \sum_{r=1}^{n} c_2^{(r)} + \cdots = F_i$$

or:

$$-\, m_i \omega^2 \sum_{r=1}^{n} c_i^{(r)} + \sum_{j=1}^{n} k_{ij} \sum_{r=1}^{n} c_j^{(r)} = F_i \tag{9.27}$$

The next step will consist of expressing the coefficients $c_i^{(r)}$ in terms of the mode shape numbers $A_i^{(r)}$ that have been defined above in connection with the free oscillation problem. We define a new coefficient, $a^{(r)}$ as:

$$c_i^{(r)} = a^{(r)}A_i^{(r)} \tag{9.28}$$

The physical significance of Equation (9.28) is that we suppose each mode excited by the external forces to have the same shape as the free oscillation mode, with an amplitude $a^{(r)}$ that remains to be determined.

Substituting Equation (9.28) into Equation (9.27), we find that the second term becomes:

$$\begin{aligned}\sum_{j=1}^{n} k_{ij} \sum_{r=1}^{n} c_j^{(r)} &= \sum_{j=1}^{n} k_{ij} \sum_{r=1}^{n} a^{(r)}A_j^{(r)} \\ &= \sum_{j=1}^{n} k_{ij}(a^{(1)}A_j^{(1)} + a^{(2)}A_j^{(2)} + \cdots) \\ &= a^{(1)} \sum_{j=1}^{n} k_{ij}A_j^{(1)} + a^{(2)} \sum_{j=1}^{n} k_{ij}A_j^{(2)} + \cdots \\ &= \sum_{r=1}^{n} a^{(r)} \sum_{j=1}^{n} k_{ij}A_j^{(r)}\end{aligned}$$

We now refer back to Equation (9.23), which states that:

$$\sum_{j=1}^{n} k_{ij}A_j^{(r)} = m_i\omega_r^2 A_i^{(r)}$$

Introducing this into the last expression above, we can write Equation (9.27) as:

$$-m_i\omega^2 \sum_{r=1}^{n} a^{(r)}A_i^{(r)} + \sum_{r=1}^{n} a^{(r)}m_i\omega_r^2 A_i^{(r)} = F_i$$

$$\sum_{r=1}^{n} a^{(r)}m_i A_i^{(r)}(\omega_r^2 - \omega^2) = F_i \tag{9.29}$$

Note that for the single degree of freedom case Equation (9.29) reduces to the form:

$$am(\omega_r^2 - \omega^2) = F$$

$$a = \frac{F/m}{\omega_r^2 - \omega^2} = \frac{F/m\omega_r^2}{1 - \left(\frac{\omega}{\omega_r}\right)^2} = \frac{F/k}{1 - \left(\frac{\omega}{\omega_r}\right)^2}$$

which checks the conclusions reached in Chapter 5 for the steady state amplitude of the single degree of freedom system with no damping.

For the multiple degree of freedom system this calculation is not so simple, because the amplitude coefficients $a^{(r)}$ occur in Equation (9.29) as the coefficients in a series. These constants can be determined, however, by expanding the exciting forces F_i into a series of the same normal functions $A_i^{(r)}$ that appear with the amplitude coefficients. We will thus write the exciting force as:

$$F_j = \sum_{r=1}^{n} f^{(r)} m_j A_j^{(r)} \tag{9.30}$$

where we use j instead of i to indicate that such an expansion could be applied in general to any one of the forces. This process of expanding the function F_j into a series is similar in principle to the expansion of a function in a Fourier series, but instead of sines and cosines we use the functions $A_j^{(r)}$ because of their appropriateness to the particular problem involved. The method of determining the unknown coefficients is analogous to that used for the Fourier coefficients.

Writing several terms of Equations (9.30) gives:

$$F_j = f^{(1)} m_j A_j^{(1)} + f^{(2)} m_j A_j^{(2)} + \cdots$$

Multiply both sides of this equation by $A_j^{(r)}$, and thus obtain:

$$F_j A_j^{(r)} = f^{(1)} m_j A_j^{(1)} A_j^{(r)} + f^{(2)} m_j A_j^{(2)} A_j^{(r)} + \cdots$$

Summing over both sides of this set of n equations gives:

$$\sum_{j=1}^{n} F_j A_j^{(r)} = f^{(r)} \sum_{j=1}^{n} m_j [A_j^{(r)}]^2 + \sum_{s=1}^{n-1} \sum_{j=1}^{n} f^{(s)} m_j A_j^{(r)} A_j^{(s)}$$

where $r \neq s$. The second term on the right becomes:

$$\sum_{s=1}^{n-1} f^{(s)} \sum_{j=1}^{n} m_j A_j^{(r)} A_j^{(s)}$$

but by the orthogonality relation of Equation (9.24):

$$\sum_{j=1}^{n} m_j A_j^{(r)} A_j^{(s)} = 0$$

and hence the only term left on the right is the single term involving $f^{(r)}$, and we find:

$$f^{(r)} = \frac{\sum_{j=1}^{n} F_j A_j^{(r)}}{\sum_{j=1}^{n} m_j A_j^{(r)2}} \tag{9.31}$$

Substituting Equations (9.31) and (9.30) into Equation (9.29) will then give the amplitude coefficients $a^{(r)}$:

$$\sum_{r=1}^{n} a^{(r)} m_i A_i^{(r)} (\omega_r^2 - \omega^2) = \sum_{r=1}^{n} f^{(r)} m_i A_i^{(r)}$$

$$a^{(r)} = \frac{f^{(r)}}{(\omega_r^2 - \omega^2)} = \frac{1}{(\omega_r^2 - \omega^2)} \frac{\sum_{j=1}^{n} F_j A_j^{(r)}}{\sum_{j=1}^{n} m_j A_j^{(r)2}} \tag{9.32}$$

The final solution for the steady state vibrations is then given by Equation (9.28) and (9.26):

$$q_i = \sum_{r=1}^{n} \frac{\sum_{j=1}^{n} F_j A_j^{(r)}}{\sum_{j=1}^{n} m_j A_j^{(r)2}} \frac{A_i^{(r)}}{(\omega_r^2 - \omega^2)} \sin \omega t \tag{9.33}$$

As a specific example illustrating the use of Equation (9.33) consider the system of three masses of Figure 9.8. Suppose that a force $F \sin \omega t$ acts on the middle mass m_2, and that we are to determine the steady state motion of the first mass m_1. Using Equation (9.33) we first substitute $i = 1$, since we wish the motion of the first mass:

$$x_1 = \sum_{r=1}^{n} \frac{\sum_{j=1}^{n} F_j A_j^{(r)}}{\sum_{j=1}^{n} m_j A_j^{(r)2}} \frac{A_1^{(r)}}{(\omega_r^2 - \omega^2)} \sin \omega t$$

We then expand the r-summation, and obtain the three terms:

$$x_1 = \left[\frac{\sum_{j=1}^{n} F_j A_j^{(1)}}{\sum_{j=1}^{n} m_j A_j^{(1)2}} \frac{A_1^{(1)}}{(\omega_1^2 - \omega^2)} + \frac{\sum_{j=1}^{n} F_j A_j^{(2)}}{\sum_{j=1}^{n} m_j A_j^{(2)2}} \frac{A_1^{(2)}}{(\omega_2^2 - \omega^2)} \right.$$
$$\left. + \frac{\sum_{j=1}^{n} F_j A_j^{(3)}}{\sum_{j=1}^{n} m_j A_j^{(3)2}} \frac{A_1^{(3)}}{(\omega_3^2 - \omega^2)} \right] \sin \omega t$$

We next expand the j-summations, noting that $F_1 = F_3 = 0$, $F_2 = F$; thus, for example:

$$\frac{\sum_{j=1}^{n} F_j A_j^{(1)}}{\sum_{j=1}^{n} m_j A_j^{(1)2}} = \frac{F A_2^{(1)}}{m[A_1^{(1)2} + A_2^{(1)2} + A_3^{(1)2}]}$$

Thus we have:

$$x_1 = \frac{F}{m} \left\{ \frac{A_1^{(1)} A_2^{(1)}}{[A_1^{(1)2} + A_2^{(1)2} + A_3^{(1)2}](\omega_1^2 - \omega^2)} \right.$$
$$+ \frac{A_1^{(2)} A_2^{(2)}}{[A_1^{(2)2} + A_2^{(2)2} + A_3^{(2)2}](\omega_2^2 - \omega^2)}$$
$$\left. + \frac{A_1^{(3)} A_2^{(3)}}{[A_1^{(3)2} + A_2^{(3)2} + A_3^{(3)2}](\omega_3^2 - \omega^2)} \right\} \sin \omega t$$

The values of the A's have already been determined in the example of Section 9.11 above, where we obtained:

$$\begin{array}{lll} A_1^{(1)} = 1 & A_1^{(2)} = 1 & A_1^{(3)} = 1 \\ A_2^{(1)} = 1.802 & A_2^{(2)} = 0.445 & A_2^{(3)} = -1.247 \\ A_3^{(1)} = 2.245 & A_3^{(2)} = -0.802 & A_3^{(3)} = 0.555 \end{array}$$

which gives directly the final answer:

$$x_1 = \frac{F}{m}\left[\frac{0.194}{(\omega_1^2 - \omega^2)} + \frac{0.241}{(\omega_2^2 - \omega^2)} - \frac{0.435}{(\omega_3^2 - \omega^2)}\right] \sin \omega t$$

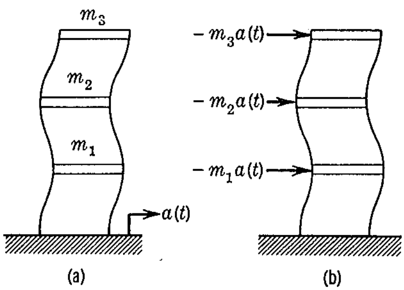
where $\omega_1^2 = 0.198 \dfrac{k}{m}$; $\omega_2^2 = 1.555 \dfrac{k}{m}$; $\omega_3^2 = 3.247 \dfrac{k}{m}$ as determined in the example of Section 9.11.

PROBLEMS

9.36. In the example of the three masses and the three springs worked out above, suppose that the force $F \sin \omega t$ acts on the right hand mass, and determine the steady state motion of the left hand mass.

9.37. Refer to the triple pendulum system of Prob. 9.34, and apply a sinusoidal horizontal force $F \sin \omega t$ to the left hand pendulum. Calculate the steady state motion of the middle pendulum.

9.38. Suppose that during an earthquake the ground supporting a shear-type building moves with a horizontal acceleration $a(t)$. (a) Show that if x_1, x_2 and x_3 are the lateral displacements of the floors relative to the moving ground, the equations of forced oscillations of the building are the same as if the ground were fixed and external exciting forces as shown in diagram (b) were applied.



PROB. 9.38

(b) If $a = a_0 \sin \omega t$, find the displacement of the first story relative to the ground, in terms of the applied motions, the natural frequencies, the masses, and the mode shape numbers $A_i^{(r)}$.

9.13 The Calculus of Variations. Many problems of classical mechanics are most naturally formulated in terms of maximum or minimum statements. Consider, for example, the famous problem of the *Brachistochrone*, as formulated by John Bernoulli in 1696.

"A particle, under the action of gravity, slides along a smooth curve which lies in a vertical plane. Find the form of the curve for which the time required for the particle to move between two given points on the curve is a minimum." As a second example, consider the statics problem of the determination of the equilibrium form of a flexible chain, supported at its ends and hanging in a vertical plane. This problem can be solved by finding the shape of the chain which makes the potential energy of the system a minimum.

Such problems have aroused the interest of some of the greatest investigators. The Brachistochrone problem was first solved independently by John Bernoulli, Newton, and Leibnitz and the development of the basic theory of such problems was carried out by Euler and Lagrange. An idea of the fundamental mathematical problem may be obtained from the analytical statement of the Brachistochrone problem. In Fig. 9.10, the particle of mass m slides down the vertical curve AB under the action of the gravity force mg, and we wish to determine the shape of the curve $y = f(x)$ such that the time from A to B is a minimum. The time required by the particle to travel an arc length ds is ds/v, and thus the total time from A to B is:

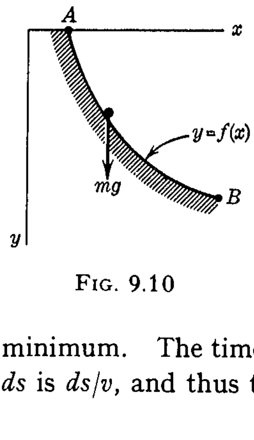


FIG. 9.10

$$t = \int_A^B \frac{ds}{v}$$

From the energy principle, the velocity of the particle will depend upon the vertical distance traveled:

$$v = \sqrt{2gy}$$

we also have $ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$ and hence the final expression for the time becomes:

$$t = \int_A^B \left[\frac{1 + \left(\frac{dy}{dx}\right)^2}{2gy}\right]^{\frac{1}{2}} dx$$

We thus see that the problem requires the minimization of a definite integral by means of a change in the form of the integrand function. We may now compare this problem with the type of minimum and maximum problem with which we are already familiar.

In ordinary maximum and minimum problems, we are generally given a function of several variables in the form:

$$y = f(x_1, x_2, \cdots)$$

and we wish to find the values of the x's for which the function has an extreme value. The necessary conditions for the solution of this problem are given by the equations:

$$\frac{\partial y}{\partial x_1} = 0; \quad \frac{\partial y}{\partial x_2} = 0; \cdots$$

In the calculus of variations we also have the problem of finding an extremum, but now the expression to be investigated no longer depends upon a finite number of independent variables, but depends instead upon the behavior of one or more dependent variables. The particular expression of this type with which we shall be concerned has the form:

$$I = \int_{X_1}^{X_2} f\left(x, y, \frac{dy}{dx}\right) dx, \quad \text{where} \quad y = \phi(x) \tag{9.34}$$

Given the function f we are to determine the function ϕ which will give I a stationary value. It will be observed that the analytical expression for the Brachistochrone problem is a special case of Equation (9.34).

The solution of this mathematical problem will not only enable us to solve many specific problems, but will also permit us to put the basic equations of motion into several new forms, which have been of great importance in the development of advanced dynamics.

9.14 Euler's Differential Equation. The basic solution of the problem defined in Equation (9.34) is given by Euler's Differential Equation, which we shall derive by a method similar to that employed by Lagrange. Given the definite integral $I = \int_A^B f(y, y', x)\, dx$, where $y = \phi(x)$, and $y' = \frac{dy}{dx}$, find $y = \phi(x)$ so that I has a stationary value.

Referring to Fig. 9.11, we note that if the curve shown there as a solid line is actually the correct $\phi(x)$ to give the minimum value to I, then the integral evaluated over some slightly different curve, such as the dotted curve, will have a slightly larger value. This slightly different curve, which has the same end points A and B as the exact curve, is called the varied path. If we imagine that at a particular value of the independent variable x we move to the varied curve, the ordinate of the varied curve at that point will be $y + \delta y$ which can be written as:

$$y + \delta y = \phi(x) + \epsilon\theta(x) \quad (9.35)$$

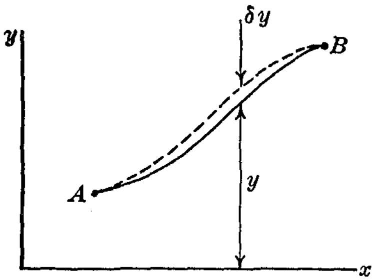


FIG. 9.11

where δy is called the variation in y and the small quantity ϵ indicates that the varied path is slightly different from the minimum path. We use the variation symbol δ to distinguish δy from the differential dy. Note that dy is the infinitesimal change in the given function $\phi(x)$ caused by an infinitesimal change dx in the independent variable x, whereas δy is a small arbitrary change in y equal to $\epsilon\theta(x)$.

As the path in Fig. 9.11 is changed from the original to the varied path, the value of the integral I will change by a small amount δI, called the variation of the integral. If $F(y, y', x)$ is the modified integrand corresponding to the varied path, we have:

$$\begin{aligned}\delta I &= \delta \int_A^B f(y, y', x)\, dx = \int_A^B F(y, y', x)\, dx - \int_A^B f(y, y', x)\, dx \\ &= \int_A^B [F(y, y', x) - f(y, y', x)]\, dx = \int_A^B \delta f(y, y', x)\, dx \qquad (9.36)\end{aligned}$$

and we see that the processes of variation and integration can be interchanged.

We next consider the way in which $f(y, y', x)$ varies as we go from the original to the varied path. At one particular value of x, y will change to $y + \delta y$, and the slope will change from y' to $y' + \delta y'$, thus:

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \qquad (9.37)$$

The slope of the varied curve is

$$\frac{d}{dx}(y + \delta y) = y' + \frac{d}{dx}(\delta y)$$

and thus $\delta y' = y' + \frac{d}{dx}(\delta y) - y' = \frac{d}{dx}(\delta y)$, which indicates that the processes of variation and differentiation can be interchanged. Equation (9.37) can now be written as:

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)$$

which, when introduced into equation (9.36) gives:

$$\delta I = \int_A^B \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\right] dx \tag{9.38}$$

Integrating by parts the second term in this integral, we obtain:

$$\int_A^B \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\, dx = \frac{\partial f}{\partial y'}\delta y \bigg|_A^B - \int_A^B \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\, dx$$

Since the varied curve coincides with the exact curve at the end-points A and B, $\delta y = 0$ at A and B and the term

$$\frac{\partial f}{\partial y'}\delta y \bigg|_A^B = 0$$

Thus Equation (9.38) becomes:

$$\delta I = \int_A^B \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\, dx$$

For a stationary value of I, $\delta I = 0$. Since δy is arbitrary and can be taken as different from zero, δI can be zero only if the expression in the square brackets is zero:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{9.39}$$

This is *Euler's Differential Equation*, and the function $y = \phi(x)$ which satisfies this equation is the desired function which will give the integral I a stationary value.

There are two special cases in which the solution to Euler's Differential Equation can be obtained with relative ease.

Case 1. If the function does not contain y, so that $I = \int_A^B f(y',x)\,dx$ then $\dfrac{\partial f}{\partial y} = 0$, and Equation (9.39) becomes:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0; \quad \frac{\partial f}{\partial y'} = \text{constant}$$

and we have obtained a general first integral of Euler's equation for this special case.

Case 2. If the function f does not contain x, so that $I = \int_A^B f(y, y',)\,dx$ Equation (9.39) may be written as:

$$y' \frac{\partial^2 f}{\partial y \partial y'} + \frac{\partial^2 f}{\partial y'^2} y'' - \frac{\partial f}{\partial y} = 0$$

multiply through this equation by y', and add and subtract the term $y'' \dfrac{\partial f}{\partial y'}$ obtaining:

$$y'^2 \frac{\partial^2 f}{\partial y \partial y'} + y'y'' \frac{\partial^2 f}{\partial y'^2} + y'' \frac{\partial f}{\partial y'} - y'' \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} y' = 0$$

the first three terms in this expression are equal to $\dfrac{d}{dx}\left(y' \dfrac{\partial f}{\partial y'}\right)$ and the last two terms are equal to $\left(-\dfrac{df}{dx}\right)$ so we have:

$$\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) - \frac{df}{dx} = 0 = \frac{d}{dx}\left[\left(y' \frac{\partial f}{\partial y'}\right) - f\right]$$

this is a perfect differential, and can be integrated directly to give:

$$y' \frac{\partial f}{\partial y'} - f = \text{constant}$$

thus giving a general first integral of Euler's equation for this special case.

EXAMPLE 1. Find the form of the plane curve $y = \phi(x)$ joining two points AB with the shortest length.

Solution. The element of length is $ds = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$ and the total length of a plane curve between A and B will be:

$$L = \int_A^B (1 + y'^2)^{\frac{1}{2}} \, dx$$

The function f in the integrand is of the form $f(y')$ and hence we can use the integrated form of Euler's equation described in Case 1 above:

$$\frac{\partial f}{\partial y'} = \text{constant}$$

$$f = (1 + y'^2)^{\frac{1}{2}}; \quad \frac{\partial f}{\partial y'} = \frac{y'}{(1 + y'^2)^{\frac{1}{2}}} = c_1; \quad y' = \sqrt{\frac{c_1^2}{1 - c_1^2}} = c_2$$

thus:

$$y' = \frac{dy}{dx} = c_2; \quad y = c_2 x + c_3$$

and the curve is a straight line.

EXAMPLE 2. We have shown previously that the solution of the Brachistochrone problem involves the minimization of the integral

$$t = \int_A^B \left[\frac{1 + y'^2}{2gy}\right]^{\frac{1}{2}} dx$$

In this problem $f = f(y, y')$ and since x is not involved the integrated form of Euler's equation discussed in Case 2 above may be used:

$$y' \frac{\partial f}{\partial y'} - f = c$$

$$f = \left(\frac{1 + y'^2}{2gy}\right)^{\frac{1}{2}}; \quad \frac{\partial f}{\partial y'} = \frac{1}{2} \frac{(1 + y'^2)^{-\frac{1}{2}}}{(2gy)^{\frac{1}{2}}} (2y')$$

so:

$$\frac{y'^2}{(1 + y'^2)^{\frac{1}{2}}(2gy)^{\frac{1}{2}}} - \frac{(1 + y'^2)^{\frac{1}{2}}}{(2gy)^{\frac{1}{2}}} = c = \frac{-1}{(1 + y'^2)^{\frac{1}{2}}(2gy)^{\frac{1}{2}}}$$

from which:

$$\frac{dy}{dx} = \sqrt{\frac{1 - 2c^2 gy}{2c^2 gy}}$$

This may be shown to be the equation of a cycloid—the curve traced by a point on the circumference of a circle as the circle rolls along a straight line. In the coordinate system of Fig. 9.12, the coordinates of the point P which traces out the cycloid are:

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

Substituting these expressions into the above equation for $\frac{dy}{dx}$, we obtain:

$$\frac{\sin\theta}{1-\cos\theta} = \left[\frac{1 - 2c^2gr(1-\cos\theta)}{2c^2gr(1-\cos\theta)}\right]^{\frac{1}{2}}$$

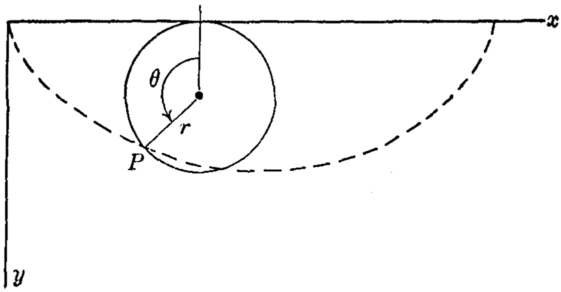


FIG. 9.12

Squaring both sides and putting

$$\sin^2\theta = (1-\cos^2\theta) = (1-\cos\theta)(1+\cos\theta)$$

this becomes:

$$\frac{1+\cos\theta}{1-\cos\theta} = \frac{1 - 2c^2gr + 2c^2gr\cos\theta}{2c^2gr(1-\cos\theta)}$$

If we put $2c^2gr = \frac{1}{2}$, this equation is identically satisfied, so that the Brachistochrone is a cycloid of radius $r = 1/4c^2g$.

It should be noted that in some problems a simplification can be introduced by an interchange of the independent and dependent variables. In the above example of the Brachistochrone, suppose that we take y as the independent variable and x as the dependent variable so that the integral becomes:

$$t = \int_A^B \left[\frac{1+x'^2}{2gy}\right]^{\frac{1}{2}} dy$$

and Euler's Differential Equation becomes:

$$\frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) - \frac{\partial f}{\partial x} = 0$$

This corresponds to choosing a varied path as shown in Fig. 9.13. In this form, the problem falls into special Case 1 above, since $\frac{\partial f}{\partial x} = 0$ and we have:

$$\frac{\partial f}{\partial x'} = c = \frac{\partial}{\partial x'}\left(\frac{1 + x'^2}{2gy}\right)^{\frac{1}{2}}$$

$$\frac{x'}{(1 + x'^2)^{\frac{1}{2}}(2gy)^{\frac{1}{2}}} = c; \quad x'^2 = c^2[(1 + x'^2)(2gy)]$$

from which: $\frac{dy}{dx} = \sqrt{\frac{1 - 2c^2gy}{2c^2gy}}$ as before.

In this problem, the use of y as the independent variable leads to a simpler solution.

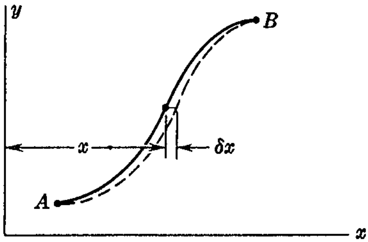
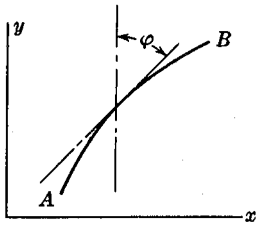


FIG. 9.13



PROB. 9.40

PROBLEMS

9.39. A flexible cable of length l hangs under its own weight from two supports on the same level a distance d apart. The shape which the cable assumes is such that the potential energy is a minimum. Set up an integral expression for the potential energy of the system, and minimize this integral to find the equation for the cable.

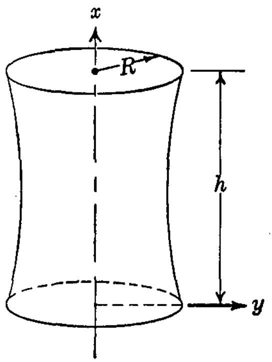
9.40. A particle moves in the x–y plane from a point A to a point B. The velocity v of the particle is variable, but is a function of y only. Show that for the path which makes the time from A to B a minimum, the angle between the tangent to the curve and the y-axis obeys the law:

$$\frac{\sin \phi}{v} = \text{constant}$$

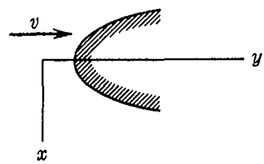
Note that this problem is the mechanical version of the optical problem of determining the path of a ray of light through a medium which has an index of refraction which varies in one direction. In the optical case the above expression is called Snell's Law.

9.41. A container of height h has a circular top and bottom, each of radius R. The sides are formed of an axially symmetrical surface of revolution. What should be the shape of the sides to give a minimum surface area?

9.42. In studying the resistance of bodies of revolution in a flowing fluid, Newton made assumptions equivalent to supposing that the pressure



PROB. 9.41



PROB. 9.42

at each point on the surface would be proportional to the square of the component of velocity normal to the surface. It is desired, under this assumption, to find the shape of the solid of revolution which would have the minimum total drag.

(a) Show that this problem involves the minimization of the integral

$$\int \frac{x}{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$$

(b) Show that the shape of the curve which minimizes the integral of part (a) must satisfy the equation

$$\frac{x\left(\dfrac{dy}{dx}\right)}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^2} = C_1$$

(c) Show that the equation of the surface of minimum drag in parametric form is:

$$x = \frac{C_1}{p}(1 + p^2)^2$$

$$y = C_1(p^2 + \tfrac{3}{4}p^4 - \log p) + C_2$$

where

$$p = \frac{dy}{dx} = y'$$

Note that the basic assumption described above applies to only a few very special cases, and that for ordinary flow conditions the results are not even a rough approximation.

9.15 Hamilton's Principle. It will be observed that Euler's Differential equation (9.39) which is the condition for minimizing a certain integral, has the same form as Lagrange's equations of motion. Writing Lagrange's equations for a conservative system in the form:

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}(T - V)\right] - \frac{\partial}{\partial q}(T - V) = 0$$

we see from Equation (9.39) that an equivalent statement is:

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \tag{9.40}$$

Equation (9.40) is an analytical statement of *Hamilton's Principle*, which says that the actual path followed by a dynamical process is such as to make the integral of the function $(T - V)$ a minimum.

For most problems of the type hitherto considered in this book, Hamilton's Principle does not lead to any particular advantage in the setting up of the equations of motion, since it is equivalent in such applications to the use of Lagrange's equations, as is shown in Example 1 below. Hamilton's Principle does have important engineering applications, however, in the setting up of the partial differential equations of motion describing infinite degree of freedom systems. In such problems as the vibrations of beams, Hamilton's Principle plays the same role in establishing the equations of motion that was played by Lagrange's equations for the finite degree of freedom systems. Typical examples of this use of Hamilton's Principle are given below in Example 2 and in the problems.

EXAMPLE 1. Find by Hamilton's Principle the equation of motion for a single degree of freedom undamped harmonic oscillator.

Solution. For such a system, $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$ thus:

$$\delta \int_{t_1}^{t_2} (\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2)\, dt = 0$$

Applying Euler's differential equation to minimize the integral, we have:

$$f = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2$$

thus:

$$\frac{d}{dt}(m\dot{x}) + kx = 0$$

$$m\ddot{x} + kx = 0$$

For problems of this kind it will be noted that Hamilton's Principle leads to exactly the same steps as the use of Lagrange's equations.

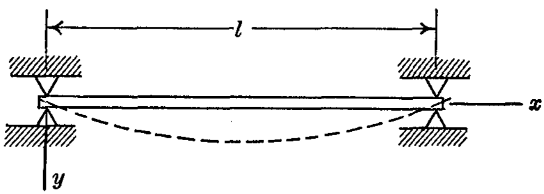


FIG. 9.14

EXAMPLE 2. A straight uniform beam having a length l, a mass per unit length μ, a cross-sectional moment of inertia I, and a modulus of elasticity E is pinned at each end as shown in Fig. 9.14. The beam performs small transverse bending oscillations in the horizontal x, y plane. Using Hamilton's Principle, find the differential equation of motion of the system.

Solution. From the theory of strength of materials we know that the potential energy of bending of the beam is:

$$V = \frac{1}{2}\int_0^l \frac{M^2}{EI}\, dx = \frac{1}{2}\int_0^l EIy''^2\, dx$$

where M is the bending moment in the beam, and $M = EI\dfrac{d^2y}{dx^2}$

$= EIy''$. The kinetic energy of the beam is $T = \frac{1}{2}\int_0^l \mu\dot{y}^2\,dx$. Hamilton's Principle in the form of Equation (9.40) then becomes:

$$\delta\int_{t_1}^{t_2}\left[\int_0^l \left(\tfrac{1}{2}\mu\dot{y}^2 - \tfrac{1}{2}EI\,y''^2\right)dx\right]dt = 0$$

Instead of applying Euler's differential equation to this problem, we shall go through the steps which lead to Euler's equation. In many problems this procedure will be more suitable than a direct substitution into Euler's equation.

Performing the variation as indicated, we obtain:

$$\int_{t_1}^{t_2}\int_0^l (\mu\dot{y}\delta\dot{y} - EIy''\delta y'')\,dx\,dt = 0$$

writing $\delta\dot{y} = \dfrac{\partial}{\partial t}(\delta y)$ and $\delta y'' = \dfrac{\partial^2}{\partial x^2}(\delta y)$ we have:

$$\int_{t_1}^{t_2}\int_0^l \left[\mu\dot{y}\frac{\partial}{\partial t}(\delta y) - EIy''\frac{\partial^2}{\partial x^2}(\delta y)\right]dx\,dt$$

Integrate the first term by parts, as follows:

$$\int_{t_1}^{t_2}\mu\dot{y}\frac{\partial}{\partial t}(\delta y)\,dt = \mu\dot{y}\delta y\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\mu\ddot{y}\,\delta y\,dt$$

Since by the definition of the varied path, $\delta y = 0$ at the two endpoints t_1 and t_2, the first part disappears.

We next integrate the second term in the integral by parts twice, as follows:

$$\int_0^l EIy''\frac{\partial^2}{\partial x^2}(\delta y)\,dx = EIy''\frac{\partial}{\partial x}(\delta y)\Big|_0^l - \int_0^l EIy'''\frac{\partial}{\partial x}(\delta y)\,dx$$

$$= EIy''\frac{\partial}{\partial x}(\delta y)\Big|_0^l - EIy'''\,\delta y\Big|_0^l + \int_0^l EIy^{IV}\,\delta y\,dx$$

In this expression, the first term is zero, since at both 0 and l the bending moment in the beam, which is proportional to y'', is zero, because the beam is supported by frictionless pins at the ends. The second term is zero since δy is zero at the two ends of the beam.

Thus, the variational equation becomes:

$$\int_{t_1}^{t_2}\int_0^l (-\mu\ddot{y} - EIy^{IV})\,\delta y\,dx\,dt = 0$$

Since δy is arbitrary, and can be taken as different from zero, the condition that the integral should vanish requires that the term in the parentheses should be zero. This gives the required partial differential equation of vibration of the beam:

$$EI\frac{\partial^4 y}{\partial x^4} + \mu\frac{\partial^2 y}{\partial t^2} = 0$$

Note that the above variational procedure not only developed the differential equation, but that the boundary conditions as well were automatically involved.

PROBLEMS

9.43. A uniform flexible string of mass per unit length μ is fastened at each end to rigid supports a distance l apart. The string is subjected to a tension force F which may be assumed to be constant for small lateral motions of the string.

(a) By considering the change of length of the string as it is deflected, show that the potential energy of the system is:

$$V = \frac{F}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx$$

where y is the deflection perpendicular to the length.

(b) Applying Hamilton's Principle as in Example (2), above, show that the partial differential equation describing small transverse motion of the string is:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

(c) Find the velocity of propagation of waves along the string.

9.44. A straight shaft of length l has a uniform circular cross section of polar moment of inertia I_p and area A. The material of the shaft has a density ρ and a modulus of elasticity in shear of G. The angle of twist ϕ of the shaft is a function of the position x along the shaft, and of the time t.

(a) Given that the total angle of twist of the shaft under a twisting moment M_t is $\phi = \dfrac{M_t l}{GI_p}$, show that the potential energy of the shaft is:

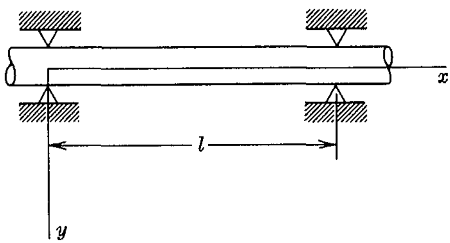
$$V = \frac{GI_p}{2}\int_0^l \left(\frac{\partial \phi}{\partial x}\right)^2 dx$$

(b) Find by Hamilton's Principle the partial differential equation describing the motion of the shaft. Assuming that both ends of the shaft are free, indicate the way in which these boundary conditions appear in the derivation.

(c) What is the speed of propagation of a torsional wave along the shaft?

9.45. In connection with the design of an oil pipe-line, the question of the effect of a fluid flowing through the pipe on the natural frequencies of transverse vibrations should be considered.* It will be assumed that a section of the pipe between two supports can be treated as a pin-ended beam. The following nomenclature will be used:

v = fluid velocity relative to the pipe, assumed constant.
m = mass per unit length of pipe plus fluid.
ρ = mass per unit length of fluid.
I = moment of inertia of pipe cross section about the neutral axis.
x = coordinate along length of pipe.
y = coordinate transverse to pipe.
l = distance between supports.
E = modulus of elasticity of pipe.



PROB. 9.45

(a) Assuming small displacements of the pipe, show that the x-component of the fluid velocity is v, and that the y-component is $\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x}$.

(b) Show that the kinetic energy of the system is:

$$T = \int_0^l \left\{\tfrac{1}{2}(m - \rho)\left(\frac{\partial y}{\partial t}\right)^2 + \tfrac{1}{2}\rho\left[v^2 + \left(\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x}\right)^2\right]\right\} dx$$

(c) Show that the potential energy of the system (strain energy of bending in the pipe) is:

$$V = \int_0^l \tfrac{1}{2}\, EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$$

(d) Using Hamilton's Principle, as in Example 2 above, derive the partial differential equation describing the transverse vibrations of the pipe.

* G. W. Housner, "Bending Vibrations of a Pipe Line Containing Flowing Fluid", *Journal of Applied Mechanics,* Vol. 19 (June 1952), p. 205.

9.16 Hamilton's Canonical Equations of Motion. For engineering applications of dynamics, Lagrange's equations in the form of Equation (9.14) or Hamilton's Principle in the form of Equation (9.40) are usually the most convenient formulation of the basic laws of motion. For certain applications in physics, however, other forms of these basic principles are often used. We shall briefly indicate the way in which some of these alternative statements are made.

We start with Lagrange's equations for a conservative system in the form: [Equation (9.11).]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Introduce the definition $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ = *generalized momentum.* (9.41)

In terms of p_i, Equation (9.11) becomes:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{9.42}$$

A new function H, the *Hamiltonian Function,* is now defined as:

$$H = \sum_i p_i\dot{q}_i - L \tag{9.43}$$

In general, the Lagrangian function L is a function of q_i, $\dot{q}_i$, and t; $L(q_i, \dot{q}_i, t)$. The $\dot{q}_i$ enters through the kinetic energy as a quadratic term, and thus Equation (9.41) above will give p_i as a linear function of $\dot{q}_i$. This set of linear equations involving p_i and $\dot{q}_i$ could in theory be solved giving $\dot{q}_i$ in terms of p_i, and hence the $\dot{q}_i$'s could in principle always be eliminated from Equation (9.43). We may thus say that H can always be expressed as a function of p_i, q_i, and t,

$$H = H(p_i, q_i, t) \tag{9.44}$$

We shall now express the differential of H in two ways, first by differentiating Equation (9.43), and then by differentiating Equation (9.44).

From equation (9.43):

$$dH = \sum_i p_i\, d\dot{q}_i + \sum_i \dot{q}_i\, dp_i - \sum_i \frac{\partial L}{\partial q_i}\, dq_i - \sum_i \frac{\partial L}{\partial \dot{q}_i}\, d\dot{q}_i - \frac{\partial L}{\partial t}\, dt$$

By virtue of Equation (9.41), the first and the fourth terms in this expression cancel, so:

$$dH = \sum_i \dot{q}_i \, dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \tag{9.45}$$

From Equation (9.44):

$$dH = \sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt \tag{9.46}$$

Equations (9.45) and (9.46) must be equal for all values of the variables, so the coefficients of the corresponding differentials can be equated, thus:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad -\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i}; \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{9.47}$$

By Equation (9.42) $\dfrac{\partial L}{\partial q_i} = \dot{p}_i$, so we can write:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{9.48}$$

Equations (9.48) are called *Hamilton's Canonical Equations of Motion.*

PROBLEMS

9.46. Show that for a conservative system the expression for Hamilton's Principle can be written as:

$$\delta \int_{t_1}^{t_2} (2T - E) \, dt = 0$$

where E is the total energy. Show that in this form Hamilton's Principle is equivalent to finding the extreme value of the integral:

$$A = 2 \int_{t_1}^{t_2} T \, dt$$

This integral is called the *action integral,* and the above statements express what is often called the *principle of least action.*

9.47. Show that if the Lagrangian function L is not a function of time, the Hamiltonian function H is constant.

9.48. (a) Starting with the following expression for the kinetic energy of a particle:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\sum_{i=1}^{3}\left(\frac{\partial\phi_i}{\partial q_1}\dot{q}_1 + \frac{\partial\phi_i}{\partial q_2}\dot{q}_2 + \frac{\partial\phi_i}{\partial q_3}\dot{q}_3\right)^2$$

where:

$$\begin{aligned} x &= \phi_1(q_1, q_2, q_3) \\ y &= \phi_2(q_1, q_2, q_3) \\ z &= \phi_3(q_1, q_2, q_3) \end{aligned}$$

show that the kinetic energy can be written as:

$$T = \frac{1}{2}\sum_{j=1}^{3}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j$$

It may be noted that since T is a homogeneous quadratic function of the $\dot{q}$'s, this result also follows directly from Euler's theorem for homogeneous functions.

(b) Combining Equations (9.41) and (9.43) above, one obtains $H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$. With this expression and the results of part (a) above show that for a conservative system, the Hamiltonian function H is equal to the total energy of the system.

Appendix I

BIBLIOGRAPHY

Along with specific references the following list includes some general books which may be of assistance to the student who desires additional explanations, illustrative examples, and problems.

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Specific Chapter References are as follows:

Chapter 1, Ref. 1, 5, 15, 16, 17, 18, 21, 27, 34
Chapter 2, Ref. 10, 12, 19, 26, 27, 28
Chapter 3, Ref. 7, 10, 12, 19, 26, 28
Chapter 4, Ref. 9, 10, 16, 19, 21, 25, 26, 27, 28, 30
Chapter 5, Ref. 4, 12, 29, 31
Chapter 6, Ref. 7, 10, 12, 19, 27, 28
Chapter 7, Ref. 7, 10, 12, 19, 26, 28, 30
Chapter 8, Ref. 10, 19, 20, 29
Chapter 9, Ref. 2, 3, 4, 6, 7, 10, 12, 14, 26, 27, 29, 31, 32, 33

The following books contain advanced material on basic dynamics which will be useful to the student who wishes to extend his knowledge past the present text: Ref. 6, 12, 14, 23, 32, 33.

Appendix II

UNITS OF MASS AND FORCE

In principle a single unit of mass with its corresponding unit of force is sufficient for all purposes. In practice, however, a number of different units may be encountered. Since units are often used without a specific statement as to their definition, misunderstandings may result.

Perhaps the greatest source of difficulty is associated with the fact that the same name "pound" is given both to a unit of force and to a unit of mass. It is customary among engineers to use the name "pound," without qualification, for a force. On the other hand, the only legal definitions existing in the United States define the pound as the unit of mass. The engineer should thus understand that any "standard weight" calibrated by the U. S. Bureau of Standards will be essentially a standard of mass, and that any measurement of the weight of a body, which is made by means of a balance or spring which has been calibrated by such a "standard weight" will be actually a measure of the mass of the body. The possible error due to a confusion of the two units might seem to be negligible, since the variation of the acceleration of gravity is small. It is conceivable, however, that this error might be of concern in the calibration of high-precision testing machines or in the weighing of very expensive materials. It should be understood that the statement that one pound-mass has a weight of one pound-force is only approximate, and the fact that the error involved is small should not be permitted to obscure the fundamental difference between the concepts of force and mass.

The following definitions will indicate the precise meanings which are to be attached to the various common terms.

Units of Mass

Standard Kilogram (kg)—The international standard of mass. The mass of a particular body in the possession of the International Committee of Weights and Measures in France.

Gram (g)—One one-thousandth part of the standard kilogram.

Pound, or *Pound-Mass* (U. S. Avoirdupois) (lb, lb-m.)—Legally defined as $\frac{1}{2.2046}$ part of the standard kilogram. The U. S. Bureau of Standards at present uses a more accurate definition which states that the pound is equal to 453.5924277 g.

British Imperial Pound—The mass of a platinum cylinder kept in the Standard's Office, England. The legal equivalent is 453.59243 g. Hence it may be considered as practically equivalent to the U. S. Standard Avoirdupois Pound.

Slug, or *Geepound*—A unit of mass having a magnitude such that a force of 1 lb applied to a body having a mass of 1 slug would result in an acceleration of 1 ft/sec.^{2}

Units of Force

Pound, *Pound-Force*, *Pound-Weight* (lb, lb-f, lb-wt)—The force required to give a mass of 1 lb an acceleration of 32.174 ft/sec.^{2}

Poundal—The force required to give a mass of 1 lb an acceleration of 1 ft/sec.^{2}

Dyne—The force required to give a mass of 1 g an acceleration of 1 cm/sec.^{2}

Gram-Weight—The force required to give a mass of 1 g an acceleration of 980.665 cm/sec.^{2} (32.174 ft/sec.^{2})

The names of the following quantities are frequently used incorrectly. The definitions given are believed to represent the most commonly accepted engineering practice.

Specific Weight—The weight per unit volume of a material.

Specific Mass—The mass per unit volume of a material.

Density—Same as Specific Mass.

Specific Gravity—The ratio of the specific mass of a material to the specific mass of some standard material. Unless otherwise stated the standard material is taken to be water at 4° C.

Appendix III

VECTOR PRODUCTS

Two different vector products are defined and used in mechanics. The *scalar* or *dot* product of two vectors is defined as the scalar quantity having a magnitude equal to the product of the magnitudes of the two vectors multiplied by the cosine of the angle between the two vectors.

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab \cos \theta$$

Scalar multiplication is both commutative and distributive, that is:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

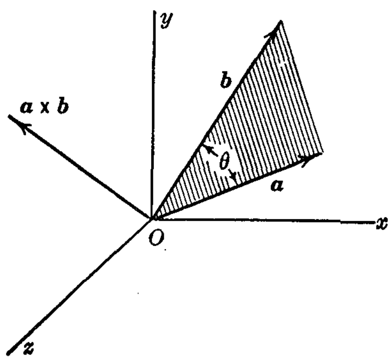
Written in terms of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ along an orthogonal coordinate system:

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$
$$\boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$$
$$\boldsymbol{a} \cdot \boldsymbol{b} = a_xb_x + a_yb_y + a_zb_z$$

since

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$$
$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{j} = 0$$

The *vector* or *cross* product of two vectors is defined as a vector whose magnitude is given by the product of the magnitudes of the two vectors



multiplied by the sine of the angle between the two vectors. The vector product is perpendicular to the plane containing the two vectors and has the direction of advance of a right-handed screw turned from the first vector to the second vector.

$$\boldsymbol{a} \times \boldsymbol{b} = ab \sin \theta \boldsymbol{e}_\perp$$

Vector multiplication is distributive:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

but is not commutative:

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

Written in terms of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we have:

$$\boldsymbol{a} \times \boldsymbol{b} = (a_yb_z - a_zb_y)\boldsymbol{i} + (a_zb_x - a_xb_z)\boldsymbol{j} + (a_xb_y - a_yb_x)\boldsymbol{k}$$

since

$$\begin{aligned} \boldsymbol{i} \times \boldsymbol{i} &= \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0 \\ \boldsymbol{i} \times \boldsymbol{j} &= -\boldsymbol{j} \times \boldsymbol{i} = \boldsymbol{k} \\ \boldsymbol{i} \times \boldsymbol{k} &= -\boldsymbol{k} \times \boldsymbol{i} = -\boldsymbol{j} \\ \boldsymbol{j} \times \boldsymbol{k} &= -\boldsymbol{k} \times \boldsymbol{j} = \boldsymbol{i} \end{aligned}$$

This may also be written as:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Appendix IV

PROPERTIES OF PLANE SECTIONS

The following symbols will be used:

A = Area
x_c, y_c = Coordinates of centroid of section in xy coordinate system.
I_{x_c}, I_{y_c} = Moment of inertia about an axis through the centroid parallel to the xy axes.
r_{x_c}, r_{y_c} = Radius of gyration of the section with respect to the centroidal axes parallel to the xy axes.
$I_{x_c y_c}$ = Product of inertia with respect to the centroidal axes parallel to the xy axes.
I_x, I_y = Moment of inertia with respect to the xy axes shown.
r_x, r_y = Radius of gyration of the section with respect to the xy axes shown.
I_{xy} = Product of inertia with respect to the xy axes shown.
I_P = Polar moment of inertia about an axis passing through the centroid.
r_P = Radius of gyration of the section about the polar axis passing through the centroid.

G marks the centroid.

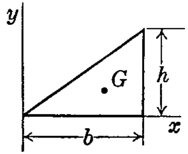
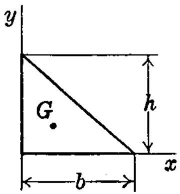
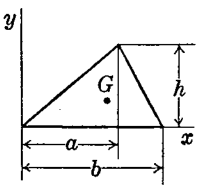
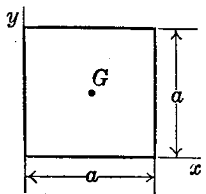
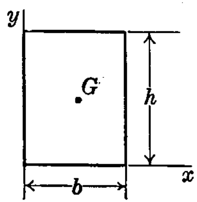
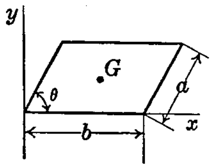
	Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
1	 Right Triangle	$A=\frac{1}{2}bh$ $x_c=\frac{2}{3}b$ $y_c=\frac{1}{3}h$	$I_{x_c}=\frac{bh^3}{36}$ $I_{y_c}=\frac{b^3h}{36}$ $I_x=\frac{bh^3}{12}$ $I_y=\frac{b^3h}{4}$	${r_{x_c}}^2=\frac{1}{18}h^2$ ${r_{y_c}}^2=\frac{1}{18}b^2$ ${r_x}^2=\frac{1}{6}h^2$ ${r_y}^2=\frac{1}{2}b^2$	$I_{x_cy_c}=\frac{A}{36}hb=\frac{h^2b^2}{72}$ $I_{xy}=\frac{A}{4}hb=\frac{h^2b^2}{8}$
2		$A=\frac{1}{2}bh$ $x_c=\frac{1}{3}b$ $y_c=\frac{1}{3}h$	$I_{x_c}=\frac{bh^3}{36}$ $I_{y_c}=\frac{b^3h}{36}$ $I_x=\frac{bh^3}{12}$ $I_y=\frac{b^3h}{12}$	${r_{x_c}}^2=\frac{1}{18}h^2$ ${r_{y_c}}^2=\frac{1}{18}b^2$ ${r_x}^2=\frac{1}{6}h^2$ ${r_y}^2=\frac{1}{6}b^2$	$I_{x_cy_c}=-\frac{A}{36}hb=-\frac{h^2b^2}{72}$ $I_{xy}=\frac{A}{12}hb=\frac{h^2b^2}{24}$
3	 Triangle	$A=\frac{1}{2}bh$ $x_c=\frac{1}{3}(a+b)$ $y_c=\frac{1}{3}h$	$I_{x_c}=\frac{bh^3}{36}$ $I_{y_c}=\frac{bh}{36}(b^2-ab+a^2)$ $I_x=\frac{bh^3}{12}$ $I_y=\frac{bh}{12}(b^2+ab+a^2)$	${r_{x_c}}^2=\frac{1}{18}h^2$ ${r_{y_c}}^2=\frac{1}{18}(b^2-ab+a^2)$ ${r_x}^2=\frac{1}{6}h^2$ ${r_y}^2=\frac{1}{6}(b^2+ab+a^2)$	$I_{x_cy_c}=\frac{Ah}{36}(2a-b)=\frac{bh^2}{72}(2a-b)$ $I_{xy}=\frac{Ah}{12}(2a+b)=\frac{bh^2}{24}(2a+b)$

	Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
4	 Square	$A=a^2$ $x_c=\frac{1}{2}a$ $y_c=\frac{1}{2}a$	$I_{x_c}=I_{y_c}=\frac{A}{12}a^2$ $I_x=I_y=\frac{A}{3}a^2$ $I_P=\frac{A}{6}a^2$	$r_{x_c}^2=r_{y_c}^2=\frac{1}{12}a^2$ $r_x^2=r_y^2=\frac{1}{3}a^2$ $r_P^2=\frac{1}{6}a^2$	$I_{x_cy_c}=0$ $I_{xy}=\frac{A}{4}a^2=\frac{a^4}{4}$
5	 Rectangle	$A=bh$ $x_c=\frac{1}{2}b$ $y_c=\frac{1}{2}h$	$I_{x_c}=\frac{bh^3}{12}$ $I_{y_c}=\frac{b^3h}{12}$ $I_x=\frac{bh^3}{3}$ $I_y=\frac{b^3h}{3}$ $I_P=\frac{bh}{12}(b^2+h^2)$	$r_{x_c}^2=\frac{1}{12}h^2$ $r_{y_c}^2=\frac{1}{12}b^2$ $r_x^2=\frac{1}{3}h^2$ $r_y^2=\frac{1}{3}b^2$ $r_P^2=\frac{1}{12}(b^2+h^2)$	$I_{x_cy_c}=0$ $I_{xy}=\frac{A}{4}bh=\frac{b^2h^2}{4}$
6	 Parallelogram	$A=ab\sin\theta$ $x_c=\frac{1}{2}(b+a\cos\theta)$ $y_c=\frac{1}{2}(a\sin\theta)$	$I_{x_c}=\frac{a^3b}{12}\sin^3\theta$ $I_{y_c}=\frac{ab}{12}\sin\theta\,(b^2+a^2\cos^2\theta)$ $I_x=\frac{a^3b}{3}\sin^3\theta$ $I_y=\frac{ab}{3}\sin\theta\,(b+a\cos\theta)^2$ $-\frac{a^2b^2}{6}\sin\theta\cos\theta$	$r_{x_c}^2=\frac{1}{12}(a\sin\theta)^2$ $r_{y_c}^2=\frac{1}{12}(b^2+a^2\cos^2\theta)$ $r_x^2=\frac{1}{3}(a\sin\theta)^2$ $r_y^2=\frac{1}{3}(b+a\cos\theta)^2$ $-\frac{1}{6}(ab\cos\theta)$	$I_{x_cy_c}=\frac{a^3b}{12}\sin^2\theta\cos\theta$

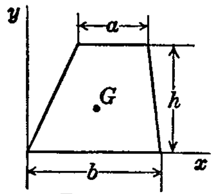
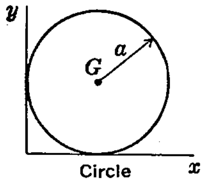
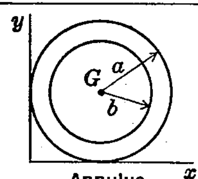
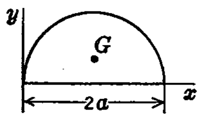
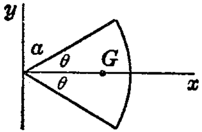
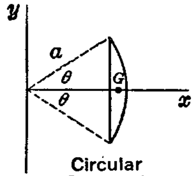
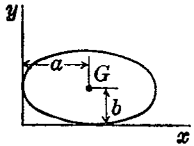
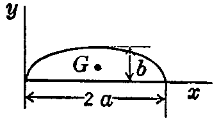
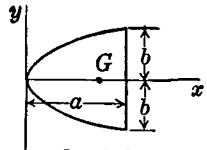
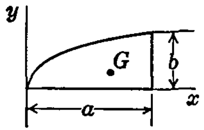
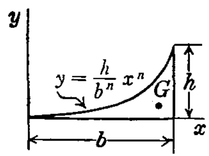
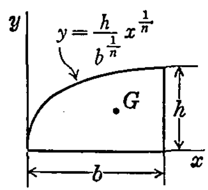
7  Trapezoid	$A = \frac{1}{2}h(a+b)$ $y_c = \frac{1}{3}h\left(\frac{2a+b}{a+b}\right)$	$I_{x_c} = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}$ $I_x = \frac{h^3(3a+b)}{12}$	${r_{x_c}}^2 = \frac{h^2(a^2+4ab+b^2)}{18(a+b)^2}$ ${r_x}^2 = \frac{h^2(3a+b)}{6(a+b)}$	
8  Circle	$A = \pi a^2$ $x_c = a$ $y_c = a$	$I_{x_c} = \frac{1}{4}\pi a^4 = I_{y_c}$ $I_x = I_y = \frac{5}{4}\pi a^4$ $I_P = \frac{1}{2}\pi a^4$	${r_{x_c}}^2 = {r_{y_c}}^2 = \frac{1}{4}a^2$ ${r_x}^2 = {r_y}^2 = \frac{5}{4}a^2$ ${r_P}^2 = \frac{1}{2}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2$
9  Annulus	$A = \pi(a^2 - b^2)$ $x_c = a$ $y_c = a$	$I_{x_c} = I_{y_c} = \frac{\pi}{4}(a^4 - b^4)$ $I_x = I_y = \frac{5}{4}\pi a^4 - \pi a^2 b^2 - \frac{\pi}{4}b^4$ $I_P = \frac{\pi}{2}(a^4 - b^4)$	${r_{x_c}}^2 = {r_{y_c}}^2 = \frac{1}{4}(a^2+b^2)$ ${r_x}^2 = {r_y}^2 = \frac{1}{4}(5a^2+b^2)$ ${r_P}^2 = \frac{1}{2}(a^2+b^2)$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2 = \pi a^2(a^2 - b^2)$
10  Semi-circle	$A = \frac{1}{2}\pi a^2$ $x_c = a$ $y_c = \frac{4a}{3\pi}$	$I_{x_c} = \frac{a^4(9\pi^2 - 64)}{72\pi}$ $I_{y_c} = \frac{1}{8}\pi a^4$ $I_x = \frac{1}{8}\pi a^4$ $I_y = \frac{5}{8}\pi a^4$	${r_{x_c}}^2 = \frac{a^2(9\pi^2 - 64)}{36\pi^2}$ ${r_{y_c}}^2 = \frac{1}{4}a^2$ ${r_x}^2 = \frac{1}{4}a^2$ ${r_y}^2 = \frac{5}{4}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = \frac{2}{3}a^4$

Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
11  Circular Sector	$A = a^2\theta$ $x_c = \dfrac{2a}{3}\dfrac{\sin\theta}{\theta}$ $y_c = 0$	$I_x = \frac{1}{4}a^4(\theta - \sin\theta\cos\theta)$ $I_y = \frac{1}{4}a^4(\theta + \sin\theta\cos\theta)$	$r_x^2 = \dfrac{1}{4}a^2\left(\dfrac{\theta - \sin\theta\cos\theta}{\theta}\right)$ $r_y^2 = \dfrac{1}{4}a^2\left(\dfrac{\theta + \sin\theta\cos\theta}{\theta}\right)$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
12  Circular Segment	$A = a^2(\theta - \frac{1}{2}\sin 2\theta)$ $x_c = \dfrac{2a}{3}\left(\dfrac{\sin^3\theta}{\theta - \sin\theta\cos\theta}\right)$ $y_c = 0$	$I_x = \dfrac{Aa^2}{4}\left[1 - \dfrac{2\sin^3\theta\cos\theta}{3(\theta - \sin\theta\cos\theta)}\right]$ $I_y = \dfrac{Aa^2}{4}\left[1 + \dfrac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$r_x^2 = \dfrac{a^2}{4}\left[1 - \dfrac{2\sin^3\theta\cos\theta}{3(\theta - \sin\theta\cos\theta)}\right]$ $r_y^2 = \dfrac{a^2}{4}\left[1 + \dfrac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
13  Ellipse	$A = \pi ab$ $x_c = a$ $y_c = b$	$I_{x_c} = \dfrac{\pi}{4}ab^3$ $I_{y_c} = \dfrac{\pi}{4}a^3b$ $I_x = \frac{5}{4}\pi ab^3$ $I_y = \frac{5}{4}\pi a^3b$ $I_P = \dfrac{\pi ab}{4}(a^2+b^2)$	$r_{x_c}^2 = \frac{1}{4}b^2$ $r_{y_c}^2 = \frac{1}{4}a^2$ $r_x^2 = \frac{5}{4}b^2$ $r_y^2 = \frac{5}{4}a^2$ $r_P^2 = \frac{1}{4}(a^2 + b^2)$	$I_{x_c y_c} = 0$ $I_{xy} = Aab = \pi a^2b^2$
14  Semi-ellipse	$A = \frac{1}{2}\pi ab$ $x_c = a$ $y_c = \dfrac{4b}{3\pi}$	$I_{x_c} = \dfrac{ab^3}{72\pi}(9\pi^2 - 64)$ $I_{y_c} = \dfrac{\pi}{8}a^3b$ $I_x = \dfrac{\pi}{8}ab^3$ $I_y = \frac{5}{8}\pi a^3b$	$r_{x_c}^2 = \dfrac{b^2}{36\pi^2}(9\pi^2 - 64)$ $r_{y_c}^2 = \frac{1}{4}a^2$ $r_x^2 = \frac{1}{4}b^2$ $r_y^2 = \frac{5}{4}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = \frac{2}{3}a^2b^2$

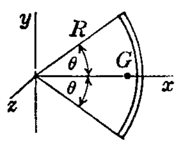
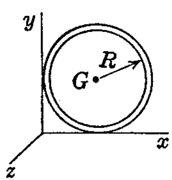
15	 Parabola	$A = \frac{4}{3}ab$ $x_c = \frac{3}{5}a$ $y_c = 0$	$I_{x_c} = I_x = \frac{4}{15}\, ab^3$ $I_{y_c} = \frac{16}{175}\, a^3 b$ $I_y = \frac{4}{7}\, a^3 b$	$r_{x_c}^{\,2} = r_x^{\,2} = \frac{1}{5}b^2$ $r_{y_c}^{\,2} = \frac{12}{175}a^2$ $r_y^{\,2} = \frac{3}{7}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
16	 Semi-parabola	$A = \frac{2}{3}ab$ $x_c = \frac{3}{5}a$ $y_c = \frac{3}{8}b$	$I_x = \frac{2}{15}\, ab^3$ $I_y = \frac{2}{7}\, a^3 b$	$r_x^{\,2} = \frac{1}{5}b^2$ $r_y^{\,2} = \frac{3}{7}a^2$	$I_{xy} = \frac{A}{4}\, ab = \frac{1}{6}\, a^2 b^2$
17	 nth degree parabola	$A = \frac{bh}{n+1}$ $x_c = \frac{n+1}{n+2}\, b$ $y_c = \frac{h}{2}\left(\frac{n+1}{2n+1}\right)$	$I_x = \frac{bh^3}{3(3n+1)}$ $I_y = \frac{hb^3}{n+3}$	$r_x^{\,2} = \frac{h^2(n+1)}{3(3n+1)}$ $r_y^{\,2} = \frac{n+1}{n+3}\, b^2$	
18	 nth degree parabola	$A = \frac{n}{n+1}\, bh$ $x_c = \frac{n+1}{2n+1}\, b$ $y_c = \frac{n+1}{2(n+2)}\, h$	$I_x = \frac{n}{3(n+3)}\, bh^3$ $I_y = \frac{n}{3n+1}\, hb^3$	$r_x^{\,2} = \frac{n+1}{3(n+3)}\, h^2$ $r_y^{\,2} = \frac{n+1}{3n+1}\, b^2$	

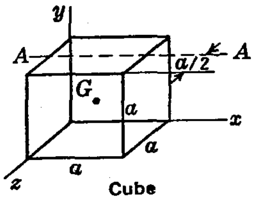
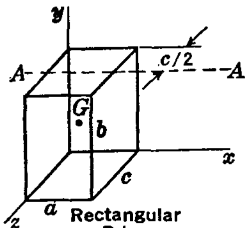
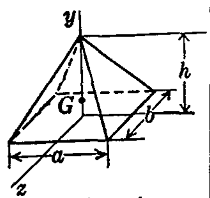
PROPERTIES OF HOMOGENEOUS BODIES

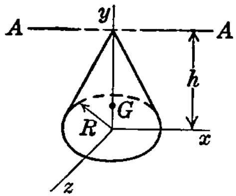
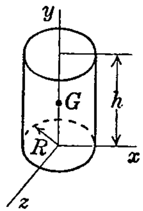
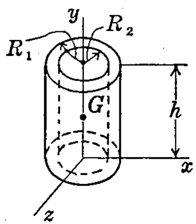
The following symbols will be used:

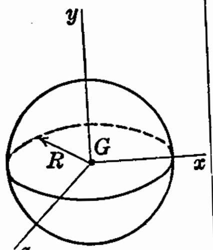
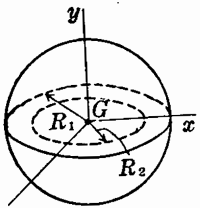
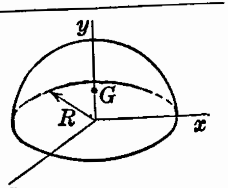
ρ = Mass density
M = Mass
x_c, y_c, z_c = Coordinates of centroid in xyz coordinate system.
$I_{x_c}, I_{y_c}, I_{z_c}$ = Moment of inertia about an axis through the centroid parallel to the xyz axes shown.
$r_{x_c}, r_{y_c}, r_{z_c}$ = Radius of gyration of the body with respect to the centroidal axes parallel to the xyz axes shown.
$I_{x_c y_c}, I_{x_c z_c}$, etc. = Product of inertia with respect to the centroidal axes parallel to the xyz axes shown.
I_x, I_y, I_z = Moment of inertia with respect to the xyz axes shown.
r_x, r_y, r_z = Radius of gyration of the body with respect to the xyz axes shown.
I_{xy}, I_{xz}, etc. = Product of inertia with respect to the xyz axes shown.
I_{AA}, r_{AA} = Moments of inertia and radii of gyration with respect to special axes shown.

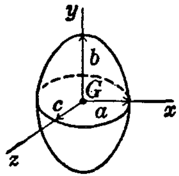
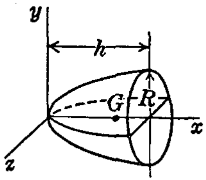
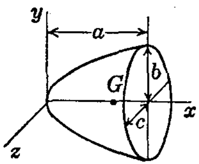
G marks the centroid

	Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
19	 Thin Rod	$M = \rho l$ $x_c = \frac{1}{2}l$ $y_c = 0$ $z_c = 0$	$I_x = I_{x_c} = 0$ $I_{y_c} = I_{z_c} = \dfrac{M}{12}l^2$ $I_y = I_z = \dfrac{M}{3}l^2$	$r_x^2 = r_{x_c}^2 = 0$ $r_{y_c}^2 = r_{z_c}^2 = \frac{1}{12}l^2$ $r_y^2 = r_z^2 = \frac{1}{3}l^2$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
20	 Thin Circular Rod	$M = 2\rho R\theta$ $x_c = \dfrac{R\sin\theta}{\theta}$ $y_c = 0$ $z_c = 0$	$I_x = I_{x_c}$ $= \dfrac{MR^2(\theta - \sin\theta\cos\theta)}{2\theta}$ $I_y = \dfrac{MR^2(\theta + \sin\theta\cos\theta)}{2\theta}$ $I_z = MR^2$	$r_x^2 = r_{x_c}^2 = \dfrac{R^2(\theta - \sin\theta\cos\theta)}{2\theta}$ $r_y^2 = \dfrac{R^2(\theta + \sin\theta\cos\theta)}{2\theta}$ $r_z^2 = R^2$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
21	 Thin Hoop	$M = 2\pi\rho R$ $x_c = R$ $y_c = R$ $z_c = 0$	$I_{x_c} = I_{y_c} = \dfrac{M}{2}R^2$ $I_{z_c} = MR^2$ $I_x = I_y = \frac{3}{2}MR^2$ $I_z = 3MR^2$	$r_{x_c}^2 = r_{y_c}^2 = \frac{1}{2}R^2$ $r_{z_c}^2 = R^2$ $r_x^2 = r_y^2 = \frac{3}{2}R^2$ $r_z = 3R^2$	$I_{x_cy_c}$, etc. $= 0$ $I_{xy} = MR^2$ $I_{xz} = I_{yz} = 0$

Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
22  Cube	$M = \rho a^3$ $x_c = \frac{1}{2}a$ $y_c = \frac{1}{2}a$ $z_c = \frac{1}{2}a$	$I_{x_c} = I_{y_c} = I_{z_c} = \frac{1}{6}Ma^2$ $I_x = I_y = I_z = \frac{2}{3}Ma^2$ $I_{AA} = \frac{5}{12}Ma^2$	${r_{x_c}}^2 = {r_{y_c}}^2 = {r_{z_c}}^2 = \frac{1}{6}a^2$ ${r_x}^2 = {r_y}^2 = {r_z}^2 = \frac{2}{3}a^2$ ${r_{AA}}^2 = \frac{5}{12}a^2$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= \frac{1}{4}Ma^2$
23  Rectangular Prism	$M = \rho abc$ $x_c = \frac{1}{2}a$ $y_c = \frac{1}{2}b$ $z_c = \frac{1}{2}c$	$I_{x_c} = \frac{1}{12}M(b^2 + c^2)$ $I_x = \frac{1}{3}M(b^2 + c^2)$ $I_{AA} = \frac{1}{12}M(4b^2 + c^2)$	${r_{x_c}}^2 = \frac{1}{12}(b^2 + c^2)$ ${r_x}^2 = \frac{1}{3}(b^2 + c^2)$ ${r_{AA}}^2 = \frac{1}{12}(4b^2 + c^2)$	$I_{x_cy_c}$, etc. $= 0$ $I_{xy} = \frac{1}{4}Mab$ $I_{xz} = \frac{1}{4}Mac$ $I_{yz} = \frac{1}{4}Mbc$
24  Right Rectangular Pyramid	$M = \frac{1}{3}\rho abh$ $x_c = 0$ $y_c = \frac{1}{4}h$ $z_c = 0$	$I_{x_c} = \frac{1}{80}M(4b^2 + 3h^2)$ $I_{y_c} = I_y = \frac{1}{20}M(a^2 + b^2)$ $I_x = \frac{1}{20}M(b^2 + 2h^2)$ $I_z = \frac{1}{20}M(a^2 + 2h^2)$	${r_{x_c}}^2 = \frac{1}{80}(4b^2 + 3h^2)$ ${r_{y_c}}^2 = {r_y}^2 = \frac{1}{20}(a^2 + b^2)$ ${r_x}^2 = \frac{1}{20}(b^2 + 2h^2)$ ${r_z}^2 = \frac{1}{20}(a^2 + 2h^2)$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

25	 Right Circular Cone	$M = \frac{1}{3}\pi\rho R^2 h$ $x_c = 0$ $y_c = \frac{1}{4}h$ $z_c = 0$	$I_{x_c} = I_{z_c} = \frac{3M}{80}(4R^2 + h^2)$ $I_{y_c} = I_y = \frac{3}{10}MR^2$ $I_x = I_z = \frac{1}{20}M(3R^2 + 2h^2)$ $I_{AA} = \frac{3}{20}M(R^2 + 4h^2)$	${r_{x_c}}^2 = {r_{z_c}}^2 = \frac{3}{80}(4R^2 + h^2)$ ${r_{y_c}}^2 = \frac{3}{10}R^2$ ${r_x}^2 = {r_z}^2 = \frac{1}{20}(3R^2 + 2h^2)$ ${r_{AA}}^2 = \frac{3}{20}(R^2 + 4h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
26	 Right Circular Cylinder	$M = \pi\rho R^2 h$ $x_c = 0$ $y_c = \frac{1}{2}h$ $z_c = 0$	$I_{x_c} = I_{z_c} = \frac{1}{12}M(3R^2 + h^2)$ $I_{y_c} = I_y = \frac{1}{2}MR^2$ $I_x = I_z = \frac{1}{12}M(3R^2 + 4h^2)$	${r_{x_c}}^2 = {r_{z_c}}^2 = \frac{1}{12}(3R^2 + h^2)$ ${r_{y_c}}^2 = {r_y}^2 = \frac{1}{2}R^2$ ${r_x}^2 = {r_z}^2 = \frac{1}{12}(3R^2 + 4h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
27	 Hollow Right Circular Cylinder	$M = \pi\rho h(R_1^2 - R_2^2)$ $x_c = 0$ $y_c = \frac{1}{2}h$ $z_c = 0$	$I_{x_c} = I_{z_c}$ $= \frac{1}{12}M(3R_1^2 + 3R_2^2 + h^2)$ $I_{y_c} = I_y = \frac{1}{2}M(R_1^2 + R_2^2)$ $I_x = I_z$ $= \frac{1}{12}M(3R_1^2 + 3R_2^2 + 4h^2)$	${r_{x_c}}^2 = {r_{z_c}}^2 = \frac{1}{12}(3R_1^2 + 3R_2^2 + h^2)$ ${r_{y_c}}^2 = {r_y}^2 = \frac{1}{2}(R_1^2 + R_2^2)$ ${r_x}^2 = {r_z}^2 = \frac{1}{12}(3R_1^2 + 3R_2^2 + 4h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

	Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
28	 Sphere	$M = \frac{4}{3}\pi\rho R^3$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{2}{5}MR^2$ $I_{y_c} = I_y = \frac{2}{5}MR^2$ $I_{z_c} = I_z = \frac{2}{5}MR^2$	$r_{x_c}^2 = r_x^2 = \frac{2}{5}R^2$ $r_{y_c}^2 = r_y^2 = \frac{2}{5}R^2$ $r_{z_c}^2 = r_z^2 = \frac{2}{5}R^2$	$I_{x_c y_c}$, etc. $= 0$
29	 Hollow Sphere	$M = \frac{4}{3}\pi\rho(R_1^3 - R_2^3)$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_x = I_y = I_z$ $= \frac{2}{5} M \frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}$	$r_x^2 = r_y^2 = r_z^2$ $= \frac{2}{5} \frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}$	I_{xy}, etc. $= 0$
30	 Hemisphere	$M = \frac{2}{3}\pi\rho R^3$ $x_c = 0$ $y_c = \frac{3}{8}R$ $z_c = 0$	$I_x = I_y = I_z = \frac{2}{5}MR^2$	$r_x^2 = r_y^2 = r_z^2 = \frac{2}{5}R^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

31  Ellipsoid	$M = \frac{4}{3}\pi\rho abc$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_x = \frac{1}{5}M(b^2 + c^2)$ $I_y = \frac{1}{5}M(a^2 + c^2)$ $I_z = \frac{1}{5}M(a^2 + b^2)$	$r_x^2 = \frac{1}{5}(b^2 + c^2)$ $r_y^2 = \frac{1}{5}(a^2 + c^2)$ $r_z^2 = \frac{1}{5}(a^2 + b^2)$	I_{xy}, etc. $= 0$
32  Paraboloid of Revolution	$M = \frac{1}{2}\pi\rho R^2 h$ $x_c = \frac{2}{3}h$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{1}{3}MR^2$ $I_{y_c} = I_{z_c} = \frac{1}{18}M(3R^2 + h^2)$ $I_y = I_z = \frac{1}{6}M(R^2 + 3h^2)$	$r_{x_c}^2 = r_x^2 = \frac{1}{3}R^2$ $r_{y_c}^2 = r_{z_c}^2 = \frac{1}{18}(3R^2 + h^2)$ $r_y^2 = r_z^2 = \frac{1}{6}(R^2 + 3h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
33  Elliptic Paraboloid	$M = \frac{1}{2}\pi\rho abc$ $x_c = \frac{2}{3}a$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{1}{6}M(b^2 + c^2)$ $I_{y_c} = \frac{1}{18}M(3c^2 + a^2)$ $I_{z_c} = \frac{1}{18}M(3b^2 + a^2)$ $I_y = \frac{1}{6}M(c^2 + 3a^2)$ $I_z = \frac{1}{6}M(b^2 + 3a^2)$	$r_{x_c}^2 = r_x^2 = \frac{1}{6}(b^2 + c^2)$ $r_{y_c}^2 = \frac{1}{18}(3c^2 + a^2)$ $r_{z_c}^2 = \frac{1}{18}(3b^2 + a^2)$ $r_y^2 = \frac{1}{6}(c^2 + 3a^2)$ $r_z^2 = \frac{1}{6}(b^2 + 3a^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

ANSWERS TO PROBLEMS

Chapter 1

1.5. $F = C\rho A v^2$

1.6. $R = \dfrac{v^2}{g}\,\phi(\theta)$;

$$R = \frac{v^2}{g}\,\phi\left(\theta, v\sqrt{\frac{k}{mg}}\right)$$

1.8. $v = \sqrt{lg}\,\phi\left(\dfrac{l}{h}\right)$; $v = C\sqrt{lg}$

1.9. $v = C\,\dfrac{D^2}{\mu}\left(\dfrac{dp}{dx}\right)$

$$v = \frac{D^2\left(\dfrac{dp}{dx}\right)}{\mu}\,\phi\left(\frac{\rho D^3\,\dfrac{dp}{dx}}{\mu^2}\right)$$

1.10. $f = \phi\left[\left(\dfrac{c}{d}\right), \left(\dfrac{p}{\mu N}\right)\right]$

1.11. $f = g\sqrt{\dfrac{\mu}{T}}\,\phi\left(\dfrac{l}{T}\,\mu g\right)$; $f = \dfrac{c}{l}\sqrt{\dfrac{T}{\mu}}$

1.12. $F = \dfrac{\sigma^2}{\rho v^2}\,\phi\left(\dfrac{\rho v^2 A^{\frac{1}{2}}}{\sigma}\right)$

1.13. $v = \sqrt{rg}\,\phi\left(\dfrac{\rho_b}{\rho_m}\right)$

1.14. (a) If:

$$\left(\frac{El^2}{P}\right)_m = \left(\frac{El^2}{P}\right)_p;$$

$$\left(\frac{a}{l}\right)_m = \left(\frac{a}{l}\right)_p;$$

$$\left(\frac{b}{l}\right)_m = \left(\frac{b}{l}\right)_p \cdots$$

then: $\left(\dfrac{\sigma l^2}{P}\right)_m = \left(\dfrac{\sigma l^2}{P}\right)_p$

(σ = stress)

(b) If:

$$\left(\frac{E}{\gamma l}\right)_m = \left(\frac{E}{\gamma l}\right)_p;$$

$$\left(\frac{a}{l}\right)_m = \left(\frac{a}{l}\right)_p;$$

$$\left(\frac{b}{l}\right)_m = \left(\frac{b}{l}\right)_p \cdots$$

then: $\left(\dfrac{\sigma}{\gamma l}\right)_m = \left(\dfrac{\sigma}{\gamma l}\right)_p$

1.15. $P_p = 6750$ lb

1.16. $F_p = \dfrac{\rho_p {l_p}^3}{\rho_m {l_m}^3} F_m$

1.17. Cannot satisfy Froudes' number and Reynolds' number simultaneously with same gravity field for model and prototype.

Chapter 2

2.3.
$$\dot{\boldsymbol{e}}_r = \dot\theta \boldsymbol{e}_\theta + \dot\phi \sin\theta\, \boldsymbol{e}_\phi$$
$$\dot{\boldsymbol{e}}_\phi = -\dot\phi\cos\theta \boldsymbol{e}_\theta - \dot\phi\sin\theta \boldsymbol{e}_r$$
$$\dot{\boldsymbol{e}}_\theta = -\dot\theta \boldsymbol{e}_r + \dot\phi\cos\theta \boldsymbol{e}_\phi$$

2.4.
$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot r \boldsymbol{e}_r + r\dot\theta \boldsymbol{e}_\theta + r\dot\phi\sin\theta \boldsymbol{e}_\phi$$
$$\begin{aligned}\boldsymbol{a} = \ddot{\boldsymbol{r}} = {} & (\ddot r - r\dot\theta^2 - r\dot\phi^2\sin^2\theta)\boldsymbol{e}_r \\ & + (2\dot\theta\dot r + r\ddot\theta - r\dot\phi^2\sin\theta\cos\theta)\boldsymbol{e}_\theta \\ & + (2\dot r\dot\phi\sin\theta + r\ddot\phi\sin\theta + 2r\dot\phi\dot\theta\cos\theta)\boldsymbol{e}_\phi\end{aligned}$$

2.6.
$$F_r = \left(\frac{mg^2}{a}\sin^2\alpha\cos^2\alpha\right)t^2$$
$$F_\phi = -mg\sin\alpha\cos\alpha$$
$$F_z = -mg\cos^2\alpha$$

2.7.
$$\begin{aligned}\dot{\boldsymbol{a}} = {} & (\dddot r - 3\dot r\dot\phi^2 - 3r\dot\phi\ddot\phi)\boldsymbol{e}_r \\ & + (r\dddot\phi + 3\ddot r\dot\phi + 3\dot r\ddot\phi - r\dot\phi^3)\boldsymbol{e}_\phi \\ & + \dddot z \boldsymbol{e}_z\end{aligned}$$

2.8. $a = \dfrac{Dd\omega^2}{4\pi h}$

2.9. $\boldsymbol{a} = -4r\dot{\phi}^2\boldsymbol{e}_r$

2.10. $\boldsymbol{v} = 56\boldsymbol{i} - 140\boldsymbol{j} - 112\boldsymbol{k}$ ft/sec

2.12. $v = r\omega$

2.13. (b) $\alpha a\boldsymbol{e}_t$; $-a\omega^2\boldsymbol{e}_n$

2.14. $a = l\omega^2 \sin\alpha\sqrt{1 + \left(\dfrac{2v}{lw}\right)^2}$

2.15. $a = 51{,}100$ ft/sec²

2.16. $\boldsymbol{a} = 5.34 \times 10^{-4}\boldsymbol{e}_\phi - 0.084\boldsymbol{e}_r - 0.049\boldsymbol{e}_\theta$ ft/sec²

2.17. $\boldsymbol{a} = -8730\boldsymbol{i} - 443\boldsymbol{j} + 264\boldsymbol{k}$ ft/sec²

2.18. N-S: $a = 0.146$ ft/sec²; E-W: $a = 0.303$ ft/sec²

2.19. $\omega = -2\Omega$

2.20. $\boldsymbol{a} = 664\boldsymbol{i} + 194\boldsymbol{j} - 94{,}800\boldsymbol{k}$ ft/sec²

2.21. $\dot{\boldsymbol{r}} = v\boldsymbol{i} + (r_0\omega + \phi_0 pd)\boldsymbol{j}$
$\ddot{\boldsymbol{r}} = (a - d\phi_0{}^2p^2 - 2\phi_0 pr_0\omega)\boldsymbol{i} + r_0\omega^2\boldsymbol{k}$

Chapter 3

3.1. $\boldsymbol{F} = -\dfrac{mv^2}{r}\boldsymbol{e}_r$

3.2. $F = 1.62$ lb; $a = 8.43$ ft/sec²

3.3. $\phi = \cos^{-1}\frac{2}{3}$; $v = \sqrt{2gr/3}$

3.4. $R_x \approx -Mr\omega^2\left(\cos\omega t + \dfrac{r}{l}\cos 2\omega t\right)$ for $\left(\dfrac{r}{l}\right)$ small

3.5. 317 ft; 23.7 sec

3.6. 23.9 ft/sec; 212.8 ft.

3.7. $3mg$

3.8. (a) -2.2 ft/sec; (b) 80.5 ft/sec

3.9. 2.61 lb sec; 102° 50′

3.10. $F = \rho A v^2$; 72.6 lb

3.11. $v_B = v_r/2$

3.12. 3.86 ft/sec

3.13. 373,000 lb

3.14. 31.0 lb

3.15. 7.76 lb

3.18. $-\dfrac{k}{2}(\sqrt{L^2 + r^2} - l)^2$

3.19. 42° 57′; 1.13 lb

3.20. $\dfrac{T^4}{8m}(A^2 + B^2)$

3.21. $k' = \dfrac{1}{\sum_i \dfrac{1}{k_i}}$

3.22. $k' = \sum_i k_i$

3.24. $k = 1525$ lb/ft

3.25. 36,900 ft/sec

3.26. $\sqrt{2gx\left(\dfrac{W_2 - W_1 - F_D}{W_1 + W_2}\right)}$

3.27. $\delta\left[1 + \sqrt{1 + \dfrac{2h}{\delta}}\right]$

3.28. $k_1 = 300$ lb/in; $k_2 = 150$ lb/in; $h = 2.4$ in; $v = 4.24$ ft/sec

3.29. $V = \frac{1}{2}\rho g A_1\left(1 + \dfrac{A_1}{A_2}\right)x^2$

3.30. $\Phi = \frac{1}{2}Ax^2 + Bxy + \frac{1}{3}Cy^3$

3.31. 16.1 in; 12.4 in

3.32. $0.472\ ka^2$

3.33. 17.3 mph

3.34. $F_r = \dfrac{e_1e_2}{r^2}$; $F_x = \dfrac{e_1e_2x}{r^3}$

3.35. $(Wk)_{\text{ext.}}/(Wk)_{\text{inext.}} = 1 - \dfrac{1}{2}\dfrac{mg}{kl}$

3.36. 20.7 ft/sec; 14.4 lb sec; 0.965 sec

3.37. $x_{\max} = x_{st}\left[1 + \sqrt{1 + \dfrac{2h}{x_{st}}}\right]$

3.38. $c = \sqrt{\dfrac{E}{\rho}}$; $v = \dfrac{\sigma}{\sqrt{E\rho}}$

3.39. $E_{\text{car.}} = \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_c v_c^2 - \frac{1}{2}\frac{(m_a v_a + m_c v_c)^2}{(m_a + m_c)}$

$\Delta E_{\text{air.}} = \frac{1}{2}m_a v_a^2 - \frac{1}{2}m_a \frac{(m_a v_a + m_c v_c)^2}{(m_a + m_c)^2}$

3.41. $F_{\text{out}} = \frac{MmG}{r^2}$

$F_{\text{in}} = \frac{MmGr}{R^3}$

3.42. $v = \sqrt{Rg}$

3.43. $a_{\max} = \frac{l\mu}{1 + \mu}$

$$v = \left[\frac{g}{l}(1 + \mu)(l^2 - a^2) - 2\mu g(l - a)\right]^{\frac{1}{2}}$$

3.44. $t = \sqrt{\frac{l}{g}}\cosh^{-1}\frac{l}{a}$

3.45. $Wk = \frac{1}{2}mr^2(\omega + \Omega)^2$

$$F_f = m_f\left\{\left[a\Omega^2 + \frac{\sqrt{2}}{4}r(\omega + \Omega)^2 + \frac{2\sqrt{2}}{\pi}r\omega(\omega + \Omega)\right]^2 + \left[\frac{2\sqrt{2}}{\pi}r\omega(\omega + \Omega) - \frac{\sqrt{2}}{4}r(\omega + \Omega)^2\right]^2\right\}^{\frac{1}{2}}$$

3.46. $x = gt^2/6$

3.48. $l = \frac{m}{\rho A}\log\frac{v_1}{v_2}$

Chapter 4

4.1. $z = \frac{W}{k}\left[t - \frac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right)\right]$

4.2. $z = \frac{m}{k}\log\cosh\sqrt{\frac{kg}{m}}\,t$

4.3. $\sqrt{\frac{W}{k}}$

4.4. 584 ft/sec; $k = \frac{W}{20}$ lb sec/ft; 73.5 ft

4.5. (a) $n < 1$; $t = \frac{v_0^{(1-n)}}{k(1 - n)}$

(b) $n < 2$

(c) $S_{t=\infty} = \frac{v_0}{k}$

4.6. 11,600 ft

4.7. $\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\Big/\cos\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{g\Delta t}{2v_0}$

4.8. $\theta = 67\frac{1}{2}°$;

$R = \frac{2\sqrt{2}}{g}v_0^2\cos^2\theta(\tan\theta - 1)$

4.9. $x = \frac{m}{k}\log\left(\frac{k\dot{x}_0}{m}t + 1\right)$

$$y = \frac{m}{k\dot{x}_0}\left(\dot{y}_0 + \frac{mg}{2k\dot{x}_0}\right) \times \log\left(\frac{k\dot{x}_0}{m}t + 1\right) - \frac{mg}{2k\dot{x}_0}t\left(\frac{k\dot{x}_0}{2m}t + 1\right)$$

4.10. $\dot{x} = \sqrt{\frac{mg\sin\alpha}{k}}\tanh\sqrt{\frac{gk\sin\alpha}{m}}\,t$

4.11. $x_{\dot{y}=0} = \frac{\dot{x}_0\dot{y}_0}{g\left(1 + \frac{k\dot{y}_0}{mg}\right)}$

4.12. 124 ft

4.14. $mgR\left(1 - \frac{1}{2}\frac{R}{r}\right)$; 24,300 ft/sec

4.15. $g_M = 5.25$ ft/sec^2

$$F = -mg\left[\left(\frac{R_E}{x}\right)^2 - \frac{m_M}{m_E}\left(\frac{R_E}{d - x}\right)^2\right]$$

$v = 17{,}800$ mph

4.16. $F_r = -C\dfrac{m}{r^3}$ (C = constant)

4.17. $t = r_0^2\sqrt{\dfrac{m}{\mu}}$

4.18. $F_r = -C\dfrac{m}{r^3}$ (C = constant, different from C in. Prob. 4.16)

4.20. $r_0 = \dfrac{k_0}{mv_0^2} \times \left(1 + \sqrt{1 + \dfrac{m^2v_0^4}{k^2}S^2}\right)$

4.23. $\dfrac{\Delta T}{T} = (1 - e^2)$

4.25. $e = 0.922$

4.27. $e \tan \theta_r = \tan \theta_i - (1 + e)\mu$ (θ measured from normal to surface)

4.28. $\theta = \tan^{-1} \mu(1 + e)$

4.29. $\Delta E = \frac{1}{2}mv^2[1 - \cos^2 \theta\{e^2 + [\tan \theta - \mu(1 + e)]^2\}]$

4.30. $|V| = 5.95$ ft/sec

4.31. Direction of approach = direction of departure for both (a) and (b)

4.32. $F_{\text{ave}} = 0.924$ lb

4.34. 23.5 in.

4.35. 0.29 ft lb

4.38. 1980 ft

4.39. 263 ft/sec

4.41. $v = \dfrac{cv_e}{kg}\left[1 - \left(\dfrac{w_0}{w_0 - ct}\right)^{-\frac{kg}{c}}\right] = 745$ ft/sec

4.42. $v = \dfrac{16}{15}v_e - \left(\dfrac{16v_e - 15v_0}{15}\right)\left(\dfrac{W_0 - wt}{W_0}\right)^{15}$

4.43. $v = \frac{1}{4}gt$

4.44. 1.88×10^{10} cm/sec classical
1.64×10^{10} cm/sec relativistic

4.48. $x_0 = \dfrac{e^2}{2m_0c^2}\dfrac{1}{\left[\dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} - 1\right]}$

4.49. 2.36×10^{-11} ft lb/fission

Chapter 5

5.1. $\tau = 2\pi\sqrt{\dfrac{l}{g}}$

5.4. $p^2 = 2g$/length of liquid column

5.5. $p^2 = 2\mu g/a$

5.6. $p^2 = \pi D^2\rho g^2/4W$

5.7. $p^2 = ag/hl$

5.8. $m\ddot{x} + c\dot{x} + 2\dfrac{pV_0A^2x}{(V_0^2 - A^2x^2)} = 0$

$f = \dfrac{1}{2\pi}\sqrt{\dfrac{2pA^2}{mV_0}}$

5.9. $p^2 = \dfrac{2ka^2}{ml^2} - \dfrac{g}{l}$

5.10. $(1 + 4a^2x^2)\ddot{x} + 4a^2x\dot{x}^2 + 2agx = 0$

$p^2 = 2ag$

5.11. $p^2 = \dfrac{p_0\gamma A}{\rho_0 l V_0}$

5.12. $m\ddot{x} + \dfrac{2T}{a}\dfrac{x}{\sqrt{1 + \left(\dfrac{x}{a}\right)^2}} + 2kx\left(1 - \dfrac{1}{\sqrt{1 + \left(\dfrac{x}{a}\right)^2}}\right) = 0$

$p^2 = 2T/a$

5.13. $p^2 = \left(\dfrac{2k}{m} - \omega^2\right)$

5.14. $p^2 = \dfrac{g\left(1 + \dfrac{A_1}{A_2}\right)}{h\left(1 + \dfrac{A_1}{A_2}\right) + \dfrac{A_1}{A_0}l}$

5.17. 3.92 in.; 2.02 cycles per second

5.20 $x_{\max} = \dfrac{V_0}{pe}$

5.21. $x = [x_0 + (\dot{x}_0 + px_0)t]e^{-pt}$

5.22. $\delta = \dfrac{\pi c a^2}{ml^2\sqrt{\dfrac{g}{l} - \left(\dfrac{ca^2}{2ml^2}\right)^2}}$

5.23. $m\dddot{x} + \dfrac{km}{c}\ddot{x} + k\dot{x} = 0$

5.24. $n = 0.191\ \text{sec}^{-1}$; $\tau = 0.905$ sec

5.25. 0.0825 ft; 0.032 ft

5.27. 0.083 ft

5.28. $n = 8.42\ \text{sec}^{-1}$

5.29. (a) $A =$

$$\frac{y_0}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}$$

(b) $A =$

$$\frac{y_0\sqrt{1 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}$$

5.30. (a) $\left(\dfrac{\omega}{p}\right) = \sqrt{1 - 2\left(\dfrac{n}{n_c}\right)^2}$

(c) $\left(\dfrac{n}{n_c}\right) = \dfrac{\sqrt{2}}{2}$

5.31. $A = \dfrac{F_0}{\sqrt{(c\omega)^2 + (m\omega^2)^2}}$

5.32. $\left(\dfrac{A}{e}\right) =$

$$\frac{\left(\dfrac{\omega}{p}\right)^2}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}$$

5.33. $W_d = \dfrac{c\,\omega^2 F_0{}^2/2}{(k - m\omega^2)^2 + (\omega c)^2}$

$$\left(\frac{n}{n_c}\right)^2 > \frac{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2}{4\left(\dfrac{\omega}{p}\right)^2}$$

5.35. $\dfrac{2ak\pi^2v^2}{(kl^2 - 4m\pi^2v^2)}\left(\cos\dfrac{l}{v}\sqrt{\dfrac{k}{m}} - 1\right)$

5.36.

$$A = \frac{2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)y_0}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}$$

5.37. $x = \dfrac{F}{k}(1 - \cos pt)$

5.40. (a) $W_d = 4FA$

(b) $c_{eq.} = \dfrac{4F}{\pi A\omega}$

(c) $A = \dfrac{F_0}{k}\dfrac{\sqrt{1 - \left(\dfrac{4}{\pi}\dfrac{F}{F_0}\right)^2}}{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]}$

5.41. $k = 135$ lb/in

5.42. 83.7%

5.43. 2.08 in

5.44. At 90 mph, 6.55% transmitted

5.45. (a) 10.8 in
(b) 2.31 lb/in
(c) $\approx 0.1\%$

5.46. 18,500 lb/in

5.47. 115 lb.; 588 lb/in

5.51. 0.008 in

5.52. 1.15 lb

5.57. Max. amp. after pulse

$$= \frac{2I}{kp\tau^2}[(p\tau)^2 + 2(1 - \cos p\tau - p\tau\sin p\tau)]^{\frac{1}{2}}$$

where I = impulse

5.58. $t < \tau$; $x = \dfrac{F_0}{kp\tau}(pt - \sin pt)$

$t > \tau$;

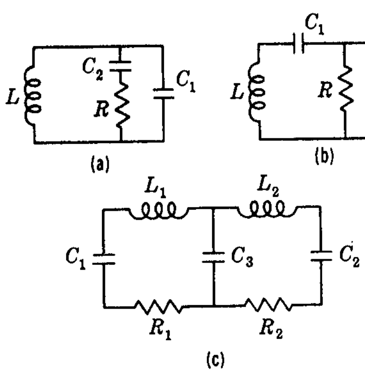
$$x = \frac{F_0}{kp\tau}[\sin p(t - \tau) + p\tau\cos p(t - \tau) - \sin pt]$$

5.59. $x = \dfrac{Wdl}{AEg}\left(1 - \cos\sqrt{\dfrac{AEg}{Wl}}\,t\right) - \frac{1}{2}dt^2 + V_0t$

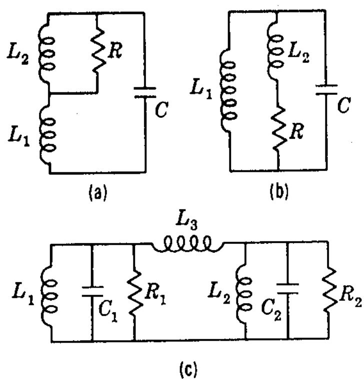
5.65.

Mechanical System	*Series, or Loop Circuit*	*Parallel, or Nodal Circuit*
m	L	C
c	R	$\frac{1}{R}$
v	i	E
k	$\frac{1}{C}$	$\frac{1}{L}$
x	Q	$\int E\,dt$
F	E	i

5.66.



5.67.



Chapter 6

6.1. $x_c = x_{oc} + \dfrac{F_x t^2}{4m}$

$y_c = y_{oc} - \dfrac{gt^2}{2}$

6.2. $l/6$

6.3. $v_r = \sqrt{\dfrac{2E(m_1 + m_2)}{m_1 m_2}}$

6.4. $\dfrac{(F - \mu m g)l}{[F - \mu(m + M)g]}$

6.5. $8m\boldsymbol{v j}$; $6.5mv^2$; $5.33mv^2$;

$\boldsymbol{v}_c = \dfrac{4}{3}\,v\boldsymbol{j}$

6.6. $\dfrac{(a - b)}{n + 1}$

6.7. $\left(S - \dfrac{Ml}{m + M}\right)$

6.9. $m\left[\left(\dfrac{l\dot{\phi}}{2}\right)^2 + (gt)^2\right]$

6.10. $\omega = \dfrac{v}{\left(\dfrac{l}{2}\right)}$; $v_c = \dot{\phi}\left(\dfrac{l}{2}\right)$

6.11. $E = \frac{1}{2}\frac{F^2}{m}(\Delta t)^2;\ \phi = \frac{F(\Delta t)}{ml}$

$v_c = \frac{F(\Delta t)}{2m}$

6.12. $v_c = \frac{1}{2}v;\ \phi = \frac{v}{l}$

6.14. $H = 4ml^2\omega$

6.15. $v_1 = \frac{r^2\omega}{r_1}$

6.16. $F_r = \frac{m_1 m_2 l}{(m_1 + m_2)}\omega^2$

6.19. $v = 2v_0$

6.20. $F = 3\mu gs$, where s = distance through which top of chain falls

6.21. $x = \frac{mv^2}{4Mg}\left[1 + \sqrt{1 + \frac{8Mga}{mv^2}}\right]$

6.22. $\frac{v_0^2}{g}\sin 2\theta$

Chapter 7

7.1. $v_B = v \sin\phi$

7.2. $v_D = \dot{x}_0(\boldsymbol{i} - \boldsymbol{j})$

7.3. $\frac{v}{\left[1 - \left(\frac{d_1}{d_2}\right)\right]}$

7.4. $\frac{\dot{x}_1}{2}\left(1 + \frac{\dot{x}_2}{\dot{x}_1}\right)$

7.5. $v_{\max} = 1.15v$;
$a_{\max} = 1.27 \times 10^{-3}v^2$ ft/sec^2
(v in ft/sec)

7.6. $\omega_2 = 22.6$ rad/sec;
$\omega_3 = 10.5$ rad/sec
$\dot{\omega}_2 = 746$ rad/sec^2;
$\dot{\omega}_3 = 945$ rad/sec^2

7.7. $\boldsymbol{v}_A = 25\left(1 + \frac{\sqrt{3}}{2}\right)\boldsymbol{i} + 12.5\boldsymbol{j}$ ft/sec

$\dot{\boldsymbol{v}}_A = -625\boldsymbol{j}$ ft/sec^2

7.8. $v_c = 430$ in/sec;
$\dot{v}_c = 29{,}500$ in/sec^2

7.9. $\boldsymbol{v}_A = 0;\ \dot{\boldsymbol{v}}_A = -\frac{Rr}{(R - r)}\dot{\theta}^2\boldsymbol{e}_r$

$\boldsymbol{v}_c = r\dot{\theta}\boldsymbol{e}_t$

$\dot{\boldsymbol{v}}_c = -\frac{r^2\dot{\theta}^2}{(R - r)}\boldsymbol{e}_r + r\ddot{\theta}\boldsymbol{e}_t$

7.10. $v_p = 3.5$ ft/sec;
$a_p = 19.5$ ft/sec^2

7.14. $\frac{1}{2}mR^2$

7.15. $\frac{1}{12}m(a^2 + b^2)$; $\frac{1}{3}m(a^2 + b^2)$

7.16. $\frac{1}{12}m(3R^2 + l^2)$

7.17. $\frac{7}{5}mR^2$

7.18. $\frac{1}{12}ml^2$

7.19. $I_y = 0.526$ lb in sec^2
$I_z = 0.114$ lb in sec^2

7.20. $\frac{3}{10}mr^2$; $\frac{1}{20}m(3r^2 + 2h^2)$

7.21. 0.0518 lb ft sec^2

7.22. $I_{xy} = \frac{1}{2}mR^2 = I_{x'y'}$
$I_{yz} = I_{xz} = 0$
$I_{x'z'} = -\frac{\sqrt{2}}{4}mRl = I_{y'z'}$

7.27. $\frac{1}{24}ml^2 \sin 2\alpha$

7.28. $\frac{1}{12}m(3R^2 + 4l^2)\sin^2\alpha + \frac{1}{2}mR^2\cos^2\alpha$

7.29. $\frac{3}{20}\frac{MR^2}{(R^2 + h^2)}(6h^2 + R^2)$

7.30. $I_{xy} = \frac{1}{96}M(7l^2 - 12R^2)\sin 2\phi$;
$I_{xz} = 0$

7.31. $I_{x'z'} = \frac{1}{8}M(R^2 - \frac{4}{3}h^2)\sin 2\alpha$;
$I_{y'z'} = 0$

7.32. $\frac{1}{8}Ma^2$

7.33. 0

7.34. $\frac{1}{8}mR^2 \sin 2\alpha$

7.35. $\boldsymbol{F}_A = 51\boldsymbol{i}$ lb;
$\boldsymbol{F}_B = -35.5\boldsymbol{i} + 50\boldsymbol{j}$ lb

7.36. $a = \frac{r}{2}\left(1 - \frac{\mu W}{F}\right)$;

$\ddot{x}_c = \frac{(F - \mu W)}{W}g$

7.37. slips at 1.34 ft/sec^2;
tips at 8.05 ft/sec^2

7.38. $\left(\frac{\mu\sqrt{3} \mp 1}{\sqrt{3} \pm \mu}\right)g$

7.39. 9610 lb at each pin

7.40. $\phi = \frac{Wal}{g(Wl + 4kh^2)}$

7.41. $d = \left(1 - \dfrac{\mu W\sqrt{3}}{2F}\right)r$

reaction force

$= \dfrac{\sqrt{3}}{2} W(1 + \mu^2)^{\frac{1}{2}}$

7.42. $\ddot{x} = g \tan \phi$

7.44. $I\theta = \frac{1}{2}At^2 + \frac{1}{6}Bt^3 - \frac{1}{12}Ct^4 + I\dot{\theta}_0 t + I\theta_0$

7.45. $\omega = \sqrt{\dfrac{4\pi NM}{I}}$

7.46. $t = \infty$; $\dfrac{I\omega_0}{2\pi k}$ revolutions

7.47. $\omega_{\max} = \dfrac{M}{k}$

7.48. $t = \dfrac{\omega_1 r_1 I_2}{\mu P r_2^2}$;

$t = \dfrac{\omega_1 r_1}{\mu P}\left(\dfrac{1}{\dfrac{r_2^2}{I_2} + \dfrac{r_1^2}{I_1}}\right)$

7.49. $\dfrac{\left(\dfrac{\omega}{p}\right)\theta_0 \sin pt}{1 - \left(\dfrac{\omega}{p}\right)^2} + \dfrac{\theta_0 \sin \omega t}{1 - \left(\dfrac{\omega}{p}\right)^2}$

7.50. 495,000 ft lb; 2350 ft lb

7.51. $\tau = 2\pi\sqrt{\dfrac{I}{RW}}$; $l = \dfrac{I}{mR}$

7.52. $a = \dfrac{\sqrt{2}}{2}r$; $\omega^2 = \dfrac{\sqrt{2}}{2}\dfrac{g}{r}$

7.53. $I_c = 45.0$ lb in sec^2

7.54. $a = \dfrac{I_0}{mR}$

7.55. $v^2 = \dfrac{2(m + M)(I + ml^2)g}{m^2 l} \times (1 - \cos \theta_0)$

7.56. $\left(I + \dfrac{WR^2}{g}\right)\ddot{\theta} = RW$

7.57. $\omega^2 = 3g/L$

7.58. $\omega = \frac{1}{3}\omega_0$; Imp. $= \frac{1}{3}\omega_0 RM$
$\Delta T = \frac{1}{6}mR^2\omega_0^2$

7.59. $\ddot{\theta} + \left(\dfrac{Wa}{2I}\sin \alpha\right)\theta = 0$

7.60. $I_2 = I_1\left[\dfrac{(f_0/f_2)^2 - 1}{(f_0/f_1)^2 - 1}\right]$

7.61. $p = \dfrac{2W}{\Delta t} \times \left[\dfrac{(1 + a^2)(\sqrt{1 + a^2} - 1)}{3gh}\right]^{\frac{1}{2}}$

7.62. 2520 lb

7.63. $\omega = \dfrac{mv}{(\frac{1}{2}M + m)r}$

7.64. $\cos^{-1}\dfrac{4}{7}$; $\omega^2 = \dfrac{4g}{7r}$

7.65. $\tan^{-1}\dfrac{\mu l^2}{(l^2 + 36a^2)}$

7.67. $r = \dfrac{ml^2 \sin 2\alpha}{m_1(L - 2a)}$

7.69. $F = \dfrac{1}{8}\dfrac{mr^2\omega^2}{l}\sin 2\alpha$

7.70. $\dfrac{1}{8}\dfrac{m\omega^2}{l}(R^2 - \frac{1}{3}h^2)\sin 2\alpha \pm \dfrac{me\omega^2}{2}$

7.71. $\dfrac{m\omega^2 ab(a^2 - b^2)}{12(a^2 + b^2)^{3/2}}$

7.72. left, 29.6 lb; right, 23.7 lb

7.73. $\dfrac{ma}{6}\sqrt{\omega^4 + \dot{\omega}^2}$

7.74. $F_{DYN.} = 9.31$ lb; $F_{ST.} = 4.94$ lb

7.75. $F_{DYN.} = \dfrac{MR^2\omega^2}{24l}$

7.76. 6.10 ft/sec

7.77. $R_A = -50$ lb (downwards)

7.78. $\mu = 0.253$

7.79. $\omega = \sqrt{\dfrac{200mgh}{3MR^2\left(3\dfrac{M}{m} + 10\right)}}$

7.80. $\ddot{\theta} = \dfrac{RW \sin \alpha}{(I + MR^2)}$; $\tan \alpha = 3\mu$

7.81. $\ddot{y} = \frac{2}{3}g$; $F = \frac{1}{3}W$

7.82. $h = \frac{7}{5}R$

7.83. $W\left(1 + \dfrac{kv^2}{gR}\cos \dfrac{v}{R}t\right)$

7.84. $\ddot{\theta} + \dfrac{5g}{7(R - r)}\theta = 0$

7.85. $l = \dfrac{I_c}{md}$

7.86. $v_c = 40.6$ ft/sec; $v_W = 81.2$ ft/sec

7.87. $$\dot{v}_2 = \frac{W_2 - 2W_3}{\left(m_2 + 4m_3 + \frac{4I_1}{R_1^2} + \frac{I_2}{R_2^2}\right)}$$

7.88. Before impact, $\dot{x}_1 = v$, $\dot{x}_2 = 0$
After impact, $\dot{x}_1 = 0$, $\dot{x}_2 = v$

7.89. $N = mg + \frac{Mg}{2} \pm \frac{Mv^2}{6R}$

7.90. $f = \frac{I_z\ddot{x}}{2R^2} \pm \frac{Mv^2}{6R}$

7.91. At piston,
$\boldsymbol{F} = -3750\boldsymbol{i} - 454\boldsymbol{j}$ lb
At crankpin,
$\boldsymbol{F} = 6130\boldsymbol{i} - 310\boldsymbol{j}$ lb

7.92. $\frac{19}{28}\frac{I^2}{m}$

7.93. 2.11 ft/sec

7.94. $\frac{1}{2}\sqrt{3gl}$

7.95. $0.193g$

7.96. $\dot{x}_c = \mu g t$ for $t \leqslant \frac{R\omega_0}{3\mu g}$

$\dot{x}_c = \frac{R\omega_0}{3}$ for $t \geqslant \frac{R\omega_0}{3\mu g}$; $\frac{\Delta E}{E} = \frac{2}{3}$

7.97. 1840 ft lb

7.98. Same direction, 2470 ft lb
Opposite direction, 831 ft lb

7.99. 111 lb

7.100. $F = \frac{vI\Omega}{Rl}$

7.101. Overturn

7.102. 5.36×10^{-10} ft lb

7.103. $a = \frac{3}{4}h\cos\alpha + \frac{k^2\Omega^2}{g}\operatorname{ctn}\alpha$

$\Omega = \frac{1}{2k\cos\alpha} \times$

$\sqrt{gh\sin\alpha(1 + 3\sin^2\alpha)}$

where

$Mk^2 = I_x\sin^2\alpha + I_z\cos^2\alpha$

7.104. $\Omega^2 > \frac{mga}{I + ma^2}$

7.105. $M_{\max} = \frac{Wl}{8}\left(1 + \frac{4rv^2}{gd^2}\right)$

7.106. $p^2 = \frac{(W \pm Ma)h}{I_0}$

7.107. 255 lb

7.108. 162 lb

7.111. $W_1' = 12.3$ lb, $\theta_1' = 243°$
$W_5' = 7.95$ lb, $\theta'_5 = 77.3°$

7.112. $\tau = 2\pi\left(\frac{I + mR^2}{kR^2}\right)^{\frac{1}{2}}$

7.113. $x = \frac{2lm\omega^2 - 2(m_1 + m)g}{2k + m\omega^2}$

7.114. $\ddot{\theta} + \left(\frac{k}{m} - \omega^2\right)\theta = 0$

Chapter 8

8.1. $t = 2l\sqrt{\frac{\rho}{E}}$

8.2. $\frac{\partial^2\phi}{\partial t^2} = \frac{G}{\rho}\frac{\partial^2\phi}{\partial x^2}$, velocity $= \sqrt{\frac{G}{\rho}}$

8.3. $V^4 = \frac{EI\omega^2}{\mu}$

8.4. velocity of waves $= \sqrt{\frac{F}{\mu}}$,

$\omega = \frac{\pi}{l}\sqrt{\frac{F}{\mu}}$

8.5. $p_s = p_0 + \frac{1}{2}\rho v_0^2$

8.6. $$v^2 = \frac{2(p_2 - p_1)}{\rho\left[1 - \left(\frac{A_1}{A_2}\right)^2\right]}$$

8.7. $v = \sqrt{v_x^2 - 2g(z - h)}$

8.8. $v = \sqrt{2\left(\frac{p_1 - p_0}{\rho}\right) + 2gh}$

8.10. $v = \sqrt{2gh}\tanh\left(\frac{1}{2l}\sqrt{2gh}\right)t$

at $t = \frac{8l}{\sqrt{2gh}}$, $v = 0.999\sqrt{2gh}$

8.12. $F = \rho v^2 A$

8.13. $F = p_1(A_1 - A_2) - \rho v_1^2 A_1\left[\frac{1}{2}\left(\frac{A_1}{A_2} + \frac{A_2}{A_1}\right) - 1\right]$

8.14. $\mathbf{F} = \rho v_1^2 A_1\left[\left(\frac{A_1}{8A_2} - 1\right)\mathbf{i} + \left(\frac{\sqrt{3}}{8}\frac{A_1}{A_2} - \frac{A_1}{4A_3}\right)\mathbf{j}\right]$

8.16. $F_B = [(p_1 - p_2) + (\rho_1 v_1^2 - \rho_2 v_2^2)]A$

8.17. $(p_2^* - p_2) = \frac{1}{2}\rho(v_1 - v_2)^2$

8.19. $F = -(p_i + \rho_i v_i^2)A_i + (p_e + \rho_e v_e^2)A_e$

8.20. $u = v\left(1 - \dfrac{1}{1 + \dfrac{\rho A v t}{m}}\right)$;

$u_t = v - \sqrt{\dfrac{\mu m g}{\rho A}}$

8.22. $Q = \dfrac{AR\omega}{\sin\alpha} \times \left[-1 + \sqrt{1 + \dfrac{2M_p \sin\alpha}{AR^3\rho\omega^2}}\right]$

8.23. $Q = 2A\sqrt{2\text{gh}}$; $Q = \dfrac{2A\sqrt{2gh}}{\cos\alpha}$

Chapter 9

9.2. $\ddot{\phi} - \sin\phi\cos\phi\,\dot{\theta}^2 = -\frac{g}{l}\sin\phi$

$\frac{d}{dt}(l^2\sin^2\phi\,\dot{\theta}) = 0$

9.3. $m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2}$

$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

9.4. (a) $m\ddot{x} - ml\ddot{\phi}\cos\phi + ml\dot{\phi}^2\sin\phi = F\sin\phi$

$ml^2\ddot{\phi} - ml\ddot{x}\cos\phi = 0$

(F = force in string)

(b) $\dot{\phi} = \frac{v}{l}$; $F = \frac{mv^2}{l}$

9.5. $\ddot{\phi} + \frac{a\omega^2}{l}\sin\phi = 0$; $p^2 = \frac{a\omega^2}{l}$

9.6. $I\ddot{\theta} + Wa\sin\theta = 0$; $p^2 = \frac{Wa}{I}$

9.7. $2\ddot{\theta} + \ddot{\phi} = -\frac{2g}{l}\theta$

$\ddot{\theta} + \ddot{\phi} = -\frac{g}{l}\phi$

9.8. $\ddot{\theta} + \frac{3a^2k}{ml^2}\theta = 0$

9.9. $p^2 = \frac{3k}{m}$

9.10. $\dfrac{dv_3}{dt} = \dfrac{\left(\dfrac{W_2}{2} - W_3\right)}{\left(\dfrac{m_2}{4} + m_3 + \dfrac{I_1}{r_1^2} + \dfrac{I_2}{4r_2^2}\right)}$

9.11. $\ddot{\phi} + \frac{g}{x_0}\phi = 0$; $\ddot{x} + \frac{k}{m}x = \frac{k}{m}x_0$

(x_0 = length of spring under gravity at $\phi = 0$)

9.12. $3\ddot{x} + (R - r)\ddot{\phi}\cos\phi - (R - r)\dot{\phi}^2\sin\phi = 0$

$\frac{3}{2}(R - r)\ddot{\phi} + \ddot{x}\cos\phi + g\sin\phi = 0$

9.13. $(R - r)^2\left[\frac{M}{3} + m + \frac{I}{r^2}\right]\ddot{\theta} = M_t$

9.14. $(M + m)\ddot{x} + m(R - r)\ddot{\phi}\sin\phi + m(R - r)\dot{\phi}^2\cos\phi + kx = 0$

$\frac{3}{2}(R - r)\ddot{\phi} + \ddot{x}\sin\phi + g\sin\phi = 0$

9.15. $(M + m)\ddot{x} + 2kx + ml\ddot{\phi} + 2ka\phi = 0$

$ml\ddot{x} + 2kax + ml^2\ddot{\phi} + (mgl + 2ka^2)\phi = 0$

9.17. $\omega^2 = \frac{1}{2}\left[\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right) \pm \sqrt{\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{4k_1k_2}{m_1m_2}}\right]$

9.18. $\omega^2 = \frac{g}{l} + \frac{ka^2}{ml^2} \pm \frac{ka^2}{ml^2}$

9.19. $\omega^2 = \frac{3}{2}\frac{k_t}{I} \pm \frac{1}{2}\sqrt{\left(\frac{3k_t}{I}\right)^2 - \frac{4k_t^2}{I^2}}$

9.20. $\omega^2 = \frac{g}{l}(2 \pm \sqrt{2})$

9.21. $A_1^{(1)} = \dfrac{\dot{x}_{10} + \dot{x}_{20}}{2\omega_1}$;

$A_1^{(2)} = \dfrac{\dot{x}_{10} - \dot{x}_{20}}{2\omega_2}$

$B_1^{(1)} = \dfrac{x_{10} + x_{20}}{2}$;

$B_1^{(2)} = \dfrac{x_{10} - x_{20}}{2}$

9.23. $\omega^2 = \dfrac{k}{m} + (2 \pm \sqrt{2})\dfrac{g}{l}$

9.24. $\omega^2 = \dfrac{F}{ml}(2 \pm 1)$

9.26. $\omega_1^2 = 3k/m$
$\omega_2^2 = 6ka^2/I$

9.27. $\omega_1^2 = 0$; $\omega_2^2 = \left(1 + \dfrac{2m_2}{3m_1}\right)\dfrac{g}{l}$

9.28. $\omega_1^2 = \dfrac{k}{m}$; $\omega_2^2 = \dfrac{1}{5}\dfrac{k}{m}$

9.29. $\omega^2 = \dfrac{(3 + \alpha)}{(2 + \alpha)}\dfrac{g}{l}$

9.30. $$m_{ij} = \sum_{k=1}^{n} m_k\left[\left(\frac{\partial x_k}{\partial q_i}\right)\left(\frac{\partial x_k}{\partial q_j}\right) + \left(\frac{\partial y_k}{\partial q_i}\right)\left(\frac{\partial y_k}{\partial q_j}\right) + \left(\frac{\partial z_k}{\partial q_i}\right)\left(\frac{\partial z_k}{\partial q_j}\right)\right]$$

9.31. $m_{11} = m$
$m_{22} = ml^2$
$m_{12} = m_{21} = ml$
$k_{11} = k$
$k_{22} = mgl$
$k_{12} = k_{21} = 0$

9.33. (a) Dynamic coupling only
(b) Static and dynamic coupling
(c) Static coupling only

9.34. (a) $m_{11} = m_{22} = m_{33} = ml^2$;
$m_{ij} = 0,\ i \neq j$
$k_{11} = k_{33} = mgl + ka^2$
$k_{22} = mgl + 2ka^2$
$k_{12} = k_{21} = k_{23} = k_{32} = -ka^2$
$k_{13} = k_{31} = 0$

(b) $\omega_1^2 = \dfrac{g}{l}$;

$\omega_2^2 = \dfrac{g}{l} + \dfrac{ka^2}{ml^2}$;

$\omega_3^2 = \dfrac{g}{l} + 3\dfrac{ka^2}{ml^2}$

(c) $A_1^{(1)} = A_2^{(1)} = A_3^{(1)} = 1$
$A_1^{(2)} = 1$; $A_2^{(2)} = 0$;
$A_3^{(2)} = -1$
$A_1^{(3)} = 1$; $A_2^{(3)} = -2$;
$A_3^{(3)} = 1$

9.36. $$x_1 = \frac{F}{m}\left[\frac{0.242}{(\omega_1^2 - \omega^2)} - \frac{0.435}{(\omega_2^2 - \omega^2)} + \frac{0.194}{(\omega_3^2 - \omega^2)}\right]\sin \omega t$$

9.37. $$\phi_2 = \frac{1}{3}\frac{F}{ml}\left[\frac{1}{(\omega_1^2 - \omega^2)} - \frac{1}{(\omega_3^2 - \omega^2)}\right]\sin \omega t$$

9.38. $$x_1 = a_0\left[\frac{(m_1A_1^{(1)} + m_2A_2^{(1)} + m_3A_3^{(1)})}{(m_1A_1^{(1)2} + m_2A_2^{(1)2} + m_3A_3^{(1)2})}\frac{A_1^{(1)}}{(\omega_1^2 - \omega^2)} + \frac{(m_1A_1^{(2)} + m_2A_2^{(2)} + m_3A_3^{(2)})}{(m_1A_1^{(2)2} + m_2A_2^{(2)2} + m_3A_3^{(2)2})}\frac{A_1^{(2)}}{(\omega_2^2 - \omega^2)} + \frac{(m_1A_1^{(3)} + m_2A_2^{(3)} + m_3A_3^{(3)})}{(m_1A_1^{(3)2} + m_2A_2^{(3)2} + m_3A_3^{(3)2})}\frac{A_1^{(3)}}{(\omega_3^2 - \omega^2)}\right]\sin \omega t$$

9.39. $\dfrac{dy}{dx} = \sqrt{\left(\dfrac{y}{c}\right)^2 - 1}$; $y = C\cosh\left(\dfrac{x}{C}\right)$

9.41. $y = C\cosh\left(\dfrac{x}{C}\right)$

9.45. (d) $EI\dfrac{\partial^4 y}{\partial x^4} + \rho v^2\dfrac{\partial^2 y}{\partial x^2} + 2\rho v\dfrac{\partial^2 y}{\partial x\,\partial t} + m\dfrac{\partial^2 y}{\partial t^2} = 0$

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