

The force on a sphere in a uniform flow with small-amplitude oscillations at finite Reynolds number

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The unsteady force acting on a sphere that is held fixed in a steady uniform flow with small-amplitude oscillations is evaluated to $O(Re)$ for small Reynolds number, Re . Good agreement is shown with the numerical results of Mei, Lawrence & Adrian (1991) up to $Re \approx 0.5$. The analytical result is transformed by Fourier inversion to allow for an arbitrary time-dependent motion which is small relative to the steady uniform flow. This yields a history-dependent force which has an integration kernel that decays exponentially for large time.

1. Introduction

Recently Mei, Lawrence & Adrian (1991, hereinafter referred to as MLA) numerically computed the unsteady force acting on a spherical particle held fixed in a fluid which has small fluctuations about its steady free-stream velocity. Specifically, the force was obtained numerically for the following imposed flow:

$$U^{\infty'}(t') = U(1 + \alpha_1 e^{-i\omega t'}), \quad (1.1)$$

with the condition $\alpha_1 \ll 1$. The primes are used to indicate dimensional quantities when there exists a corresponding non-dimensional quantity elsewhere in the paper. The Reynolds number, Re , based on the particle radius, a , and free-stream velocity, U , ranged from zero up to 50 in their numerical study. In the low-frequency limit, their results indicated that the force has a much shorter memory than that predicted by the Basset history integral from the unsteady Stokes solution.

Later, Mei & Adrian (1992, hereinafter referred to as MA) evaluated the force analytically at small Reynolds number and low frequency, ω , for the above imposed flow. A matched asymptotic solution was used in the limit $Sl_\omega \ll Re \ll 1$, where Sl_ω is the Strouhal number ($a\omega/U$). The results agreed well with the previous numerical study of MLA in this limit. Based on the results from both the numerical and analytical studies, a modified expression for the history force was proposed in the time domain. It had an integration kernel that decayed as t^{-2} at large time for both small and finite Reynolds numbers, as opposed to the $t^{-\frac{1}{2}}$ decay, associated with the Basset term for zero Reynolds number.

In the present study, we extend the above analytical results to arbitrary frequency (or Sl_ω), maintaining the requirement of $Re \ll 1$. This is accomplished by making use of a previously obtained expression from Lovalenti & Brady (1993) for the unsteady force acting on a particle in arbitrary motion (relative to the fluid) accurate to $O(Re)$.

It is derived from the general reciprocal theorem for the Navier–Stokes equations through the use of a uniformly valid asymptotic expansion for the flow field. When the result is applied to the motion given by (1.1), it is found that the force agrees with both the analytical results of MA and the numerical results of MLA up to $Re \approx 0.5$. However, when the expression is transformed to account for arbitrary time-dependent motion, a history-dependent force with an integration kernel that decays exponentially at large time is obtained, in contrast to the proposed expression of MA which decays algebraically.

In what follows, we first derive the force expression in the frequency domain for the flow given by (1.1) and compare it to the results of MLA and MA. Next, in §3, we generalize the expression to arbitrary time-dependent motion through Fourier inversion and evaluate its behaviour at large time. We conclude in §4 with a discussion of the results.

2. Evaluation of the force expression in the frequency domain

For a fixed spherical particle in a rectilinear imposed flow, $U^\infty(t)$, the hydrodynamic force derived in Lovalenti & Brady (1993) reduces to

$$\begin{aligned}
 F^H(t) = & 6\pi U^\infty(t) + 2\pi ReSl \dot{U}^\infty(t) \\
 & + \frac{9}{2}(ReSl\pi)^{\frac{1}{2}} \left\{ \int_0^1 \int_{-\infty}^t \left[\frac{2}{3} U^\infty(t) \right. \right. \\
 & \left. \left. - \left\{ \frac{1}{A^2} (e^{-A^2x^2} - e^{-A^2}) \right\} U^\infty(s) \right] \frac{ds}{(t-s)^{\frac{3}{2}}} dx \right\}, \quad (2.1)
 \end{aligned}$$

where

$$A(t,s) = \frac{1}{2} \left(\frac{Re}{Sl} \right)^{\frac{1}{2}} \frac{\int_s^t U^\infty(q) dq}{(t-s)^{\frac{1}{2}}}. \quad (2.2)$$

The Reynolds number and Strouhal number are defined by

$$Re = \frac{aU_c}{\nu}, \quad Sl = \frac{a/U_c}{\tau_c}, \quad (2.3)$$

where U_c and τ_c are the characteristic velocity and timescale of the imposed flow and ν is the kinematic viscosity of the fluid. The force, $F^H(t)$, has been non-dimensionalized by $a\mu U_c$, where μ is the viscosity of the fluid. The first term of (2.1) is the steady Stokes drag; the second represents a combination of the added mass and the force due to the accelerating imposed flow (which would have been exerted on the fluid displaced by the sphere); and the last term is a new history integral: it reduces to the steady Oseen correction for steady motion, and to the Basset history integral for short-time unsteady motion.

For the flow given by (1.1), we let $U_c = U$ and $\tau_c = \omega^{-1}$, allowing the dimensionless imposed flow to be expressed as

$$U^\infty(t) = (1 + \alpha_1 e^{-it}). \quad (2.4)$$

If we use this flow in the force expression (2.1) and take the limit of $\alpha_1 \ll 1$, we obtain to

$O(\alpha_1)$ after some tedious, but straightforward, manipulation and change of variables

$$\begin{aligned}
 F^H(t) = & 6\pi \left(1 + \frac{3}{8}Re + \alpha_1 e^{-it} \right) - 2\pi i Re Sl_\omega (\alpha_1 e^{-it}) \\
 & + \frac{9}{2} Re \pi^{\frac{1}{2}} \alpha_1 e^{-it} \int_0^1 \int_0^\infty \left(\frac{1 - e^{is\gamma_\omega}}{s\gamma_\omega} \right) \left(\frac{3 e^{-sx^2} - e^{-s}}{s} - e^{-s} \right) \frac{ds}{s^{\frac{3}{2}}} dx \\
 & + \frac{9}{8} Re \pi \alpha_1 e^{-it} \int_0^1 \int_0^\infty \left[\frac{2}{3} - \frac{e^{is\gamma_\omega}}{s} (e^{-sx^2} - e^{-s}) \right] \frac{ds}{s^{\frac{3}{2}}} dx, \quad (2.5)
 \end{aligned}$$

where $\gamma_\omega = 4Sl_\omega/Re$, and $Sl_\omega = a\omega/U$. We note that by taking the limit of small α_1 we have linearized the relationship between the time-dependent part of the velocity and the force, which will allow for Fourier inversion to the time domain in the next section. The above integrations were carried out using *Mathematica* to obtain

$$\begin{aligned}
 F^H(t) = & 6\pi \left(1 + \frac{3}{8}Re + \alpha_1 e^{-it} \right) - 2\pi i Re Sl_\omega (\alpha_1 e^{-it}) \\
 & + 6\pi Re \alpha_1 e^{-it} \frac{2^{\frac{1}{2}}(1-i)(\gamma_\omega + i)^{\frac{3}{2}} - 2i}{4\gamma_\omega}. \quad (2.6)
 \end{aligned}$$

If we expand this expression for small γ_ω (i.e. for small frequency such that $Sl_\omega \ll Re$), the force to $O(Sl_\omega)$ is

$$\begin{aligned}
 F^H(t) = & 6\pi \left(1 + \frac{3}{8}Re + \alpha_1 e^{-it} \right) - 2\pi i Re Sl_\omega (\alpha_1 e^{-it}) \\
 & + 6\pi Re \alpha_1 e^{-it} \left(\frac{3}{4} - i \frac{3 Sl_\omega}{4 Re} \right). \quad (2.7)
 \end{aligned}$$

This expression agrees with the analytical result of MA. In addition, further terms in the low-frequency expansion are in integer powers of the frequency; the even powers are associated with the real coefficients of the imposed flow ($\alpha_1 e^{-it}$) and the odd powers with the imaginary coefficients.

To compare (2.6) with the numerical work of MLA, we define the following quantities based on their equivalence to those in MLA:

$$D_{1RAC} = \text{Re} \left\{ Re \frac{2^{\frac{1}{2}}(1-i)(\gamma_\omega + i)^{\frac{3}{2}} - 2i}{4\gamma_\omega} \right\} - \frac{3}{4} Re, \quad (2.8)$$

$$A_{1I} = -\text{Im} \left\{ Re \frac{2^{\frac{1}{2}}(1-i)(\gamma_\omega + i)^{\frac{3}{2}} - 2i}{4\gamma_\omega} \right\}. \quad (2.9)$$

Here, D_{1RAC} represents the real part of the frequency-dependent drag coefficient and A_{1I} is the imaginary part of the frequency-dependent drag coefficient excluding the $-2\pi i Re Sl_\omega$ -term, both of which are nondimensionalized by $6\pi\mu a$. In figures 1(a,b) and 2(a,b) these quantities are plotted as a function of γ_ω for various Reynolds numbers, with the numerical data from MLA included for comparison. The same quantities scaled by the Reynolds number are presented as well to show that the results may be collapsed on a single curve for small Re . The figures show good agreement of the analytical and numerical results up to $Re \approx 0.5$. This might appear somewhat surprising given that the force expression is valid strictly for the limit of infinitesimally small Reynolds number, its accuracy being only to $O(Re)$. We note, however, that a similar finding was made by Maxworthy (1965) who determined that the experimentally observed terminal settling velocity of spheres was adequately predicted by the $O(Re)$ -accurate Oseen approximation up to $Re \approx 0.4$.

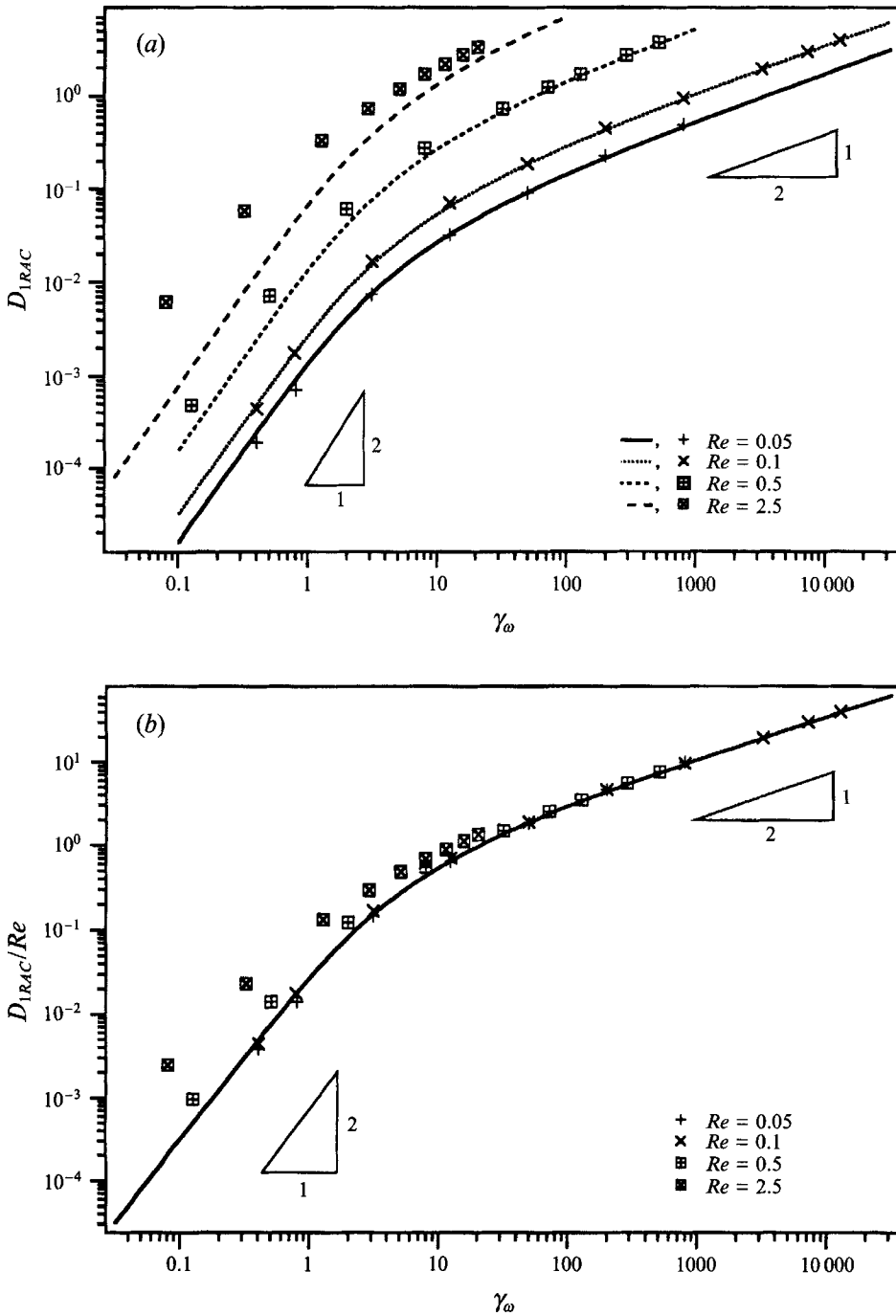


FIGURE 1. The real part of the acceleration-dependent drag coefficient for small-amplitude oscillations about a uniform flow past a sphere as a function of the dimensionless frequency at various Reynolds numbers, (a) unscaled; (b) scaled. The lines are the analytical result (2.8) and the symbols are the numerical results of MLA.

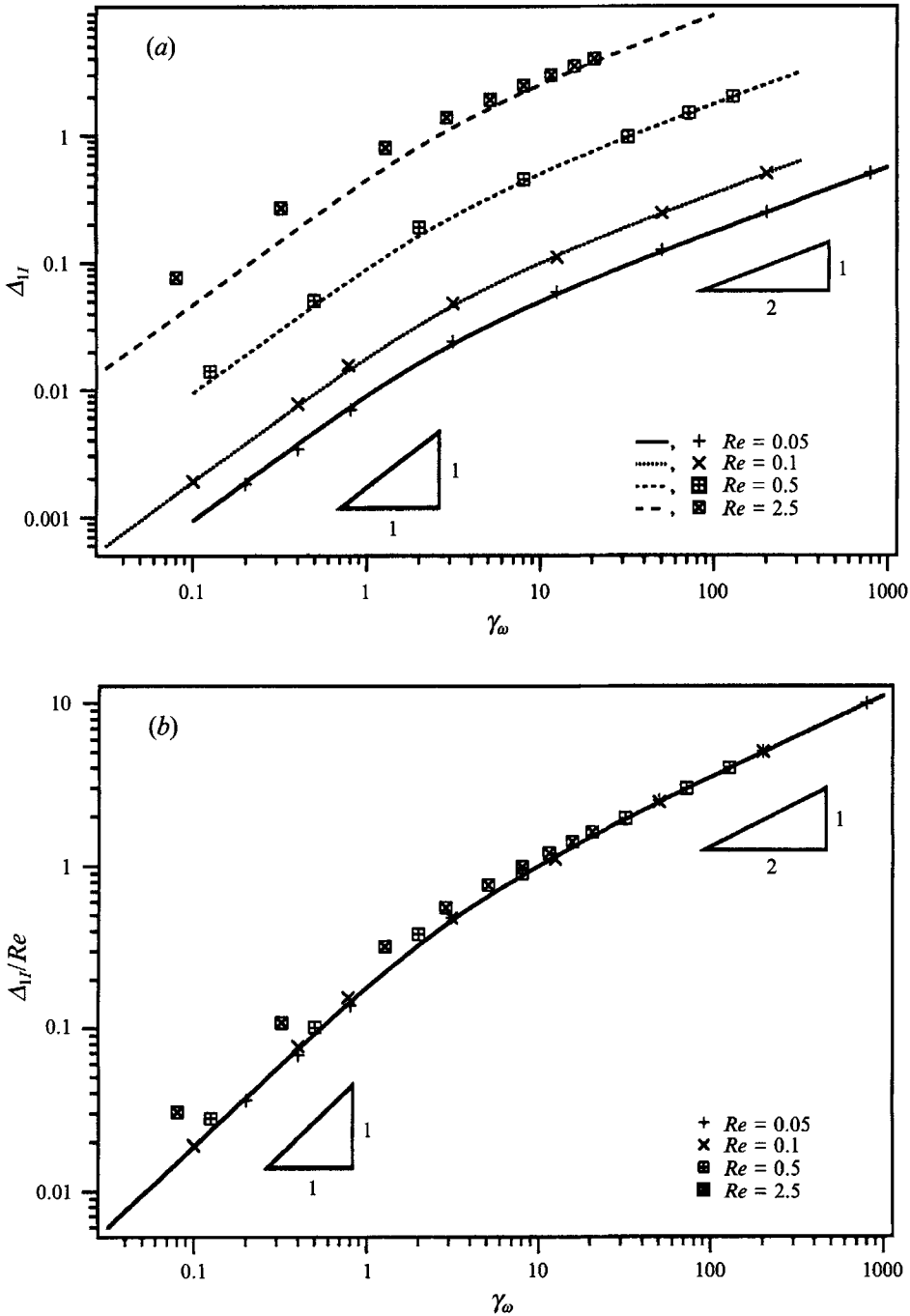


FIGURE 2. The difference between the imaginary part of the acceleration-dependent drag coefficient and $-2\pi i Re S l_\omega$ for small-amplitude oscillations about a uniform flow past a sphere as a function of the dimensionless frequency at various Reynolds numbers, (a) unscaled; (b) scaled. The lines are the analytical result (2.9) and the symbols are the numerical results of MLA.

3. Generalization of the force expression to arbitrary time-dependent motion

In order to evaluate the force for a small general time-dependent flow, we must consider α_1 as the Fourier transform of a small unsteady velocity, $U_1(t)$, which is superimposed on the steady uniform flow U , under the condition that $U_1(t) \ll U$ for all t . Then α_1 is a function of ω and is related to $U_1(t)$ by

$$U_1(t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_1 e^{-i\omega t'} d\omega, \quad \alpha_1(\omega) = \int_{-\infty}^{\infty} U_1(s') e^{-i\omega s'} ds'. \tag{3.1}$$

Thus the α_1 -dependent part of the force expression (2.6) may be readily transformed to the time domain by integration with respect to ω to obtain

$$F^H(t) = 6\pi (1 + U_1(t) + \frac{3}{8} Re (1 + 2U_1(t)) + F'(t)) + 2\pi Re Sl \dot{U}_1(t), \tag{3.2}$$

where

$$F'(t) = \frac{Re}{2\pi} \int_{-\infty}^{\infty} \dot{U}_1(s) \int_{-\infty}^{\infty} \frac{2 - 3\gamma_\omega i + 2^{\frac{1}{2}}(1+i)(\gamma_\omega + i)^{\frac{3}{2}}}{4\gamma_\omega^2} e^{-i\gamma_\omega(t-s)/\gamma} d\gamma_\omega ds. \tag{3.3}$$

Here Sl is as defined in (2.3) and $\gamma = 4Sl/Re$.

The γ_ω -integration in the expression for $F'(t)$ may be simplified by contour integration. The branch cut for the square root in the complex γ_ω -plane originates at $\gamma_\omega = -i$ and extends along the negative imaginary axis to $-\infty$. The boundedness of the integrand in (3.3), particularly at the origin, means that its integration along any closed contour not crossing the branch cut must be zero. Therefore, the appropriate contours for $s > t$ and $s < t$ are in the upper and lower half-planes, respectively. The radius of the semicircular portions of the contours are taken to the limit of infinity, and it can be seen that there is no contribution from the integration along these parts of the contours. As expected, this implies that there is no contribution to the integral when $s > t$. When $s < t$, the γ_ω -integration reduces to two integrals along each side of the branch cut:

$$F'(t) = \frac{Re}{2\pi} \int_{-\infty}^t \dot{U}_1(s) \left[- \int_{-i-\epsilon}^{-i\infty-\epsilon} \frac{2 - 3\gamma_\omega i + 2^{\frac{1}{2}}(1+i)(\gamma_\omega + i)^{\frac{3}{2}}}{4\gamma_\omega^2} e^{-i\gamma_\omega(t-s)/\gamma} d\gamma_\omega - \int_{-i\infty+\epsilon}^{-i+\epsilon} \frac{2 - 3\gamma_\omega i + 2^{\frac{1}{2}}(1+i)(\gamma_\omega + i)^{\frac{3}{2}}}{4\gamma_\omega^2} e^{-i\gamma_\omega(t-s)/\gamma} d\gamma_\omega \right] ds, \tag{3.4}$$

where ϵ is an infinitesimally small, real, positive number. If we set $x = i\gamma_\omega - 1$ this expression simplifies to

$$F'(t) = \frac{Re}{2\pi} \int_{-\infty}^t G(t-s) \dot{U}_1(s) ds. \tag{3.5}$$

The integration kernel for this history force is given by

$$G(t) = e^{-t/\gamma} \int_0^\infty \frac{x^{\frac{3}{2}}}{(1+x)^2} e^{-xt/\gamma} dx = e^{-t/\gamma} \Gamma\left(\frac{5}{2}\right) \Psi\left(\frac{5}{2}, \frac{3}{2}, t/\gamma\right), \tag{3.6}$$

where Ψ is a confluent hypergeometric function, sometimes known as the Kummer function (Abramowitz & Stegun 1972).

The asymptotic properties of $G(t)$ for small and large time are

$$G(t) = \left(\frac{\gamma\pi}{t}\right)^{\frac{1}{2}} - \frac{3}{2}\pi + O\left(\left(\frac{t}{\gamma}\right)^{\frac{1}{2}}\right), \quad \frac{t}{\gamma} \ll 1, \tag{3.7}$$

$$G(t) = \left[\frac{3\pi^{\frac{1}{2}}}{4} \left(\frac{\gamma}{t}\right)^{\frac{5}{2}} + O\left(\left(\frac{\gamma}{t}\right)^{\frac{7}{2}}\right)\right] e^{-t/\gamma}, \quad \frac{t}{\gamma} \gg 1. \tag{3.8}$$

In dimensional time these limits are $t' \ll v/U^2$ and $t' \gg v/U^2$, where v/U^2 represents the time it takes vorticity to diffuse out to, or be convected through, the Oseen distance v/U . The integration kernel behaves as that in the Basset history integral for small time, but shows exponential decay for large time. Note that the second term of (3.7) will result in the cancelling of the $\frac{3}{4}Re U_1(t)$ -term in the other part of the force expression (3.2) when the timescale of the motion is small. We note also that the behaviour for large time is in exact agreement with Lovalenti & Brady (1993) wherein the temporal response was observed for the force when the velocity made a step change from one non-zero velocity to another.

4. Discussion of results

The reason MA obtained the algebraic decay t^{-2} instead of exponential decay for their integration kernel can be explained as follows: Their result is based on the inversion of a function that interpolates only the one-term asymptotic forms of the imaginary part of the history force in the low- and high-frequency limits. The problem with this is that the one term in the low-frequency limit, $-\frac{3}{4}Sl_\omega i$, is insufficient to predict the long-time behaviour of the integration kernel. Indeed, when inverted for time-dependent motion this term would yield the acceleration at the current time, which has no history dependence. Thus, their resultant integration kernel depends critically on the choice of interpolating functions; one can obtain a different decay by choosing a different interpolating function. In addition, by their own principle of causality, the imaginary part of the history force must be an odd function of the frequency. However, if their interpolated expression is expanded for low frequency, an expansion in all powers of the frequency is obtained, not just the odd powers.

It is interesting to note that the force *does* decay as t^{-2} for a step change from a *zero* velocity, as can be observed from the result of Sano (1981). This distinction in decay rates is the result of the difference between the physical processes of the growth of the Oseen wake into essentially irrotational fluid, which is associated with algebraic decay, and the modification of the wake already established to infinite length, which is associated with exponential decay. In the case here, the wake clearly has been established by the uniform bulk flow U . Once the disturbance created by the small unsteady flow has diffused through the viscous Stokes region surrounding the particle, it is balanced exponentially fast by modification of the wake structure through convective transport mechanisms.

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