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When Are Quantum Systems Operationally Independent?

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Abstract

We propose some formulations of the notion of “operational independence” of two subsystems S_1, S_2 of a larger quantum system S and clarify their relation to other independence concepts in the literature. In addition, we indicate why the operational independence of quantum subsystems holds quite generally, both in nonrelativistic and relativistic quantum theory.

1 Introduction

The aim of this note is to propose mathematically well defined formulations of the notion of “operational independence” of two subsystems S_1, S_2 of a larger quantum system S and to clarify their relation to other independence concepts in the mathematical physics literature. In addition, we shall indicate why the operational independence of quantum subsystems holds quite generally, both in nonrelativistic and relativistic quantum theory.

Intuitively, operational independence of subsystems S_1 and S_2 expresses the notion that any two physical operations (measurements, state preparations *etc*) which can be carried out on S_1 and S_2 *separately* can also be carried out *jointly* as a single operation on system S .

It will be seen that operational independence can be given different technical formulations within the context of operator algebraic models of quantum systems. If the observables of quantum systems S_1, S_2 and S are represented by selfadjoint elements of

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C^* -subalgebras $\mathcal{A}_1, \mathcal{A}_2$ of a C^* -algebra \mathcal{A} , then S_1 and S_2 are called *operationally C^* -independent in \mathcal{A}* if any two completely positive, unit preserving maps T_1 and T_2 on \mathcal{A}_1 and \mathcal{A}_2 , respectively, have a joint extension to a completely positive, unit preserving map T on \mathcal{A} (Definition 6). Completely positive maps T satisfying $T(I) \leq I$ are called *operations* in the physics literature, since they can be used to represent physical operations carried out on the quantum systems [10, 21]. If the observables of the quantum systems in question are represented by von Neumann algebras, then it is natural to require the operations T_1, T_2 and T to be normal (continuous in the σ -weak topology) – the resulting definition is operational W^* -independence (Definition 7). Requiring that the extension T factors across the subalgebras (and preserves faithfulness) leads to Definitions 8 and 9.

In this paper we shall explain the relations of these notions to the already established notions of subsystem independence in the literature and, in so doing, provide some useful alternative characterizations of operational independence. In addition, we shall be able to demonstrate that the strongest form of operational independence formulated here obtains quite generally in nonrelativistic quantum mechanics and in relativistic quantum field theory.

We outline the structure of the paper. Section 2 recalls some notions of independence which have been investigated in the literature and which are relevant from the perspective of operational independence. Section 3 recalls the concept of operation as a completely positive map on C^* -, resp. W^* -, algebras together with some basic properties of completely positive maps. Section 4 formulates the definitions of operational independence in terms of completely positive maps and establishes their logical relations with the notions described in Section 2. Finally, in Section 5 the relation to a further, previously studied independence property called the split property is explained, and this relation is used to show that operational independence holds widely in quantum theory.

2 Some notions of independence

Throughout the paper \mathcal{A} denotes a unital C^* -algebra, $\mathcal{A}_1, \mathcal{A}_2$ are assumed to be C^* -subalgebras of \mathcal{A} (with common unit I). $\mathcal{A}_1 \vee \mathcal{A}_2$ will denote the smallest C^* -subalgebra of \mathcal{A} containing both \mathcal{A}_1 and \mathcal{A}_2 . \mathcal{N} denotes a von Neumann algebra, and $\mathcal{N}_1, \mathcal{N}_2$ will be von Neumann subalgebras of \mathcal{N} (with common unit). $\mathcal{N}_1 \vee \mathcal{N}_2$ will denote the smallest von Neumann algebra in \mathcal{N} containing both \mathcal{N}_1 and \mathcal{N}_2 . If \mathcal{N} is a von Neumann algebra acting on the Hilbert space \mathcal{H} , then \mathcal{N}' represents its commutant, the set of all bounded operators on \mathcal{H} which commute with all elements of \mathcal{N} . $\mathcal{S}(\mathcal{A})$ is the state space of the C^* -algebra \mathcal{A} . (For the operator algebraic notions see [31], [20] or [3].) For a Hilbert space \mathcal{H} , the set of all bounded operators on \mathcal{H} is denoted by $\mathcal{B}(\mathcal{H})$.

Since there are different quantitative and qualitative aspects to the notion of independent subsystems, it is natural that there be many theory dependent formulations of such independence. We discuss only a few of these here. The following technical definitions of independence were formalized in the context of algebraic quantum theory in a comprehensive review up to 1990 of the hierarchy of independence concepts and their non-trivial logical interrelations [28]. See [28] for a discussion of their operational meaning and their history. For more recent developments, see [14, 22, 18].

Definition 1. A pair $(\mathcal{A}_1, \mathcal{A}_2)$ of C^* -subalgebras of a C^* -algebra \mathcal{A} is called *C^* -independent* if for any state ϕ_1 on \mathcal{A}_1 and for any state ϕ_2 on \mathcal{A}_2 there exists a state ϕ on \mathcal{A} such that

both

$$\begin{aligned}\phi(X) &= \phi_1(X) \quad \text{for any } X \in \mathcal{A}_1 \\ \phi(Y) &= \phi_2(Y) \quad \text{for any } Y \in \mathcal{A}_2\end{aligned}$$

obtain.

Definition 2. A pair $(\mathcal{A}_1, \mathcal{A}_2)$ of C^* -subalgebras of a C^* -algebra \mathcal{A} is called C^* -independent in the product sense if the map $\eta(XY) \doteq X \otimes Y$ extends to an C^* -isomorphism of $\mathcal{A}_1 \vee \mathcal{A}_2$ with $\mathcal{A}_1 \otimes \mathcal{A}_2$, where $\mathcal{A}_1 \otimes \mathcal{A}_2$ denotes the tensor product of \mathcal{A}_1 and \mathcal{A}_2 with the minimal C^* -norm (see [31, 20, 13]).

If \mathcal{A} is faithfully represented on a Hilbert space \mathcal{H} , then the minimal norm referred to here is the ordinary operator norm in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \simeq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

Definition 3. A pair $(\mathcal{N}_1, \mathcal{N}_2)$ of von Neumann subalgebras of the von Neumann algebra \mathcal{N} is called W^* -independent if for any normal state¹ ϕ_1 on \mathcal{N}_1 and for any normal state ϕ_2 on \mathcal{N}_2 there exists a normal state ϕ on \mathcal{N} such that both

$$\begin{aligned}\phi(X) &= \phi_1(X) \quad \text{for any } X \in \mathcal{N}_1 \\ \phi(Y) &= \phi_2(Y) \quad \text{for any } Y \in \mathcal{N}_2\end{aligned}$$

obtain.

Definition 4. A pair $(\mathcal{N}_1, \mathcal{N}_2)$ of von Neumann subalgebras of the von Neumann algebra \mathcal{N} is called W^* -independent in the product sense if for any normal state ϕ_1 on \mathcal{N}_1 and for any normal state ϕ_2 on \mathcal{N}_2 there exists a normal product state ϕ on \mathcal{M} extending ϕ_1 and ϕ_2 , i.e. a normal state ϕ on \mathcal{N} such that

$$\phi(XY) = \phi_1(X)\phi_2(Y) \quad \text{for any } X \in \mathcal{N}_1, Y \in \mathcal{N}_2.$$

The above independence notions are not independent logically. Here we collect some results on their interrelations. Note that only C^* -independence in the product sense requires that the algebras mutually commute. The apparent asymmetry between the definitions of C^* -, resp. W^* -, independence in the product sense will be resolved below (for mutually commuting von Neumann algebras acting on a separable Hilbert space).

Proposition 1.

1. If $\mathcal{A}_1, \mathcal{A}_2$ are commuting, then the C^* -independence in the product sense of $(\mathcal{A}_1, \mathcal{A}_2)$ implies the C^* -independence of $(\mathcal{A}_1, \mathcal{A}_2)$, but the converse is false [28].
2. W^* -independence of a pair of arbitrary von Neumann algebras implies C^* -independence of the pair [28, 14], but the converse is false. In fact, examples of pairs of von Neumann algebras which do not mutually commute have been found which are C^* -independent but not W^* -independent. But if $\mathcal{N}_1, \mathcal{N}_2$ are commuting von Neumann algebras acting on a separable Hilbert space, then the C^* -independence of $(\mathcal{N}_1, \mathcal{N}_2)$ implies the W^* -independence of the pair [14], so that for such pairs C^* -independence is equivalent to W^* -independence.

¹These are the states which can be represented by a density matrix. Hence, in general, physicists tacitly restrict their attention to normal states.

3. The W^* -independence in the product sense of $(\mathcal{N}_1, \mathcal{N}_2)$ implies the W^* -independence of $(\mathcal{N}_1, \mathcal{N}_2)$, but the converse is false [28].
4. If $\mathcal{N}_1, \mathcal{N}_2$ are commuting, then the W^* -independence in the product sense of $(\mathcal{N}_1, \mathcal{N}_2)$ implies the C^* -independence in the product sense of $(\mathcal{N}_1, \mathcal{N}_2)$, but the converse is false [28, 14]. (This is further discussed below.)

Note that if $\mathcal{A}_1, \mathcal{A}_2$ are commuting C^* -algebras, then the extension state ϕ in Definition 1 may be chosen to be a product state [24], *i.e.*

$$\phi(XY) = \phi(X)\phi(Y) = \phi_1(X)\phi_2(Y),$$

for all $X \in \mathcal{A}_1, Y \in \mathcal{A}_2$. The corresponding assertion for W^* -independence is false [28]. Indeed, in that context one has the following theorem.

Proposition 2 ([30]). *Let $\mathcal{N}_1, \mathcal{N}_2$ be commuting factor von Neumann algebras acting on a common Hilbert space \mathcal{H} . Then the map $\eta(XY) \doteq X \otimes Y$ extends to a W^* -isomorphism of $\mathcal{N}_1 \vee \mathcal{N}_2$ with the W^* -tensor product $\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$ if and only if there exists a normal product state on $\mathcal{N}_1 \vee \mathcal{N}_2$.*

In fact, the assumption that the algebras be factors may be dropped if the normal product state is required to have central support I , the identity map on \mathcal{H} [11]. Hence, one has the following result.

Proposition 3. *Let $\mathcal{N}_1, \mathcal{N}_2$ be commuting von Neumann algebras acting on a separable Hilbert space. Then $(\mathcal{N}_1, \mathcal{N}_2)$ is W^* -independent in the product sense if and only if there exists a faithful normal product state on $\mathcal{N}_1 \vee \mathcal{N}_2$.*

Proof. Let $(\mathcal{N}_1, \mathcal{N}_2)$ be W^* -independent in the product sense. Since the Hilbert space on which the algebras act is separable, there exist faithful normal states ϕ_1, ϕ_2 on $\mathcal{N}_1, \mathcal{N}_2$, respectively [31, Prop. II.3.19]. But then $\phi_1 \otimes \phi_2$ is a faithful normal state on $\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$ [31, Cor. IV.5.12]. If $\eta : \mathcal{N}_1 \vee \mathcal{N}_2 \rightarrow \mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$ is the hypothesized W^* -isomorphism, then $(\phi_1 \otimes \phi_2) \circ \eta$ is a faithful normal product state on $\mathcal{N}_1 \vee \mathcal{N}_2$. For the converse, see [30, 11].

♠

An analogous characterization of C^* -independence in the product sense was proven in [14].

Proposition 4 ([14]). *Let $\mathcal{A}_1, \mathcal{A}_2$ be commuting subalgebras of a C^* -algebra \mathcal{A} acting on a separable Hilbert space. Then $(\mathcal{A}_1, \mathcal{A}_2)$ is C^* -independent in the product sense if and only if there exists a faithful product state on $\mathcal{A}_1 \vee \mathcal{A}_2$.*

These results resolve the asymmetry between the definitions of C^* -, resp. W^* -, independence in the product sense, at least in the indicated important special case. It therefore follows that for a pair of commuting von Neumann algebras acting on a separable Hilbert space, W^* -independence in the product sense implies C^* -independence in the product sense. However, the converse is false — see below.

3 Positive and completely positive maps

Recall that a linear map $T: \mathcal{A} \rightarrow \mathcal{B}$ can be extended to a linear map $T_n: M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ (here $M_n(\mathcal{A})$ is the set of n by n matrices with entries which are elements from the C^* -algebra \mathcal{A}) by

$$T_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} T(a_{11}) & \cdots & T(a_{1n}) \\ \cdot & \cdot & \cdot \\ T(a_{n1}) & \cdots & T(a_{nn}) \end{pmatrix}.$$

Definition 5. T is *completely positive* if T_n is positive for every $n \in \mathbb{N}$. A completely positive map $T: \mathcal{A} \rightarrow \mathcal{A}$ satisfying $T(I) \leq I$ is called an *operation* [10, 21]. An operation T such that $T(I) = I$ is said to be *nonselective*. An operation T on a von Neumann algebra \mathcal{N} is called *normal* if it is σ -weakly continuous. A positive linear map $T: \mathcal{A} \rightarrow \mathcal{B}$ is *faithful* if $T(X) > 0$ whenever $\mathcal{A} \ni X > 0$.

The dual T^* of a nonselective operation defined by

$$T^*: \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A}) \quad T^*\phi \doteq \phi \circ T$$

maps the state space $\mathcal{S}(\mathcal{A})$ of \mathcal{A} into itself. If T is a normal nonselective operation on the von Neumann algebra \mathcal{N} , then T^* takes normal states to normal states.

Operations are the mathematical representatives of physical operations, *i.e.* physical processes which take place as a result of physical interactions with the quantum system. (For a detailed interpretation of operations see [21].) A state on \mathcal{A} is a completely positive unit preserving map from \mathcal{A} to \mathbb{C} [2]. So, if ϕ is a state on \mathcal{A} , then

$$\mathcal{A} \ni X \mapsto T(X) = \phi(X)I \in \mathcal{A} \tag{1}$$

is a nonselective operation in the sense of the above definition, which is canonically associated with the state and which may be interpreted as the preparation of the system into the state ϕ . Further examples of operations are provided by measurements. In particular, if one measures a quantum system with observable algebra $\mathcal{B}(\mathcal{H})$ for the value of a (possibly unbounded) observable Q with purely discrete spectrum $\{\lambda_i\}$ and corresponding spectral projections P_i , then according to the ‘‘projection postulate’’ this measurement can be represented by the operation T defined as

$$\mathcal{B}(\mathcal{H}) \ni X \mapsto T(X) = \sum_i P_i X P_i \in \mathcal{B}(\mathcal{H}). \tag{2}$$

T is a normal nonselective operation.

A classic result characterizing certain completely positive maps was established in [25].

Proposition 5 (Stinespring’s Representation Theorem). $T: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a completely positive linear map from a C^* -algebra \mathcal{A} into $\mathcal{B}(\mathcal{H})$ if and only if it has the form

$$T(X) = V^* \pi(X) V \quad X \in \mathcal{A},$$

where $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ is a representation of \mathcal{A} on the Hilbert space \mathcal{K} and $V: \mathcal{H} \rightarrow \mathcal{K}$ is a bounded linear map. If \mathcal{A} is a von Neumann algebra and T is normal, then π can be chosen to be a normal representation.

So, in particular, C^* -homomorphisms are completely positive. A corollary of Stinespring's theorem was proven by Kraus [21].

Proposition 6 (Kraus' Representation Theorem). $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a normal operation if and only if there exist bounded operators W_i on \mathcal{H} such that

$$T(X) = \sum_i W_i^* X W_i \quad \sum_i W_i^* W_i \leq I.$$

Compare with equation (2).

It is important in Stinespring's theorem that T takes its value in the set of all bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space. This is related to the fact that operations defined on a subalgebra of an arbitrary C^* -algebra are *not*, in general, extendible to an operation on the larger algebra [2]. Indeed, a C^* -algebra \mathcal{B} is said to be *injective* if for any C^* -algebras $\mathcal{A}_1 \subset \mathcal{A}$ every completely positive unit preserving linear map $T_1: \mathcal{A}_1 \rightarrow \mathcal{B}$ has an extension to a completely positive unit preserving linear map $T: \mathcal{A} \rightarrow \mathcal{B}$. It was shown in [2] that $\mathcal{B}(\mathcal{H})$ is injective.

4 Operational independence

In the light of these considerations, the following generalizations of C^* - and W^* -independence are natural.

Definition 6. A pair $(\mathcal{A}_1, \mathcal{A}_2)$ of C^* -subalgebras of C^* -algebra \mathcal{A} is *operationally C^* -independent in \mathcal{A}* if any two nonselective operations on \mathcal{A}_1 and \mathcal{A}_2 , respectively, have a joint extension to a nonselective operation on \mathcal{A} ; *i.e.* if for any two completely positive unit preserving maps

$$T_1: \mathcal{A}_1 \rightarrow \mathcal{A}_1 \quad , \quad T_2: \mathcal{A}_2 \rightarrow \mathcal{A}_2,$$

there exists a completely positive unit preserving map

$$T: \mathcal{A} \rightarrow \mathcal{A}$$

such that

$$\begin{aligned} T(X) &= T_1(X) && \text{for all } X \in \mathcal{A}_1 \\ T(Y) &= T_2(Y) && \text{for all } Y \in \mathcal{A}_2. \end{aligned}$$

Definition 7. A pair $(\mathcal{N}_1, \mathcal{N}_2)$ of von Neumann subalgebras of a von Neumann algebra \mathcal{N} is *operationally W^* -independent in \mathcal{N}* if any two *normal* nonselective operations on \mathcal{N}_1 and \mathcal{N}_2 , respectively, have a joint extension to a *normal* nonselective operation on \mathcal{N} .

Since operations defined on a subalgebra need not be extendible to a larger algebra in general, it is important in Definitions 6 and 7 that operational independence of subalgebras is defined with respect to some fixed larger algebra. Note, however, that, here and below, this joint extension then has further extensions to arbitrary superalgebras, as long as the range of the first extension is interpreted as mapping into an injective algebra, which remains the fixed range of the further extensions.

Operational C^* -independence expresses the notion that any operation (measurement, state preparation *etc*) on system S_1 is co-possible with any such operation on system S_2

(if these systems are represented by C^* -algebras — similarly for W^* -algebras). Given a nonselective operation T , its dual T^* takes states into states; hence, the content of operational C^* -and W^* -independence also can be formulated in terms of changes of states of the systems involved: Operational C^* -independence of $(\mathcal{A}_1, \mathcal{A}_2)$ entails the feature that any *transition* of state ϕ_1 of \mathcal{S}_1 into state ψ_1 is compatible with any *transition* ϕ_2 of \mathcal{S}_2 into state ψ_2 . That is to say, these two transitions can take place as a transition of a single state ϕ of \mathcal{S} into state ψ . Operational W^* -independence has a similar interpretation in terms of transitions between *normal* states on the respective von Neumann algebras.

In analogy with C^* -and W^* -independence in the product sense, the following strengthened versions of operational C^* -and W^* -independence seem natural.

Definition 8. A pair $(\mathcal{A}_1, \mathcal{A}_2)$ of C^* -subalgebras of a C^* -algebra \mathcal{A} is *operationally C^* -independent in \mathcal{A} in the product sense* if any two (faithful) nonselective operations on \mathcal{A}_1 and \mathcal{A}_2 , respectively, have a joint extension to a (faithful) nonselective operation on \mathcal{A} which is a *product* across \mathcal{A}_1 and \mathcal{A}_2 ; *i.e.* if for any two (faithful) completely positive unit preserving maps

$$T_1: \mathcal{A}_1 \rightarrow \mathcal{A}_1 \quad , \quad T_2: \mathcal{A}_2 \rightarrow \mathcal{A}_2 ,$$

there exists a (faithful) completely positive unit preserving map

$$T: \mathcal{A} \rightarrow \mathcal{A}$$

such that

$$T(X) = T_1(X) \quad \text{for all } X \in \mathcal{A}_1 \tag{3}$$

$$T(Y) = T_2(Y) \quad \text{for all } Y \in \mathcal{A}_2 \tag{4}$$

$$T(XY) = T(X)T(Y) \quad X \in \mathcal{A}_1 \quad Y \in \mathcal{A}_2 \tag{5}$$

Definition 9. A pair $(\mathcal{N}_1, \mathcal{N}_2)$ of von Neumann subalgebras of a von Neumann algebra \mathcal{N} is *operationally W^* -independent in \mathcal{N} in the product sense* if any two (faithful) *normal* nonselective operations on \mathcal{N}_1 and \mathcal{N}_2 , respectively, have a joint extension to a (faithful) *normal* nonselective operation T on \mathcal{N} which is a *product* across \mathcal{N}_1 and \mathcal{N}_2 in the sense of eq. (5).

We first remark that in Definition 9 the *prima facie* additional requirement that faithful operations are extended by faithful operations is superfluous in the case of states. In other words, W^* -independence in the product sense *entails* that faithful states can be extended by faithful product states (cf. the proof of Prop. 3). This is not true in the case of states in Definition 8 [19]. The status of this additional requirement is under investigation in the case of general operations [19]. The assumption is added here for reasons which will become apparent below.

States provide special cases of operations, yet C^* -and W^* -independence are *not*, strictly speaking, special cases of operational C^* -and W^* -independence. Indeed, C^* -and W^* -independence require a narrower class of operations on \mathcal{S}_1 and \mathcal{S}_2 to have a joint extension, but the joint extension must belong, in turn, to that narrower class of operations (the states). On the other hand, operational C^* -and W^* -independence require a larger class of partial operations to have a joint extension, but the extension can be in that larger class of operations. Thus C^* -and W^* -independence on one hand, and operational C^* -and W^* -independence on the other, are *prima facie* not related in a straightforward

manner. Let us examine this relationship more closely. Assume that $(\mathcal{A}_1, \mathcal{A}_2)$ is operationally C^* -independent in \mathcal{A} . Let ϕ_1 and ϕ_2 be two states on \mathcal{A}_1 and \mathcal{A}_2 , respectively. As mentioned earlier, the two maps

$$T_1(X) = \phi_1(X)I, X \in \mathcal{A}_1, \quad (6)$$

$$T_2(Y) = \phi_2(Y)I, Y \in \mathcal{A}_2, \quad (7)$$

are completely positive unit preserving maps on \mathcal{A}_1 and \mathcal{A}_2 , respectively, so by assumption, T_1 and T_2 have a joint extension T to \mathcal{A} . This T need not be associated with a state; however, for *any* state ϕ on \mathcal{A} , the state $T^*\phi$ on \mathcal{A} is clearly an extension of both ϕ_1 and ϕ_2 . It is clear that similar reasoning remains valid if the states ϕ_1, ϕ_2 and ϕ are assumed to be normal states on operationally W^* -independent von Neumann subalgebras \mathcal{N}_1 and \mathcal{N}_2 of \mathcal{N} . What is more, if operational independence in the product sense obtains, then one has

$$T^*\phi(XY) = \phi(T(XY)) = \phi(T(X)T(Y)) = \phi(T_1(X)T_2(Y)) = \phi_1(X)\phi_2(Y), \quad (8)$$

for all $X \in \mathcal{A}_1, Y \in \mathcal{A}_2$, and for *any* state $\phi \in \mathcal{S}(\mathcal{A})$. We observe that operational independence in the product sense thereby entails the existence of operations which prepare the quantum system presented in any initial (normal) state into a product state yielding any two prescribed (normal) partial states. This is a remarkable property; therefore it is noteworthy that operational independence in the product sense can be verified in rather general circumstances (see the next section). In light of these remarks, we have a series of propositions; the proofs of the first two are now immediate.

Proposition 7. *Operational C^* -independence of $(\mathcal{A}_1, \mathcal{A}_2)$ in \mathcal{A} entails the C^* -independence of the pair $(\mathcal{A}_1, \mathcal{A}_2)$.*

Proposition 8. *Operational W^* -independence of \mathcal{N}_1 and \mathcal{N}_2 in \mathcal{N} entails the W^* -independence of the pair $(\mathcal{N}_1, \mathcal{N}_2)$.*

Note that in Propositions 7 and 8 the algebras $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{N}_1, \mathcal{N}_2)$ are *not* assumed to be commuting.

Before proceeding to the next results, we need the following proposition. A proof of most, but not all, of the assertions in this proposition using the Stinespring representation theorem can be found in [13, Lemma 2.5]. We present an alternative argument here which also establishes the remaining points.

Proposition 9 ([13]). *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ be unital C^* -algebras and let $T : \mathcal{A}_1 \rightarrow \mathcal{B}_1$ and $S : \mathcal{A}_2 \rightarrow \mathcal{B}_2$ be (faithful) completely positive maps. Then $T \otimes S : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{B}_1 \otimes \mathcal{B}_2$ is a (faithful) completely positive map. If $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ are von Neumann algebras and T and S are normal, then $T \otimes S : \mathcal{A}_1 \overline{\otimes} \mathcal{A}_2 \rightarrow \mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$ is normal.*

Proof. That $T \otimes S$ is completely positive, resp. normal, under the stated conditions is a consequence of [31, Prop. IV.4.23, Prop. IV.5.13]. So let S and T be faithful and $0 \neq A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Let $\hat{I}_{\mathcal{A}_1}$, resp. $\hat{I}_{\mathcal{A}_2}$ etc, denote the identity map on \mathcal{A}_1 , resp. \mathcal{A}_2 etc. These maps are completely positive.

First, consider the case $\mathcal{A}_2 = \mathcal{B}_2$. By [31, Thm. IV.4.9] there exist $\phi_1 \in \mathcal{S}(\mathcal{A}_1)$, $\phi_2 \in \mathcal{S}(\mathcal{A}_2)$, such that $(\phi_1 \otimes \phi_2)(AA^*) \neq 0$. Let T_1, T_2 be the completely positive maps

defined in (6), (7). Since $\hat{I}_{\mathcal{A}_1} \otimes T_2$ is completely positive and AA^* is positive, one must have $(\hat{I}_{\mathcal{A}_1} \otimes T_2)(AA^*) \geq 0$. And since

$$(\phi_1 \otimes \phi_2)(\hat{I}_{\mathcal{A}_1} \otimes T_2)(AA^*) = (\phi_1 \otimes \phi_2)(AA^*) \neq 0,$$

one must also have $(\hat{I}_{\mathcal{A}_1} \otimes T_2)(AA^*) \neq 0$. One therefore concludes $(\hat{I}_{\mathcal{A}_1} \otimes T_2)(AA^*) > 0$. Note that $(\hat{I}_{\mathcal{A}_1} \otimes T_2)(AA^*)$ can be naturally identified with a strictly positive element of \mathcal{A}_1 as follows. Given the state ϕ_2 on \mathcal{A}_2 , one has the left slice map $L : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{A}_1$ which satisfies

$$L\left(\sum_i X_i \otimes Y_i\right) = \sum_i \phi_2(Y_i)X_i.$$

This map is completely positive [3, II.9.7.1], and one has $(\hat{I}_{\mathcal{A}_1} \otimes T_2)(AA^*) = L(AA^*) \otimes I_{\mathcal{A}_2}$, where $I_{\mathcal{A}_2}$ is the unit in \mathcal{A}_2 . Therefore, $L(AA^*) > 0$. But then

$$(\hat{I}_{\mathcal{B}_1} \otimes T_2) \circ (T \otimes \hat{I}_{\mathcal{A}_2})(AA^*) = (T \circ L(AA^*)) \otimes I_{\mathcal{A}_2} > 0,$$

since T is faithful. This entails that $(T \otimes \hat{I}_{\mathcal{A}_2})(AA^*) \neq 0$ and thus $T \otimes \hat{I}_{\mathcal{B}_2}$ is faithful (recall $I_{\mathcal{A}_2} = I_{\mathcal{B}_2}$ here). A similar argument implies that $\hat{I}_{\mathcal{A}_1} \otimes S$ is faithful in the case $\mathcal{A}_1 = \mathcal{B}_1$.

In the general case, one notes that $T \otimes S = (T \otimes \hat{I}_{\mathcal{B}_2}) \circ (\hat{I}_{\mathcal{A}_1} \otimes S)$, and the proposition follows. ♠

An immediate consequence of this observation is given next.

Proposition 10. *Let $\mathcal{A}_1, \mathcal{A}_2$ be mutually commuting C^* -algebras acting on a separable Hilbert space. The pair $(\mathcal{A}_1, \mathcal{A}_2)$ is C^* -independent in the product sense if and only if it is operationally C^* -independent in $\mathcal{A}_1 \vee \mathcal{A}_2$ in the product sense.*

Proof. Let $(\mathcal{A}_1, \mathcal{A}_2)$ be C^* -independent in the product sense, so there exists a C^* -isomorphism $\eta : \mathcal{A}_1 \vee \mathcal{A}_2 \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ such that $\eta(XY) = X \otimes Y$, for all $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$. If T_i is a (faithful) completely positive unit preserving map on \mathcal{A}_i , $i = 1, 2$, then $T_1 \otimes T_2$ is a (faithful) completely positive unit preserving map on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Thus, $(T_1 \otimes T_2) \circ \eta$ is such a map on $\mathcal{A}_1 \vee \mathcal{A}_2$ and satisfies all the conditions required to establish the operational C^* -independence in $\mathcal{A}_1 \vee \mathcal{A}_2$ in the product sense of $(\mathcal{A}_1, \mathcal{A}_2)$.

Conversely, let $(\mathcal{A}_1, \mathcal{A}_2)$ be operationally C^* -independent in $\mathcal{A}_1 \vee \mathcal{A}_2$ in the product sense. There exist faithful states ϕ_1, ϕ_2 on $\mathcal{A}_1, \mathcal{A}_2$, respectively (there exist such states on \mathcal{A}_1'' and \mathcal{A}_2'' by [31, Prop. II.3.19] — just restrict these to \mathcal{A}_1 and \mathcal{A}_2 , respectively), so that T_1, T_2 defined as in (6) and (7) are faithful operations on $\mathcal{A}_1, \mathcal{A}_2$, respectively. By hypothesis, there exists a faithful joint product extension T on $\mathcal{A}_1 \vee \mathcal{A}_2$. Choosing the state ϕ in equation (8) to be faithful on $\mathcal{A}_1 \vee \mathcal{A}_2$, one then has a faithful product state on $\mathcal{A}_1 \vee \mathcal{A}_2$. Prop. 4 completes the proof. ♠

Of course, a similar argument yields the analogous result in the W^* -case.

Proposition 11. *Let $\mathcal{N}_1, \mathcal{N}_2$ be mutually commuting von Neumann algebras acting on a separable Hilbert space. The pair $(\mathcal{N}_1, \mathcal{N}_2)$ is W^* -independent in the product sense if and only if it is operationally W^* -independent in $\mathcal{N}_1 \vee \mathcal{N}_2$ in the product sense.*

In light of Propositions 1, 10 and 11, we can then conclude that operational W^* -independence in the product sense is strictly stronger than operational C^* -independence in the product sense. In fact, choosing \mathcal{N}_1 to be the hyperfinite type III factor² and $\mathcal{N}_2 = \mathcal{N}_1'$, the pair $(\mathcal{N}_1, \mathcal{N}_2)$ is C^* -independent in the product sense, but it is not W^* -independent in the product sense [28, 14]. (This situation actually arises in relativistic quantum field theory — cf. *e.g.* [28].)

Proposition 12. *Let $\mathcal{N}_1, \mathcal{N}_2$ be mutually commuting von Neumann algebras acting on a separable Hilbert space. For the pair $(\mathcal{N}_1, \mathcal{N}_2)$, operational W^* -independence in $\mathcal{N}_1 \vee \mathcal{N}_2$ in the product sense implies operational C^* -independence in $\mathcal{N}_1 \vee \mathcal{N}_2$ in the product sense, but the converse is false.*

5 Operational independence and the split property

In this section we discuss the relation of operational independence with a further well studied independence property and use this relation to demonstrate that operational W^* -independence in the product sense holds quite generally in both nonrelativistic and relativistic quantum theory. The independence property in question is a strengthening of W^* -independence in the product sense.

Definition 10. A pair $(\mathcal{N}_1, \mathcal{N}_2)$ of von Neumann subalgebras acting on a Hilbert space \mathcal{H} is called *W^* -independent in the spatial product sense* if the map

$$XY \rightarrow X \otimes Y \quad X \in \mathcal{N}_1 \quad Y \in \mathcal{N}_2$$

extends to a spatial isomorphism of $\mathcal{N}_1 \vee \mathcal{N}_2$ with $\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$, *i.e.* there exists a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ such that $UXYU^* = X \otimes Y$ for all $X \in \mathcal{N}_1, Y \in \mathcal{N}_2$.

In general, W^* -independence in the spatial product sense is strictly stronger than W^* -independence in the product sense [11]. However, there are many commonly met situations in which they are equivalent [11, Thm. 1, Cor. 1], in particular when either of the von Neumann algebras is a factor or either is of type III. W^* -independence in the spatial product sense is, in turn, known to be equivalent to an important structure property of inclusions of von Neumann algebras, which has been intensively studied for the purposes of both abstract operator algebra theory and algebraic quantum field theory.

Proposition 13 ([5]). *For a mutually commuting pair $(\mathcal{N}_1, \mathcal{N}_2)$ of von Neumann algebras, the following are equivalent.*

1. *There exists a type I factor \mathcal{M} such that $\mathcal{N}_1 \subset \mathcal{M} \subset \mathcal{N}_2'$.*
2. *$(\mathcal{N}_1, \mathcal{N}_2)$ is W^* -independent in the spatial product sense.*

Although according to the usage introduced in [12] we should say that the pair $(\mathcal{N}_1, \mathcal{N}_2')$ is split, it is for our purposes more convenient to say that a pair $(\mathcal{N}_1, \mathcal{N}_2)$ of von Neumann algebras is *split* if condition (1) in the previous proposition holds.

As a consequence of the results discussed above, it is now evident that operational W^* -independence in the product sense obtains in many physically relevant settings. In order

²See [20, 31] for a description of the Murray–von Neumann classification of von Neumann algebras and subsequent refinements. See also [23] for a discussion of the necessity and physically relevant consequences of the various types of von Neumann algebras in quantum theory.

not to lengthen this note unduly, we shall make some brief comments and not formulate specific theorems.³

In nonrelativistic quantum mechanics, the algebras of observables are typically type I factors; therefore in that setting mutually commuting algebras of observables are necessarily split. Hence, such pairs of algebras are operationally W^* -independent in the product sense.

In relativistic quantum theory [1, 16], where the algebra of observables $\mathcal{A}(\mathcal{O})$ carries the interpretation of the algebra generated by all observables measurable in the spacetime region \mathcal{O} , the local algebras $\mathcal{A}(\mathcal{O})$ are typically type III von Neumann algebras [15, 8]. Hence, for spacelike separated spacetime regions $\mathcal{O}_1, \mathcal{O}_2$ (for which $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ mutually commute), the operational W^* -independence in the product sense of $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is *equivalent* to the pair being split. In [7, 32] it has been shown that, in the presence of the additional structures present in algebraic quantum field theory, the split property is equivalent to the local preparability of arbitrary normal states on the local algebras; this latter involves a special case of the operation (6) (cf. also [28, Thm. 3.13] for a formulation which does not require those additional structures). Hence, the equivalences we have established above are not unexpected.

The split property has been verified for all *strictly* spacelike separated⁴ (precompact, convex) regions $\mathcal{O}_1, \mathcal{O}_2$ in a number of physically relevant quantum field models, both interacting and noninteracting [5, 26].⁵ Moreover, the split property for all strictly spacelike separated (precompact, convex) regions $\mathcal{O}_1, \mathcal{O}_2$ has also been shown to be a consequence of a condition (nuclearity) which expresses the requirement that the energy–level density for any states essentially localized in a bounded spacetime region cannot grow too fast with the energy and assures that the given model is thermodynamically well–behaved (*e.g.* thermal equilibrium states exist for all temperatures [6, 9]). Hence, for such regions the pair $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ of observable algebras typically satisfies operational W^* -independence in the product sense. On the other hand, in general, pairs $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ associated with regions which are spacelike separated and tangent are not W^* -independent in the product sense [27, 28] (although they are W^* -independent) and therefore not operationally W^* -independent in the product sense. Moreover, pairs $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ associated with certain unbounded spacelike separated regions (*e.g.* wedges) cannot be split [5] and thus are not operationally W^* -independent in the product sense.

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³However, some of the matters discussed in this section are treated in more detail in [29].

⁴The regions remain spacelike separated even under translation by a sufficiently small neighborhood of the origin.

⁵It is known to fail in some physically pathological models, such as models with noncompact global gauge group and models of free particles such that the number of species of particles grows rapidly with mass [12].

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