SUBGROUP DISTORTION IN METABELIAN AND FREE NILPOTENT GROUPS

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Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

May 2011

Nashville, Tennessee

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To Adam,

my partner through it all.

ACKNOWLEDGMENTS

The individual who deserves my foremost gratitude is my advisor, Alexander Olshanskii. You truly lead by example, and you inspire with your brilliance, patience, and diligence.

Thanks are due as well to the other members of my committee. To Mark Sapir, you are always willing to discuss mathematics, and your course taught me a lot about asymptotic group theory. To Denis Osin, you introduced me to many interesting mathematical questions, and your generosity aided me on several occasions. To Guoliang Yu, I enjoyed learning from your metaphors, which are intuitive and unforgettable. To Senta Victoria Greene, thank you for agreeing to join my committee at the last minute!

I would also like to acknowledge my friends and fellow graduate students whose support and enthusiasm helped me immensely. In particular, to Matt Calef, Xuemei Chen, Justin Fitzpatrick, Michael Hull, Casey Leonetti, Thomas Sinclair, Anneliese Spaeth, and Hang Wang.

This work was partially supported by the National Science Foundation.

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CHAPTER I

INTRODUCTION

I.1 History

The mathematical area that this thesis focuses on is asymptotic group theory. Over the last 5 years, I have been learning about several different functions, all of which are "large-scale" invariants of the groups they are associated to. Although the foundations for asymptotic group theory were laid a long time ago, the subject as it is known today was introduced by Gromov, in 1987, in [Gr]. In his subsequent monograph, [Gr2], the so-called Gromov's program was introduced. The goal of this program is to classify all finitely generated groups with the word metric up to quasi-isometry. This includes the study of properties of a group that do not change under quasi-isometry. Some specific such ideas that will be introduced in this dissertation include the growth function of a finitely generated group. Another direction for research is the idea of using geometric techniques to study quasi-isometry invariants of groups in order to obtain algebraic results. For instance, in the case of free nilpotent groups, we are able to give an algebraic description of the collection of subgroups for which there exists a quasi-isometric embedding. The excellent paper [Dr] contains a description of this kind of background and more.

The building blocks for all of this theory was put into place long before the 1980's, although as mentioned above, that was the time that the field of geometric group theory started to develop as it is known today. In the 1910's, Max Dehn posed the word problem for groups. Dehn's algorithm was created to solve the word problem for fundamental groups of surfaces (in particular closed orientable surfaces of genus at least two). Small cancellation theory was introduced by Grindlinger in the 1960's, and further developed by Lyndon and Schupp. It was later seen that Dehn's algorithm solves the word problem in certain small cancellation groups. The study of van Kampen diagrams has been invaluable in modern combinatorial group theory, which studies presentations of groups. These kinds of ideas, combined with the fundamental notion of the Cayley graph and word metric of a group, are all the foundations of the type of mathematics I have been working on. For instance, the Dehn function, named after Max Dehn, is of interest to many people. One particularly remarkable result in this area is that of Birget, Rips and Sapir in [SBR] describing which functions can be Dehn functions of finitely presented groups.

Although not included in this dissertation, I have considered the study of Dehn functions in [D]. There it was proved that every finitely generated nilpotent group of nilpotency class 2 is isomorphically embeddable into a group with a quadratic Dehn function. Also, that the central product of $\frac{n(n-1)}{2}$ copies of an n-generated, 2-nilpotent group has quadratic isoperimetric function. This generalizes the work done in [OS2].

There are many connections indeed between topology, geometry and group theory. However, the field of geometric group theory encompasses even more connections that that. There are also important bridges with the study of algorithmic problems. For instance, another function which will be studied in detail in this thesis is the distortion function associated to a particular embedding of one finitely generated group into another. The distortion function has deep connections with the (algorithmic) membership problem. It was observed in [Gr2] and proved in [F] that for a finitely generated subgroup H of a finitely generated group G

with solvable word problem, the membership problem is solvable in H if and only if the distortion function of H in G, $\Delta_H^G(l)$, is bounded by a recursive function.

I.2 Motivation

This thesis will introduce several important ideas from geometric group theory. The main types of groups that we will study are nilpotent and solvable. These kinds of groups are generalizations of abelian groups. In particular, we will study free nilpotent and free metabelian groups, as well as have a discussion of free solvable groups. We will also study wreath products of finitely generated abelian groups. The kinds of asymptotic (large scale geometric) notions which we will associate to these groups are the relative growth function and the distortion function. We will also consider semigroups.

Let us turn our attention to distortion. The main motivation for studying the distortion function is twofold. First of all, it is interesting because of its connections with the algorithmic membership problem, just as the study of Dehn functions is motivated by the word problem. Also, it is interesting to study from the point of view of geometric group theory because it provides yet another way to view (extrinsic) geometry of groups, and because it is an asymptotic invariant.

There has been a wide range of work done on distortion in finitely generated groups. For instance, the complete description of length functions on subgroups of finitely generated and finitely presented groups can be found in [O] and [OS]. This answered a question posed by Gromov. Other interesting finitely generated groups with fractional distortion are constructed in [Br]. In [Ge], an example of a subgroup whose distortion is stronger than any multi-exponential function is constructed. In [U], it is shown that there are certain (free solvable) groups for which the membership problem is not solvable, leading to the existence of embeddings with no recursive upper bound on distortion. In [Os2], the formula for distortion in finitely generated nilpotent groups and nilpotent Lie groups is obtained.

Note that wreath products A wr B where A is abelian play a very important role in group theory for many reasons. Given any product $G = CD$ with abelian normal subgroup C, then any two homomorphisms from $A \to C$ and $B \to D$ (uniquely) extend to a homomorphism from A wr B to G. Also, if B is presented as a factor-group F/N of a k-generated free group F, then the maximal extension $F/[N,N]$ of B with abelian kernel is canonically embedded in $\mathbb{Z}^k$ wr B (see [M].) Wreath products of abelian groups give an inexhaustible source of examples and counter-examples in group theory.

For instance, the group $\mathbb{Z}$ wr $\mathbb{Z}$ is the simplest example of a finitely generated (though not finitely presented) group containing a free abelian group of infinite rank. In [GS] the group $\mathbb{Z}$ wr $\mathbb{Z}$ is studied in connection with diagram groups and in particular with Thompson's group. In the same paper, it is shown that for $H_d = (\cdots(\mathbb{Z} \text{ wr } \mathbb{Z}) \text{ wr } \mathbb{Z})\cdots \text{ wr } \mathbb{Z})$, where the group $\mathbb{Z}$ appears d times, there is a subgroup $K \leq H_d \times H_d$ having distortion function $\Delta_K^{H_d \times H_d}(l) \succeq l^d$. In contrast to the study of these iterated wreath products, here we obtain polynomial distortion of arbitrary degree in the group $\mathbb{Z}$ wr $\mathbb{Z}$ itself. In [C] the distortion of $\mathbb{Z}$ wr $\mathbb{Z}$ in Baumslag's metabelian group (cf. [B]) is shown to be at least exponential, and an undistorted embedding of $\mathbb{Z}$ wr $\mathbb{Z}$ in Thompson's group is constructed.

We also consider the relative growth function associated to a subgroup of a finitely generated group. The notion of relative growth was first introduced by Grigorchuk in [G2]. The relative growth of subgroups in solvable and linear groups was studied by Osin in [Os]. He provided a description of relative growth functions of cyclic subgroups in solvable groups, up to a rougher equivalence relation then the one we will use in this dissertation. He also provided a negative example to the following question: Is it true that the relative growth functions of subgroups in solvable groups (linear groups) are either polynomial or exponential? However, the growth of any finitely generated subgroup of a free solvable groups is either exponential or polynomial. In this dissertation, it is the connections between the relative growth of cyclic subgroups, and the corresponding distortion function of the embedding that is studied.

I.3 Results

One of the results of this dissertation is an understanding of some connections between relative growth of cyclic subgroups in finitely generated groups and distortion, and is summarized in the following.

Theorem I.3.1.

1. *There exists a cyclic subgroup G of a two generated group H such that $\Delta_G^H(r)$ is not bounded above by any recursive function, yet $f_{rel}(r)$ is $o(r^2)$.*
2. *For any cyclic subgroup G of a finitely generated group H such that $\Delta_G^H(r)$ is not bounded above by any recursive function, we have that $f_{rel}(n)$ cannot be bounded from above by any function of the form $\frac{r^2}{g(r)}$ where the effective limit of $g(r)$ is infinity.*

A large part of this dissertation is focused on computing distortion in free nilpotent groups and wreath products of cyclic groups. A classification of subgroup distortion in finitely generated free nilpotent groups is given by the following theorem.

Theorem I.3.2. *Let F be a free m-generated, c-nilpotent group. A subgroup H in F is undistorted if and only if H is a retract of a subgroup of finite index in F.*

When the undistorted subgroup H is normal in F we may further refine our classification.

Corollary I.3.3. *Let H be a nontrivial normal subgroup of the free m-generated, c-nilpotent group F, and assume that $c \geq 2$. Then H is undistorted if and only if $[F : H] < \infty$.*

This led to the question of distortion in free metabelian and free solvable groups. As will be explained in more detail later, the question of distortion in free solvable groups is not good for study, because there exist subgroups with distortion greater than any recursive funciton. In the course of studying distortion in metabelian groups, the effects of subgroup distortion in the wreath products A wr $\mathbb{Z}$, where A is finitely generated abelian were considered. This is due to the fact, which will be elaborated on later, that distortion in free metabelian groups is similar to that in wreath products of free abelian groups. I was able to prove the following.

Theorem I.3.4. *Let A be a finitely generated abelian group.*

1. *For any finitely generated subgroup $H \leq A$ wr $\mathbb{Z}$ there exists $m \in \mathbb{N}$ such that the distortion of H in A wr $\mathbb{Z}$ is*
$$\Delta_H^{A\,wr\,\mathbb{Z}}(l) \approx l^m.$$

2. *If A is finite, then $m = 1$; that is, all subgroups are undistorted.*

3. *If A is infinite, then for every $m \in \mathbb{N}$, there is a 2-generated subnormal subgroup H of A wr $\mathbb{Z}$ having distortion function*
$$\Delta_H^{A\ wr\ \mathbb{Z}}(l) \approx l^m.$$

In the future, I would like to expand on these ideas to a larger class of wreath products, with the ultimate goal of obtaining a further understanding of distortion in solvable groups. This goal and others will be discussed in more detail in the section on Future Work at the end of this manuscript.

In terms of semigroup theory, my work was motivated by the work done for groups by Olshanskii in [O], [O2] and by Olshanskii and Sapir in [OS]. I described for a given semigroup S, which functions $l : S \to \mathbb{N}$ can be realized up to equivalence as length functions $g \mapsto |g|_H$ by embedding S into a finitely generated semigroup H. I also provided a complete description of length functions of a given finitely generated semigroup with enumerable set of relations inside a finitely presented semigroup.

CHAPTER II

PRELIMINARIES

We begin by providing a wealth of background information, including all necessary definitions and notation, as well as more motivation on the subject. As mentioned in the introduction, to understand infinite groups in general, it can be useful to study their geometry, up to quasi-isometry. It is this understanding of the geometric properties of a group that can help us ultimately understand its structure. We begin with some background in basic group theory.

II.1 Basic Group Theory

First we introduce the most basic notion of a nilpotent group. We will use the notation that for elements x, y of a group, $[x,y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. We also use the notation that for a group G, the derived subgroup is $G' = [G,G] = \mathrm{gp}\langle\{[g,h] : g,h \in G\}\rangle$ and the center is $Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$.

Definition II.1.1. Let G be a group. Then G is called nilpotent of nilpotency class c, or c-nilpotent, if the descending central series

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots \supseteq G_{c+1} = \{1\}$$

terminates and satisfies $G_c \neq 1$, where by definition $G = G_1$, and $G_i = [G_{i-1}, G]$, for all $i \geq 2$. Note that G_i is also sometimes denoted by $\gamma_i(G)$, and $G_2 = G'$. We will use all this notation interchangeably.

Definition II.1.2. Let G be a group. Then G is called solvable of solvability class c, or c-solvable, if the derived series

$$G = G^{(1)} \supseteq G^{(2)} \supseteq G^{(3)} \supseteq \cdots \supseteq G^{(c+1)} = \{1\}$$

terminates and satisfies $G^{(c)} \neq 1$, where by definition $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$, for all $i \geq 2$.

Lemma II.1.3. *Let x, y, z be elements of any group. Then $[xy,z] = [x,z]^y[y,z]$ and $[x,yz] = [x,z][x,y]^z$.*

Proof. The computations are simple:

$$[xy,z] = y^{-1}x^{-1}z^{-1}xyz = y^{-1}x^{-1}z^{-1}xzyy^{-1}z^{-1}yz = [x,z]^y[y,z]$$

and

$$[x,yz] = x^{-1}z^{-1}y^{-1}xyz = x^{-1}z^{-1}xzz^{-1}x^{-1}y^{-1}xyz = [x,z][x,y]^z.$$

□

Some commutator identities hold in nilpotent groups. In [H] special cases of the following facts are discussed. We provide detailed proofs.

Lemma II.1.4. *If G is c-nilpotent, then the following identities hold:*

1.

$$[xy,z]=[x,z][y,z] \textit{ and } [x,yz]=[x,z][y,z] \textit{ if } z\in G_{c-1} \tag{II.1}$$

2.

$$[x_1,\dots,yz,\dots,x_c]=[x_1,\dots,y,\dots,x_c][x_1,\dots,z,\dots,x_c] \tag{II.2}$$

3.

$$[x_1^{n_1},\dots,x_c^{n_c}]=[x_1,\dots,x_c]^{n_1\cdots n_c} \textit{ for } n_1,\dots,n_c\in\mathbb{N}. \tag{II.3}$$

Proof.

1. Because G is c-nilpotent, $G_c\subset Z(G)$. Therefore, by Lemma II.1.3, if $z\in G_{c-1}$ then $[xy,z]=[x,z]^y[y,z]=[x,z][y,z]$.

2. We proceed by induction. By Equation (II.1), we have that the identity holds in case $c=2$. Observe that $[x_1,\dots,yz,\dots,x_c]=[[x_1,\dots,yz,\dots],x_c]$. By induction hypothesis, this equals $[[x_1,\dots,y,\dots],x_c][[x_1,\dots,z,\dots],x_c]$. By Lemma II.1.3, this expression equals $[x_1,\dots,y,\dots,x_c]^y[x_1,\dots,z,\dots,x_c]$. Because $\gamma_c(G)\subseteq Z(G)$, we see that

$$[x_1,\dots,y,\dots,x_c]^y[x_1,\dots,z,\dots,x_c]=[x_1,\dots,y,\dots,x_c][x_1,\dots,z,\dots,x_c].$$

3. By equation (II.1) we have that $[x_1^{n_1},\dots,x_c^{n_c}]=[x_1,x_2^{n_2}\dots,x_c^{n_c}][x_1^{n_1-1},\dots,x_c^{n_c}]$ which, by induction on n_1, equals $[x_1,x_2^{n_2}\dots,x_c^{n_c}]^{n_1}$. By induction on c,

$$[x_2^{n_2}\dots,x_c^{n_c}]=[x_2,\dots,x_c]^{n_2\cdots n_c}.$$

Therefore,

$$[x_1^{n_1},\dots,x_c^{n_c}]=[x_1,x_2^{n_2}\dots,x_c^{n_c}]^{n_1}=[x_1,[x_2,\dots,x_c]^{n_2\cdots n_c}]^{n_1}.$$

By equation (II.1) we have that

$$[x_1,[x_2,\dots,x_c]^{n_2\cdots n_c}]^{n_1}=([x_1,x_2,\dots,x_c][x_1,[x_2,\dots,x_c]^{n_2\cdots n_c-1}])^{n_1}.$$

By induction on $n_2\cdots n_c$, the previous expression equals $[x_1,\dots,x_c]^{n_1\cdots n_c}$. □

Next we will look at free objects in the class of nilpotent groups, and their properties.

Definition II.1.5. A free n-generated class c nilpotent group $G_{n,c}$ is a c-nilpotent group with generators $y_1,\dots y_n$ defined by the following universal property: given an arbitrary d-nilpotent group H for $d\le c$ and elements $h_1,\dots,h_n\in H$ there is a unique homomorphism $\phi:G_{n,c}\to H: y_i\mapsto h_i$ for all $i=1,\dots,n$. We will occasionally use the alternative notation $F_{n,c}$ to denote the free n-generated, c-nilpotent group.

This generalizes the notion of absolutely free group.

Definition II.1.6. An n-generated (absolutely) free group is a group F with generators $x_1,\dots,x_n$ satisfying the following universal property: given an arbitrary group H and elements $h_1,\dots,h_n\in H$ there is a unique homomorphism $\phi:F\to H: x_i\mapsto h_i$ for all $i=1,\dots,n$.

A free group is so named for two reasons. First, one is “free” to choose a homomorphism using the universal property. Also, a free group is one in which no relations hold between the generators, so it is “free” of relations. This is in contrast to a free nilpotent group, which is not free of relations. However, the only relations present are those arising from the fact that the group is nilpotent. Observe that a free abelian group simply means a free 1-nilpotent group. Free objects in other varieties (e.g. that of solvable groups) are defined analogously to Definition II.1.6.

Definition II.1.7. A linearly independent subset of a free abelian group which generates the group is called a basis. The number of elements in a basis is called the rank of the free abelian group. All free abelian groups are isomorphic to a direct sum of $\mathbb{Z}$. The number of copies equals the rank of the free abelian group.

Let us look at some examples.

Example II.1.8.

1. We have that $G_{2,2} \cong UT_3(\mathbb{Z})$, where $UT_3(\mathbb{Z})$ is the group of all 3×3 unitriangle matrics with entries in the integer ring. Moreover, the group $UT_3(\mathbb{Z})$ is also isomorphic to the 3-dimensional Heisenberg group $\mathscr{H}^3 = \langle a,b,c | [a,b] = c, [a,c] = [b,c] = 1\rangle$. It is easy to compute that this group has center $\langle c\rangle_\infty$ coinciding with its commutator subgroup. It can be realized as a linear group under the identification

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Consider the $2n+1$-dimensional Heisenberg group

$$\mathscr{H}^{2n+1} = \langle p_1,\dots p_n, q_1,\dots q_n, z | [p_i,q_j] = z^{\delta_{ij}}, [p_i,z] = [q_j,z] = 1\rangle.$$

For each $n \geq 1$ this is a 2-nilpotent group.

Proposition II.1.9. *Let G_n be an n-generated free 2-nilpotent group generated by $\{y_1,\dots y_n\}$, and F_n an n-generated absolutely free group. Then the following hold:*

1. $G_n \cong F_n/[[F_n,F_n],F_n]$.

2. The derived subgroup $[G_n,G_n]$ is equal to the center of G_n. In particular, the derived subgroup is abelian.

3. $[G_n,G_n]$ is free abelian of rank $\frac{n(n-1)}{2}$ with basis $\{[y_i,y_j]\}_{i<j}$.

4. The abelianization $G_n/[G_n,G_n]$ is also free abelian, of rank n.

Proof.

1. Let $H = F_n/[[F_n,F_n],F_n]$. Then clearly H is 2-nilpotent. Let F_n be generated by $x_1,\dots,x_n$. Then by definition of G_n, there is a homomorphism $\phi : G_n \to H$ taking each y_i to $x_i[[F_n,F_n],F_n]$. We must prove that ϕ is a bijection. By Von Dyck’s theorem, because G_n is an n-generated group satisfying the defining relations of H, there is a surjective homomorphism $\psi : H \to G_n : \psi(x_i[[F_n,F_n],F_n]) = y_i\ \forall i$. This is clearly the inverse map to ϕ, hence both maps are isomorphisms.

2. The fact that $G_n' \subseteq Z(G_n)$ follows directly from the definition of 2-nilpotent. When $n = 2$ the reverse inclusion holds because, as remarked in Example II.1.8, a presentation of G_2 can be given by $\langle a,b,c | [a,b] = c, [a,c] = [b,c] = 1\rangle$, and the center of this group is explicitly computed to be $\langle c\rangle$, which is equal to the commutator subgroup. Now let $n > 2$ and suppose by way of contradiction that there exists an element $w \in Z(G_n) - G_n'$. Write $w = \prod_{j=1}^{k} y_{i_j}^{\varepsilon_j}$, $k \in \mathbb{N}, \varepsilon_j \in \mathbb{Z}$. Then because w is not an element of G_n', the exponent sum for some y_{i_j} is nonzero. Without loss of generality that the exponent sum of y_1 is nonzero. By freeness, there is a homomorphism from G_n to G_2 taking y_1 to y_1, y_2 to y_2 and y_i to 1 for all $i > 2$. Under this homomorphism, w goes to a word w' in $Z(G_2)$, such that the exponent sum of y_1 in w' is nonzero, which is a contradiction.

3. The fact that the given set generates G_n' follows from the definition of derived subgroup, together with Lemma II.1.3 and the fact that for any $x, y \in G_n$ we have $[x,y]^{-1} = [y,x]$. Thus is suffices to show the generating set is linearly independent. Suppose by way of contradiction that there is a linear combination $[x_1,x_2]^{a_{1,2}}[x_1,x_3]^{a_{1,3}} \cdots [x_{n-1},x_n]^{a_{n-1,n}}$ equal to 1 with some integer coefficient $a_{i,j} \neq 0$. Consider the homomorphism from G_n onto G_2 taking x_i to x_i, x_j to x_j and x_k to 1 if $k \neq i, j$. Then in G_2 we have that $1 = [x_i,x_j]^{a_{i,j}}$, which is a contradiction.

4. A free abelian group of rank n can be given by the presentation $A_n = \langle a_1,\ldots,a_n | [a_i,a_j]_{i<j}\rangle$. Therefore, by Von Dyck's Theorem, because $G_n/[G_n,G_n]$ is a group which also satisfies these relations, there is an epimorphism $\varepsilon : A_n \to G_n/[G_n,G_n] : a_i \mapsto y_i[G_n,G_n]$. Because G_n is free 2-nilpotent, and A_n is abelian, there is also a homomorphism $\psi : G_n \mapsto A_n : y_i \mapsto a_i$. We have that $G_n/\ker(\psi)$ is abelian, hence by definition of derived subgroup, $[G_n,G_n] \subseteq \ker(\psi)$. Therefore the function $G_n/[G_n,G_n] \to A_n : g[G_n,G_n] \mapsto \psi(g)$ is well-defined, and an inverse to ε. Therefore ε is an isomorphism. □

Definition II.1.10. Let C be a class of groups (e.g. finite, cyclic, free). Then a group G is said to be virtually in the class C if G has a subgroup of finite index in C.

The following definition is important because it is satisfied by all finitely generated nilpotent groups (cf. [B2]).

Definition II.1.11. A group is said to be polycyclic if there exists a finite subnormal series $1 = G_0 \leq G_1 \leq \cdots \leq G_l = G$ such that the factor G_{i+1}/G_i is cyclic, for all $i = 0,\ldots,l-1$. Such a series is called a polycyclic series.

The following elementary Lemma will also be useful later. It is useful when proving that subgroups of nilpotent groups are of finite index.

Lemma II.1.12. *If G is a finitely generated nilpotent group and $H \leq G$, then if some positive power of each element of a set of generators of G lies in H, then $[G : H] < \infty$ and a positive power of every element of G lies in H.*

A proof of this fact can be found in [B2], Lemma 2.8.

We will use the following definitions later in our study of undistorted subgroups in free nilpotent groups.

Definition II.1.13. Let G be any group, and H a subgroup of G. A map $r : G \to H$ is a retract if r is a group homomorphism and $r \restriction_H = \mathrm{id}_H$. In this case, we also say that H is a retract of G.

Definition II.1.14. A group G is a semidirect product of H and N, written $G = H \lambda N$ if $N \trianglelefteq G, H \leq G, H \cap N = \{1\}$, and $HN = G$.

Lemma II.1.15. *A subgroup H of a group G is a retract if and only if G is a semidirect product of H and some normal subgroup N of G.*

Proof. First, suppose that H is a retract of G under a homomorphism $\phi : G \to H$ such that $\phi \restriction_H = \mathrm{id}_H$. Let $N = \ker(\phi)$. Then $N \trianglelefteq G$. We will show that $G = HN$. Let $g \in G$. Then observe that $g = \phi(g)\phi(g^{-1})g$ where $\phi(g) \in H$, and as we will show, $\phi(g^{-1})g \in N$. Indeed, $\phi(\phi(g^{-1})g) = \phi(\phi g^{-1})\phi(g) = \phi(g^{-1})\phi(g) = 1$ because $\phi(g^{-1}) \in H$ so $\phi(\phi g^{-1}) = \phi(g^{-1})$. Now suppose that $G = H\lambda N$. First consider the map $\psi : H \hookrightarrow G \twoheadrightarrow G/N : h \mapsto h \mapsto hN$. Observe that ψ is an isomorphism of H with G/N. Clearly ψ is a homomorphism, by its defintion. Also, $h \in \ker\psi$ if and only if $h \in N \cap H$ which occurs if and only if $h = 1$, and so ψ is monomorphism. To see ψ is surjective, take any $gN \in G/N$, which we may write as$g = hn$ where $h \in H, n \in N$, so $gN = hN = \psi(h)$. Next, let $\phi = \psi^{-1} : G/N \to H$. Then define $\alpha : G \to H : \alpha(g) = \phi(gN)$. We claim that α is a retract. It is a homomorphism because it is a composition of two homomorphisms. If $h \in H$ then $\alpha(h) = \phi(hN) = h$, so α yields the identity map when restricted to H, as required. □

II.2 Asymptotic Group Theory

A main tool for studying large scale geometry of groups and metric spaces is the notion of quasi-isometry.

Definition II.2.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A quasi-isometry between X and Y is a map $q : X \to Y$ such that:

1. There exists constants $\lambda > 0, L \geq 0$ such that

$$\frac{1}{\lambda} d_X(x,y) - L \leq d_Y(q(x), q(y)) \leq \lambda d_X(x,y) + L$$

for all $x, y \in X$.

2. There exists $\infty > D \geq 0$ such that for any $y \in Y$ we have $d_Y(y, q(X)) \leq D$.

Example II.2.2. It is well known that $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric under the natural inclusion map (each with the usual metric).

The notion of quasi-isometry is more flexible than the notion of isometry, and aims to capture information about the large-scale geometry of a space. As in Example II.2.2, the quasi-isometry confirms our intuition that $\mathbb{R}$ and $\mathbb{Z}$ look the same "from an infinite distance". The objects we wish to study will be invariant under quasi-isometry.

Lemma II.2.3. *The Cayley graphs of a group G with respect to two different generating sets are quasi-isometric.*

Proof. Let $S_1 = \{g_1, \dots g_n\}$ and $S_2 = \{h_1, \dots h_m\}$ be two symmetric generating sets of G, and consider the respective Cayley graphs (G, d_1) and (G, d_2). Let $\gamma_1, \gamma_2 \in G$. Then let $k = d_1(\gamma_1, \gamma_2) = \min\{K | \gamma_1^{-1}\gamma_2 = g_{i_1} \dots g_{i_K}, g_{i_j} \in S_1, \forall i_j\}$. For each $i = 1, \dots, n$ write $g_i = \prod_{j=1}^{s_i} h_{t_{i,j}}$ where each $h_{t_{i,j}} \in S_2$. Then

$$\gamma_1^{-1}\gamma_2 = g_{i_1} \dots g_{i_K} = \prod_{q=1}^{K} (\prod_{j=1}^{s_{i_q}} h_{t_{i,j}}).$$

Therefore, $d_2(\gamma_1, \gamma_2) \leq c_1 k$ where $c_1 = \max\{s_1, \dots s_m\}$. Therefore, $d_2(\gamma_1, \gamma_2) \leq c_1 d_1(\gamma_1, \gamma_2)$. And by symmetry, we obtain the reverse inequality. This shows that the identity map is a quasi-isometry. $\square$

II.2.1 Subgroup Distortion

Another notion which will be investigated in this thesis is that of distortion of a subgroup.

Definition II.2.4. Let M be a subgroup of a group G, where M and G are generated by the finite sets S and T, respectively. Then the distortion function of M in G is defined as

$$\Delta_M^G : \mathbb{N} \to \mathbb{N} : n \mapsto \max\{|w|_S : w \in M, |w|_T \leq n\},$$

where $|w|_S$ denotes the word length of w in M with respect to the finite generating set S, and $|w|_T$ is defined similarly.

We will only study subgroup distortion up to a natural equivalence relation. First, we define an ordering on the set of all monotone functions from $\mathbb{N} \to \mathbb{N}$.

Definition II.2.5. We say that $f \preceq g$ if there exists $C > 0$ such that $f(l) \leq Cg(Cl)$ for all $l \geq 0$. We say two functions are equivalent, written $f \approx g$, if $f \preceq g$ and $g \preceq f$.

This equivalence relation preserves the asymptotic behaviour of the function. In particular, it would identify all quadratic functions, but distinguish a quadratic function from a cubic, or a polynomial of any other degree. The distortion function does not depend on the choice of S and T, if studied up to equivalence. This follows from the proof of Lemma II.2.3. Moreover, if $[G : H] < \infty$ then H is undistorted in G. This follows by [Al] because $[G : H] < \infty$ implies that H embeds quasi-isometrically into G.

If M is infinite, then the distortion function is at least linear, so one may use the expression $Cg(Cl + C) + Cl + C$ in Definition II.2.5 of equivalence rather than $Cg(Cl)$, without changing the equivalence class of the distortion function. We will use this in some of our later estimates.

Remark II.2.6. Suppose there exists a subsequence of $\mathbb{N}$ given by $\{l_i\}_{i \in \mathbb{N}}$ where $l_i < l_{i+1}$ for $i \geq 1$. If there exists $c > 0$ such that $\frac{l_{i+1}}{l_i} \leq c$, for all $i \geq 1$, and $f(l_i) \geq g(l_i)$, then $f \succeq g$.

Definition II.2.7. The subgroup M of G is said to be undistorted if $\Delta_M^G(n) \approx n$.

If a subgroup M is not undistorted, then it is said to be distorted, and its distortion refers to the equivalence class of $\Delta_M^G(n)$. The distortion function measures the difference in the metrics induced by generators of G and M. Intuitively, a subgroup M of a group G is highly distorted if one must travel a long distance in the Cayley graph of M whereas traveling between the same points in G takes a relatively short distance.

It is also true that any retraction of a group is undistorted. To see this, one takes a generating set of the subgroup H to be the images under the retract of a finite set of generators for the big group G. Let $\phi : G \to H$ be a retraction. Let $\{g_1, \ldots, g_n\}$ be a finite generating set for G. Select the finite generating set $\{\phi(g_1), \ldots, \phi(g_n)\}$ for H. Then if $h \in H$, we have $h = g_{i_1} \cdots g_{i_m}$ so $h = \phi(h) = \phi(g_{i_1}) \cdots \phi(g_{i_m})$ so $|h|_H \leq |h|_G$, which implies that H is undistorted in G.

Example II.2.8.

1. Consider the three-dimensional Heisenberg group

$$\mathscr{H}^3 = \langle a, b, c | c = [a, b], [a, c] = [b, c] = 1 \rangle.$$

Consider the cyclic subgroup $M = \langle a \rangle_\infty$. This subgroup is undistorted, because it is a retract.

2. Consider $\mathscr{H}^3$ again, and this time consider the cyclic subgroup $N = \langle c \rangle_\infty$. This time, N is distorted and in fact it has at least quadratic distortion. To see why this is true, notice that the word c^{n^2} has quadratic length in N, but that in $\mathscr{H}^3$, we have

$$c^{n^2} = [a, b]^{n^2} = [a^n, b^n]$$

which has at most linear length. Observe that $M \cong N \cong \mathbb{Z}$, so distortion depends heavily on the embedding; i.e. it is not well-defined to ask whether the integers are distorted in $\mathscr{H}^3$. However, whenever we speak on the distortion of a specific subgroup, we will always understand what the underlying embedding is.

3. Consider the Baumslag-Solitar Group $\mathrm{BS}(1,2) = \langle a, b | bab^{-1} = a^2 \rangle$. This group may be concretely recognized as a matrix group under

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

It has cyclic subgroup $\langle a \rangle$ with at least exponential distortion. Indeed,

$$a^{2^n} = (a^2)^{2^{n-1}} = (bab^{-1})^{2^{n-1}} = [(bab^{-1})^2]^{2^{n-2}} = [ba^2b^{-1}]^{2^{n-2}} =$$

$$= [b^2ab^{-2}]^{2^{n-2}} = \cdots = b^n ab^{-n}.$$

The following result of Osin will be very useful to us later on.

Proposition II.2.9. *Let G be a finitely generated nilpotent group, and M a subgroup of G. Let M^0 be the collection of elements of M having infinite order. For $m \in M^0$, let the weight of m in G be defined by*

$$v_G(m) = \max\{k | \langle m \rangle \cap G_k \neq \{1\}\}$$

and similarly for $v_M(m)$. Then

$$\Delta_M^G(n) \approx n^r$$

where
$$r = \max_{m \in M^0} \frac{v_G(m)}{v_M(m)}.$$

A proof of this fact can be found in [Os2], Theorem 2.2.

Corollary II.2.10. *If G is nilpotent of class c and M is cyclic, then*

$$\Delta_M^G(n) \approx n^d$$

where $d \in \mathbb{N}$ and $d \le c$.

Because we are only interested in the equivalence class of distortion functions, we will sometimes use the "big-O" notation to describe asymptotic behaviour. We record here its precise definition.

Definition II.2.11. Let f, g be defined on some subset of the real numbers. We say $f(x) = O(g(x))$ if there exist $x_0 \in \mathbb{R}, M > 0$, such that $|f(x)| \le M|g(x)|$ for all $x > x_0$.

We recall a couple of other similar notions.

Definition II.2.12. We say that a function $f(r)$ is $o(g(r))$ if $\lim_{r\to\infty} \frac{f(r)}{g(r)} = 0$.

Definition II.2.13. The effective limit of a function $g(r)$ is infinity if there is an algorithm that given an integer C computes $N = N(C)$ such that $g(r) \ge C$ for every $r > N$.

II.2.2 Relative Growth

Finally, we discuss some background information on relative subgroup growth.

Definition II.2.14. Let G be a finitely generated group finitely generated by T with any subgroup H. The relative growth function of H in G is

$$f_{rel} : \mathbb{N} \to \mathbb{N} : r \to \#\{g \in H : |g|_T \le r\} = \#(B_G(r) \cap H).$$

It is natural to use a slightly different equivalence relation to study relative growth functions. We also use a different notation for elements of $\mathbb{N}$, normally writing "r" rather than "n" or "l". For two nondecreasing functions $f, g : \mathbb{N} \to \mathbb{N}$, say that f does not exceed g up to equivalence if there exists a constant c so that for all $r \in \mathbb{N}$ we have $f(r) \le g(cr)$, and that f is equivalent to g if both f does not exceed g and g does not exceed f, up to equivalence. If we define the relative growth function to be the equivalence class of f_{rel} above, then it becomes independent of the choice of finite generating set. This is because if S and T and two finite generating sets for G, then for the constant $c = \max\{|s|_T : s \in S\}$ we have that $\{g \in G : |g|_S \le r\} \subseteq \{g \in G : |g|_T \le cr\}$. If one considers infinite subgroups only, then one may use the equivalence relation defined earlier in Definition II.2.5 for distortion instead and obtain the same equivalence class of relative subgroup growth. Because we will compare relative growth and distortion functions to one another, and because the latter equivalence relation is more natural for studying subgroup distortion, we will choose to utilize it.

We provide some examples, found in [Os], which can be compared with Example II.2.8.

Example II.2.15. 1. Let G be the 3-dimensional Heisenberg group $G = \mathscr{H}^3 = \langle a,b,c | c = [a,b], [c,a] = [c,b] = 1\rangle$ and $H = \mathrm{gp}\langle c\rangle$. Then $g_G(r) \approx r^4, g_H(r) \approx r$, and $f_{rel}(r) \approx r^2$.

2. Let G be the Baumslag-Solitar Group $BS(1,2) = \langle a,b | bab^{-1} = a^2\rangle$. Let H be the cyclic subgroup $\langle a\rangle_\infty$. Then $f_{rel}(r) \approx 2^r$.

Observe that in Example II.2.15 Parts (1) and (2) we have that the relative growth of the cyclic subgroups under consideration and the distortion function are the same.

The relative growth function can be studied in contrast with the usual growth functions of H (if H is also fintiely generated) and G defined respectively as: $g_H(r) = \#B_H(r)$ and $g_G(r) = \#B_G(r)$. It is clear that $f_{rel}(r) \preceq g_G(r)$ and that when $H = G$ that $f_{rel}(r) \approx g_G(r)$. Also, if $K \leq H \leq G$ then there are two relative growth functions, $f_1(r)$, the relative growth of K in G, and $f_2(r)$, the relative growth of H in G. In this case we have that $f_1(r) \preceq f_2(r)$.

Observe further that if H is a finitely generated subgroup of a finitely generated group G, then we have that

$$f_{rel}(r) \succeq g_H(r). \tag{II.4}$$

This follows because $B_H(r) \subseteq B_G(cr) \cap H$ where c is a constant depending only on the choice of finite generating sets.

Some remarkable results have been obtained regarding the usual growth function. One says that a finitely generated group G with finite generating set S has polynomial growth if $g_G(r) \preceq r^d$ for some $d \in \mathbb{N}$. A group G is said to have exponential growth if for some $d \in \mathbb{N}$, $g_G(r) \succeq d^r$. Because of the fact that $g_G(r) \leq (2\#S+1)^r$, we see that the regular and relative growth functions are always at most exponential. It was proved by Wolf in [W] that that if G is a finitely generated nilpotent group, then G has polynomial growth. The degree of polynomial growth in nilpotent groups is computed by Bass in [B] and is given by the following explicit formula

$$d = \sum_{k\geq 1} k \cdot \mathrm{rk}(G_k/G_{k+1}), \tag{II.5}$$

where rk represents the rank of an abelian group, and G_k the terms of the descending central series for G. With respect to Gromov's program which was described earlier, solvable and nilpotent groups arise as well. The famous theorem of Gromov says that a finitely generated group G has polynomial growth only if it is virtually nilpotent (see [Gr3]). In particular, a group which is quasi-isometric to a nilpotent group is itself virtually nilpotent. However, solvable groups do not enjoy this kind of rigidity: it is proved in [Dy] that a group which is quasi-isometric to a solvable group may itself not be virtually solvable. Moreover, there are examples of groups of intermediate growth; that is, groups whose growth function is neither polynomial nor exponential (see [G]).

CHAPTER III

RELATIVE GROWTH AND DISTORTION

The main result of this section is the following, which sheds light on the connections between the relative growth of cyclic subgroups, and the corresponding distortion function of the embedding. It will be proved in Section III.3.

Theorem. I.3.1

1. *There exists a cyclic subgroup G of a two generated group H such that $\Delta_G^H(r)$ is not bounded above by any recursive function, yet $f_{rel}(r)$ is $o(r^2)$.*
2. *For any cyclic subgroup G of a finitely generated group H such that $\Delta_G^H(r)$ is not bounded above by any recursive function, we have that $f_{rel}(n)$ cannot be bounded from above by any function of the form $\frac{r^2}{g(r)}$ where the effective limit of $g(r)$ is infinity.*

The result is interesting in light of the fact that, such as in Examples II.2.8 and II.2.15, the distortion and relative growth can both be equal.

III.1 Connections with Distortion

Here we would like to understand some of the connections between the relative growth of a finitely generated subgroup in a finitely generated group, and the distortion function $\Delta_H^G(r)$.

Lemma III.1.1. *Suppose that K is a cyclic subgroup of a finitely generated group G. If the distortion of K in G is not linear, then the relative growth function of K is also not linear.*

Proof. Let the cyclic subgroup K be generated by an element a. We assume that distortion is not linear, and will show that the relative growth is also not linear. By hypothesis on distortion, we have that for any d there exists a l so that $\Delta_K^G(l) > dl$. Letting $\Delta_K^G(l) = \max\{|m| : |a^m|_G \leq l\} = |m_0|$ for some $m_0 \in \mathbb{Z}$ we have that $|a^{m_0}|_G \leq l \leq \frac{|m_0|}{d}$. Rephrasing, we may say that for every $\varepsilon > 0$, we can find a^m so that $|a^m|_G \leq \varepsilon m$. Let us fix $m = m(\varepsilon)$. Consider any a^l. Write $l = km + r$ where $0 \leq r < m$. Let $c = c(m) = \max\{|a^r|_G : 0 \leq r < m\}$. Then $a^l = (a^m)^k a^r$ and so $|a^l|_G \leq k|a^m|_G + c \leq l\varepsilon + c$. Because l was arbitrary, it follows that the relative growth function of K is at least $\varepsilon^{-1}l + C$ for $C = C(\varepsilon)$. Because ε was arbitrarily small, the relative growth function is not bounded from above by any linear function. $\square$

Lemma III.1.2. *If H is a finitely generated subgroup of a finitely generated group G, then $f_{rel}(r) \approx r$ implies that $\Delta_H^G(r)$ must also be linear. That is to say, if the embedding is distorted, then the relative growth is non-linear.*

Proof. It follows from the assumption that $f_{rel}(r)$ is linear that $g_H(r)$ is also linear, because $f_{rel}(r) \succeq g_H(r)$, by Equation (II.4). Therefore, by Gromov's Theorem, we have that H is virtually nilpotent. By Bass's formula, $1 = \mathrm{rk}(H/H')$, which implies that H' is finite and H is virtually cyclic: there exists an infinite cyclic subgroup K of H with finite index. Suppose by way of contradiction that the embedding of H to G is distorted. Then because $[H:K] < \infty$, the embedding of K to G is also distorted. This implies that $\frac{\Delta_K^G(r)}{r}$ is unbounded. Therefore, because K is cyclic, it follows from Lemma III.1.1 the relative growth of K in G also has $\frac{f_{rel}(r)}{r}$ unbounded. This is a contradiction to the hypothesis that the relative growth of H in G is linear. $\square$

Lemma III.1.3. *Let G be a finitely generated group, and let H be the infinite cyclic subgroup generated by an element $a \in G$. Then*

$$\Delta_H^G(r) \succeq f_{rel}(r).$$

Proof. We have that

$$\Delta_H^G(r) = \max\{|a^k|_H : |a^k|_G \le r\} = \max\{|k| : |a^k|_G \le r\} = |k_0|$$

for some $k_0 \in \mathbb{Z}$. Then if $a^m \in H$ has $|a^m|_G \le r$ we have that $|m| \le |k_0|$, by definition of distortion. That is, if we consider the set

$$S = \{1, a, a^{-1}, a^2, a^{-2}, \ldots, a^{k_0}, a^{-k_0}\}$$

we have that $\{a^m \in H : |a^m|_G \le r\} \subseteq S$. Therefore,

$$f_{rel}(r) \le \#S = 2|k_0| + 1 \approx \Delta_H^G(r)$$

as required. $\square$

Note that such a relationship between relative growth and distortion cannot hold in a larger class of subgroups than infinite cyclic. For even if $H = G$, then $\Delta_H^G(n) \approx n$, while $f_{rel}(n) = g_H(n)$ is strictly greater than linear if H is not virtually cyclic.

Combining Lemmas III.1.1 and III.1.3, we have proved the following.

Proposition III.1.4. *A cyclic subgroup of a finitely generated group is undistorted if and only if it has linear relative growth.*

Again we remark on the relationship between distortion and relative growth. If one has a subgroup H of a group G which is distorted, then there is at least one element in $B_G(n) \cap H$ having large length in H. This does not always imply that there are at least $|g|_H$ other elements in $B_G(n) \cap H$. One reason for this fact is that distortion can be superexponential, whereas relative growth is always at most exponential, as explained above.

III.2 Embeddings and Relative Growth

We recall the following result of Olshanskii (see [O]).

Theorem III.2.1. *Let $l : \mathbb{Z} \to \mathbb{N}$ be a function satisfying:*

- *(C1) $l(n) = l(-n), n \in \mathbb{Z}; l(n) = 0$ if and only if $n = 0$*

- *(C2)* $l(n+m) \leq l(n) + l(m)$
- *(C3) There exists* $a > 0$ *such that* $\#\{i \in \mathbb{Z} : l(i) \leq r\} \leq a^r$ *for any* $r \in \mathbb{N}$.

Then there exists a two-generated group H *and an element* $g \in H$ *such that*

$$|g^n|_H \approx l(n).$$

We refer to conditions $(C1), (C2)$, and $(C3)$ as the (C) condition.

Remark III.2.2. We may translate the geometric group theoretic functions into different terms as follows. Suppose that $l : \mathbb{Z} \to \mathbb{N}$ satisfies the (C) condition, so that we have an embedding $\langle g \rangle \to H$ as in Theorem III.2.1. Then the relative growth of the cyclic group in H is given by

$$f_{rel}(r) = \#\{n \in \mathbb{Z} : l(n) \leq r\},$$

and the distortion is

$$\Delta^H_{\langle g \rangle}(r) = \max\{n \in \mathbb{N} : l(n) \leq r\}.$$

III.3 Proof of Theorem I.3.1

III.3.1 Constructing a Cyclic Subgroup with Prescribed Data

We begin by introducing some lemmas that will be used in the proof of Part (1) of Theorem I.3.1.

Lemma III.3.1. *There exist increasing sequences* $\{a_i\}_{i \in \mathbb{N}}, \{n_i\}_{i \in \mathbb{N}}$ *of natural numbers satisfying the following properties for all* $i \geq 2$.

- $a_1 = n_1 = 1$
- $a_i \geq 2^{i+3} n_{i-1}$
- $n_i > n_{i-1} a_i / a_{i-1}$
- $n_{i-1} \mid n_i$
- *There does not exist a recursive function* f *such that* $n_i \leq f(a_i)$ *for every* i.

Proof. We will use induction to define a choice of sequences that satisfies all required conditions. We use that the set of recursive functions is countable. Denote it by $\{f_i\}_{i \in \mathbb{N}}$. Suppose that a_{i-1} and n_{i-1} have been defined. Let $a_i = 2^{i+3} n_{i-1} + 1$. Let

$$n_i = \max\{n_{i-1}\left(\lceil \frac{n_{i-1} a_i}{a_{i-1}} \rceil + 1\right), n_{i-1}\left(\max_{j \leq i}(f_j(a_i))\right)\}.$$

$\square$

We will construct an embedding of a cyclic subgroup with the required properties, by exploiting Theorem III.2.1 and using the sequence defined in Lemma III.3.1.

Define a function $l : \mathbb{Z} \to \mathbb{N}$ by the formula $l(0) = 0$ and for nonzero $n \in \mathbb{Z}$

$$l(n) = \min\{a_{i_1} + \cdots + a_{i_s} | n = \pm n_{i_1} \pm \cdots \pm n_{i_s} \text{ for some } i_1, \ldots, i_s \in \mathbb{N}\} \tag{III.1}$$

Lemma III.3.2. *The function l defined in Equation* (III.1) *satisfies the* (C) *condition of Theorem III.2.1.*

Proof. Observe that for each $n \in \mathbb{Z}, n = n \cdot 1 = n \cdot n_1$, so the function is defined. To see that the condition $(C1)$ is satisfied, select $n \in \mathbb{Z}$. Without loss of generality, $n \neq 0$. Let $l(n) = a_{i_1} + \cdots + a_{i_s}$, so that there is an expression $n = \pm n_{i_1} \pm \cdots \pm n_{i_s}$. This implies that $-n = -(\pm n_{i_1} \pm \cdots \pm n_{i_s})$ and so by definition of l, we have that $l(-n) \leq l(n)$. Equality holds by symmetry. The $(C2)$ condition is similarly easy: let $l(n) = a_{i_1} + \cdots + a_{i_s}$ and $l(m) = a_{j_1} + \cdots + a_{j_t}$. Then one expression representing $n + m$ is $\pm n_{i_1} \pm \cdots \pm n_{i_s} \pm n_{j_1} \pm \cdots \pm n_{j_t}$ which implies that $l(n+m) \leq l(n) + l(m)$.

□

Therefore, Theorem III.2.1 implies that there is an embedding of a cyclic subgroup $\langle g \rangle$ into a two-generated group H with $|g^n|_H \approx l(n)$.

We would like to obtain some lemmas which will provide useful estimates for computing the relative growth function associated to this embedding. Let r be a natural number. We want to be able to compute $\#\{n : l(n) \leq r\}$. Suppose that n is in this set, let

$$l(n) = a_{i_1} + \cdots + a_{i_s}$$

and consider the corresponding minimal presentation given by

$$n = \pm n_{i_1} \pm \cdots \pm n_{i_s}.$$

Lemma III.3.3. *This expression has no summands with subscript greater than or equal to j, where j is defined by the property: $a_{j-1} \leq r < a_j$.*

This is true, since otherwise $l(n) \geq a_j > r$. Therefore, we may rewrite the expression as

$$n = k_1 n_1 + \cdots + k_{j-1} n_{j-1}.$$

We may assume that $j \geq 3$ in the following, since eventually we will let r become very large, and with it, so will j.

Remark III.3.4. Observe that $|k_{j-1}| \leq \frac{r}{a_{j-1}}$, for otherwise, $l(n) \geq |k_{j-1}| a_{j-1} > r$. For the same reason, $|k_{j-2}|$ also does not exceed $\frac{r}{a_{j-2}}$.

Lemma III.3.5. *For any $2 \leq i < j-1$ we have that $|k_i| < \frac{n_{i+1}}{n_i}$.*

Proof. We will show that $|k_{j-2}| \leq \frac{n_{j-1}}{n_{j-2}}$. For if by way of contradiction, there is $s \geq 0$ such that $|k_{j-2}| = \frac{n_{j-1}}{n_{j-2}} + s$, then

$$n = k_1 n_1 + \cdots \pm |k_{j-2}| n_{j-2} + k_{j-1} n_{j-1} =$$

$$k_1 n_1 + \cdots \pm \left(\frac{n_{j-1}}{n_{j-2}}\right) n_{j-2} \pm s n_{j-2} + k_{j-1} n_{j-1} =$$

$$k_1 n_1 + \cdots \pm s n_{j-2} + (k_{j-1} \pm 1) n_{j-1}.$$

Then we have that $s a_{j-2} + |k_{j-1} \pm 1| a_{j-1} \leq s a_{j-2} + |k_{j-1}| a_{j-1} + a_{j-1}$. We will show, contrary to minimality, that $a_{j-1} + s a_{j-2} < |k_{j-2}| a_{j-2}$. The right hand side equals $\frac{n_{j-1}}{n_{j-2}} a_{j-2} + s a_{j-2}$. We are done because $a_{j-1} < \frac{n_{j-1} a_{j-2}}{n_{j-2}}$. Similarly, the above arguments works for any i. □

We proceed with the proof of Theorem I.3.1 Part (1).

Proof. We will show that the embedding $\langle g \rangle \hookrightarrow H$ obtained by applying Theorem III.2.1 to the function l of Equation (III.1) satisfies the required properties.

By the choice of the sequences $\{a_i\}, \{n_i\}$ the embedding has distortion function which is not bounded by a recursive function. This follows because $\Delta^H_{\langle g \rangle}(a_i) \geq n_i$ by definition, and by construction, there is no recursive function satisfying this property.

Now we will show that the relative growth function $f(r) = \#\{n : l(n) \leq r\}$ is $o(r^2)$. Taking into account the signs, we have by Remark III.3.4 and Lemma III.3.5 that the number of values of n with $l(n) \leq r$ is at most the product over the number of values of k_j, namely:

$$\frac{2(r+1)}{a_{j-1}} \frac{2(r+1)}{a_{j-2}} \frac{2n_{j-2}}{n_{j-3}} \frac{2n_{j-3}}{n_{j-4}} \cdots \frac{2n_2}{n_1} < \frac{r^2 2^{j+2} n_{j-2}}{a_{j-1} a_{j-2}} < \frac{r^2}{a_{j-2}},$$

by the choice of a_i in Lemma III.3.1. Now we have that $\lim_{r\to\infty} a_{j-2} = \infty$ by the choice of $j = j(r)$ as in Lemma III.3.3. Therefore, $f(r)$ is $o(r^2)$. □

III.3.2 Producing Bounds on Relative Growth

We now introduce some notation and lemmas towards proving Theorem I.3.1 Part (2).

Let $G = \langle g \rangle \leq H$ where H is finitely generated. Consider the length function corresponding to the embedding given by $l : \mathbb{N} \to \mathbb{N} : l(r) = |g^r|_H$.

Lemma III.3.6. *Suppose that the distortion function $\Delta^H_G(r)$ is not bounded from above by any recursive function. Then for any $g(r)$ with effective limit infinity, we cannot have $20 l(n) \geq g(n)$ for all $n > N$.*

Proof. Suppose by way of contradiction that $20 l(n) \geq g(n)$ for all $n > N$ and some N. Then the effective limit of $l(n)$ is also infinity, and so given any C, one can effectively compute $N(C)$, such that $l(n) > C$ for any $n \geq N(C)$. But this means that the distortion function $\Delta^H_G(r)$ is bounded from above by the recursive function $N(r)$ of r, a contradiction. □

Remark III.3.7. By Lemma III.3.6, there exists an infinite sequence $n_1 = 1 < n_2 < n_3 < \ldots$ such that $20 l(n_i) < g(n_i)$. In addition we may assume by choosing a subsequence that for all $i > 1$ we have

- $n_i > i^2(n_1 + \cdots + n_{i-1})$.
- $n_{j+1} > n_j^2$

Denote $l(n_i)$ by a_i. Let us consider the numbers n of the form $k_1n_1 + ... + k_in_i$ where

$$0 \leq k_j < \frac{n_{j+1}((j+1)^2 - 1)}{n_j(j+1)^2}, \text{ for } j \in \{1, \dots, i\}. \tag{III.2}$$

Lemma III.3.8. *Different coefficients with this condition define diferent sums.*

Proof. This is true because otherwise for some $j \leq i$ and $m > 0$, we will have $mn_j = m_{j-1}n_{j-1} + \cdots + m_1n_1$ where for each $s \leq j$, $|m_s| < (1 - \frac{1}{(s+1)^2})\frac{n_{s+1}}{n_s}$ by choice of coefficients in Equation (III.2). Then we have that

$$mn_j < n_j\left(1 - \frac{1}{j^2}\right) + n_{j-2}\left(1 - \frac{1}{(j-1)^2}\right) + \cdots + n_2\left(1 - \frac{1}{(2)^2}\right) <$$

$$n_j\left(1 - \frac{1}{j^2}\right) + \frac{n_j}{j^2} = n_j$$

by the choice of n_j, a contradiction. □

Lemma III.3.9. *If we assume in addition that,*

$$k_{i-1} < \frac{n_i}{3n_{i-1}}, k_i \leq \frac{r}{3a_i} \text{ and } r = n_i,$$

then we have that $l(n) \leq r$.

Proof. By the properties of l, together with the additional assumptions stated in Lemma III.3.9, we have that

$$l(k_1n_1 + \cdots + k_in_i) \leq \frac{a_1n_2}{3n_1} + \cdots + \frac{a_ir}{3a_i} + \frac{a_{i-1}n_i}{3n_{i-1}}.$$

We have that $\frac{a_ir}{3a_i} = \frac{r}{3}$ and that $\frac{a_{i-1}n_i}{3n_{i-1}} = \frac{a_{i-1}r}{3n_{i-1}} \leq \frac{r}{3}$ because $a_j \leq n_j$. Finally, observe that

$$\frac{a_{j-1}n_j}{n_{j-1}} < \frac{a_jn_{j+1}}{6n_j}$$

for each j by the choice of n_{j+1}. Therefore,

$$\frac{a_1n_2}{n_1} + \cdots + \frac{a_{i-1}n_i}{3n_{i-1}} < (1 + \frac{1}{6} + \frac{1}{6^2} + \cdots + \frac{1}{6^{i-2}})\frac{a_{i-1}n_i}{n_{i-1}} \leq \frac{2r}{3}.$$

Therefore, $l(n) \leq r$. □

We now proceed with the proof of Theorem I.3.1 Part (2).

Proof. Let r be fixed. Lemmas III.3.8 and III.3.9 together with the choice of $k_1, \dots, k_i$ imply that the number of n's with such $l(n) \leq r$ is at least

$$\left(1-\frac{1}{4}\right)\left(\frac{n_2}{n_1}\right)\left(1-\frac{1}{9}\right)\left(\frac{n_3}{n_2}\right)\cdots\left(1-\frac{1}{i-1^2}\right)\left(\frac{n_{i-1}}{n_{i-2}}\right)\left(\frac{1}{3}\frac{n_i}{n_{i-1}}\right)(\frac{r}{3a_i}) >$$

$$\prod_{j=2}^{i-1}\left(1-\frac{1}{j^2}\right)\frac{r^2}{9a_i} > \frac{r^2}{20a_i}.$$

This follows because $r = n_i$ and the product $\prod_{j=2}^{\infty}(1-\frac{1}{j^2})$ converges to $\frac{1}{2}$. Hence the value of the corresponding relative growth function of $\mathbb{Z}$ at $r = n_i$ is at least

$$\frac{r^2}{20a_i} = \frac{r^2}{20l(n_i)} > \frac{r^2}{g(n_i)}$$

by the choice of n_i. Thus this function $f_{rel}(r)$ is not bounded from above by any $\frac{r^2}{g(r)}$ where g is a function with effective limit infinity, because the g we started with was arbitrary. $\square$

CHAPTER IV

DISTORTION IN FREE NILPOTENT GROUPS

IV.1 Introduction

IV.1.1 Background and Preliminaries

The primary notion which will be investigated in this portion of the dissertation is that of distortion of a subgroup, which has been defined previously.

Observe that if $M \leq H \leq G$ and both M is undistorted in H as well as H is undistorted in G, then M must also be undistorted in G; this follows from the definition of distortion. Suppose that we fix finite generating sets of M, H, and G. Let $\Delta_M^H(n) = an+b$ and $\Delta_H^G(n) = cn+d$ for some $a,b,c,d > 0$. Let $w \in M$ realize $|w|_M = \Delta_M^G(n) = \max\{|w|_M : w \in M, |w|_G \leq n\}$. Then because $w \in M \subseteq H$, and $|w|_G \leq n$, we have that $|w|_H \leq \Delta_H^G(n) = an+b$. Moreover, $|w|_M \leq \Delta_M^H(an+b)$ by definition. By hypothesis, this equals $c(an+b)+d$. Therefore, $\Delta_M^G(n) = |w|_M \leq (ca)n + (cb+d) \approx n$.

In this section, we will be studying free nilpotent groups. Note that free nilpotent groups are torsion-free. See, for example, [B2].

Remark IV.1.1. *We remind the reader that we use the notation that the commutator $[x_1,x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and inductively define higher commutators by $[x_1,\dots,x_i] = [x_1,[x_2,\dots,x_i]]$, for $i \geq 3$. The descending central series of a group G is defined inductively as: $\gamma_1(G) = G$ and $\gamma_i(G) = [G,\gamma_{i-1}(G)]$. With this notation we have that the free nilpotent group $G_{n,c}$ has presentation given by $R/\gamma_{c+1}(R)$ where R is the absolutely free group of rank m.*

IV.1.2 Statement of Main Results

The main result of this note is the following. It will be proved in Section IV.3.

Theorem. I.3.2 *Let F be a free m-generated, c-nilpotent group. A subgroup H in F is undistorted if and only if H is a retract of a subgroup of finite index in F.*

When the undistorted subgroup H is normal in F we may further refine our classification.

Corollary. I.3.3 *Let H be a nontrivial normal subgroup of the free m-generated, c-nilpotent group F, and assume that $c \geq 2$. Then H is undistorted if and only if $[F:H] < \infty$.*

IV.2 Facts on Nilpotent Groups

We record several well known facts about nilpotent and free nilpotent groups which will be used in the proof of Theorem I.3.2. For instance, nilpotent groups possess special commutator identities, as discussed in Lemma II.1.4 of the Preliminaries Section.

Lemma IV.2.1. *If G is any finitely generated nilpotent group, and $H \leq G$ then $[G : HG'] < \infty$ implies $[G : H] < \infty$.*

In [H], a special case of this Lemma is proved. The more general result of Lemma IV.2.1 follows by a simple argument.

Proof. We will proceed by induction on c. If $c = 1$ the claim is obvious. Now suppose the claim is true for any group of nilpotency class $d < c$ and let G be $c$nilpotent with $[G : HG'] < \infty$. By induction, $[G : HG_c] < \infty$. It suffices to show that $[HG_c : H] < \infty$. By Lemma II.1.12, it suffices to show that a positive power of any generator of HG_c lies in H. The group HG_c is generated by elements of H as well as some c-long commutators of generators of G. Consider such a generator $[f_1, \ldots, f_c]$ where each f_j for $1 \leq j \leq c$ is a generator of G. Because $[G : HG'] < \infty$ we know that for each j there exists $k_j > 0$ with $f_j^{k_j} \in HG'$. Let $k = \prod_j k_j$. Then for each j, $f_j^k = u_j v_j \in HG'$, where $u_j \in H, v_j \in G'$. By Lemma II.1.4

$$[f_1, \ldots, f_c]^k = [u_1 v_1, \ldots, u_c v_c] = [u_1, \ldots, u_c] \in H.$$

□

The following result of Magnus will help us in proving Theorem I.3.2.

Proposition IV.2.2. *Let R be an absolutely free group. For $1 \neq x \in R$, let the weight of x, $w(x) = m$, be the first natural number such that $x \in \gamma_m(R)$ but $x \notin \gamma_{m+1}(R)$. Then for nontrivial elements x_1 and x_2 having respective weights λ_1 and λ_2, we have that the weight of $x = [x_1, x_2]$ equals $\lambda_1 + \lambda_2$ if $\lambda_1 \neq \lambda_2$. Moreover, $w(x) > \lambda_1 + \lambda_2$ if and only if the subgroup generated by x_1 and x_2 is also generated by some $\overline{x_1}, \overline{x_2}$ with weights λ_1 and $\lambda_1 + \mu$, respectively, where $\mu > 0$ and in this case, the weight of x is $2\lambda_1 + \mu$.*

A proof of Proposition IV.2.2 can be found in [M2].

Lemma IV.2.3. *If $c > 1$ and F is free c-nilpotent, then the centralizer of an element $x_1 \notin F'$ is of the form $\gamma_c(F) \times \langle a \rangle$, where $a \notin F'$.*

Proof. Let R be an absolutely free group with the same number of generators as F. As mentioned in Remark IV.1.1, we have that $F = R/\gamma_{c+1}(R)$. An element x_2 is contained in the centralizer of x_1 in F, $C_F(x_1)$, if and only if $x = [x_1, x_2] = 1$ in F if and only if $x \in \gamma_{c+1}(R)$. That is, if considered as words in the absolutely free group R, $w(x) \geq c + 1$. If $w(x_2) = 1$ then by Proposition IV.2.2, and with notation as in Proposition IV.2.2, $w(x) \geq c + 1$ which is equivalent to saying that $2 + \mu \geq c + 1$; i.e. $1 + \mu \geq c$. This means that $\mathrm{gp}\langle x_1, x_2 \rangle = \mathrm{gp}\langle \overline{x_1}, \overline{x_2} \rangle$ where $w(\overline{x_1}) = 1$ and $w(\overline{x_2}) = 1 + \mu \geq c$, which occurs if $\overline{x_2} \in \gamma_c(R)$. Observe that if $w(x_2) \neq 1$ then by Proposition IV.2.2, $w(x) = w(x_2) + 1 \geq c + 1$ hence $w(x_2) \geq c$ which implies that $x_2 \in \gamma_c(R)$. Therefore, we have that $x_2 \in \mathrm{gp}\langle \overline{x_1} \rangle \times \gamma_c(R)$, with the understanding that in case $w(x_2) \neq 1$ we take $\overline{x_1} = x_1$ and $\overline{x_2} = x_2$.

Hence, the image $x_2 \gamma_{c+1}(R)$ in F belongs to $\langle \overline{x_1} \gamma_{c+1}(R) \rangle \times (\gamma_c(R)/\gamma_{c+1}(R))$. The product is direct: the intersection is trivial because $c > 1$ implies that $\langle \overline{x_1} \rangle \cap \gamma_c(R) \subseteq \langle \overline{x_1} \rangle \cap \gamma_2(R) = \{1\}$ because $w(x_1) = 1$. Let

$\langle y\gamma_{c+1}(R)\rangle$ be the unique maximal cyclic subgroup of the free nilpotent group F containing $x_1\gamma_{c+1}(R)$. This subgroup is the isolator of the cyclic subgroup. We will show that

$$\langle y\gamma_{c+1}(R)\rangle \times (\gamma_c(R)/\gamma_{c+1}(R))$$

is the centralizer of x_1. One inclusion has already been shown. It suffices to observe that $y \in C_F(x_1)$. This follows because there exists $n \in \mathbb{Z}$ with $y^n\gamma_{c+1}(R) = x_1\gamma_{c+1}(R)$. □

Proposition IV.2.4. *Let F be a free m-generated, c-nilpotent group with free generators $a_1,\dots a_m$, for $c \geq 1$. Suppose $b_1,\dots b_k \in F$ are such that $\{b_1F',\dots b_kF'\}$ is a linearly independent set in the free abelian group F/F' then $K := gp\langle b_1,\dots b_k\rangle$ is free c-nilpotent.*

For a proof of Proposition IV.2.4 refer to [N].

IV.3 Undistorted Subgroups in Free Nilpotent Groups

From this point on, all notation is fixed. Let F be a free m-generated, c-nilpotent group with free generators $a_1,\dots,a_m$, for $c \geq 1$. Suppose that H is any nontrivial subgroup of F. Consider the group HF'/F'. Being a subgroup of the free abelian group F/F', it is free abelian itself. Denote the free generators of HF'/F' by $b_1F',\dots b_kF'$, where each $b_i \in H$, so $k = \text{rank}(HF'/F')$. Without loss of generality, $k > 0$, for if $k = 0$ then $H \subset F'$ so by Proposition II.2.9, H is a distorted subgroup in F. We can assume further that $b_1,\dots b_k, a_{k+1},\dots a_m$ are independent modulo F'.

Let $D = \text{gp}\langle a_1,\dots a_k\rangle$. Consider the map $r : F \to D$:

$$r(a_i) = \begin{cases} a_i & \text{if } i \leq k, \\ 1 & \text{if } i > k. \end{cases}$$

Then r is a retraction of F. This is clear: r is a homomorphism because F is free, and r restricted to D is the identity map. Let $N = \ker(r)$.

Lemma IV.3.1. *We have $[F : HN] < \infty$.*

Proof. The elements $b_1,\dots,b_k,a_{k+1},\dots,a_m$ generate a subgroup S of finite index in F. This follows because the elements $b_1,\dots,b_k,a_{k+1},\dots,a_m$ are linearly independent modulo F', so

$$SF'/F' = \text{gp}\langle b_1F',\dots,b_kF',a_{k+1}F',\dots,a_mF',F'\rangle$$

is free abelian of rank m and is a subgroup of F/F'. Therefore we have that $[F/F' : SF'/F'] < \infty$, which implies that $[F : SF'] < \infty$. Hence by Lemma IV.2.1, we have that $[F : S] < \infty$. Because N is generated by $a_{k+1},\dots,a_m$ and H contains $b_1,\dots,b_k$, then HN contains S, so $[F : HN] < \infty$.

□

The following Lemmas are working towards proving that for H undistorted, $H \cap N = \{1\}$, which would essentially complete the proof of Theorem I.3.2.

Lemma IV.3.2. *If $H \cap N \neq \{1\}$ then $N \cap H \cap \gamma_c(F) \neq \{1\}$.*

Proof. Observe that the Lemma is true in case $c = 1$, so in the proof we assume that $c \geq 2$. Because H is nilpotent group, and $H \cap N$ is nontrivial normal subgroup, we must have $Z(H) \cap H \cap N \neq \{1\}$. Observe that

$$Z(H) = (\cap_{h \in H} C_F(h)) \cap H \leq C_F(b_1) \cap H$$

which by Lemma IV.2.3 has the form $(\gamma_c(F) \times \langle a \rangle) \cap H$ where $a \notin F'$. Now observe that $H \cap N \leq F'$. This follows because the image of H in F/F' is generated by $\{b_1, \ldots, b_k\}$ and the image of N in F/F' is generated by $\{a_{k+1}, \ldots, a_m\}$, so because the set $\{b_1, \ldots, b_k, a_{k+1}, \ldots, a_m\}$ is independent, the intersection $HF'/F' \cap NF'/F' = 1$, so $(H \cap N)F'/F' = 1$ which implies that $H \cap N \subset F'$. Thus we have

$$Z(H) \cap N \leq (\gamma_c(F) \times \langle a \rangle) \cap F' \leq \gamma_c(F).$$

Therefore, there is a nontrivial element in $Z(H) \cap N \cap \gamma_c(F)$ as required. $\square$

Lemma IV.3.3. *If $N \cap H \cap \gamma_c(F) \neq \{1\}$ then H is distorted.*

Proof. Let $1 \neq u \in N \cap H \cap \gamma_c(F)$. We will show that that $\langle u \rangle \cap \gamma_c(H) = \{1\}$. For if $u^r \in \gamma_c(H)$ for some $0 \neq r \in \mathbb{Z}$, then u^r is a product of c-long commutators of the from $[y_1, \ldots, y_c]^{\pm 1}$ where y_i is either one of $b_1, \ldots, b_k$ or an element of F' since H is generated by $b_1, \ldots, b_k$ and $F' \cap H$. But if one of the y_i's belongs to F', then the commutator is trivial because it is a $c+1$-long commutator in F. It follows that $u^r \in \mathrm{gp}\langle b_1, \ldots, b_k \rangle \cap N$.

By Lemma IV.3.1, the subgroup $S = \mathrm{gp}\langle b_1, \ldots, b_k, a_{k+1} \ldots, a_m \rangle$ has finite index in F. This implies by Lemma II.1.12 that

$$[r(F) : r(S)] = [D : \mathrm{gp}\langle r(b_1), \ldots, r(b_k) \rangle] < \infty.$$

Therefore, we also have that

$$[D/D' : \mathrm{gp}\langle r(b_1), \ldots, r(b_k) \rangle D'/D'] < \infty$$

and so $\{r(b_1)D', \ldots, r(b_k)D'\}$ is linearly independent in the free abelian group of rank k, D/D'. By Proposition IV.2.4 we have that both

$$\mathrm{gp}\langle r(b_1), \ldots, r(b_k) \rangle \text{ and } \mathrm{gp}\langle b_1, \ldots, b_k \rangle$$

are free k-generated, c-nilpotent groups. This implies that the intersection $\mathrm{gp}\langle b_1, \ldots, b_k \rangle \cap N$ is trivial, because $N = \ker(r)$.

Hence $\langle u \rangle \cap \gamma_c(H) = \{1\}$ and $1 \neq u \in \gamma_c(F)$. It follows by Propsotion II.2.9 that the distortion of the cyclic subgroup $\langle u \rangle$ in F is greater than its distortion in H. Thus H cannot be undistorted in F. $\square$

Corollary IV.3.4. *If $H \cap N \neq \{1\}$, then H is distorted.*

Proof. This follows directly from Lemmas IV.3.2 and IV.3.3. $\square$

Now we proceed with the proof of Theorem I.3.2.

Proof. As mentioned in Section IV.1, every retract of a subgroup having finite index in any group G is undistorted. Conversely, if H is undistorted in F then by Corollary IV.3.4 we have that $H \cap N = \{1\}$. Then by Lemma IV.3.1, H is a retract of the subgroup HN of finite index in F, as required. $\square$

We also proceed with the proof of Corollary I.3.3.

Proof. We use the notation already established in this Section. Observe that if $k = m$ then we have by definition of k that $[F : H] < \infty$. If by way of contradiction we suppose that $k < m$, then by Corollary IV.3.4, H undistorted implies that $H \cap N = \{1\}$. It follows by the normality of H and N and the fact that $b_1 \in H$ and $a_m \in N$ that $[b_1, a_m] = 1$. On the other hand, by Proposition IV.2.4, we have that $\mathrm{gp}\langle b_1, a_m \rangle$ is free nilpotent of class at least 2, a contradiction. $\square$

IV.4 Examples and Discussion

Example IV.4.1. In the formulation of Theorem I.3.2, one may not replace "retract of a subgroup of finite index" by "finite index subgroup in a retract", although this is true in some cases (e.g. $\langle a^2 \rangle$ in $\mathscr{H}^3$).
For a counterexample, consider the cyclic subgroup $H = \langle a^2[a,b]^3 \rangle$ of the free 2-generated, 2-nilpotent group $F = \langle a, b | [a,[a,b]] = [b,[a,b]] = 1 \rangle$. Since no non-trivial power of the generator of H is in F', it follows that $H \cap F' = \{1\}$. Therefore, by Proposition II.2.9, H is undistorted in F. By Theorem I.3.2, we know that H is a retract of a subgroup of finite index in F. Following the steps of the proof, we arrive at the subgroup $M = \langle a^2[a,b]^3, b \rangle$.
However, it should be remarked that H is not a subgroup of finite index in a retraction of F. First, observe that H is not a retraction itself. For, if by way of contradiction there were such a homomorphism $\phi : F \to H$, then we have equations $\phi(a) = (a^2[a,b]^3)^n$ and $\phi(b) = (a^2[a,b]^3)^m$ as well as $a^2[a,b]^3 = \phi(a)^2[\phi(a), \phi(b)]^3$. But this set of equations has no solutions, even modulo F'. Next, observe that H is not a proper subgroup of finite index in any $K \leq F$. This follows because H is a maximal cyclic subgroup in a torsion-free nilpotent group.

Example IV.4.2. Freeness is necessary for the formulation. For instance, consider the case of non-free 5-dimensional Heisenberg group $F = \mathscr{H}^5$ defined by the presentation

$$\langle x, y, u, v, z | [x,y] = [u,v] = z, [x,z] = [y,z] = [u,z] = [v,z] = 1 \rangle.$$

Then by Lemma II.2.9, $H = \mathscr{H}^3$ is an undistorted subgroup of F. However, as we will show, H is not a retract of any subgroup K of finite index in F. For if by way of contradiction, $[F : K] < \infty$ and H is a retract of K, then we would have that the Dehn functions $f_H \preceq f_K \approx f_F$ which implies that $n^3 \preceq n^2$. These facts about Dehn functions are well known as mentioned earlier and the reader may see [A] or [OS2] for more information about the Dehn function of $\mathscr{H}^5$ and [Ge] for more information on the Dehn function of $\mathscr{H}^3$.

The following result is a direct implication of the proof of Theorem I.3.2.

Corollary IV.4.3. *Every undistorted subgroup H of F is "almost a retract" in the following sense: there exists a normal subgroup $N \leq F$ such that HN is of finite index in F and $H \cap N = \{1\}$.*

Corollary IV.4.4. *The undistorted subgroup H of F is virtually free $c-$nilpotent.*

Proof. With the notation of Section IV.3, we have that $r(H)$ contains the free subgroup $K = \mathrm{gp}\langle r(b_1), \ldots, r(b_k)\rangle$. Because $[D : K] < \infty$ and $K \leq r(H)$ it follows that $[r(H) : K] < \infty$. Finally, because $H \cap N = \{1\}$ we have that $r(H) \cong H/(H \cap N) \cong H$. $\square$

Example IV.4.5. There are undistorted subgroups of free nilpotent groups that are not free. For example, consider again the 3-dimensional Heisenberg group $\mathscr{H}^3$ and its subgroup $H = \mathrm{gp}\langle a^2, b, [a,b]\rangle$. Then H is undistorted because it is of finite index in $\mathscr{H}^3$. Moreover, H is not free because $H' = \langle [a^2, b]\rangle$ and so H/H' contains the nontrivial torsion element $[a,b]$. However, the group H is virtually free, as it contains the free nilpotent subgroup $\langle a^2, b\rangle$ of finite index.

CHAPTER V

DISTORTION IN WREATH PRODUCTS OF ABELIAN GROUPS

We study the effects of subgroup distortion in the wreath products A wr $\mathbb{Z}$, where A is finitely generated abelian. We show that every finitely generated subgroup of A wr $\mathbb{Z}$ has distortion function equivalent to some polynomial. Moreover, for A infinite, and for any polynomial l^k, there is a 2-generated subgroup of A wr $\mathbb{Z}$ having distortion function equivalent to the given polynomial. Also a formula for the length of elements in arbitrary wreath product H wr G easily shows that the group $\mathbb{Z}_2$ wr $\mathbb{Z}^2$ has distorted subgroups, while the lamplighter group $\mathbb{Z}_2$ wr $\mathbb{Z}$ has no distorted (finitely generated) subgroups.

V.1 Introduction

Here we study the effects of distortion in various subgroups of the wreath products $\mathbb{Z}^k$ wr $\mathbb{Z}$, for $0 < k \in \mathbb{Z}$, and more generally, in A wr $\mathbb{Z}$ where A is finitely generated abelian. The main results are as follows. Note that as opposed to previous sections of this dissertation, where the variable for functions from $\mathbb{N} \to \mathbb{N}$ was called "n", for the remainder of this section, we use the convention that they are called "l".

Theorem. I.3.4 *Let A be a finitely generated abelian group.*

1. *For any finitely generated subgroup $H \leq A$ wr $\mathbb{Z}$ there exists $m \in \mathbb{N}$ such that the distortion of H in A wr $\mathbb{Z}$ is*
$$\Delta_H^{A\, wr\, \mathbb{Z}}(l) \approx l^m.$$

2. *If A is finite, then $m = 1$; that is, all subgroups are undistorted.*

3. *If A is infinite, then for every $m \in \mathbb{N}$, there is a 2-generated subnormal subgroup H of A wr $\mathbb{Z}$ having distortion function*
$$\Delta_H^{A\, wr\, \mathbb{Z}}(l) \approx l^m.$$

Theorem I.3.4 will be proved in Section V.11.

The following will be explained in Subsection V.2.3.

Corollary V.1.1. *For every $m \in \mathbb{N}$, there is a 2-generated subgroup H of the free n-generated metabelian group $S_{n,2}$ having distortion function*
$$\Delta_H^{S_{n,2}}(l) \succeq l^m.$$

Corollary V.1.2. *If we let the standard generating set for $\mathbb{Z}$ wr $\mathbb{Z}$ be $\{a,b\}$, then the subgroup $H = \langle b, [\cdots[a,b],b],\cdots,b]\rangle$, where the commutator is $(m-1)$-fold is $m-1$ subnormal with distortion l^m. In particular, the subgroup $\langle [a,b],b\rangle$ is normal, isomorphic to the whole group $\mathbb{Z}$ wr $\mathbb{Z}$, and has quadratic distortion.*

Corollary V.1.2 follows from the proof of Theorem I.3.4. Because the subgroup $\langle [a,b],b\rangle$ of $\mathbb{Z}$ wr $\mathbb{Z}$ is normal, it follows by induction that the distorted subgroup H is subnormal.

Remark V.1.3. There are distorted embeddings from the group $\mathbb{Z}$ wr $\mathbb{Z}$ into itself as a normal subgroup. For example, the map defined on generators by $b \mapsto b, a \mapsto [a,b]$ extends to an embedding, and the image is a quadratically distorted subgroup by Corollary V.1.2. By Lemma V.2.5, $\mathbb{Z}$ wr $\mathbb{Z}$ is the smallest example of a metabelian group embeddable to itself as a normal subgroup with distortion.

Corollary V.1.4. *There is a distorted embedding of $\mathbb{Z}$ wr $\mathbb{Z}$ into Thompson's group F.*

Under the embedding of Remark V.1.3, $\mathbb{Z}$ wr $\mathbb{Z}$ embeds into itself as a distorted subgroup. It is proved in [GS] that $\mathbb{Z}$ wr $\mathbb{Z}$ embeds to F. Therefore, Corollary V.1.4 is true.

It is interesting to contrast Theorem I.3.4 part (2) with the following, which will be discussed in Section V.4. Throughout this paper, we use the convention that $\mathbb{Z}_n$ represents the finite group $\mathbb{Z}/n\mathbb{Z}$.

Proposition V.1.5. *If we consider the group $G = \mathbb{Z}_p$ wr $\mathbb{Z}^k$ for p prime, then there exists a finitely generated subgroup H of G with distortion at least l^k.*

V.2 Background and Preliminaries

V.2.1 Subgroup Distortion

Here we provide some examples of distortion as well as some basic facts to be used later on.

Example V.2.1.

1. Consider the three-dimensional Heisenberg group $\mathscr{H}^3 = \langle a,b,c | c = [a,b], [a,c] = [b,c] = 1\rangle$. It has cyclic subgroup $\langle c\rangle_\infty$ with quadratic distortion, which follows from the equation $c^{l^2} = [a^l, b^l]$.

2. The Baumslag-Solitar Group $BS(1,2) = \langle a,b | bab^{-1} = a^2\rangle$ has cyclic subgroup $\langle a\rangle_\infty$ with at least exponential distortion, because $a^{2^l} = b^l ab^{-l}$.

However, there are no similar mechanisms distorting subgroups in $\mathbb{Z}$ wr $\mathbb{Z}$. Therefore, a natural conjecture would be that free metabelian groups or the group $\mathbb{Z}$ wr $\mathbb{Z}$ do not contain distorted subgroups. This conjecture was brought to the attention of the author by Denis Osin. The result of Theorem I.3.4 shows that the conjecture is not true.

The following facts are well-known and easily verified. When we discuss distortion functions, it is assumed that the groups under consideration are finitely generated.

Lemma V.2.2.

1. If $H \le G$ and $[G:H] < \infty$ then $\Delta_H^G(l) \approx l$.

2. If $H \le K \le G$ then $\Delta_H^K(l) \preceq \Delta_H^G(l)$.

3. If $H \le K \le G$ then $\Delta_H^G(l) \preceq \Delta_K^G((\Delta_H^K(l))$.

4. If H is a retract of G then $\Delta_H^G(l) \approx l$.

5. If G is a finitely generated abelian group, and $H \le G$, then $\Delta_H^G(l) \approx l$.

V.2.2 Wreath Products

We consider the wreath products A wr B of finitely generated groups $A = \mathrm{gp}\langle S\rangle = \langle\{y_1,\dots,y_s\}\rangle$ and $B = \mathrm{gp}\langle T\rangle = \langle\{x_1,\dots,x_t\}\rangle$. We introduce the notation that A wr B is the semidirect product $W\lambda B$, where W is the direct product $\times_{g\in B}A_g$, of isomorphic copies A_g of the group A. We view elements of W as functions from B to A with finite support, where for any $f\in W$, the support of f is $\mathrm{supp}(f) = \{g\in B : f(g)\neq 1\}$. The (left) action $\circ$ of B on W by automorphisms is given by the following formula: for any $f\in W, g\in B$ and $x\in B$ we have that $(g\circ f)(x) = f(xg)$.

Any element of the group A wr B may be written uniquely as wg where $g\in B, w\in W$. The formula for multiplication in the group A wr B is given as follows. For $g_1,g_2\in B, w_1,w_2\in W$ we have that $(w_1g_1)(w_2g_2) = (w_1(g_1\circ w_2))(g_1g_2)$. In particular, B acts by conjugation on W in the wreath product: $gwg^{-1} = g\circ w$.

Therefore the wreath product is generated by the subgroups A_1 and B. In what follows, the subgroup A_1 is identified with A, and so $A_g = gAg^{-1}$, and $S\cup T$ is a finite set of generators in A wr B. In particular, $\mathbb{Z}$ wr $\mathbb{Z}$ is generated by a and b where a generates the left (passive) infinite cyclic group and b generates the right (active) one.

Here we observe that a finitely generated abelian subgroup of $G = A$ wr B with finitely generated abelian A and B is undistorted. It should be remarked that the author is aware that the proof of the fact that abelian subgroups of $\mathbb{Z}^k$ wr $\mathbb{Z}$ are undistorted is available in [GS]. In that paper it is shown that $\mathbb{Z}^k$ wr $\mathbb{Z}$ is a subgroup of the Thompson group F, and that every finitely generated abelian subgroup of F is undistorted. However, our observation is elementary and so we include it.

Lemma V.2.3. *Let A and B be finitely generated abelian groups. Then every finitely generated abelian subgroup H of A wr B is undistorted.*

Proof. It follows from the classification of finitely generated abelian groups G that every subgroup S is a retract of a subgroup of finite index in G, and so we are done if H is a subgroup of A or B, or if $H\cap W = \{1\}$, by Lemma V.2.2. Therefore we assume that $H\cap W\neq\{1\}$. Since H is abelian, this implies that the the factor-group HW/W is finite. Then it suffices to prove the lemma for $H_1 = H\cap W$ since $[H:H_1]\leq\infty$. Because H_1 is finitely generated, it is contained in a finite product of conjugate copies of A. That is to say, $H_1\subset A'$ for a wreath product A' wr $B' = W\lambda B'$ where B' has finite index in B. We are now reduced to our earlier argument, thus completing the proof. □

Remark V.2.4. In fact, under the assumptions of Lemma V.2.3, H is a retract of a subgroup having finite index in A wr B.

We now return to one of the motivating ideas of this paper, and complete the explanation of Remark V.1.3.

Lemma V.2.5. *The group $\mathbb{Z}$ wr $\mathbb{Z}$ is the smallest metabelian group which embeds to itself as a normal distorted subgroup in the following sense. For any metabelian group G, if there is an embedding $\phi : G\to G$ such that $\phi(G)\trianglelefteq G$ and $\phi(G)$ is a distorted subgroup in G, then there exists some subgroup H of G for which $H\cong\mathbb{Z}$ wr $\mathbb{Z}$.*

Proof. By Lemma V.2.2, we have that the group $G/\phi(G)$ is infinite, else $\phi(G)$ would be undistorted. Being a finitely generated solvable group, $G/\phi(G)$ must have a subnormal factor isomorphic to $\mathbb{Z}$. Because $\phi(G) \cong G$, one may repeat this argument to obtain a subnormal series in G with arbitrarily many infinite cyclic factors. Therefore, the derived subgroup G' has infinite (rational) rank.

Since the group $B = G/G'$ is finitely presented, the action of B by conjugation makes G' a finitely generated left B module. Hence, $G' = \langle B \circ C \rangle$ for some finitely generated $C \leq G'$. Because it is a finitely generated abelian group, $B = \langle b_k \rangle \cdots \langle b_1 \rangle$ is a product of cyclic groups. Therefore for some i we have a subgroup $A = \langle \langle b_{i-1} \rangle \cdots \langle b_1 \rangle \circ C \rangle$ of finite rank in G' but $\langle \langle b_i \rangle \circ A \rangle$ has infinite rank. Then A has an element a such that the $\langle b_i \rangle$-submodule generated by a has infinite rank, and so it is a free $\langle b_i \rangle$-module. It follows that a and b, where $b_i = bG'$, generate a subgroup of the form $\mathbb{Z}$ wr $\mathbb{Z}$. □

V.2.3 Connections with Free Solvable Groups

In [M], Magnus shows that if $F = F_k$ is an absolutely free group of rank k with normal subgroup N, then the group $F/[N,N]$ embeds into $\mathbb{Z}^k$ wr $F/N = \mathbb{Z}^k$ wr G. This wreath product is a semidirect product $W \lambda G$ where the action of G by conjugation turns W into a free left $\mathbb{Z}[G]$-module with k generators. For more information in an easy to read exposition, refer to [RS].

Remark V.2.6. The monomorphism $\alpha : F/[N,N] \to \mathbb{Z}^k$ wr G is called the Magnus embedding.

Because the subgroup W of $G = \mathbb{Z}$ wr $\mathbb{Z} = W \lambda \mathbb{Z}$ is abelian, we also use additive notation to represent elements of W.

Remark V.2.7. In the case of $\mathbb{Z}$ wr $\mathbb{Z} = \langle a \rangle$ wr $\langle b \rangle$, we use module language to write any element as

$$w = \sum_{i=-\infty}^{\infty} {}^{\circ}\, m_i (b^i \circ a) = f(x)a \text{ where } f(x) = \sum_{i=-\infty}^{\infty} {}^{\circ}\, m_i x^i$$

is a Laurent polynomial in x, and the sum is finite, indicated by the $^\circ$ symbol.

Lemma V.2.8. *Consider the group $\mathbb{Z}$ wr $\mathbb{Z} = W \lambda \langle b \rangle$. Let $1 \neq w \in W, x \notin W$. Then $gp\langle w, x \rangle \cong \mathbb{Z}$ wr $\mathbb{Z}$ under the group monomorphism $a \mapsto w$, $b \mapsto b$.*

Proof. This follows because in this case W is a free module with one generator a over the domain $\mathbb{Z}[\langle b \rangle]$, $w = ra$ for some $r \in \mathbb{Z}[\langle b \rangle]$, and the mapping $x \to rx$ $(x \in W)$ is an injective module homomorphism. □

We let $S_{k,l}$ denote the k-generated derived length l free solvable group.

Lemma V.2.9. *If $k, l \geq 2$, then the group $S_{k,l}$ contains a subgroup isomorphic to $\mathbb{Z}$ wr $\mathbb{Z}$.*

Proof. It suffices to show that the free metabelian group of rank 2, $S_{2,2}$, contains a subgroup isomorphic to $\mathbb{Z}$ wr $\mathbb{Z}$. This follows because for any $H \leq S_{2,2}$ we may use the Nielsen-Schrier Theorem to identify $H \leq F_k^{(l-2)}/F_k^{(l)} \leq F_k/F_k^{(l)} \cong S_{k,l}$.

Let $S_{2,2}$ have free generators x, y. Because $\mathbb{Z}$ wr $\mathbb{Z}$ is metabelian, we have a homomorphism

$$\phi : S_{2,2} \to \mathbb{Z} \text{ wr } \mathbb{Z} : x \mapsto a, y \mapsto b,$$

where a, b are the usual generators of $\mathbb{Z}$ wr $\mathbb{Z}$. Let H be the subgroup of $S_{2,2}$ generated by $[x,y]$ and y. Then H maps to the subgroup $L = gp\{[a,b], b\}$ which is isomorphic to $\mathbb{Z}$ wr $\mathbb{Z}$ by Lemma V.2.8. It follows that the normal closure of $[x,y]$ in $S_{2,2}$ is itself a free $\mathbb{Z}[\langle y \rangle]$-module, and H is isomorphic to L. $\square$

It should be noted that by results of [S], the group $\mathbb{Z}$ wr $\mathbb{Z}^2$ can not be embedded into any free metabelian or free solvable groups.

As mentioned in the Introduction, subgroup distortion has connections with the membership problem.

By Theorem 2 of [U], the membership problem for free solvable groups of length greater than two is undecidable. Therefore, because of the connections between subgroup distortion and the membership problem, we restrict our primary attention to the case of free metabelian groups. It is worthwhile to note that the membership problem for free metabelian groups is solvable (see [Ro]).

Lemma V.2.9 motivates us to study distortion in $\mathbb{Z}$ wr $\mathbb{Z}$ in order to better understand distortion in free metabelian groups. Distortion in free metabelian groups is similar to distortion in wreath products of free abelian groups, by Lemma V.2.9 and the Magnus embedding. In particular, if $k \geq 2$ then

$$\mathbb{Z} \text{ wr } \mathbb{Z} \leq S_{k,2} \leq \mathbb{Z}^k \text{ wr } \mathbb{Z}^k.$$

Thus by Lemma V.2.2, given $H \leq \mathbb{Z}$ wr $\mathbb{Z}$ we have

$$\Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l) \preceq \Delta_H^{S_{k,2}}(l).$$

This explains Corollary V.1.1. On the other hand, given $L \leq S_{k,2}$ then we have

$$\Delta_L^{S_{k,2}}(l) \preceq \Delta_L^{\mathbb{Z}^k \text{ wr } \mathbb{Z}^k}(l).$$

Based on this discussion, we ask the following. An answer would be helpful in order to more fully understand subgroup distortion in free metabelian groups.

Question V.2.10.

What effects of subgroup distortion are possible in $\mathbb{Z}^k$ wr $\mathbb{Z}^k$ for $k > 1$?

V.3 Canonical Forms and Word Metric

Here we aim to further understand how the length of an element of a wreath product A wr B depends on the canonical form of this element.

Let us start with $G = \mathbb{Z}^k \text{ wr } \mathbb{Z} = W\lambda\langle b \rangle$, where $\mathbb{Z}^k = \text{gp}\{a_1, \ldots, a_k\}$. We will use the notation that $(w)_i$ equals the conjugate bwb^{-i} for $i \in \mathbb{Z}$ and $w \in W$. We remark that as opposed to previous sections where conjugation was performed in the opposite order, we will use the convention in this section that commutators are $[x,y] = xyx^{-1}y^{-1}$ and conjugation is $x^y = yxy^{-1}$ for x, y group elements. This convention is based on our decision to use a left action to define our wreath products.

Remark V.3.1. By the definition of $\mathbb{Z}^k$ wr $\mathbb{Z}$, arbitrary element in $\mathbb{Z}^k$ wr $\mathbb{Z} = \mathrm{gp}\langle a_1, \ldots, a_k, b\rangle$ is (in module notation for the abelian subgroup W) of the form

$$wb^t = \left(\sum_{i=1}^{k} f_i(x)a_i\right) b^t,$$

where $f_i(x)$ are Laurent polynomials. The form is unique.

The normal form described in Remark V.3.2 for elements of A wr $\mathbb{Z}$, where A is a finitely generated abelian group, is necessary to obtain a general formula for computing the word length.

Remark V.3.2. Arbitrary element of A wr $\mathbb{Z}$ may be written in a normal form, following [CT], as

$$((u_1)_{\iota_1} + \cdots + (u_N)_{\iota_N} + (v_1)_{-\varepsilon_1} + \cdots + (v_M)_{-\varepsilon_M})\, b^t$$

where $0 \leq \iota_1 < \cdots < \iota_N, 0 < \varepsilon_1 < \cdots < \varepsilon_M$, and $u_1, \ldots, u_N, v_1, \ldots, v_M$ are elements in $A - \{1\}$.

The following formula for the word length in A wr $\mathbb{Z}$ is given in [CT].

Lemma V.3.3. *Given an element in A wr $\mathbb{Z}$ having normal form as in Remark V.3.2, its length is given by the formula*

$$\sum_{i=1}^{N} |u_i|_A + \sum_{i=1}^{M} |v_i|_A + \min\{2\varepsilon_M + \iota_N + |t - \iota_N|, 2\iota_N + \varepsilon_M + |t + \varepsilon_M|\}.$$

where $|*|_A$ *is the length in the group* A.

The formula from Lemma V.3.3 becomes more intelligible if one extends it to wreath products A wr B of arbitrary finitely generated groups. We want to obtain such a generalization in this section since we consider non-cyclic active groups in Section V.4. We fix the notation that, with respect to the symmetric generating set $T = T^{-1}$, the Cayley graph $\mathrm{Cay}(B)$ is defined as follows. The set of vertices is all elements of G. For any $g \in G, t \in T$, g and gt are joined by an edge pointing from g to gt whose label is t.

Any $u \in A$ wr B can be expressed as follows:

$$(b_1 \circ a_1) \ldots (b_r \circ a_r) g \tag{V.1}$$

where $g \in B, w = (b_1 \circ a_1) \ldots (b_r \circ a_r) \in W, 1 \neq a_j \in A, b_j \in B$ and for $i \neq j$ we have $b_i \neq b_j$. The expression $(V.1)$ is unique, up to a rearrangement of the (commuting) factors $b_j \circ a_j$.

For any $u = wg \in A$ wr B with canonical form as in Equation $(V.1)$ we consider the set P of paths in the Cayley graph $\mathrm{Cay}(B)$ which start at 1, go through every vertex $b_1, \ldots, b_r$ and end at g. We introduce the notation that

$$\mathrm{reach}(u) = \min\{||p|| : p \in P\},$$

$$\mathrm{route}(a) = \text{ the particular } p \in P \text{ realizing } \mathrm{reach}(u) = ||p||.$$

We also define the norm of any such representative w of W by

$$||w||_A = \sum_{j=1}^{r} |a_j|_S.$$

We have the following formula for word length, which generalizes that given for the case where $B = \mathbb{Z}$ in the paper [CT].

Theorem V.3.4. *For any element $u = wg \in A$ wr B, we have that*

$$|wg|_{S,T} = ||w||_A + \text{reach}(u)$$

where $u = (b_1 \circ a_1)\ldots(b_r \circ a_r)g$ is the canonical form of Equation $(V.1)$.

Proof. We will use the following pseudo-canonical (non-unique) form in the proof. This is just the expression of Equation $(V.1)$ but without the assumption that all b_j are distinct.

For any element $u \in A$ wr B which is expressed in pseudo-canonical form we may define a quantity depending on the given factorization by

$$\Psi((b_1 \circ a_1)\ldots(b_r \circ a_r)g) = \sum_{j=1}^{r} |a_j|_S + |b_1|_T + |b_1^{-1}b_2|_T + \cdots + |b_{r-1}^{-1}b_r|_T + |b_r^{-1}g|_T.$$

First we show that for u in canonical form $(V.1)$ it holds that $|u|_{S,T} \geq ||w||_A + \text{reach}(u)$.

By the choice of generating set $\{S,T\}$ of A wr B, we have that any element $u \in A$ wr B may be written as

$$u = g_0 h_1 g_1 \cdots h_m g_m \tag{V.2}$$

where $m \geq 0, g_i \in B, h_j \in A, g_0$ and g_m can be trivial, but all other factors are non-trivial. We may choose the expression (V.2) so that $|u|_{S,T} = \sum_{j=1}^{m} |h_j|_S + \sum_{i=0}^{m} |g_i|_T$. Observe that we may use the expression from Equation $(V.2)$ to write

$$u = (x_1 \circ h_1)\ldots(x_m \circ h_m)g \tag{V.3}$$

where $g = g_0 \ldots g_m$ and $x_j = g_0 \ldots g_{j-1}$, for $j = 1,\ldots,m$.

Then we have by definition that for the pseudo-canonical form $(V.3)$,

$$\Psi((x_1 \circ h_1)\ldots(x_m \circ h_m)g) = \sum_{j=1}^{m} |h_j|_S + |x_1|_T + |x_1^{-1}x_2|_T + \cdots + |x_{m-1}^{-1}x_m|_T + |x_m^{-1}g|_T$$

$$= \sum_{j=1}^{m} |h_j|_S + \sum_{i=0}^{m} |g_i|_T = |u|_{S,T}. \tag{V.4}$$

It is possible that in the form of Equation $(V.3)$, some $x_i = x_j$ for $1 \leq i \neq j \leq m$. When taking u to the canonical form $wg = (b_1 \circ a_1)\dots(b_r \circ a_r)g$ of Equation $(V.1)$, we claim that

$$||w||_A \leq \sum_{j=1}^{m} |h_j|_S \tag{V.5}$$

and that

$$\text{reach}(u) \leq |x_1|_T + |x_1^{-1}x_2|_T + \cdots + |x_{m-1}^{-1}x_m|_T + |x_m^{-1}g|_T. \tag{V.6}$$

Obtaining the canonical form requires a finite number of steps of the following nature. We take an expression such as

$$(x_1 \circ h_1)\dots(x_i \circ h_i)\dots(x_i \circ h_j)\dots(x_m \circ h_m)$$

and replace it with

$$(x_1 \circ h_1)\dots(x_i \circ h_i h_j)\dots(x_{j-1} \circ h_{j-1})(x_{j+1} \circ h_{j+1})\dots(x_m \circ h_m).$$

The assertion of Equation $(V.5)$ follows because

$$|h_i h_j|_S \leq |h_i|_S + |h_j|_S.$$

Equation $(V.6)$ is true because

$$|x_{j-1}^{-1}x_{j+1}|_T \leq |x_{j-1}^{-1}x_j|_T + |x_j^{-1}x_{j+1}|_T,$$

which implies that

$$|b_1|_T + |b_1^{-1}b_2|_T + \cdots + |b_{r-1}^{-1}b_r|_T + |b_r^{-1}g|_T \leq |x_1|_T + |x_1^{-1}x_2|_T + \cdots + |x_{m-1}^{-1}x_m|_T + |x_m^{-1}g|_T.$$

Finally, we have that

$$\text{reach}(u) \leq |b_1|_T + |b_1^{-1}b_2|_T + \cdots + |b_{r-1}^{-1}b_r|_T + |b_r^{-1}g|_T,$$

because the right hand side is the length of a particular path in P: the path which travels from 1 to b_1 to $b_2, \dots,$ to b_r to g. It follows that the length of this path is at least as large as the length of route(u).

Thus for a canonical form $u = (b_1 \circ a_1)\dots(b_r \circ a_r)g$ we see by Equations $(V.4)$, $(V.5)$ and $(V.6)$ that

$$||w||_A + \text{reach}(u) \leq \Psi((x_1 \circ h_1)\dots(x_m \circ h_m)g) = |u|_{S,T}.$$

To obtain the reverse inequality, take $u = (b_1 \circ a_1)\dots(b_r \circ a_r)g$ in A wr B in canonical form. By the definition, route(u) will be a path that starts at 1, goes in some order directly through all of $b_1, \dots, b_r$, and ends at g.

We may rephrase this to say that for some $\sigma \in \text{Sym}(s)$, there is a path $p = \text{route}(u)$ in P such that

$|p|_T = |b_{\sigma(1)}|_T + |b^{-1}_{\sigma(1)} b_{\sigma(2)}|_T + ... + |b^{-1}_{\sigma(r-1)} b_{\sigma(r)}|_T + |b^{-1}_{\sigma(r)} g|_T$. In other words,

$$\text{reach}(u) = |b_{\sigma(1)}|_T + |b^{-1}_{\sigma(1)} b_{\sigma(2)}|_T + ... + |b^{-1}_{\sigma(r-1)} b_{\sigma(r)}|_T + |b^{-1}_{\sigma(r)} g|_T.$$

Moreover, in the wreath product we have that

$$u = (b_{\sigma(1)} \circ a_{\sigma(1)}) \cdots (b_{\sigma(r)} \circ a_{\sigma(r)}) g = b_{\sigma(1)} a_{\sigma(1)} b^{-1}_{\sigma(1)} b_{\sigma(2)} a_{\sigma(2)} \cdots b^{-1}_{\sigma(r-1)} b_{\sigma(r)} a_{\sigma(r)} b^{-1}_{\sigma(r)} g.$$

This implies that

$$|u|_{S,T} \leq |b_{\sigma(1)}|_T + |a_{\sigma(1)}|_S + |b^{-1}_{\sigma(1)} b_{\sigma(2)}|_T + \cdots + |a_{\sigma(r)}|_S + |b^{-1}_{\sigma(r)} g|_T$$

$$= \sum_{j=1}^{r} |a_j|_S + \text{reach}(u) = ||w||_A + \text{reach}(u).$$

□

V.4 Distortion in $\mathbb{Z}_p$ wr $\mathbb{Z}^k$

We begin with the following result, the proof of which exploits the formula of Theorem V.3.4.

Proposition V.4.1. *The group $\mathbb{Z}_2$ wr $\mathbb{Z}^2$ contains distorted subgroups.*

This is interesting in contrast to the case of $\mathbb{Z}_2$ wr $\mathbb{Z}$ which has no effects of subgroup distortion. The essence in the difference comes from the fact that the Cayley graph of $\mathbb{Z}$ is one-dimensional, and that of $\mathbb{Z}^2$ is asymptotically two-dimensional, which gives us more room to create distortion using Theorem V.3.4.

We will use the following notation in the case of $G = \mathbb{Z}_2$ wr $\mathbb{Z}^2$: a generates the passive group of order 2 while b and c generate the active group $\mathbb{Z}^2$.

The canonical form of Equation $(V.1)$ will be denoted by

$$((g_1 + \cdots + g_k)a)g$$

for $g_1, \ldots, g_k$ distinct elements of $\mathbb{Z}^2$ and $g \in \mathbb{Z}^2$. We may do this because any nontrivial element of $\mathbb{Z}_2$ is just equal to the generator a. The proof of the following lemma is similar to the proof of Lemma V.2.8.

Lemma V.4.2. *Let $H \leq G$ be generated by a nontrivial element $w \in W$ as well as the generators b, c of $\mathbb{Z}^2$. Then $H \cong G$.*

We know that $W = \bigoplus_{g \in \mathbb{Z}^2} \langle g \circ a \rangle$ is a free module over $\mathbb{Z}_2[\mathbb{Z}^2]$. Therefore, we may think of W as being the Laurent polynomial ring in two variables, say, x for b and y for c. We can use the module language to express any element as $w = f(x,y)a = (x^{i_1} y^{j_1} + \cdots + x^{i_k} y^{j_k})a$, where for $p \neq q$ we have that $x^{i_p} y^{j_p} \neq x^{i_q} y^{j_q}$. This corresponds to the canonical form $w = (g_1 + \cdots + g_k)a$ where $g_p = b^{i_p} c^{j_p}$ for $p = 1, \ldots, k$.

We now have all the required facts to prove Proposition V.4.1.

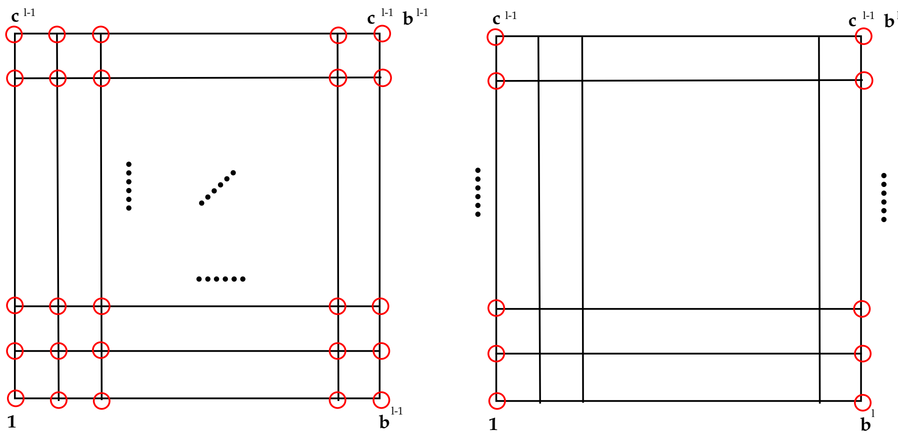

Figure V.4.1: Figure 1: The l^2 vertices (left) and the rectangle with perimeter $2l+2(l-1)$ (right)

Proof. of Proposition V.4.1: Let $G = \mathbb{Z}_2 \text{ wr } \mathbb{Z}^2 = \text{gp}\langle a,b,c\rangle$ and $H = \text{gp}\langle b,c,w\rangle$ where $w = [a,b] = (1+x)a$. By Lemma V.4.2 we have that $H \cong G$. Let

$$f_l(x) = \sum_{i=0}^{l-1} x^i \text{ and } g_l(x) = (1+x)f_l(x).$$

The element $f_l(x)f_l(y)w \in H$ is in canonical form, when written in the additive group notation as $\sum_{i,j=0}^{l-1} b^i c^j \circ w$.

By Theorem V.3.4, we have that its length in H is at least l^2+l^2 since the support of it has cardinality l^2, and the length of arbitrary loop going through l^2 different vertices is at least l^2.

Now we compute the length of $f_l(x)f_l(y)w$ in G. We have that

$$f_l(x)f_l(y)w = (1+x)f_l(x)f_l(y)a = g_l(x)f_l(y)a = \left[\sum_{i=0}^{l-1}(y^i + y^i x^l)\right]a.$$

Theorem V.3.4 shows that $|f_l(x)f_l(y)w|_G = 2l+2(l-1)+2l$. This is because the shortest path in $\text{Cay}(\mathbb{Z}^2)$ starting at 1, passing through $1, c, \ldots, c^{l-1}$ and $b^l, cb^l, \ldots, c^{l-1}b^l$ and ending at 1 is given by traversing the perimeter of the rectangle, and so gives the length of $2(l-1)+2l$.

Therefore the subgroup H is at least quadratically distorted. $\square$

Remark V.4.3. The subgroup H is not normal in G because the element aca^{-1} is not in H.

The proof of Proposition V.4.1 can be generalized as follows. Consider the group $G = \mathbb{Z}_p \text{ wr } \mathbb{Z}^k = \text{gp}\langle a, b_1, \ldots, b_k\rangle$ for p prime and $k > 1$. Then the subgroup $H = \text{gp}\langle w, b_1, \ldots, b_k\rangle$ where $w = (1-x_1)\cdots(1-x_{k-1})a = [\ldots[a,b_1],b_2],\ldots b_{k-1}]$ has distortion at least l^k. This is a restatement of Proposition V.1.5.

By (the analogue of) Lemma V.4.2 we have that $H \cong G$ and so we can compute lengths using Theorem V.3.4. Consider the element $f_l(x_1)\cdots f_l(x_k)w$ in H. Then it has length in H at least equal to $l^k + l^k$ because the path in $\text{Cay}(\mathbb{Z}^k)$ arising from Theorem V.3.4 would need to pass through at least l^k vertices: ${b_1}^{\alpha_1}\cdots {b_k}^{\alpha_k}$

for $\alpha_i \in \{0,\dots,l-1\}, i=1,\dots,k$. In the group G,

$$f_l(x_1)\cdots f_l(x_k)w = g_l(x_1)\cdots g_l(x_{k-1})f_l(x_k)a.$$

This has linear length, which follows because the vertices of the support are placed along the edges of a k-dimensional parallelotope, such that the length of any edge of the parallelotope is at most l.

V.5 Structure of Some Subgroups of A wr $\mathbb{Z}$

Lemma V.5.1. *Let G be a group having normal subgroup W and cyclic $G/W = \langle bW \rangle$. Then any finitely generated subgroup H of G may be generated by elements of the form $w_1b^t, w_2, \dots, w_s$ where $w_i \in W$.*

The proof is elementary and follows from the assumption that G/W is cyclic.

Remark V.5.2. It follows that if A is finitely generated abelian, then any finitely generated subgroup in A wr $\mathbb{Z} = W\lambda\langle b\rangle$ can be generated by elements $w_1b^t, w_2, \dots, w_s$ where $w_i \in W$.

Definition V.5.3. For A a fixed finitely generated abelian group and any $t > 0$, the group L_t is the subgroup of A wr $\mathbb{Z}$ generated by the subgroup W and by the element b^t.

The following discussion will be used in later sections.

Lemma V.5.4. *If A is a fixed r generated abelian group then $L_t \cong A^t$ wr $\mathbb{Z}$.*

The statement follows from Remark V.3.1.

Lemma V.5.5. *For any $w \in W$ there is an automorphism $L_t \to L_t$ identical on W such that $wb^t \to b^t$, provided $t \neq 0$.*

This follows because the actions by conjugation of b^t and wb^t on W coincide.

Lemma V.5.6. *Let H be a finitely generated subgroup of A wr $\mathbb{Z}$ not contained in W, where A is finitely generated abelian. Then the distortion of H in A wr $\mathbb{Z}$ is equivalent to the distortion of a subgroup H' in A' wr $\mathbb{Z}$ where b (the generator of $\mathbb{Z}$) is contained in H', and $A' \cong A^t = A + \cdots + A$ is also finitely generated abelian.*

Proof. By Lemma V.5.1 the generators of H may be chosen to have the form $w_0b^t, w_1, \dots, w_s$ where $w_i \in W$. Therefore, for this value of t we have that H is a subgroup of L_t. Using the isomorphisms of Lemmas V.5.4 and V.5.5 we have that H is a subgroup of A^t wr $\mathbb{Z} = A'$ wr $\mathbb{Z}$ generated by the image of $b^tw_0, w_1, \dots, w_s$ under the two isomorphisms: elements $b, x_1, \dots, x_s$. Finally, because $[A \text{ wr } \mathbb{Z} : L_t] < \infty$ we have by Lemma V.2.2 that the distortion of H in A wr $\mathbb{Z}$ is equivalent to the distortion of its image in A^t wr $\mathbb{Z}$. $\square$

Definition V.5.7. Let H be a subgroup of A wr $\mathbb{Z} = W\lambda\langle b\rangle$ where A is finitely generated abelian. We call H "a subgroup with b" if the generators of H may be given by $b, w_1, \dots, w_s$ for $w_i \in W$.

V.6 Distortion in $\mathbb{Z}^k$ wr $\mathbb{Z}$

The main results of this paper deal with distortion in finitely generated subgroups of wreath products of the form A wr $\mathbb{Z}$, where A is finitely generated abelian. In the case where A is free abelian, we may reduce computations to certain subgroups that are easier to understand.

V.6.1 Some Modules

We will need the following auxiliary remarks about module theory. The following is well known (see also [FS]).

Lemma V.6.1. *The ring $F[\langle b\rangle]$ is a principal ideal ring if F is a field.*

Lemma V.6.2. *Let F be a field, and suppose that $\overline{W}$ and $\overline{V}$ are free modules over $F[\langle b\rangle]$ of respective ranks r and $l \leq k$. Then these free $F[\langle b\rangle]$-modules $\overline{V}$ and $\overline{W}$ have bases $e'_1,\dots,e'_l$ and $f'_1,\dots,f'_r$ respectively such that*

$$e'_i = u'_i f'_i, i = 1,\dots,l$$

for some $u'_i \in F[\langle b\rangle]$.

Proof. The statement of Lemma V.6.2 is a result from module theory. It follows because by Lemma V.6.1 $\overline{W}$ is a free module over a prinicipal ideal ring with submodule $\overline{V}$. See for instance, [Bo].

□

We are now able to prove the following special case of Theorem I.3.4 Part (2).

Lemma V.6.3. *If p is a prime, then any finitely generated subgroup H of $G = \mathbb{Z}_p^k$ wr $\mathbb{Z}$ is undistorted.*

Proof. One may assume that the subgroup H is infinite, so by Lemma V.5.6 one may assume that H is a subgroup of $L = \mathbb{Z}_p^l$ wr $\mathbb{Z} = W\lambda\langle b\rangle$ $(l = kt)$ with b. By Lemma V.2.2, it suffices to show that H has finite index in a retract K of a subgroup L of finite index in G.

Since p is a prime, that $\mathbb{Z}_p$ is a field. This implies by Lemma V.6.1 that the ring $R = \mathbb{Z}_p[\langle b\rangle]$ is a principal ideal ring.

Let $V = H \cap W$. Then V is a free R-module, being a submodule of the free module W over the PIR R. Just as in Lemma V.6.2, we have that V and W have bases $e_1,\dots,e_m$ and $f_1,\dots,f_l$ respectively, for $m \leq l$ such that

$$e_i = g_i f_i, i = 1,\dots,m \tag{V.7}$$

for some polynomials $g_i \in R\backslash 0$. Thus we can choose the generators for L and H to be $\{b, f_1,\dots,f_l\}$ and $\{b, e_1,\dots,e_m\}$, respectively, and H is a subgroup of the retract K of L, where K is isomorphic to $\mathbb{Z}_p^m$ wr $\mathbb{Z}$ and is generated by $\{b, f_1,\dots,f_m\}$. Now V is a submodule of the $\mathbb{Z}_p[\langle b\rangle]$-module W' generated by $\{f_1,\dots,f_m\}$, and the factor-module W'/V is a direct sum of cyclic modules $\langle f_i\rangle/\langle g_i f_i\rangle$. Hence W'/V is finite since it is easy to see that each $\langle f_i\rangle/\langle g_i f_i\rangle$ has finite order at most $p^{\deg g_i}$. Since the subgroup H contains b, the index of H in K is also finite.

□

We return to our discussion of module theory. Let $H \leq \mathbb{Z}^r \text{ wr } \mathbb{Z}$ be generated by b, as well as any elements $w_1, \ldots, w_k \in W$. Let V be the normal closure of $w_1, \ldots, w_k$ in $\mathbb{Z}^r \text{ wr } \mathbb{Z}$, i.e., the $\mathbb{Z}[\langle b \rangle]$-submodule of W generated by $w_1, \ldots, w_k$. Let $\overline{V} = V \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\overline{W} = W \otimes_{\mathbb{Z}} \mathbb{Q}$. Observe $\overline{W}$ and $\overline{V}$ are free modules over $\mathbb{Q}[\langle b \rangle]$ of respective ranks r and $l \leq k$.

Remark V.6.4. It follows from Lemma V.6.2 that there exist $0 < m, n \in \mathbb{Z}$ with $(me_i') = u_i(nf_i')$ where $e_i = me_i' \in V, f_i = nf_i' \in W, u_i \in \mathbb{Z}[\langle b \rangle]$. Moreover, the modules generated by $\{e_1, \ldots, e_l\}$ and $\{f_1, \ldots, f_r\}$ are free.

Remark V.6.5. There is a bijective correspondence between the set of finitely generated $\mathbb{Z}[\langle b \rangle]$ submodules N of $\mathbb{Z}[\langle b \rangle]^r$ and the set of subgroups $K = N\langle b \rangle$ of $\mathbb{Z}^r \text{ wr } \mathbb{Z}$ such that the finite set of generators of K is of the form $b, w_1, \ldots, w_k$, $w_i \in W$.

Remark V.6.6. Let V_1 and W_1 be generated as submodules over $\mathbb{Z}[\langle b \rangle]$ by the elements from Remark V.6.4: $e_1, \ldots, e_l$ and $f_1, \ldots, f_r$ respectively. Let H_1 and G_1 be subgroups of $\mathbb{Z}^r \text{ wr } \mathbb{Z}$ generated by $\{b, V_1\}$ and $\{b, W_1\}$ respectively. It follows by Remark V.6.4 that that $G_1 \cong \mathbb{Z}^r \text{ wr } \mathbb{Z}$ and $H_1 \cong \mathbb{Z}^l \text{ wr } \mathbb{Z}$.

Remark V.6.7. Observe that under the correspondence of Remark V.6.5 each generator of the group H_1 is in the normal closure of only one generator of G_1. That is, for each i, $e_i = u_i f_i$ for $u_i \in \mathbb{Z}[\langle b \rangle]$ means that there exist expressions $e_i = g_i(x) f_i$ where $g_i(x) = \sum_{p=1}^{t_i} n_{i,p} x^{j_{i,p}}$.

Definition V.6.8. We will call subgroups of $\mathbb{Z}^r \text{ wr } \mathbb{Z}$ or $\mathbb{Z}_p^r \text{ wr } \mathbb{Z}$ generated by b and w_i from different submodules $\mathbb{Z}[\langle b \rangle]a_i$ or $\mathbb{Z}_p[\langle b \rangle]a_i$ "special".

Lemma V.6.9. *There exists* $0 < n', m' \in \mathbb{N}$ *so that* $n'W \subset W_1 \subset W$, *and* $m'V \subset V_1 \subset V$.

Proof. By Remark V.6.5 we have that V is a finitely generated $\mathbb{Z}[\langle b \rangle]$ module with generators $w_1, \ldots, w_k$. For each w_i, we have that the element $w_i \otimes 1 \in \overline{V}$. Therefore, by Lemma V.6.2, there are $\lambda_{i,j} \in \mathbb{Q}[\langle b \rangle]$ so that $w_i = \sum_{j=1}^{l} \lambda_{i,j} e_j'$. First observe that $mw_i = \sum_{j=1}^{l} \lambda_{i,j} e_j$, because $e_i = me_i' \in V$.

Next, there exists $M_i \in \mathbb{N}$ so that $M_i m w_i = \sum_{j=1}^{l} \mu_{i,j} e_j \in V_1$ where $\mu_{i,j} \in \mathbb{Z}[\langle b \rangle]$. Let $m' = M_1 \ldots M_k m$. Then for any $v \in V$, we have that $v = \sum_{i=1}^{k} v_i w_i$ where $v_i \in \mathbb{Z}[\langle b \rangle]$, and therefore, $m'v \in V_1$ as required. A similar argument works for obtaining n'. □

Lemma V.6.10. *Let* $\mathbb{Z}^r \text{ wr } \mathbb{Z} = G = W \lambda \langle b \rangle$ *and let* $K = \langle\langle w_1, \ldots, w_k \rangle\rangle^G \leq G$ *be the normal closure of elements* $w_i \in W$. *Suppose that there exists* $n \in \mathbb{N}$ *and a finitely generated subgroup* $K' \leq K$ *so that* $nK \leq K'$. *Then*

$$\Delta^G_{\langle b, K' \rangle}(l) \approx \Delta^G_{\langle b, K \rangle}(l).$$

Proof. We will use the notation that $K_1 = \text{gp}\langle K, b \rangle, K_1' = \text{gp}\langle K', b \rangle, K_1'' = \text{gp}\langle nK, b \rangle$. Observe that the mapping $\phi : G \to G : b \to b, w \to nw$ for $w \in W$ is an injective homomorphism which restricts to an isomorphism $K_1 \to K_1''$. An easy computation which uses Lemma V.3.3 and the definition of ϕ shows that for any $g \in K_1$, we have that

$$|g|_G \leq |\phi(g)|_G \leq n|g|_G \tag{V.8}$$

where the lengths are computed in G with respect to the usual generating set $\{a_1, \ldots, a_r, b\}$.

Observe that under the map ϕ we have that

$$\text{for } x \in K_1, |x|_{K_1} = |\phi(x)|_{K_1''}, \tag{V.9}$$

where the lengths in K_1'' are computed with respect to the images under ϕ of a fixed generating set of K_1.

By their definitions, we have the embeddings

$$K_1'' \leq K_1' \leq K_1 \overset{\phi}{\hookrightarrow} K_1''. \tag{V.10}$$

By Equation (V.10) there exists $k' > 0$ depending only on the chosen generating sets of K_1 and K_1' so that

$$\text{for any } x \in K_1', |x|_{K_1} \leq k'|x|_{K_1'}. \tag{V.11}$$

It also follows by by Equation (V.10) that there exists a constant $k > 0$ depending only on the chosen generating sets of K_1'' and K_1' so that

$$\text{for any } x \in K_1'', |x|_{K_1'} \leq k|x|_{K_1''}. \tag{V.12}$$

First we show that $\Delta^G_{K_1''}(l) \preceq \Delta^G_{K_1}(l)$.

Let $g \in K_1''$ be such that $|g|_G \leq l$ and $|g|_{K_1''} = \Delta^G_{K_1''}(l)$. Then there exists $g' \in K_1$ such that $\phi(g') = g$. Therefore, it follows that $\Delta^G_{K_1''}(l) = |g|_{K_1''} = |\phi(g')|_{K_1''} = |g'|_{K_1} \leq \Delta^G_{K_1}(l)$. The first and second equalities follow by definition, the third by Equation (V.9), and the inequality is true because by Equation (V.8) we have that $|g'|_G \leq |\phi(g)|_G = |g|_G \leq l$.

We claim that $\Delta^G_{K_1}(l) \preceq \Delta^G_{K_1'}(l)$.

Let $g \in K_1$ be such that $|g|_{K_1} = \Delta^G_{K_1}(l)$. Then $|g|_{K_1} \leq |\phi(g)|_{K_1} \leq k'|\phi(g)|_{K_1'} \leq k'\Delta^G_{K_1'}(nl)$, which follows from Equations (V.8), (V.11) and by definition.

On the other hand, we will show that $\Delta^G_{K_1'}(l) \preceq \Delta^G_{K_1''}(l)$. Let $g \in K_1'$ be such that $|g|_{K_1'} = \Delta^G_{K_1'}(l)$. Then $|g|_{K_1'} \leq |\phi(g)|_{K_1'} \leq k|\phi(g)|_{K_1''} \leq k\Delta^G_{K_1''}(nl)$, which follows from Equations (V.8), (V.12) and by definition.

Therefore, we have that $\Delta^G_{K_1}(l) \preceq \Delta^G_{K_1'}(l) \preceq \Delta^G_{K_1''}(l) \preceq \Delta^G_{K_1}(l)$. □

Lemma V.6.11. *Let H be a subgroup of $\mathbb{Z}^r$ wr $\mathbb{Z}$ with b. Then the distortion of H in $\mathbb{Z}^r$ wr $\mathbb{Z}$ is equivalent to the distortion of a special subgroup. Recall by Definition V.6.8 that this means H is generated by elements $b, w_1, \ldots, w_k$ where $k \leq r$, $\mathbb{Z}^r$ wr $\mathbb{Z} = gp\langle b, a_1, \ldots, a_r\rangle$ and each w_i is in the normal closure of one a_i only.*

This follows from the results of Section V.6.1. Recall that the special subgroup H_1 of the group G_1 was defined in Lemma V.6.6, and these groups were associated to the given $H \leq G$. It follows from Lemmas V.6.9 and V.6.10 that the distortion functions

$$\Delta^G_{G_1}(l) \approx \Delta^G_G(l) \approx l \text{ and } \Delta^G_{H_1}(l) \approx \Delta^G_H(l).$$

V.7 The Case of A wr $\mathbb{Z}$

In this section, we will reduce distortion in subgroups of A wr $\mathbb{Z}$ where A is finitely generated abelian to that in subgroups of $\mathbb{Z}^k$ wr $\mathbb{Z}$ only. By the previous section, we have reduced the problem of studying distortion in $\mathbb{Z}^k$ wr $\mathbb{Z}$ to the study of special subgroups.

Here we recall some basic similarities and differences between the groups $\mathbb{Z}_n^k$ wr $\mathbb{Z}$ and $\mathbb{Z}^k$ wr $\mathbb{Z}$. Let $G = \mathbb{Z}_n^k$ wr $\mathbb{Z}$, for $n \geq 2, k \geq 1$.

Although the notion of equivalence has only been defined for functions from $\mathbb{N}$ to $\mathbb{N}$, we would like to define a notion of equivalence for functions on a finitely generated group. We say that two functions $f, g : G \to \mathbb{N}$ are equivalent if there exists $C > 0$ such that for any $x \in G$ we have

$$\frac{1}{C} f(x) - C \leq g(x) \leq C f(x) + C.$$

If there is a function $f : G \to \mathbb{N}$ such that $f \approx |\cdot|_G$, then for any subgroup H of G, $\Delta_H^G(l) \approx \max\{|x|_H : x \in H, f(x) \leq l\}$.

Lemma V.7.1. *For any $g \in G$, the following function $f : G \to \mathbb{N}$ is equivalent to the length in G. Using the notation of Remark V.3.2, we have that*

$$f(g) = |t| + \varepsilon_M + \iota_N \approx |g|_G.$$

Proof. First let $g \in G$ have normal form as in the statement of Lemma V.3.2. Then by Lemma V.3.3 it follows that

$$|g|_G \leq (N+M)(n-1) + 2(\iota_N + \varepsilon_M) + |t| \leq (\iota_N + 1 + \varepsilon_M)(n-1) + 2(\iota_N + \varepsilon_M) + |t|$$

$$\leq (n+1)(\iota_N + \varepsilon_M) + |t| + (n-1) \leq C f(g) + C,$$

where $C = n+1$. The computations follow from the definitions, as well as the fact that $\varepsilon_M \geq M, \iota_N \geq N-1$ and the length in $\mathbb{Z}_n^k$ of each u_i, v_j is bounded from above by $n-1$. On the other hand, observe that $|g|_G \geq \max\{|t|, \iota_N + \varepsilon_M\}$. Therefore, $2|g|_G \geq f(g)$, so the two functions are equivalent. $\square$

Lemma V.7.2. *Let A be a finitely generated abelian group and consider $G = A$ wr $\mathbb{Z} = A$ wr $\langle b \rangle$. If H is a finitely generated subgroup of G, then there exists k so that the distortion of H in G is equivalent to that of a finitely generated subgroup in $\mathbb{Z}^k$ wr $\mathbb{Z}$.*

Proof. There exists a series

$$A = A_0 > A_1 > \cdots > A_m \cong \mathbb{Z}^k$$

for $k \geq 0$ where A_{i-1}/A_i has prime order for $i = 1, \ldots, m$.

We induct on m. If $m = 0$, then $A \cong \mathbb{Z}^k$ and the claim holds.

Now let $m > 0$. Observe that A_1 is a finitely generated abelian group with a series $A_1 > \cdots > A_m \cong \mathbb{Z}^k$ of length $m-1$. Therefore, by induction, any finitely generated subgroup in $G_2 = A_1$ wr $\mathbb{Z}$ has distortion equivalent to that of a finitely generated subgroup in $\mathbb{Z}^k$ wr $\mathbb{Z}$, for some k.

By Lemma V.6.3, all finitely generated subgroups of $G_1 = (A/A_1)$ wr $\mathbb{Z}$ are undistorted. Denote the natural homomorphism by $\phi : G \to G_1$. Let

$$U = \bigoplus_{\langle b \rangle} A_1 = \ker(\phi).$$

Observe that $U \cdot \langle b \rangle \cong G_2$. The product is semidirect because U is a normal subgroup which meets $\langle b \rangle$ trivially, and it is isomorphic to the wreath product by definition: the action of b on the module $\bigoplus_{\langle b \rangle} A_1$ is the same.

Let H be a finitely generated subgroup in G. Suppose that H is not contained in W. It follows in this case by Lemmas V.5.1 and Lemma V.5.6 that H is a subgroup with b.

Let $R = \mathbb{Z}[\langle b \rangle]$. Observe that R is a Noetherian ring. This follows from basic algebra because $\mathbb{Z}$ is a commutative Noetherian ring. Therefore, W is a finitely generated module over the Noetherian ring R, hence is Noetherian itself. Thus, the R-submodule $H \cap U$ is finitely generated. Let $\{w'_1, \ldots, w'_r\}$ generate $H \cap U$ as a R-module. Let $\{b, w_1, \ldots, w_s\}$ be a set of generators of H modulo U; that is, the images of these elements generate the subgroup $H_1 = HU/U \cong H/H \cap U$ of G_1. Then the set $\{b, w_1, \ldots, w_s, w'_1, \ldots, w'_r\}$ generates H. Furthermore, the collection $\{b, w'_1, \ldots, w'_r\}$ generates the subgroup $H_2 = (H \cap U) \cdot \langle b \rangle$ of G_2.

Let $g \in H$ have $|g|_G \leq l$. Then the image $g_1 = \phi(g)$ in G_1 belongs to H_1, because $g \in H$, and has length $|g|_{G_1} \leq l$ by Lemma V.7.1 and definition of ϕ and G_1. It follows by Lemma V.6.3 that H_1 is undistorted in G_1. Therefore, there exists a linear function $f : \mathbb{N} \to \mathbb{N}$ (which does not depend on the choice of g) such that $|g_1|_{H_1} \leq f(l)$. That is to say, there exists a product P of at most $f(l)$ of the chosen generators $\{b, w_1, \ldots, w_s\}$ of H_1 such that $P = g_1^{-1}$ in H_1. Taking preimages, we obtain that $gP \in U$.

Because H is a subgroup of G, there exists a constant c depending only on the choice of finite generating set of H such that for any $x \in H$ we have that

$$|x|_G \leq c|x|_H. \tag{V.13}$$

It follows by Equation (V.13) that

$$|gP|_G \leq |g|_G + |P|_G \leq |g|_G + c|P|_H \leq l + cf(l). \tag{V.14}$$

Observe that $gP \in H_2$. This follows because $gP \in U$ by construction, and $g \in H$ by choice. Further, $P \in H$ because it is a product of some of the generators of H. Since $H_2 = (H \cap U) \cdot \langle b \rangle$ we see that $gP \in H_2$. Using the fact that G and G_2 are wreath products together with the length formula in Lemma V.3.3, we have that for any $x \in G_2$,

$$|x|_{G_2} \leq |x|_G. \tag{V.15}$$

By induction, the finitely generated subgroup H_2 of G_2 has distortion function $F(l)$ equivalent to that of a finitely generated subgroup $\tilde{H}_2$ in $\mathbb{Z}^k$ wr $\mathbb{Z}$ for some k. That is, $F(l) = \Delta_{H_2}^{G_2}(l) \approx \Delta_{\tilde{H}_2}^{\mathbb{Z}^k \text{ wr } \mathbb{Z}}(l)$. In particular, for any $x \in H_2$,

$$|x|_{H_2} \leq F(|x|_{G_2}). \tag{V.16}$$

Since $gP \in H_2$, we have that

$$|gP|_{H_2} \leq F(|gP|_{G_2}) \leq F(|gP|_G) \leq F(l + cf(l)).$$

The first inequality follows from Equation (V.16), the second from Equation (V.15), and the last from Equation (V.7).

Because $H_2 \leq H$ there is a constant k such that for any $x \in H_2, |x|_H \leq k|x|_{H_2}$.

Combining all previous estimates, we compute that

$$|g|_H \leq |gP|_H + |P|_H \leq k|gP|_{H_2} + f(l) \leq kF(l + cf(l)) + f(l).$$

Thus, at this point we have shown that $\Delta_H^G(l) \preceq F(l) = \Delta_{H_2}^{G_2}(l)$, since f is linear. On the other hand, $\Delta_{H_2}^G(l) = \Delta_H^G(l)$ by Lemma V.6.10. By Lemma V.2.2 we have that $\Delta_{H_2}^{G_2}(l) \preceq \Delta_{H_2}^G(l)$ and so $\Delta_H^G(l) \approx \Delta_{H_2}^{G_2}(l) \approx \Delta_{\tilde{H}_2}^{\mathbb{Z}^k \text{ wr } \mathbb{Z}}(l)$.

If the subgroup H had been abelian, it follows by induction that it is undistorted, because the finitely generated group $H \cap U$ is also abelian, and so its distortion in G_2 is linear. $\square$

V.7.1 Estimating Word Length

We need to establish a looser way of estimating lengths in $\mathbb{Z}^r$ wr $\mathbb{Z}$, $r \geq 1$ than the formula introduced in Lemma V.3.3.

Lemma V.7.3. *Let $\mathbb{Z}^r$ wr $\mathbb{Z}$ have standard generating set $\{a_1, \ldots, a_r, b\}$. Let $H \leq \mathbb{Z}^r$ wr $\mathbb{Z}$ be a special subgroup with generators $b, w_1, \ldots, w_k$.Then H is isomorphic to $\mathbb{Z}^k$ wr $\mathbb{Z}$.*

This follows from what has been established already. Each w_i generates a free cyclic $\mathbb{Z}[\langle b \rangle]$ submodule. By hypothesis, all w_i's are in different direct summands, so they generate a free $\mathbb{Z}[\langle b \rangle]$ module of rank k.

We will only consider special subgroups of $\mathbb{Z}^r$ wr $\mathbb{Z}$. Such a subgroup H has generators $b, w_1, \ldots, w_k$ where $w_i \in W$, and further, for each $i = 1, \ldots, k$ we have that

$$w_i = r_i(x)a_i \text{ where } r_i(x) = \sum_{j=0}^{t_i} d_{i,j}x^j. \tag{V.17}$$

This follows without loss of generality by conjugating by a power of b. Then for any element $g \in H$, we may write

$$g = \left(\sum_{i=1}^{k} f_i(x)w_i\right)b^n \text{ where } f_i(x) = \sum_{q=s_i}^{s_i+p_i} z_{i,q}x^q \tag{V.18}$$

for some $s_i, z_{i,q} \in \mathbb{Z}, p_i \geq 0$. In the generators of $\mathbb{Z}^r$ wr $\mathbb{Z}$ we may also write this element as

$$\left(\sum_{i=1}^{k} g_i(x)a_i\right)b^n \text{ where } g_i(x) = r_i(x)f_i(x) = \sum_{j=s_i}^{s_i+p_i+t_i} y_{i,j}x^j, \tag{V.19}$$

for some $y_{i,j} \in \mathbb{Z}$. For this element, consider the norms

$$e(g) = \sum_{i=1}^{k} \sum_{j=s_i}^{s_i+p_i+t_i} |y_{i,j}| \text{ and } e_H(g) = \sum_{i=1}^{k} \sum_{q=s_i}^{s_i+p_i} |z_{i,q}|.$$

Letting $\iota = \max_i\{t_i + s_i + p_i, 0\}, \varepsilon = \min_i\{s_i, 0\}, \iota_H = \max_i\{s_i + p_i, 0\}$ we define $u_H(g) = \iota_H - \varepsilon$ and $u(g) = \iota - \varepsilon$.

Consider the function

$$\delta(l) = \max\{e_H(g) : g \in H \cap W, e(g) \le l \text{ and } u(g) \le l\}.$$

The following Lemma shows that we may simplify computations of word length in special subgroups.

Lemma V.7.4. *Let $H \le \mathbb{Z}^r$ wr $\mathbb{Z}$ be special, given by generators of the form described in Equation* (V.17). *Then we have that*

$$\Delta_H^{\mathbb{Z}^r \, wr \, \mathbb{Z}}(l) \approx \delta(l).$$

Proof. Recall that by Lemma V.3.3 as well as Lemma V.7.3, we have the following formulas. For $g \in H$ with the notation established above, we have that: $|g|_H = e_H(g) + \min\{-2\varepsilon + \iota_H + |n - \iota_H|, 2\iota_H - \varepsilon + |n - \varepsilon|\}$ and $|g|_{\mathbb{Z}^r \text{ wr } \mathbb{Z}} = e(g) + \min\{-2\varepsilon + \iota + |n - \iota|, 2\iota - \varepsilon + |n - \varepsilon|\}$.

The following inequality follows from the definitions:

$$\max\{e(g), u(g), |n|\} \le |g|_{\mathbb{Z}^r \text{ wr } \mathbb{Z}}. \tag{V.20}$$

Similarly, we have that

$$|g|_H \le e_H(g) + 2u_H(g) + |n| \text{ and } |g|_{\mathbb{Z}^r \text{ wr } \mathbb{Z}} \le e(g) + 2u(g) + |n|. \tag{V.21}$$

Observe that for $g \in H \cap W$ we have that

$$|g|_H \ge \max\{e_H(g), u_H(g)\}. \tag{V.22}$$

Observe that

$$\max\{u_H(g) : g \in H, u(g) \le l\} \le l. \tag{V.23}$$

Thus,

$$\Delta_H^{\mathbb{Z}^r \text{ wr } \mathbb{Z}}(l) \le \max\{e_H(g) : g \in H, e(g) \le l, u(g) \le l\} + \max\{2u_H(g) : g \in H, u(g) \le l\}$$

$$+ \max\{|n| : g \in H, |n| \le l\} \le \delta(l) + 3l.$$

The first inequality follows from Equation (V.20), the second from Equation (V.21).

On the other hand, we have that

$$\Delta_H^{\mathbb{Z}\text{ wr }\mathbb{Z}}(l) \geq \max\{e_H(g) : g \in H \cap W, e(g) \leq l/4, u(g) \leq l/4\}$$

$$-\max\{u_H(g) : g \in H \cap W, e(g) \leq l/4, u(g) \leq l/4\} \geq \delta(l/4) - l/4.$$

The first inequality follows from Equation (V.21), the second from Equation (V.22), and the third from Equation (V.23).

Thus $\Delta_H^{\mathbb{Z}^r\text{ wr }\mathbb{Z}}(l)$ and $\delta(l)$ are equivalent. □

V.7.2 Tame Subgroups

In this Subsection, we will be able to further reduce the study of distortion of special subgroups to that of two generated subgroups of $\mathbb{Z}$ wr $\mathbb{Z}$.

Definition V.7.5. We call a subgroup of $\mathbb{Z}$ wr $\mathbb{Z}$ generated by b and $w = h(x)a \in W$ where $h(x) \in \mathbb{Z}[x]$ has nonzero constant term a "tame" subgroup. We will fix the notation that the polynomial $h(x) = d_0 + \cdots + d_t x^t$ where $d_0, d_t \neq 0$.

Lemma V.7.6. *Let $H \leq \mathbb{Z}^r$ wr $\mathbb{Z} = G$ be a special subgroup. Let $H_i = gp\langle b, w_i\rangle$ and $G_i = gp\langle b, a_i\rangle$ for $i = 1, \ldots, k$. Then we have that*

$$\Delta_H^G(l) \approx \max\{\Delta_{H_i}^{G_i}(l)\}_{i=1,\ldots,k}.$$

Further, we may assume without loss of generality that each H_i is a tame subgroup.

Proof. Observe that $H_i \hookrightarrow H$ is an undistorted embedding, due to that fact that H_i is a retract of H (and similarly for $G_i \hookrightarrow G$). Therefore, by Lemma V.2.2 we have that

$$\Delta_{H_i}^{G_i}(l) \preceq \Delta_{H_i}^G(l) \preceq \Delta_H^G(l),$$

for every i. To prove the other inclusion, we will apply Lemma V.7.4. We know, using the notation of Lemma V.7.4, that

$$\Delta_H^G(l) \approx \delta(l) = \max\{e_H(g) : g \in H \cap W, e(g) \leq l \text{ and } u(g) \leq l\} = \sum_{i=1}^{k} \sum_{q=s_i}^{s_i+p_i} |z_{i,q}|$$

for some $g \in H \cap W$ which is equal to $\sum_{i=1}^k f_i(x)w_i$ and $f_i(x) = \sum_{q=s_i}^{s_i+p_i} z_{i,q}x^q$.

Let the number $j \in \{1, \ldots, k\}$ be so that

$$\max_{i=1\ldots,k}\{\sum_{q=s_i}^{s_i+p_i} |z_{i,q}|\} = \sum_{q=s_j}^{s_j+p_j} |z_{j,q}|.$$

Denote this maximum value by M_j. We have that $\Delta_H^G(l) \approx \delta(l) = \sum_{i=1}^k \sum_{q=s_i}^{s_i+p_i} |z_{i,q}| \leq kM_j$. We will show that $M_j \leq \Delta_{H_j}^{G_j}(l)$, and therefore that $\Delta_H^G(l) \leq kM_j \leq k\Delta_{H_j}^{G_j}(l) \leq k\max\{\Delta_{H_i}^{G_i}(l)\}_{i=1,\ldots,k}$. Consider the element $f_j(x)w_j \in H_j \cap W$. Let $w_j = r_j(x)a_j$, where $r_j(x) = \sum_{i=0}^{t_j} d_{j,i}x^i$ and let $g_j(x) = f_j(x)r_j(x) = \sum_{i=s_j}^{s_j+p_j+t_j} y_{j,i}x^i$.

Then $\sum_{i=s_j}^{s_j+p_j+t_j} |y_{j,i}| \leq l$, by hypothesis that g satisfies $e(g) \leq l$. It also follows that $\max\{t_j+s_j+p_j,0\} \leq \max\{t_i+s_i+p_i,0\}_{i=1,\ldots,k}$ and $\min\{s_j,0\} \geq \min\{s_i,0\}_{i=1,\ldots,k}$, and so $\max\{t_j+s_j+p_j,0\} - \min\{s_j,0\} \leq l$. Therefore, by Lemma V.7.4 applied to the subgroup H_j of G_J, we have that $M_j \leq \Delta_{H_j}^{G_j}(l)$ as desired. □

V.8 Distortion of Polynomials

In order to understand distortion in tame subgroups of $\mathbb{Z}$ wr $\mathbb{Z}$, we will introduce the notion of the distortion of a polynomial.

Definition V.8.1. Let R be a subring of a field with a real valuation, and consider the polynomial ring $R[x]$. We will define the norm function $S : R[x] \to \mathbb{R}^+$ which takes any $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ to $S(f) = \sum_{i=0}^{n} |a_i|$. For any $h \in R[x]$ and $c > 0$, we define the distortion of the polynomial h from $\mathbb{N}$ to $\mathbb{N}$ by

$$\Delta_{h,c}(l) = \sup\{S(f) : \deg(f) \leq cl, \text{ and } S(hf) \leq cl\}. \tag{V.24}$$

Remark V.8.2. Taking into account the inequality $S(hf) \leq cl$, one can easily find some explicit upper boundes $C_i = C_i(h,c,l)$ for the modules of the coefficient at x^i of $f(x)$ in Formula (V.8.1), starting with the lowest coefficients. Therefore the supremum in Equation (V.24) is finite. Furthermore, if $R = \mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$ then the supremum is taken over a compact set of polynomials of bounded degree with bounded coefficients, and since S is a continuous function, one may replace sup by max in Definition V.8.1.

Note that the distortion does not depend on the constant c, up to equivalence, and so we will consider $\Delta_h(l)$. We will eventually show that the distortion of a tame subgroup is equivalent to the distortion of the polynomial associated to its generator in W.

V.8.1 Connections Between Subgroup and Polynomial Distortion

The following fact makes concrete our motivation for studying distortion of polynomials.

Lemma V.8.3. *Let H be a tame subgroup, where as usual (cf. Definition V.7.5) $H = \langle b, w\rangle \leq \mathbb{Z}$ wr $\mathbb{Z}$ where $w = h(x)a$ for $h = d_0 + \cdots + d_t x^t \in \mathbb{Z}[x]$. Then*

$$\Delta_h(l) \approx \Delta_H^{\mathbb{Z}\, wr\, \mathbb{Z}}(l).$$

We break the proof of Lemma V.8.3 into two smaller lemmas, each demonstrating one inequality.

Lemma V.8.4. *With all notation as in Lemma V.8.3, we have that $\Delta_H^{\mathbb{Z}\, wr\, \mathbb{Z}}(l) \preceq \Delta_h(l)$.*

Proof. By Lemma V.7.4, we have that $\Delta_H^{\mathbb{Z}\,\mathrm{wr}\,\mathbb{Z}}(l) \approx \delta(l) = \max\{e_H(g) : g \in H \cap W, e(g) \leq l, u(g) \leq l\}$. Let $g = f(x)w \in H \cap W$ be so that $\delta(l) = e_H(g)$. Write $f(x) = \sum_{q=s}^{s+p} z_q x^q$. There exists $n \in \mathbb{Z}$ so that $g_1 = b^n g b^{-n} \in H$ and $g_1 = f_1(x)w$ where $f_1(x)$ is a regular polynomial. It is easy to check that $e_H(g) = e_H(g_1), e(g) = e(g_1)$ and $u(g_1) \leq u(g)$. Now observe that $\deg(f_1) \leq u(g_1) \leq u(g) \leq l$ and $S(hf_1) = e(g_1) = e(g) \leq l$. Therefore, $\Delta_h(l) \succeq S(f_1) = e_H(g_1) = e_H(g) \approx \Delta_H^{\mathbb{Z}\,\mathrm{wr}\,\mathbb{Z}}(l)$. □

Lemma V.8.5. *With all notation as in Lemma V.8.3, we have that $\Delta_H^{\mathbb{Z}\, wr\, \mathbb{Z}}(l) \succeq \Delta_h(l)$*

Proof. By Lemma V.2.8 we have that $H \cong \mathbb{Z}$ wr $\mathbb{Z}$ under the isomorphism $b \mapsto b, w \mapsto a$. We fix the notation that $w_i = b^i w b^{-i}$. Let $\Delta_h(l) = S(f)$ where $f(x) = \sum_{q=0}^{l} z_q x^q$. We have that in the subgroup H, $f(x)h = \sum_{q=0}^{l} z_q w_q$, so by Lemma V.3.3 we have that

$$|f(x)h|_H \geq \sum_{q=0}^{l} |z_q| = S(f) = \Delta_h(l).$$

On the other hand, in $\mathbb{Z}$ wr $\mathbb{Z} = \langle a, b \rangle$ we have that $f(x)w = f(x)h(x)a$. Let $f(x)h(x) = \sum_{i=0}^{l+t} y_i x^i$. Then

$$f(x)h(x)a = \sum_{j=0}^{t+l} y_j a_j.$$

Therefore by Lemma V.3.3,

$$|f_l(x)w|_{\mathbb{Z} \text{ wr } \mathbb{Z}} = \sum_{j=0}^{t+l} |y_j| + 2(t+l) = S(fh) + 2(l+t) \leq 3l + t.$$

Therefore, the distortion of H in $\mathbb{Z}$ wr $\mathbb{Z}$ is at least $\Delta_h(l)$. □

V.9 Lower Bounds on Polynomial Distortion

For the rest of this section and the next, although we are motivated by studying groups, we are only discussing polynomials. Given any polynomial $h = \sum_{j=0}^{t} d_j x^j \in \mathbb{Z}[x], d_o, d_t \neq 0$, we are able to compute the equivalence class of its distortion function.

Lemma V.9.1. *The distortion of h with respect to the ring of polynomials over $\mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$ is bounded from below by $l^{\kappa+1}$, up to equivalence, where c is a complex root of h of multiplicity κ and modulus one.*

Proof. Let c be a complex root of h of multiplicity κ and modulus 1. That is,

$$h(x) = (x-c)^{\kappa} \tilde{h}(x)$$

over $\mathbb{C}$. Let

$$f_l(x) = x^{l-1} + cx^{l-2} + \cdots + c^{l-2}x + c^{l-1}.$$

Then the product

$$h(x) f_l^{\kappa+1}(x) = (x^l - c^l)^{\kappa} \tilde{h}(x) f_l(x)$$

satisfies $S(hf_l^{\kappa+1})$ is $O(l)$, because $f_l(x)$ is $O(l)$. On the other hand, because $|c| = 1$, we have that $S(f_l^{\kappa+1}) \geq |f_l(c)^{\kappa+1}| = l^{\kappa+1}$. This implies that if $c \in \mathbb{R}$; i.e. $c = \pm 1$, then $\Delta_h(l) \succeq l^{\kappa+1}$, where the distortion is considered over $\mathbb{C}$, $\mathbb{R}$ or over $\mathbb{Z}$.

We will show that a similar computation holds over $\mathbb{R}$ or over $\mathbb{Z}$ even in the case when $c \in \mathbb{C} - \mathbb{R}$. Let $\bar{c}$

be the complex conjugate of c. By hypothesis that $c \notin \mathbb{R}$ we know that $\bar{c} \neq c$. Then $\bar{c} = c^{-1}$ is a root of $h(x)$ of multiplicity κ as well, and

$$h(x) = (x-c)^{\kappa}(x-\bar{c})^{\kappa}H(x)$$

where $H(x)$ has real coefficients. Consider the product $f_l(x)\bar{f}_l(x)$, where

$$\bar{f}_l(x) = x^{l-1} + \bar{c}x^{l-2} + \cdots + \bar{c}^{l-1}.$$

A simple calculation shows that each of the coefficients of this product is a sum of the form

$$\sum_{i+j=\kappa} c^i \bar{c}^j = \sum_{i+j=\kappa} c^{i-j} = c^{\kappa} + c^{\kappa-2} + \cdots + c^{-\kappa}.$$

This is a geometric progression with common ratio $c^2 \neq 1$. Therefore, $S(\bar{f}_l) \leq \frac{2}{|1-c^2|}$ and therefore $S(f_l\bar{f}_l)$ is $O(S(f_l)) = O(l)$. This computation implies that the products

$$h(x)f_l^{\kappa+1}(x)\bar{f}_l^{\kappa+1}(x) = (x^l - c^l)^{\kappa}(x^l - \bar{c}^l)^{\kappa}H(x)f_l(x)\bar{f}_l(x)$$

have the sum of modules of coefficients which are $O(l)$.

The polynomial $f_l^{\kappa+1}(x)\bar{f}_l^{\kappa+1}(x)$ has real coefficients. There is a polynomial $F_l(x)$ with integer coefficients such that each coefficient of $F_l(x) - f_l^{\kappa+1}(x)\bar{f}_l^{\kappa+1}(x)$ has modulus at most one. Thus $S(hF_l)$ is also $O(l)$.

We will show that the sums of modules of coefficients of $F_l(x)$ grow at least as $l^{\kappa+1}$ on a subsequence from Remark II.2.6. It suffices to obtain the same property for $f_l^{\kappa+1}(x)\bar{f}_l^{\kappa+1}(x)$. Since $|c| = 1$, we have that the sum of modules of the coefficients of $f_l^{\kappa+1}(x)\bar{f}_l^{\kappa+1}(x)$ is at least

$$|f_l^{\kappa+1}(c)\bar{f}_l^{\,\kappa+1}(c)| = l^{\kappa+1}|\bar{f}_l^{\,\kappa+1}(c)|.$$

We will show that there exists a subsequence $\{l_i\}$ so that on this sequence,

$$|\bar{f}_{l_i}^{\kappa+1}(c)| \geq \frac{1}{2}.$$

We have that

$$\bar{f}_l(c) = c^{l-1} + c^{l-2}\bar{c} + \cdots + \bar{c}^{l-1} = c^{l-1} + c^{l-3} + \cdots + c^{1-l}$$

because $\bar{c} = c^{-1}$. Therefore $|\bar{f}_l(c)| = |1 + c^2 + \cdots + c^{2l-2}|$ and similarly, $|\bar{f}_{l+1}(c)| = |1 + c^2 + \cdots + c^{2l}|$. One of these two numbers must be at least one half because $|\bar{f}_l(c) - \bar{f}_{l+1}(c)| = |c^{2l}| = 1$. Thus either l or $l+1$ can be included in the sequence $\{l_i\}$, and all required properties are shown. □

V.10.1 Some Linear Algebra

In order to obtain upper bounds on distortion of polynomials we require some facts from linear algebra. Fix an integer $k \geq 1$ and let $n > 0$ be arbitrary.

Lemma V.10.1. *Let $Y_1, \ldots, Y_n, C_2, \ldots, C_n$ be $k \times 1$ column vectors. Suppose that the sum of the modules of all coordinates of $C_2, \ldots, C_n$ is bounded by some constant c, and that the modulus of each coordinate of Y_1 and Y_n is also bounded by c. Suppose further we have the relationship*

$$Y_i = AY_{i-1} + C_i, i = 2, \ldots, n \tag{V.25}$$

where A is a $k \times k$ matrix, in Jordan normal form, having only one Jordan block. Then the modulus of each coordinate of arbitrary $Y_i, 2 \leq i \leq n-1$ is bounded by dcn^{k-1} where d depends on A only. In the case where the eigenvalue of A does not have modulus one, the modulus of each coordinate of arbitrary $Y_i, 2 \leq i \leq n-1$ is bounded by cd, where d depends on A only. All matrix entries are assumed to be complex.

Proof. Let λ be the eigenvalue of A, so that $A = \begin{pmatrix} \lambda & 0 & 0\ldots & 0 \\ 1 & \lambda & 0\ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0\ldots & 1 & \lambda \end{pmatrix}$.

We will consider cases.

- First suppose that $|\lambda| < 1$.

From Formula (V.25) we derive:

$$\begin{aligned} Y_i &= A(AY_{i-2} + C_{i-1}) + C_i = (A)^2 Y_{i-2} + AC_{i-1} + C_i = \cdots \\ &= (A)^{i-1} Y_1 + (A)^{i-2} C_2 + \cdots + AC_{i-1} + C_i. \end{aligned} \tag{V.26}$$

The following formula for A^r is well-known because A is assumed to be a Jordan block; it may also be checked easily using induction. We have that

$$A^r = \begin{pmatrix} \lambda^r & 0 & 0\ldots & 0 \\ r\lambda^{r-1} & \lambda^r & 0\ldots & 0 \\ \frac{r(r-1)}{2!}\lambda^{r-2} & r\lambda^{r-1} & \lambda^r\ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{r!}{(r-(k-1))!(k-1)!}\lambda^{r-(k-1)}\ldots & \frac{r(r-1)}{2!}\lambda^{r-2} & r\lambda^{r-1} & \lambda^r \end{pmatrix},$$

with the understanding that if $r < k-1$, any terms of the form $\binom{r}{j}\lambda^{r-j}$ where $r < j$ are 0. Arbitrary nonzero element of the matrix A^r is of the form $\binom{r}{j}\lambda^{r-j}$ for some $j \leq k-1$. Let $a_{s,t}(r)$ denote the s,t entry of A^r.

Then $a_{s,t}(r)$ is either zero or of the form $\binom{r}{j}\lambda^{r-j}$ for some $0 \le j \le k-1$ depending on s and t. Then

$$\sum_{r=1}^{i} |a_{s,t}(r)| \le \sum_{r=1}^{\infty} |a_{s,t}(r)| = \sum_{r=1}^{\infty} |\binom{r}{j}\lambda^{r-j}|$$

which is a constant depending on A and not on i, because the series on the right is convergent when $|\lambda| < 1$. Let

$$c_1 = \max_{1\le s,t\le k}\{\sum_{r=1}^{\infty} |a_{s,t}(r)|\}.$$

Let $\bar{A}$ be the $k \times k$ matrix whose s,t entry is $\sum_{r=1}^{\infty} |a_{s,t}(r)|$, and the column $\bar{C}$ be obtained by placing in the j^{th} row the sum of the modules of the entries of the j^{th} row of all C_i and Y_1. Then every entry of $\bar{C}$ is bounded by $2c$. Observe that the modulus of every entry in the right side of (V.26) is bounded by an entry of $\bar{A}\bar{C}$, which is in turn bounded by $2cc_1$, which does not include any power of n at all.

- Let $|\lambda| > 1$.

Because λ^{-1} is an eigenvalue of A^{-1}, there exists a decomposition $A^{-1} = SJS^{-1}$ where

$$J = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0\dots & 0 \\ 1 & \frac{1}{\lambda} & 0\dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0\dots & 1 & \frac{1}{\lambda} \end{pmatrix}.$$

Letting $Y_i' = S^{-1}Y_i$ and $C_i' = S^{-1}C_i$ we see by Equation (V.25) that

$$Y_{n-r}' = J^r Y_n' + J^r C_n' + J^{r-1}C_{n-1}' + \cdots + JC_{n-r+1}',$$

for $r = 1,\dots,n-2$. Observe that the sum of modules of coordinates of Y_{n-r}' is less than or equal to ksc, where s depends on S (and hence on A) only. Similarly, the sum of all modules of all coordinates of $C_2',\dots,C_n'$ is bounded above by ksc. This case now follows just as the previous one to obtain constant upper bounds on the modules of the entries in $Y_2',\dots,Y_{n-1}'$. Finally, the modulus of any coordinate of Y_{n-r} is bounded by ks times the modulus of a coordinate of Y_{n-r}'.

- Let $|\lambda| = 1$.

  In this case, we have that

$$|\binom{r}{j}\lambda^{r-j}| = \binom{r}{j} = \frac{r(r-1)\cdots(r-(j-1))}{j!}$$

$$\le r(r-1)\cdots(r-(j-1)) \le r^j \le n^{k-1}.$$

It follows from Equation $(V.26)$ that every entry of Y_i is bounded above by $2cn^{k-1}$. □

Lemma V.10.2. *Let $Y_1,\dots,Y_n,C_2,\dots,C_n$ be $k\times 1$ column vectors. Suppose that the sum of the modules of all coordinates of $C_2,\dots,C_n$ is bounded by some constant c, and that the modulus of each coordinate of Y_1 and Y_n is also bounded by c. Suppose further we have the relationship*

$$Y_i = AY_{i-1} + C_i, i = 2,\dots,n$$

where A is a $k\times k$ matrix. Then the modulus of each coordinate of arbitrary $Y_i, 2\le i\le n-1$ is bounded by $dcn^{\kappa-1}$ where d depends on A only, and $\kappa\le k$ is the maximal size of any Jordan block of the Jordan form of A having eigenvalue with modulus one.

Proof. There exists a Jordan decomposition, $A = SA'S^{-1}$.

Let $S^{-1} = (s_{i,j})_{1\le i,j\le k}$ and let $s = \max|s_{i,j}|$. Then for $C_i' = S^{-1}C_i$ and $Y_i' = S^{-1}Y_i$ we have that

$$Y_i' = A'Y_{i-1}' + C_i'. \tag{V.27}$$

By hypothesis, the sum of the modules of all coordinates of $C_2',\dots,C_n'$ is bounded by $ksc = c'$ and the coordinates of Y_1' and Y_n' are bounded by c' as well. As we will explain, our problem can be reduced to the similar problem for Y_i' in (V.27). Suppose that the modules of coordinates of every Y_i' are bounded by $dc'n^{\kappa-1}$ where d depends on A only. Then, letting $S = (\tilde{s_{i,j}})_{1\le i,j\le k}$ and $\tilde{s} = \max|\tilde{s_{i,j}}|$ we have by definition of Y_i' that arbitrary element of Y_i has modulus bounded above by $k\tilde{s}dc'n^{\kappa-1} = d'cn^{\kappa-1}$ where $d' = k^2s\tilde{s}d$ only depends on A', as required.

Lemma V.10.1 says that if A' has only one Jordan block, then the bound is constant if the eigenvalue does not have modulus one. Otherwise, we have in this case that $k=\kappa$ and the claim is true. If there is more than one Jordan block present in A', the problem is decomposed into at most k subproblems, each with only one Jordan block of size smaller than k. Again, we are done by Lemma V.10.1. □

We will use Lemma V.10.2 to prove the following fact, which requires establishing some notation prior to being introduced. Let $d_0,\dots,d_t\in\mathbb{Z}$ where $d_0,d_k\neq 0$. Let the $(n+k)\times n$ matrix M have j^{th} column, for $j=1,\dots,n$, given by $[0,\dots,0,d_0,d_1,\dots,d_k,0,\dots,0]^T$, where d_0 first appears as the j^{th} entry in this j^{th} column. Given the matrix M, we may also construct the matrix

$$A = \begin{pmatrix} 0 & 1 & 0\dots & 0 \\ 0 & 0 & 1\dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0\dots & 0 & 1 \\ a_1 & a_2\dots & a_{k-1} & a_k, \end{pmatrix} \tag{V.28}$$

where $a_j = -\frac{d_{k-j+1}}{d_0}$, for $j=1,\dots,k$. Let κ be the maximal size of a Jordan block of the Jordan form of A having eigenvalue with modulus one.

Lemma V.10.3. *Suppose that $X = [x_1,x_2,\dots,x_n]^T$ is a solution to the system of equations $MX = B$, where $B = [b_1,b_2,\dots,b_{n+k}]^T$. Then it is possible to bound the modules of all coordinates $x_1,\dots,x_n$ of the vector*

X such that $|x_i| \leq cbn^{\kappa-1}$ *where* $b = \sum_j \{|b_j|\}$ *for* $1 \leq j \leq n+k$ *and the constant* c *depends upon* $d_0, \ldots, d_k$ *only.*

Prior to proving Lemma V.10.3 we prove an easier special case.

Lemma V.10.4. *It is possible to bound the coordinates* $x_1, \ldots, x_k$ *of the vector* X *from Lemma V.10.3 from above by* $b\tilde{\gamma}$ *where* $b = \sum_j \{|b_j|\}$ *and* $\tilde{\gamma} = \tilde{\gamma}(d_0, \ldots, d_{k-1})$.

Proof. By Cramer's Rule, we have the explicit formula that

$$|x_i| = \left| \frac{\det(L_i)}{\det(L)} \right|$$

where L is the $k \times k$ upper left submatrix of M corresponding to the first k equations, and L_i is obtained by replacing column i in L with $[b_1, \ldots, b_k]^T$. Because $\det(L) = d_0^k$, it suffices to show that the desired bounds exist for $\det(L_i)$; that is, we must show that there exists a constant $\tilde{\gamma}$ depending on $d_0, \ldots, d_{k-1}$ only such that $|\det(L_i)| \leq b\tilde{\gamma}$ for $i = 1, \ldots, k$. By expanding along the i^{th} column in L_i, we find that

$$\det(L_i) = \pm b_1 f_1(d_0, \ldots, d_{k-1}) \pm b_2 f_2(d_0, \ldots, d_{k-1}) \pm \cdots \pm b_k f_k(d_0, \ldots, d_{k-1}),$$

where for each $i = 1, \ldots, k$, f_i is a function of $d_0, \ldots, d_{k-1}$ only obtained as the determinant of a submatrix containing none of $b_1, \ldots, b_k$. The proof is complete by the triangle inequality. $\square$

Note that the $|x_j|$ for $j = n-k+1, \ldots, n$ are similarly bounded by $b\overline{\gamma}$ for the same b and some $\overline{\gamma} = \overline{\gamma}(d_0, \ldots, d_{k-1})$ as in Lemma V.10.4. It is clear according to Lemma V.10.4 that we may assume that $|x_i| \leq b\gamma$ for the same $\gamma = \gamma(d_0, \ldots, d_{k-1})$ for all $i = 1, \ldots, k, n-k+1, \ldots, n$.

We proceed with the Proof of Lemma V.10.3.

Proof. It suffices to obtain upper bounds for $|x_i|$ when $n-k \geq i \geq k+1$.

For such indices, we have that

$$d_k x_{i-k} + d_{k-1} x_{i+1-k} + \cdots + d_0 x_i = b_i.$$

In other words,

$$x_i = \xi_i + a_1 x_{i-k} + a_2 x_{i+1-k} + \cdots + a_k x_{i-1},$$

where $\xi_i = \frac{b_i}{d_0}$ and $a_j = -\frac{d_{k-j+1}}{d_0}$. Let $X_i = [x_{i-k+1}, \ldots, x_i]^T$ and let $\Xi_i = [0, \ldots, 0, \xi_i]^T$. Then for the matrix A of Equation (V.28) we have the recursive relationship

$$X_i = AX_{i-1} + \Xi_i$$

for $i = k, \ldots, n$. Observe that the matrix A depends on $d_0, \ldots, d_k$ only, and that the sum of modules of the entries in all Ξ_i are bounded by $\frac{b}{|d_0|}$.

We see by Lemma V.10.4 that Lemma V.10.2 applies to our situation. Therefore, the modules of coordinates of arbitrary X_i, $k+1 \leq i \leq n-k$ are bounded by $dc(n-k+1)^{\kappa-1} \leq dcn^{\kappa-1}$, where d depends only on $d_0,\ldots,d_k$, $c = \max\{\frac{b}{|d_0|}, \gamma b\}$. □

Lemma V.10.5. *Let $h(x) = d_0 + \cdots + d_t x^t$, where $d_0, d_t \neq 0$. Then the distortion of h is at most $l^{\kappa+1}$ where κ is the maximal size of a Jordan block of the Jordan form of*

$$A = \begin{pmatrix} 0 & 1 & 0\ldots & 0 \\ 0 & 0 & 1\ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0\ldots & 0 & 1 \\ -\frac{d_t}{d_0} & -\frac{d_{t-1}}{d_0}\ldots & -\frac{d_2}{d_0} & -\frac{d_1}{d_0} \end{pmatrix}$$

of Equation (V.28) *with eigenvalue having modulus one.*

Proof. Consider any $f = \sum_{q=s}^{s+p} z_q x^q$ as in Definition V.8.1. Then consider $hf = \sum_{j=s}^{s+p+t} y_j x^j$. The coefficients y_j are given by the matrix equation $MZ = Y$, where $Z = [z_s,\ldots,z_{s+p}]^T, Y = [y_s,\ldots,y_{s+p+t}]^T$ and

$$M = \begin{pmatrix} d_0 & 0 & 0 & \ldots & 0 \\ d_1 & d_0 & 0 & \ldots & 0 \\ d_2 & d_1 & d_0 & \ldots & 0 \\ \vdots & \ldots & \ddots & \ddots & \vdots \\ d_t & d_{t-1} & \ldots & d_1\ldots & 0 \\ 0 & d_t & \ldots & d_2\ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & d_t & d_{t-1} \\ 0 & \ldots & 0 & 0 & d_t \end{pmatrix}$$

is an $(p+t+1)\times(p+1)$ matrix.

By Lemma V.10.3 we have that for each $q = s,\ldots,s+p$ that $|z_q| \leq cy(p+1)^{\kappa-1}$ where $c = c(d_0,\ldots,d_t), y = \sum_j |y_j| \leq l$. Therefore,

$$\Delta_h(l) \leq S(f) = \sum_{q=s}^{s+p} |z_q| \leq c(l+1)^{\kappa+1}.$$

□

The following Lemma shows that the upper and lower bounds are the same, and so we can compute exactly the distortion of a polynomial.

Lemma V.10.6. *Let $h(x) = d_0 + \cdots + d_t x^t$ be a polynomial in $\mathbb{Z}[x]$. Then the distortion of h is equivalent to a polynomial. Further, the degree of this polynomial is exactly one plus the maximal multiplicity of a root of $h(x)$ having modulus one.*

Proof. Lemma V.9.1 shows that the distortion is bounded from below by the one plus the maximal multiplicity of a root of $h(x)$ having modulus one. We will show that the upper bounds of one greater than the maximal size of a Jordan block of the Jordan form of A with eigenvalue having modulus one obtained in Remark V.10.5 are the same as these lower bounds. The characteristic polynomial of the matrix A equals $x^t + \frac{d_1}{d_0}x^{t-1} + \cdots + \frac{d_{t-1}}{d_0}x + \frac{d_t}{d_0}$. Therefore, $d_0 h(x)$ is the characteristic polynomial for A^{-1}. If c is an eigenvalue of A with modulus one, then c^{-1} is an eigenvalue of A^{-1} also having modulus one. Therefore any Jordan block for A corresponding to an eigenvalue c with modulus one has size which does not exceed the multiplicity of the root c^{-1} in the polynomial h. The roots c and c^{-1} where $|c| = 1$ have equal multiplicities in h. □

Lemma V.10.7. *Any finitely generated subgroup H of A wr $\mathbb{Z}$ where A is finitely generated abelian has distortion equivalent to the distortion of a tame subgroup of $\mathbb{Z}$ wr $\mathbb{Z}$.*

This follows by combining what has already been established: Lemmas V.7.2, V.7.6, V.6.11, V.10.6.

V.11 Proof of the Main Theorem

Theorem I.3.4 Part (1) follows from Lemmas V.10.7, V.8.3 and V.10.6.

It follows by Lemma V.7.2 that all finitely generated subgroups in A wr $\mathbb{Z}$ where A is finite abelian are undistorted. For in this case, $k = 0$ and so $F(l)$ is linear. Therefore, Theorem I.3.4 Part (2) is also proved.

Now we complete the proof of Theorem I.3.4, Part (3). Let A be a finitely generated abelian group of rank k. Consider the 2-generated subgroup $H \leq \mathbb{Z}$ wr $\mathbb{Z}$ constructed as follows. Let $m \in \mathbb{N}$. Consider $h(x) = (1-x)^{m-1}$. Then the distortion of the polynomial h is seen to be equivalent to l^m, by Lemma V.10.6. By Lemma V.8.3, this means that the 2-generated subgroup $\langle b, (1-x)^{m-1}a\rangle = H_m \leq \mathbb{Z}$ wr $\mathbb{Z}$ has distortion $\Delta_{H_m}^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l) \approx l^m$. The subgroup $\mathbb{Z}$ wr $\mathbb{Z}$ is a retract of A wr $\mathbb{Z}$ if A is infinite. Therefore, the distortion of H_m in $\mathbb{Z}$ wr $\mathbb{Z}$ and in A wr $\mathbb{Z}$ are equivalent by Lemma V.2.2.

Remark V.11.1. If we adopt the notation that the commutator $[a,b] = aba^{-1}b^{-1}$, then we see that in $\mathbb{Z}$ wr $\mathbb{Z}$, the element of W corresponding to the polynomial $(1-x)^{m-1}a$ is $[\cdots[a,b],b],\cdots,b]$ where the commutator is $(m-1)$-fold. This explains Corollary V.1.2.

CHAPTER VI

LENGTH FUNCTIONS FOR SEMIGROUPS

VI.1 Introduction

Following the work done in [O] for groups, we describe, for a given semigroup S, which functions $l : S \to \mathbb{N}$ can be realized up to equivalence as length functions $g \mapsto |g|_H$ by embedding S into a finitely generated semigroup H. We also, following the work done in [O2] and [OS], provide a complete description of length functions of a given finitely generated semigroup with enumerable set of relations inside a finitely presented semigroup.

It has recently come to my attention that some of these results have been independently obtained in [E], by using the same method of proof. However, we still include them in this thesis, because at the time I was working on them, I did not know they had been proved already. These results are included in the dissertation with the knowledge of Ershov.

VI.1.1 Preliminaries

Let S be an arbitrary semigroup (without signature identity element) with a finite generating set $\mathscr{A} = \{a_1, \dots, a_m\}$.

Definition VI.1.1. The length of an element $g \in S$ is $|g| = |g|_{\mathscr{A}}$ is the length of the shortest word over the alphabet $\mathscr{A}$ which represents the element g, where for any word W in $\mathscr{A}$ we define its length $||W||$ to be the number of letters in W.

Observe that if the semigroup S is embedded into another finitely generated semigroup H with a generating system $\mathscr{B} = \{b_1, \dots, b_k\}$, then for any $g \in S$ we have

$$|g|_{\mathscr{B}} \leq c|g|_{\mathscr{A}} \tag{VI.1}$$

with the constant $c = \max\{|a_1|_{\mathscr{B}}, \dots, |a_m|_{\mathscr{B}}\}$ independent of g. Motivated by inequality (VI.1), we introduce the following notion of equivalence.

Definition VI.1.2. Let $l_1, l_2 : S \to \mathbb{N} = \{1, 2, 3, \dots\}$. We say that l_1 and l_2 are equivalent, $l_1 \approx l_2$, if there exist constants $c_1, c_2 > 0$ such that

$$c_1 l_1(g) \leq l_2(g) \leq c_2 l_1(g)$$

for all $g \in S$.

The discussion above implies that the word length in S does not depend up to equivalence on the choice of finite generating set.

We will also be considering a semigroup analogue of the notion of distortion. We say that an embedding of one semigroup H with finite generating system $\mathscr{B}$ into another semigroup R with finite generating system

$\mathscr{T}$ is undistorted if

$$(|\cdot|_{\mathscr{T}})\restriction_H \approx |\cdot|_{\mathscr{B}}.$$

Otherwise, the embedding is distorted. The notion is clearly independent of the choice of finite generating sets $\mathscr{B}$ and $\mathscr{T}$.

Definition VI.1.3. The semigroup S is said to have the presentation

$$\langle X | A_i = B_i, i \in I \rangle$$

in terms of generators and defining relations if the set S is the quotient of the free semigroup on the set X by the congruence relation generated by the set $\{A_i = B_i\}_{i \in I}$.

Two words x and y are equal in S if and only if there is a finite chain

$$w_0 \to w_1 \to \cdots \to w_k$$

where $w_0 \equiv x, w_1 \equiv y$ and $w_{j-1} \to w_j$ means that the word w_j was obtained from w_{j-1} by replacing some subword of the form $A_i = B_i, i \in I$.

VI.1.2 Statement of Main Results

The main goal of this note is to prove an analog of Theorem 1 in [O] for semigroups. The necessary conditions for distortion functions of semigroups are as follows. The main result of this article is the sufficiency of said conditions.

Lemma VI.1.4. *Let S be a semigroup and $l : S \to \mathbb{N}$ a function defined by some embedding of the semigroup S into a semigroup H with a finite generating system $\mathscr{B} = \{b_1, \ldots, b_k\}$; that is, $l(g) = |g|_{\mathscr{B}}$. Then*

(D1) $l(gh) \le l(g) + l(h)$ for all $g, h \in S$;

(D2) There exists a positive number a such that card$\{g \in S : l(g) \le r\} \le a^r$ for any $r \in \mathbb{N}$.

Proof. The condition $(D1)$ is obvious. To prove the condition $(D2)$ it will suffice to take $a = k+1$. This follows because the number of all words in $\mathscr{B}$ having length $\le r$ is not greater than $(k+1)^r$. □

We establish the notation that the (D) condition refers to conditions $(D1)$ and $(D2)$ of Lemma VI.1.4.

Theorem VI.1.5. *1. For any semigroup and any function $l : S \to \mathbb{N}$ satisfying the (D) condition, there is an embedding of S into a 2-generated semigroup H with generating set $\mathscr{B} = \{b_1, b_2\}$, such that the function $g \to |g|_{\mathscr{B}}$ is equivalent to the function l.*

2. For any semigroup S and any function $l : S \to \mathbb{N}$ satisfying the (D) condition, there is an embedding of S into a finitely generated semigroup K with finite generating set $\mathscr{C}$ such that the function $g \to |g|_{\mathscr{C}}$ is equal to the function l.

Corollary VI.1.6. *1. Let g be an element such that g generates as infinite subsemigroup in a semigroup H with finite generating set $\mathscr{B} = \{b_1, \ldots, b_k\}$; i.e. card$\{g^n\}_{n\in\mathbb{N}} = \infty$. Denote $l(i) = |g^i|_{\mathscr{B}} = |g^i|$ for $i \in \mathbb{N}$. Then*

(C1) $l(i+j) \le l(i) + l(j)$ for all $i, j \in \mathbb{N}$ (l is subadditive);

(C2) There exists a positive number a such that card$\{i \in \mathbb{N} : l(i) \le r\} \le a^r$ for any $r \in \mathbb{N}$.

2. *For any function $l : \mathbb{N} \to \mathbb{N}$, satisfying conditions $(C1)$ and $(C2)$, there is a 2-generated semigroup H and an element $h \in H$ such that $|h^i|_H \approx l(i)$.*

3. *For any function $l : \mathbb{N} \to \mathbb{N}$, satisfying conditions $(C1)$ and $(C2)$, there is a finitely generated semigroup K and an element $k \in K$ such that $|k^i|_K = l(i)$.*

We observe that the main result of [O2] also holds for semigroups.

Theorem VI.1.7. *Let l be a computable function with properties $(D1)-(D2)$ on a semigroup S. Suppose further that S has enumerable set of defining relations. Then S can be isomorphically embedded into some finitely presented semigroup R in such a way that the function l is equivalent to the restriction of $|\ |_R$ to S.*

This Theorem will be proved in Section VI.4.

Example VI.1.8. Because the function $l : \mathbb{N} \to \mathbb{N} : i \mapsto \lceil i^{\pi - e} \rceil$ is computable (π and e being computable numbers) and satisfies the (D) condition, we have by Theorem VI.1.7 that there exists a finitely presented semigroup R and an element $r \in R$ such that $|r^i|_R \approx l(i)$.

Theorem VI.1.7 fails to provide a complete description of length functions of a given finitely generated semigroup with enumerable set of relations inside finitely presented semigroups. In [OS], the corresponding question was answered for groups, by extending the (D) condition. We obtain a semigroup analog of the main result in [OS] as follows.

We use the notation that F_m is an absolutely free semigroup of rank m. Given an m-generated semigroup S, and a function $l : S \to \mathbb{N}$, we may obtain the natural lift function $l^* : F_m \to \mathbb{N}$.

Definition VI.1.9. Let S be an m-generated semigroup, and $l : S \to \mathbb{N}$. We say that l satisfies condition $(D3)$ if there exists a natural number n and a recursively enumerable set $T \subset F_m \times F_n$ such that

1. $(v_1, u), (v_2, u) \in T$ for some words v_1, v_2, u then v_1 and v_2 represent the same element in S.

2. If v_1 and v_2 represent the same element in S then there exists an element u such that $(v_1, u), (v_2, u) \in T$.

3. $l^*(v) = \min\{||u|| : (v, u) \in T\}$ for every $v \in F_m$.

Theorem VI.1.10. *Let S be a finitely generated subsemigroup of a finitely presented semigroup H. Then the corresponding length function on S satisfies conditions $(D1)-(D3)$. Conversely, for every finitely generated semigroup S and function $l : S \to \mathbb{N}$ satisfying conditions $(D1)-(D3)$, there exists an embedding of S into a finitely presented semigroup H such that the length function $g \to |g|_H$ is equivalent to l, in the sense of Definition VI.1.2.*

This Theorem will be proved in Section VI.4.

When S has solvable word problem, the condition $(D3)$ can be replaced by a simpler condition.

Definition VI.1.11. The graph of a function $l^* : F_m \to \mathbb{N}$ is the set $\{(w, l^*(w)) : w \in F_m\}$. A pair (w,k) is said to lie above the graph of l^* if $l^*(w) \leq k$.

We observe that the following result of [OS] also holds in the semigroup setting. In fact, the proof uses no special properties of groups such as existence of identity element or inverses and so goes through immediately and directly.

Theorem VI.1.12. *Let S be a finitely generated semigroup with decidable word problem. Then the function $l : g \mapsto |g|_H$ given by an embedding of S into a finitely presented semigroup H satisfies the conditions $(D1)-(D2)$ as well as the following condition:*

$(D3')$ The set of pairs above the graph of l^* is recursively enumerable.

Conversely, for every function $l : S \to \mathbb{N}$ satisfying $(D1), (D2)$ and $(D3')$, there exists an embedding of S into a finitely presented semigroup H such that the corresponding length function on S is equivalent (in the sense of Definition VI.1.2) to l.

The following Corollary follows from Theorem VI.1.12 and reminds us of the statement of Corollary VI.1.6.

Corollary VI.1.13.

1. *Let g be an element generating an infinite subsemigroup in a finitely presented semigroup H with generating set $\mathscr{B} = \{b_1, \ldots, b_k\}$. Denote $l(i) = |g^i|_{\mathscr{B}} = |g^i|$ for $i \in \mathbb{N}$. Then*
   - *(C1) $l(i+j) \leq l(i) + l(j)$ for all $i, j \in \mathbb{N}$ (l is subadditive);*
   - *(C2) There exists a positive number a such that card$\{i \in \mathbb{N} : l(i) \leq r\} \leq a^r$ for any $r \in \mathbb{N}$.*
   - *(C3) The set of natural pairs above the graph of l is recursively enumerable.*
2. *Conversely, For any function $l : \mathbb{N} \to \mathbb{N}$, satisfying conditions $(C1)-(C3)$, there is a finitely presented semigroup H and an element $g \in H$ such that $|g^i|_H \approx l(i)$.*

VI.2 Exponential Sets of Words

Definition VI.2.1. Let $\mathscr{X}$ be a set of words over the alphabet
$\mathscr{A} = \{a_1, \ldots, a_m\}$. We call $\mathscr{X}$ exponential if there are constants N and $c > 1$ such that

$$\text{card}\{X \in \mathscr{X} : ||X|| \leq i\} \geq c^i$$

for every $i \geq N$.

Definition VI.2.2. A collection $\mathscr{Y}$ of words satisfies the overlap property if whenever $Y,Z \in \mathscr{Y}$ we have that

$$Y \text{ is not a proper subword of } Z \text{ and} \tag{VI.2}$$

$$U \text{ nonempty, } Y \equiv UV \text{ and } Z \equiv WU \text{ implies } Y \equiv U \equiv Z \tag{VI.3}$$

where $\equiv$ represents letter-for-letter equality.

Lemma VI.2.3. *There exists an exponential set of words in the alphabet $\{b_1, b_2\}$ satisfying the overlap property of Definition VI.2.2.*

Proof. Consider the set $\mathscr{M}$ of all words

$$\{b_1^3 V b_2^3 : V \equiv b_2 V' b_1 \text{ contains neither } b_1^3 \text{ nor } b_2^3 \text{ as a subword.}\}$$

This set does satisfiy the overlap property of Definition VI.2.2. Condition (VI.2) is satisfied because if $Y,Z \in \mathscr{M}$ and Y is a subword of $Z \equiv W_1 Y W_2$, then we have that b_1^3 is a prefix of both Y and Z. However, the only time that b_1^3 can occur in a word in $\mathscr{M}$ is at the very beginning. Therefore, W_1 is empty. Similarly, W_2 is empty. Condition (VI.3) is satisfied because if $Y \equiv UV$ and $Z \equiv WU$ then the prefix of U must be b_1 and the suffix of U must be b_2, say $U = b_1 U' b_2$. This implies that $b_1 U' b_2 V \equiv W b_1 U' b_2 \equiv b_1^3 V' b_2^3$ for some V'. Therefore, $U \equiv b_1^3 V'' b_2^3$, for some V'' which implies that both V and W are empty.
We will verify that $\mathscr{M}$ is an exponential set. Consider the set

$$M_i = \{x \in \mathscr{M} : ||x|| \leq i\}.$$

Consider a word $x \equiv b_1^3 b_2 b_2^{\beta_1} b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_2^{\alpha_n} b_1^{\beta_2} b_1 b_2^3$ where $\beta_j \in \{0,1\}$ for $j = 1,2$ and $\alpha_j \in \{1,2\}$ for $j \in \{1,\ldots,n\}$, and $n = \frac{i-10}{2}$. Such a word has $||x|| \leq 10 + 2n = i$ so $x \in M_i$. If $i > N = 12$, then there exists $c > 1$ satisfying $2^{\frac{i-6}{2i}} > c$. This implies that $\text{card}(M_i) \geq 2^{\frac{i-6}{2}} \geq c^i$ for all $i \geq N$. □

Lemma VI.2.4. *Let $\mathscr{M}$ be an exponential set satisfying the overlap property. Suppose $V \equiv X_1 X_2 \cdots X_t \equiv SY_1 Y_2 \cdots Y_m T$ where $m,t \geq 1$ and $X_n, Y_j \in \mathscr{M}$ for all $1 \leq n \leq t, 1 \leq j \leq m$. Then there exists an $i \leq t$ such that $S \equiv X_1 \cdots X_{i-1}, T \equiv X_{i+m} \cdots X_t$ and $Y_j \equiv X_{-1+i+j}$ for $j = 1,\ldots,m$.*

Proof. Because $X_1 X_2 \cdots X_t \equiv SY_1 Y_2 \cdots Y_m T$ is letter-for-letter equality, we know that the first letter, u, in Y_1 also occurs in X_i for some i. We proceed by considering cases. If u is also the first letter in X_i then either X_i is a subword of Y_1 or vice-versa. In either of these cases, by condition (VI.2), we have that $X_i \equiv Y_1$. Now suppose that u is not the first letter in X_i. If Y_1 is a subword of X_i then we apply condition (VI.2) again. Otherwise, a suffix of X_i must equal a prefix of Y_1, which implies by condition (VI.3) that $X_i \equiv Y_1$. Now consider Y_2. We know that the first letter of Y_2 must also be the first letter of X_{i+1}. Therefore, one is a subword of the other, so by condition (VI.2) we obtain that $Y_2 \equiv X_{i+1}$. The same argument shows that $Y_j \equiv X_{-1+i+j}$ for $j = 1,\ldots,m$ hence $T \equiv X_{i+m} \cdots X_t$ and $S \equiv X_1 \cdots X_{i-1}$. □

Lemma VI.2.5. *Let $\mathscr{M}$ be an exponential set of words over a finite alphabet $\{a_1,\ldots,a_m\}$. Then for a given function $l : S \to \mathbb{N}$ satisfying the (D) condition, there is a constant $d = d(\mathscr{M},l)$ such that there exists an injection $S \to \mathscr{M} : g \mapsto X_g \in \mathscr{M}$ satisfying*

$$l(g) \leq ||X_g|| < dl(g), g \in S. \tag{VI.4}$$

Proof. A proof can be found in [O] for the case where words are considered in a positive alphabet and hence it holds for semigroups as well. □

VI.3 Constructing the Embedding

We begin by fixing some notation. Let $\mathscr{M}$ be the exponential set of words in the alphabet $\mathscr{B} = \{b_1,b_2\}$ obtained in Lemma VI.2.3. Let S be a semigroup and $l : S \to \mathbb{N}$ a function satisfying the (D) condition. Let $d = d(\mathscr{M},l)$ and $\mathscr{X} = \{X_g\}_{g\in S} \subset \mathscr{M}$ be the constant and exponential subset guaranteed by Lemma VI.2.5 and satisfying the inequality (VI.4).

The semigroup S is a homomorphic image of the free semigroup F_S with basis $\mathscr{A} = \{x_g\}_{g\in S}$ under the epimorphism $\varepsilon : x_g \mapsto g$. Let $\rho = \ker(\varepsilon)$. Therefore, $S \cong F_S/\rho$, and ρ provides all relations which hold in S. Let

$$R = \{(x_h, x_{h'}x_{h''}) : h = h'h'' \text{ in } S\}.$$

Then R represents the relations of S arising from its multiplication table.

Lemma VI.3.1. *The semigroup S has presentation $\langle \mathscr{A} | R\rangle$.*

Proof. We must show that the congruence ρ coincides with $R^\sharp$, the unique smallest congruence relation on F_S containing R. By definition of kernel the congruence ρ equals

$$\{(x_{g_1}\cdots x_{g_n}, x_{h_1}\cdots x_{h_m}) \in F_S \times F_S : \varepsilon(x_{g_1}\cdots x_{g_n}) = \varepsilon(x_{h_1}\cdots x_{h_n})\} =$$

$$\{(x_{g_1}\cdots x_{g_n}, x_{h_1}\cdots x_{h_m}) : g_1\cdots g_n = h_1\cdots h_m \text{ in } S\}.$$

To see that $\rho \supseteq R^\sharp$ is easy, because ρ clearly contains R, and hence ρ is at least as large as the smallest congruence on F_S containing R. Conversely, take an arbitrary element $(x_{g_1}\cdots x_{g_n}, x_{h_1}\cdots x_{h_m}) \in \rho$. We will show this element is also in $R^\sharp$. Because $R^\sharp$ is an equivalence relation, it suffices to show that $(x_{g_1}\cdots x_{g_n}, x_g) \in R^\sharp$, where $g = g_1\cdots g_n$ in S. By induction, we may assume that $(x_{g_1}\cdots x_{g_{n-1}}, x_{g_1\cdots g_{n-1}}) \in R^\sharp$. Then because $R^\sharp$ is left compatible, we have that $(x_{g_1}\cdots x_{g_n}, x_{g_1\cdots g_{n-1}}x_{g_n}) \in R^\sharp$. By definition of R we also have that $(x_{g_1\cdots g_{n-1}}x_{g_n}, x_g) \in R^\sharp$. Therefore, $(x_{g_1}\cdots x_{g_n}, x_g) \in R^\sharp$ as required. □

Consider the following commutative diagram:

$$\begin{tikzcd}
F_S \arrow[rr, "\varepsilon(x_g)=g"] \arrow[d, "\beta(x_g)=X_g"'] & & S \arrow[d, "\gamma(g)=X_g\xi"] \\
F(b_1,b_2) \arrow[rr, "\overline{\varepsilon}"'] & & H=F(b_1,b_2)/\xi
\end{tikzcd}$$

where ξ is the unique smallest congruence relation on the free semigroup $F(b_1,b_2)$ containing the set $\beta R = \{(X_h, X_{h'}X_{h''}) : h = h'h'' \text{ in } S\}$ of defining relations of H, and $\overline{\varepsilon}$ is the natural epimorphism. Observe that γ may be well-defined by the formula $\gamma\varepsilon = \overline{\varepsilon}\beta$; i.e. $\gamma := \overline{\varepsilon}\beta\varepsilon^{-1}$. This definition is independent of the choice of $\varepsilon^{-1}(g)$ for $g \in S$; in particular, we may select representative $\varepsilon^{-1}(g) = x_g$. This is because if we have two representatives, $\varepsilon^{-1}(g) = x_g = x_{g_1}x_{g_2}\cdots x_{g_n}$ then $\varepsilon(x_g) = \varepsilon(x_{g_1}x_{g_2}\cdots x_{g_n}) = \varepsilon(x_{g_1})\cdots\varepsilon(x_{g_n})$ so $g = g_1\cdots g_n$ in S. One computes that $\overline{\varepsilon}\beta(x_{g_1}x_{g_2}\cdots x_{g_n}) = \overline{\varepsilon}(X_{g_1}X_{g_2}\cdots X_{g_n}) = X_{g_1}X_{g_2}\cdots X_{g_n}\xi$ and $\overline{\varepsilon}\beta(x_g) = \overline{\varepsilon}X_g = X_g\xi$. By definition, $X_{g_1}X_{g_2}\cdots X_{g_n}\xi = X_g\xi$ if $(X_{g_1}X_{g_2}\cdots X_{g_n}, X_g) \in \xi$. By induction, we may assume that $(X_{g_1}\cdots X_{g_{n-1}}, X_{g_1\cdots g_{n-1}}) \in \xi$. Then because ξ is left compatible, we have that $(X_{g_1}\cdots X_{g_n}, X_{g_1\cdots g_{n-1}}X_{g_n}) \in \xi$. By definition of βR we also have that $(X_{g_1\cdots g_{n-1}}X_{g_n}, X_g) \in \xi$. Therefore, $(X_{g_1}\cdots X_{g_n}, X_g) \in \xi$ as required.

Lemma VI.3.2. *The map β is injective.*

Proof. Suppose $x_{g_1}\cdots x_{g_n}, x_{h_1},\cdots x_{h_m} \in F_S$ and $\beta(x_{g_1}\cdots x_{g_n}) = \beta(x_{h_1},\cdots x_{h_m})$. Then $X_{g_1}\cdots X_{g_n} = X_{h_1}\cdots X_{h_m}$. Because $X_{g_1}\cdots X_{g_n}$ and $X_{h_1}\cdots X_{h_m}$ are words in the free group $F(b_1,b_2)$ the equality must in fact be letter-for-letter. Therefore by Lemma VI.2.4, we have that $n = m$ and $X_{g_i} \equiv X_{h_i}$ for $i = 1,\ldots,n$. By Lemma VI.2.5 the map $S \to F(b_1,b_2) : g \mapsto X_g$ is injective, hence $g_1 = h_1,\ldots,g_n = h_n$ so $x_{g_1}\cdots x_{g_n} = x_{h_1},\cdots x_{h_m}$. □

Lemma VI.3.3. *The map γ is injective.*

Proof. Suppose that $g, g' \in S$ and $\gamma(g) = \gamma(g')$. We will show that $g = g'$. Since $\gamma = \overline{\varepsilon}\beta\varepsilon^{-1}$, we have that $\overline{\varepsilon}\beta x_g = \overline{\varepsilon}\beta x_{g'}$ which implies that $\overline{\varepsilon}X_g = \overline{\varepsilon}X_{g'}$. Thus by definition of $\overline{\varepsilon}$ we have that $(X_g, X_{g'}) \in \xi$ which means that there is a finite chain

$$X_g = X_{k_0} \to X_{k_1} \to X_{k_2} \cdots \to X_{k_m} = X_{g'}$$

where each $\to$ is obtained by applying a defining relation. Every X_{k_i} is a product of elements of the form X_h where $h \in S$. Each time we apply a defining relation, we replace one X_h with $X_{h'}X_{h''}$ or vice-versa, where $h = h'h''$ in S. Therefore, for each X_{k_i}, the product of subscripts equals the same element of S; in particular, $g = g'$ as required. □

Let H_S be the free subsemigroup of $F(b_1,b_2)$ with free generating set $\{X_g\}_{g\in S}$. We know that H_S is free by Lemma VI.3.2, because $H_S = \mathrm{im}\beta \cong F_S/\ker\beta \cong F_S$. As $\overline{\varepsilon}$ is an epimorphism, we can consider the system $\mathscr{B} = \{b_1, b_2\}$ to be a generating set for the semigroup H which contains the isomorphic copy $\gamma(S)$ of S, by Lemma VI.3.3.

By an H_S-word we mean any word of the form $W(X_g,\dots,X_h)$. Any H_S-word can be rewritten as a word in the letters b_1 and b_2.

The following is an important ingredient in the proof of Theorem VI.1.5 Part (1).

Lemma VI.3.4. *For any H_S-word U, there is an H_S-word V such that $\overline{\varepsilon}(V)=\overline{\varepsilon}(U)$ and $||V||\le ||W||$ for any word W with $\overline{\varepsilon}(W)=\overline{\varepsilon}(U)$.*

Proof. It suffices to show that if a word W satisfies $\overline{\varepsilon}(W)=\overline{\varepsilon}(U)$ then W must be an H_S-word. Because $W=U$ in H there is a finite chain

$$U=U_0\to U_1\to\cdots\to U_m=W$$

where each $\to$ is obtained by applying a defining relation in H. Suppose by induction that at the n^{th} step we have $U_n\to U_{n+1}$ where the H_S-word $U_n\equiv X_{l_1}\cdots X_{l_t}$ for $l_1,\dots,l_t\in S$. Therefore we have that $X_{l_1}\cdots X_{l_t}\equiv T'X_hT=T'X_{h'}X_{h''}T\equiv U_{n+1}$ for some words T,T' where the defining relation applied was $X_h=X_{h'}X_{h''}$ for $h=h'h''$ in S. By Lemma VI.2.4, both T and T' are H_S-words. Thus so is U_{n+1}, and by induction, W. □

Proof. of Theorem VI.1.5 Part 1:
By Lemma VI.3.3 we may identify S with its image $\gamma(S)\subset H$. The equalities

$$g=\gamma(g)=\overline{\varepsilon}\beta\varepsilon^{-1}(g)=\overline{\varepsilon}\beta(x_g)=\overline{\varepsilon}(X_g)$$

and the inequalities (VI.4) yield

$$|g|_{\mathscr{B}}\le dl(g) \tag{VI.5}$$

for $d>0$ and for any $g\in S\subset H$. To obtain the opposite estimate, we consider an element $g\in S$ and apply Lemma VI.3.4 to the H_S-word $U\equiv X_g$. For a word W of minimum length representing the element X_g, and for the H_S-word V from Lemma VI.3.4, we have

$$|g|_{\mathscr{B}}=||W||\ge ||V||. \tag{VI.6}$$

By definition of H_S there exists a unique decomposition of the H_S-word V as a product $V\equiv X_{g_1}X_{g_2}\cdots X_{g_s}$ for some $g_j\in S$. Because $V=W$ in H we have that $(X_{g_1}\cdots X_{g_s},X_g)\in\xi$ which implies by previous arguments that $g=g_1\cdots g_s$ in the subsemigroup S of H. Taking into account the inequalities (VI.4) we conclude that $||X_{g_j}||\ge l(g_j)$. Hence, by the condition (D_1) we have that

$$||V||=\sum_{j=1}^{s}||X_{g_j}||\ge\sum_{j=1}^{s}l(g_j)\ge l(g).$$

Therefore, $|g|_{\mathscr{B}}\ge l(g)$, by (VI.6). This, together with inequality (VI.5), completes the proof. □

The following Lemma will essentially prove Theorem VI.1.5 Part 2. We fix notation as in the Theorem: S is a finitely generated semigroup, and $l:S\to\mathbb{N}$ satisfies the (D) condition.

Lemma VI.3.5. *There is an exponential set of words $\mathscr{N}$ over a finite alphabet $\mathscr{C}$ satisfying the overlap property such that there is an injection $S \to \mathscr{N} : g \mapsto X_g$ satisfying*

$$l(g) = ||X_g||. \tag{VI.7}$$

Proof. Let a be the integer arising from condition $(D2)$ for the given function l. Let $\mathscr{C} = \{c_1, \dots, c_{a+2}\}$. It suffices to produce a set of words $\mathscr{N}$ satisfying the overlap property and subject to

$$\text{card}\{y \in \mathscr{N} : ||y|| = i\} \geq a^i.$$

For if this is satisfied, then for every $g \in S$ we may find a distinct word of length $l(g)$ from our exponential set satisfying the overlap property. The same argument as that given in the proof of Lemma VI.2.3 shows that the set

$$\mathscr{N} = \{c_1 v(c_2, \dots, c_{a+1}) c_{a+2}\}$$

where v is an arbitrary word in $c_2, \dots, c_{a+1}$ does satisfy the required properties. □

Remark VI.3.6. Observe that Theorem VI.1.5 Part 2 follows from Lemma VI.3.5 by replacing the set $\mathscr{M}$ by $\mathscr{N}$ and the inequalities (VI.4) by equality (VI.7) everywhere in the proof of Theorem VI.1.5 Part 1.

VI.4 Embedding to Finitely Presented Semigroups

In this section we will prove Theorems VI.1.7 and VI.1.10.

We begin with an undistorted analogue of Murskii's embedding theorem.

Theorem VI.4.1. *Let H be a semigroup with a finite generating set $\mathscr{B}$ and a recursively enumerable set of (defining) relations. Then there exists an isomorphic embedding of H in some finitely presented semigroup R with generating set $\mathscr{T}$ without distortion.*

Observe that Theorem VI.1.7 follows immediately from Theorem VI.1.5, Part 1, Theorem VI.4.1 and the assumption that S has recursively enumerable set of defining relations.

Although an undistorted semigroup analog of Murskii's embedding appears in [Bi], that Theorem makes additional assumptions regarding time complexity of the word problem in H. It is not clear to the author whether a simple proof of Theorem VI.4.1 may be extracted from [Bi].

To prove Theorem VI.4.1 we will instead use such an embedding which was invented in [Mu], and show that it is undistorted.

Proof. of Theorem VI.4.1. Let $P \in H$ and W is a word representing the image of P in R under the embedding. We have by [Mu] Lemma 3.3 that if a word P in the alphabet $\mathscr{B}$ is equal in R to a word W in the alphabet $\mathscr{T}$ then it is possible to represent W in the form

$$W \equiv P_0 U_1 P_1 U_2 \cdots U_l P_l$$

such that

1. All P_i's are words in the alphabet $\mathscr{B}$;
2. One can delete some subwords from every U_i and obtain a word U_i', which by Lemma 3.1 in [Mu] has subword $\tilde{R}_i$, where R_i are words in the alphabet $\mathscr{B}$ and $\tilde{R}_i$ is are not words in the alphabet $\mathscr{B}$, but $||\tilde{R}_i|| = ||R_i||$ for all i.
3. The word $P_0R_1P_1\cdots R_lP_l$ is equal to P in H.

This implies that

$$|P|_{\mathscr{B}} \leq |P_0R_1P_1\cdots R_lP_l|_{\mathscr{B}} \leq |P_0|_{\mathscr{B}} + |R_1|_{\mathscr{B}} + \cdots + |P_l|_{\mathscr{B}}$$

$$\leq ||P_0|| + ||R_1|| + \cdots + ||P_l|| = ||P_0|| + ||\tilde{R}_1|| + \cdots + ||P_l||$$

$$\leq ||P_0|| + ||U_1'|| + \cdots + ||P_l|| \leq ||P_0|| + ||U_1|| + \cdots + ||P_l|| = ||W||.$$

Indeed, we have that $||\tilde{R}_i|| \leq ||U_i'||$ because $\tilde{R}_i$ is a subword of U_i' for all i. Similarly, because U_i' is obtained from U_i by deleting subwords, we have $|U_i| \geq |U_i'|$ for all i. Since W is any word equal to P in R, the above inequalities hold in particular when $||W|| = |P|_{\mathscr{T}}$ so we have that $|P|_{\mathscr{B}} \leq |P|_{\mathscr{T}}$, which shows that the embedding is undistorted. □

We proceed with consideration of Theorem VI.1.10, in particular towards establishing notation to be used in the proof.

Let S be a finitely generated semigroup with generating set $\mathscr{A} = \{a_1, \ldots, a_m\}$. For any $k > 0$, let F_k denote the free semigroup of rank k.

Suppose that a function $l : S \to \mathbb{N}$ satisfies conditions $(D1)-(D3)$. Let $\pi : F_m \to S$ be the natural projection. By hypothesis, there exists a recursively enumerable set T satisfying Properties $(1), (2)$, and (3) of Condition $(D3)$. Let U be the natural projection of T onto F_n. Let $\phi : U \to F_m$ such that $v = \phi(u)$ if $(v,u) \in T$ and (v,u) is the first pair in the enumeration of T whose second component is u.

By Lemma VI.2.3 there exists an exponential set of words $\mathscr{M}$ over the alphabet $\{x_1, x_2\}$ satisfying the overlap condition of Definition VI.2.2. For the word length function $F_n \to \mathbb{N}$, there exists by Lemma VI.2.5 a constant d and an injection $\psi : F_n \to \mathscr{M} \subset F_2 = F(x_1, x_2) : u \to X_u$ satisfying

$$||u|| \leq ||X_u|| < d||u||. \tag{VI.8}$$

We may chose the function ψ to be recursive. Begin by putting an order (e.g. ShortLex) on U. Then for every u starting with the shortest we select the smallest word X_u satisfying equation (VI.8) and such that $X_u \neq X_{u'}$ if $u' < u$.

Let $F(V)$ be the free semigroup with basis $V = \{x_v\}_{v \in F_m}$. Consider the natural epimorphism defined on generators by $\zeta : F(V) \to S : \zeta(x_v) = \pi(v)$. Define the free semigroup $F(Y)$ with basis $Y = \{y_u\}_{u \in U}$. Let $\eta : F(Y) \to F(V)$ be defined by $\eta(y_u) = x_{\phi(u)}$. Then the product $\varepsilon = \zeta\eta$ is an epimorphism because by Parts (1) and (2) of Condition $(D3)$, for any $v \in F_m$ there is $(v', u) \in T$ such that $\phi(u) = v'$ and $\pi(v') = \pi(v)$. Therefore, there is a presentation $S = \langle Y | \mathscr{R} \rangle$ defined by the isomorphism $S \cong F(Y)/\ker(\varepsilon)$.

Define a homomorphism $\beta : F(Y) \to F_2 : \beta(y_u) = \psi(u) = X_u$. Let ξ be the unique smallest congruence relation on the free semigroup F_2 containing the set $\beta(\mathscr{R}) = \{(\beta(a), \beta(b)) : (a,b) \in \mathscr{R}\}$. Let $\overline{\varepsilon}$ the natural epimorphism of F_2 onto $H = F_2/\xi$. Let $\gamma : S \to H$ be defined by $\gamma = \overline{\varepsilon}\beta\varepsilon^{-1}$. There is also a map $F(V) \to F_m : x_v \mapsto v$. Consider the commutative diagram defined by all these maps:

$$\begin{tikzcd}
F(V) \arrow[r] \arrow[dr, "\zeta"] & F_m \arrow[d, "\pi"] \\
F(Y) \arrow[u, "\eta"] \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & S \arrow[d, "\gamma"] \\
F_2 \arrow[r, "\overline{\varepsilon}"] & H
\end{tikzcd}$$

Lemma VI.4.2. *The map β is injective.*

Proof. This fact is proved exactly similarly to Lemma VI.3.2. The application of Lemma VI.2.4 is still valid, because our set $\mathscr{M} \supset \{X_u\}_{u\in U}$ is exponential and satisfies the overlap property. Moreover, we have that the map $U \to F_2 : u \to X_u$ is injective. These are the only facts used in the proof of Lemma VI.3.2. $\square$

Lemma VI.4.3. *The map γ is a well-defined monomorphism.*

Proof. The fact that γ does not depend on the choice of preimage under ε of $g \in S$ follows exactly as the proof of the same fact in Section VI.3. If we had two preimages for one element, say $\varepsilon(y_u) = \varepsilon(y_{u'}) = g \in S$ then $(y_u, y_{u'}) \in N$ which implies that $(X_u, X_{u'}) \in \xi$. This implies that γ does not depend on the choice of preimage of $g \in S$, and that γ is a homomorphism.

Moreover, γ is injective. The proof is similar to that of Lemma VI.3.3. Suppose that $g, g' \in S$ and $\gamma(g) = \gamma(g')$. We will show that $g = g'$. Since $\gamma = \overline{\varepsilon}\beta\varepsilon^{-1}$, we have that $\overline{\varepsilon}\beta y_u = \overline{\varepsilon}\beta y_{u'}$, where y_u and $y_{u'}$ are preimages of g and g', respectively under ε; i.e. $\varepsilon(y_u) = \zeta\eta(y_u) = \zeta(x_{\phi(u)}) = \pi\phi(u) = g$. This implies that $\overline{\varepsilon}X_u = \overline{\varepsilon}X_{u'}$. Thus by definition of $\overline{\varepsilon}$ we have that $(X_u, X_{u'}) \in \xi$ which means that there is a finite chain

$$X_u = X_{k_0} \to X_{k_1} \to X_{k_2} \cdots \to X_{k_m} = X_{u'}$$

where each $\to$ is obtained by applying a defining relation. Each time we apply a defining relation we replace $X_{k_i} = \prod_{j=1}^{s_i} X_{u_{i,j}}$ with $X_{k_{i+1}} = \prod_{j=1}^{s_{i+1}} X_{u_{i+1,j}}$. Since $X_{k_i} = \beta(\prod_{j=1}^{s_i} y_{u_{i,j}})$ and $X_{k_{i+1}} = \beta(\prod_{j=1}^{s_{i+1}} y_{u_{i+1,j}})$ and by definition of ξ we conclude that $\varepsilon(\prod_{j=1}^{s_i} y_{u_{i,j}}) = \varepsilon(\prod_{j=1}^{s_{i+1}} y_{u_{i+1,j}})$. Since this holds true at every step, we have in particular that $\varepsilon(y_u) = \varepsilon(y_{u'})$; i.e. $g = g'$. $\square$

Lemma VI.4.4. *The semigroup H is recursively presented.*

Proof. The set of defining relations for H is $\xi = \beta(\mathscr{R})$. Because the map $\psi : F_n \to F_2$ was chosen to be recursive, and by definition of β, it suffices to show that the relations $\mathscr{R}$ are recursively enumerable. First observe that the set of relations of S in generators $\{a_1, \ldots, a_m\}$ is recursively enumerable. We have by Condition $(D3)$, Parts (1) and (2) that $v_1(a_1, \ldots, a_m) = v_2(a_1, \ldots, a_m)$ in S if and only if both $(v_1, u), (v_2, u) \in T$ for some u. Therefore, because T is recursively enumerable, so is the set of relations of S. We enumerate

S by putting (v_1, v_2) in the list as soon as we get $(v_1, u), (v_2, u)$ in the enumeration of all members of T. Then we have that $(w(y_{u_1}, \dots, y_{u_s}), w'(y_{u'_1}, \dots, y_{u'_t})) \in \ker(\varepsilon)$ if and only if

$$\zeta w(x_{\phi u_1}, \dots, x_{\phi u_s}) = \zeta w'(x_{\phi u'_1}, \dots, x_{\phi u'_t}) \text{ in } S. \tag{VI.9}$$

Thus we have to enumerate such pairs (w, w'). To do this, we enumerate all variables of the form $x_{\phi u}$ with $u \in U$. This is possible by definition of ϕ and U and by the fact that T is recursively enumerable. Next, we enumerate all pairs $(w(x_{v_1}, \dots, x_{v_s}), w'(x_{v'_1}, \dots, x_{v'_t}))$ with $\zeta(w) = \zeta(w')$. This is possible because $\zeta(w) = \zeta(w')$ if and only if $\pi(w(v_1, \dots, v_s)) = \pi(w'(v'_1, \dots, v'_t))$ if and only if

$$\begin{aligned} & w(v_1(a_1, \dots, a_m), \dots, v_s(a_1, \dots, a_m)) = \\ & w'(v'_1(a_1, \dots, a_m), \dots, v'_t(a_1, \dots, a_m)). \end{aligned} \tag{VI.10}$$

We have already seen that the set of all relations of S is recursively enumerable. Given any relation in S in generators $\{a_1, \dots, a_m\}$, we may find all possible $w, w', v_1, \dots, v'_t$ such that the relation may be presented as it is written in equation (VI.10). There is an algorithm which can do this because the lengths of possible $w, w', v_1, \dots, v'_t$ are bounded by the length of the given relation of S. To complete the proof it suffices to compare these two lists to obtain a list of all pairs (w, w') satisfying equation (VI.9). Since S is recursively enumerable, and the graph of ϕ is recursively enumerable, it suffices to recursively enumerate the set of all pairs of words $(w(x_{v_1}, \dots, x_{v_s}), w'(x_{v'_1}, \dots, x_{v'_t}))$ with $\zeta(w) = \zeta(w')$. But $\zeta(w) = \zeta(w')$ if and only if $\pi(w(v_1, \dots, v_s)) = \pi(w'(v'_1, \dots, v'_t))$. But we have already seen that the set of such words is recursively enumerable. $\square$

Proof. of Theorem VI.1.10.

We first suppose that S is a semigroup with finite generating set $\mathscr{A} = \{a_1, \dots, a_m\}$ and that a function $l : S \to \mathbb{N}$ satisfies conditions $(D1)-(D3)$. Lemmas VI.4.3 and VI.4.4 show that there is an embedding $S \to H$ to a recursively presented and 2-generated semigroup. By Theorem VI.4.1, it suffices to prove that the function $l : S \to \mathbb{N}$ is equivalent to the word length on H restricted to S. Let $g = \pi(v) \in S$. By Part (3) of Condition $(D3)$, there exists a word $u \in F_n$ such that $||u|| = l(g)$. Let $v' = \phi(u)$. We have that $\pi(v) = \pi(v')$ by Part (1) of Condition $(D3)$ and by the definition of ϕ. Then $\varepsilon(y_u) = \pi(\phi(u)) = \pi(v') = \pi(v) = g$. Therefore, by definition we have that $\gamma(g) = \overline{\varepsilon}\beta(y_u) = \overline{\varepsilon}(X_u)$, and so

$$|\gamma(g)|_H \leq ||X_u|| \leq d||u|| = dl(g). \tag{VI.11}$$

The reverse inequality follows exactly from the arguments of the Proof of Theorem VI.1.5 Part (1), which only uses the overlap property, Lemma VI.4.2 and the replacing of inequalities $(VI.4)$ by $(VI.11)$ and the definition of H_S by the free semigroup $\{X_u\}_{u \in U}$.

To prove the converse, suppose that S is a subsemigroup of H with generating set $\mathscr{B} = \{b_1, \dots, b_m\}$. We must show that

$$l : S \to \mathbb{N} : l(g) = |g|_{\mathscr{B}}$$

satisfies condition $(D3)$. Since H is finitely presented, the collection $T \subset F_m \times F_n$ defined by

$$T = \{(v,u) : v(a_1,\ldots,a_m) = u(b_1,\ldots,b_n) \text{ in } H\}$$

is recursively enumerable. Condition $(D3)$ Part (1) is satisfied because if $(v_1,u),(v_2,u) \in T$ then $v_1(a_1,\ldots,a_m) = u(b_1,\ldots,b_n)$ in H, and $v_2(a_1,\ldots,a_m) = u(b_1,\ldots,b_n)$ in H. Therefore, since the map $S \to H$ is an injection, we have that $v_1(a_1,\ldots,a_m) = v_2(a_1,\ldots,a_m)$ in S. To see that Condition $(D3)$, Part (2) is satisfied, suppose $v_1 = v_2$ in S and let $v_1(a_1,\ldots,a_m) \in H$. Then we may write v_1 with respect to the generating set $\mathscr{B}$ of H; that is, there exists $u \in H$ with $v_1(a_1,\ldots,a_m) = u(b_1,\ldots,b_n)$. Now consider words $u(x_1,\ldots,x_n) \in F_n, v_1(y_1,\ldots,y_m) \in F_m$, where F_n has basis $\{x_1,\ldots,x_n\}$ and F_m has basis $\{y_1,\ldots,y_n\}$. We have that $(v_1,u),(v_2,u) \in T$ because $u(b_1,\ldots,b_n) = v_1(a_1,\ldots,a_m) = v_2(a_1,\ldots,a_m)$. To see that condition $(D3)$ Part (3) is satisfied, let $v = v(y_1,\ldots,y_m) \in F_m$. Then

$$l^*(v) = l(v(a_1,\ldots,a_m)) = |v(a_1,\ldots,a_m)|_{\mathscr{B}}$$

$$= \min\{||u|| : u = v \text{ in } H\} = \min\{||u|| : (v,u) \in T\}.$$

$\square$

CHAPTER VII

FUTURE DIRECTIONS

VII.1 Distortion in Some Wreath Products

Two ultimate goals I have for my future research are as follows. First, I would like to obtain an algebraic classification of finitely generated solvable groups all of whose subgroups are undistorted. Secondly, I would like to find a universal upper bound on subgroup distortion in metabelian groups. As mentioned earlier, it is provide in [Ro] that finitely generated subgroups of finitely generated free metabelian groups have solvable membership problem, which means that a recursive upper bound on distortion exists. It would be interesting to be able to compute it. I have many smaller questions, which will be described in detail below, and all of which relate in some way to these two larger goals.

Because I would like to more fully understand distortion in metabelian groups, I propose to study what effects of subgroup distortion are possible in $\mathbb{Z}^k$ wr $\mathbb{Z}^k$ for $k > 1$. We were able to compute subgroup distortion in groups of the from A wr $\mathbb{Z}$, where A is finitely generated abelian (in [DO]). Perhaps some of the work already described in this thesis could generalize to lead to an upper bounds on the types of distortion which are possible in wreath products when the active group is free abelian of rank greater than one.

A natural question to ask in response to what has already been proved is: in what larger classes of groups can I generalize the methods of [DO] to study distortion? I am particularly interested in the following specific cases: $\mathbb{Z}^m$ wr $\mathbb{Z}^n$ for $m \geq 1, n > 1$, as described above, as well as $\mathbb{Z}$ wr F_n, where F_n is a free group of rank $n > 1$. All of these groups are generalizations of the cases I have already studied. The case of the free group was mentioned to me by Thomas Sinclair; he said an answer to this question on the geometry of wreath products of the integers with a free group would be of interest to those working in von Neumann algebras.

Some other questions arise from the methods employed to study distortion in wreath products of abelian groups. I wonder whether the module theory could be made to work in the case of, say, $\mathbb{Z}$ wr $\mathbb{Z}^2$. In this case, $\mathbb{Q}[\mathbb{Z}^2]$ is like a polynomial ring in two variables, and so it is no longer a principal ideal ring, which seems to be an obstacle. I also wonder if the proof of Theorem I.3.4 Part (1) could be deconstructed to understand distortion in more general direct products of groups.

We use the Magnus embedding here primarily as a tool to explain that the study of wreath products of abelian groups motivates the understanding of distortion in free metabelian groups. However, it would be interesting to determine whether the Magnus embedding itself is undistorted. This problem was mentioned to me by both Denis Osin and Alexander Olshanskii.

VII.2 Embedding to Finitely Presented Groups

Another research objective that I would like to explore is the idea of embedding wreath products into finitely presented groups. This was suggested to me by Sean Cleary. In [C], the distortion of the group $\mathbb{Z}$ wr $\mathbb{Z}$ is studied in Baumslag's metabelian group, which was constructed in [B2]. The group $\mathbb{Z}$ wr $\mathbb{Z}$ is finitely generated but not finitely presented. However, it is possible to embed it (with exponential distortion) into Baumslag's finitely generated group. I would like to use some of the ideas of [DO] and [C] to understand other emebddings of wreath products into finitely presented groups, and the possible effects of subgroup distortion.

VII.3 Relation with Thompson's Group

In the paper [GS], Guba and Sapir ask the following question: does Thompson's group contain a subgroup with super-polynomial distortion? Perhaps the methods developed by myself and Olshanskii in [DO] can work to solve this problem. Also in [GS], it is mentioned that the membership problem is still open for Thompson's group. Therefore, it would be interesting to study iterated wreath products of the form $(\cdots(\mathbb{Z} \text{ wr } \mathbb{Z}) \text{ wr } \mathbb{Z})\cdots \text{ wr } \mathbb{Z})$, which embed into Thompson's group. I would like to understand whether these groups have finitely generated subgroups with exponential distortion (or even distortion not bounded from above by a recursive function.)

VII.4 Relative Subgroup Growth

With regards to the results obtained on subgroup growth, it would be interesting to consider cyclic subgroups of finitely generated groups which do have recursive distortion function. Perhaps some of the same methods can be used to understand how small the relative growth can be (i.e. almost linear?) if the distortion is some prescribed recursive function, such as exponential or superexponential. It is also worth considering whether we can obtain similar results for semigroups as well.

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