TOPICS ON SHIFT-INVARIANT SPACES WITH EXTRA INVARIANCE

By

Haichao Wang

Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August, 2011

Nashville, Tennessee

Approved:

Professor Akram Aldroubi

Professor Doug Hardin

Professor Alexander Powell

Professor Mitchell Wilkes

Professor Dechao Zheng

To my parents,

and

my beloved wife, Lujun

ACKNOWLEDGMENTS

I would like to express my gratitude to my advisor, Professor Akram Aldroubi, for his insight and patience, and especially for his never-ending encouragement and professional advice. I would also like to thank all the faculty members of my committee, especially Professor Alex Powell for many helpful and inspiring conversations. Additionally, I thank Professor Ilya Krishtal, Professor Qiyu Sun and Professor Romain Tessera for their insight and collaboration. Also my thanks go to all my friends in the graduate program at Vanderbilt. Lastly I would like to thank Akram Aldroubi, Fanfei Chong, Jacqueline Davis and Doug Hardin for proofreading this paper.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

A shift-invariant space (SIS) is a closed subspace V of $L^2(\mathbb{R})$, such that if $f \in V$, then $f(\cdot + k) \in V$ for all $k \in \mathbb{Z}$. Such spaces have many applications in numerical analysis, multiresolution analysis (MRA) and wavelet theory (see e.g. [AG01, AST05, Bow00, CS07, UB00] and the references therein).

One example of a shift-invariant space is the Paley-Wiener space of functions that are bandlimited to $[-1/2, 1/2]$:

$$PW(\mathbb{R}) = \big\{f \in L^2(\mathbb{R}) : \mathrm{supp}\widehat{f} \subseteq [-\frac{1}{2}, \frac{1}{2}]\big\}.$$

It is easy to see that $PW(\mathbb{R})$ is actually not only a shift-invariant space, it is invariant under arbitrary translation, i.e., if $f \in PW(\mathbb{R})$, then $f(x+\alpha) \in PW(\mathbb{R})$ for all $\alpha \in \mathbb{R}$.

One method for constructing SISs is as follows:

Definition I.0.1. Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be a vector function with $\phi_1, \ldots, \phi_r$ in $L^2(\mathbb{R})$. The SIS $V(\Phi)$ generated by Φ is given by

$$V(\Phi) = \{\sum_{k\in\mathbb{Z}} C^T(k)\Phi(\cdot + k) : C \in (\ell^2)^r\}$$

where $C = (c_1, \cdots, c_r)^T$ for each $c_i = \{c_i(k)\} \in \ell^2$. In other words, every function $f \in V(\Phi)$ is of the form $\sum_{i=1}^r \sum_{k\in\mathbb{Z}} c_i(k)\phi_i(\cdot + k)$. In this case we say that the space $V(\Phi)$ is *finitely generated* by $\Phi = \{\phi_1, \cdots, \phi_r\}$ and the functions $\phi_1, \cdots, \phi_r$ are called *generators*. If $\phi_i(\cdot + k)$ and $\phi_j(\cdot + l)$ are orthogonal for $(i, k) \neq (j, l)$ and $||\phi_i||_2 = 1$ for all i, then $\phi_1, \cdots, \phi_r$ are called *orthonormal* *generators*. If Φ only consists of a single function ϕ, we say that $V(\phi)$ is a *principal shift-invariant space* (PSIS). One can show that $PW(\mathbb{R}) = V(\text{sinc})$, where $\text{sinc}(x) = \sin \pi x / \pi x$.

For an example of a PSIS, consider a function ϕ such that its Fourier transform $\widehat{\phi}(\xi) = \chi_{[0,1)} + \chi_{[2,3)}$. We can compute that $\phi(x) = \text{sinc}(x)(e^{\pi i x} + e^{5\pi i x})$. Then the PSIS $V(\phi)$, generated by the single function ϕ, is given by

$$V(\phi) = \{ \sum_{k=-\infty}^{\infty} c_k \phi(\cdot + k) : \{c_k\}_{k=-\infty}^{\infty} \in \ell^2 \}.$$

In this example, $V(\phi)$ is not only shift invariant, but also $\frac{1}{2}\mathbb{Z}$-invariant, i.e., if $h \in V(\phi)$, then we have that $h(\cdot + \frac{1}{2}) \in V(\phi)$.

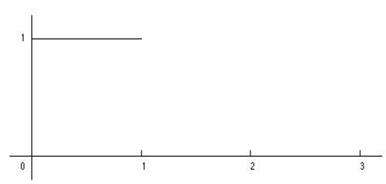

Figure I.1: $\beta^0 = \chi_{[0,1)}$.

Other important examples are the SIS generated by B-splines. Specifically, the B-spline β^0 of order 0 which is the characteristic function on $[0, 1)$, i.e. $\beta^0 = \chi_{[0,1)}$. The B-spline β^s of order s is defined to be the $(s+1)$-fold convolution of the function β^0, i.e. $\beta^s = \beta^{s-1} * \beta^0$. It is not hard to check that each β^s has compact support and is a piecewise polynomial of degree s in C^{s-1} with integer breakpoints. For any s, the SIS V generated by $\{\beta^i : i \leq s\}$ is the space of all piecewise polynomials of degree less than or equal to s with the integer breakpoints. This space is used in numerical analysis, signal processing and many other applications. From the construction, one can easily show that V is shift invariant. However, it is not $\frac{1}{n}\mathbb{Z}$-invariant for any $n > 1$.

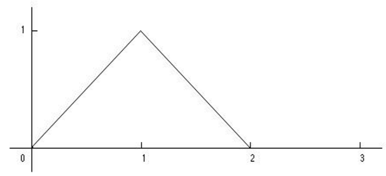

Figure I.2: $\beta^1 = \beta^0 * \beta^0$.

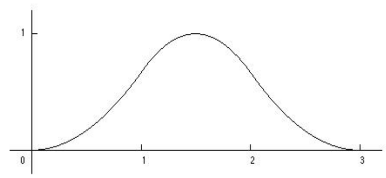

Figure I.3: $\beta^2 = \beta^0 * \beta^1$.

From all the examples above, we can see that some SISs are only shift invariant(invariant under $\mathbb{Z}$) while others have more invariance structure, for example invariance under $\frac{1}{n}\mathbb{Z}$ for some $n > 1$ or even invariance under all the real numbers. In general, a finitely generated SIS need not possess any invariance other than translation by integers. Shift-invariant spaces with additional invariance have been used in the study of wavelet analysis and sampling theory [Web00, CS03, HL09], and have been completely characterized in [ACHKM10] for $L^2(\mathbb{R})$ and in [ACP09] for $L^2(\mathbb{R}^n)$. For a subspace V of $L^2(\mathbb{R})$, let

$$\tau(V) := \Big\{ t \in \mathbb{R} \Big| \ f(\cdot - t) \text{ belong to } V \text{ for all } f \in V \Big\}. \qquad \text{(I.0.1)}$$

For any closed subspace V of L^2, one may verify that $\tau(V)$ is a closed additive subgroup of $\mathbb{R}$, and hence $\tau(V)$ is either $\{0\}$, or $\mathbb{R}$, or $\alpha\mathbb{Z}$ for some $\alpha > 0$. It can be shown that [ACHKM10] for any finitely generated shift-invariant space $V(\Phi)$,

$$\tau(V(\Phi)) = \mathbb{R} \text{ or } \tau(V(\phi)) = \frac{1}{n}\mathbb{Z} \text{ for some } n \in \mathbb{N}. \tag{I.0.2}$$

We say that a shift-invariant space V has additional invariance if $\tau(V) \supsetneq \mathbb{Z}$. It is well-known that the Paley-Wiener space PW is invariant under all translations. Thus,

$$\tau(PW) = \mathbb{R}.$$

A closed subspace V of L^2 with $\tau(V) = \mathbb{R}$ is usually known as a *translation-invariant space*. The fact that the space PW of bandlimited functions is also translation-invariant ($\tau(PW) = \mathbb{R}$) makes it useful for modeling signals and images. However, it is known that any function ϕ that generates a Riesz basis for PW has slow spatial-decay in the sense that $\phi \notin L^1(\mathbb{R})$, e.g., $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$. We can show that this slow spatial-decay property for the generator of a translation-invariant PSIS is not unique to the space of bandlimited functions PW. This is exactly the reason we want to consider $\frac{1}{n}\mathbb{Z}$-invariant SISs. We want to construct SISs that are close to being translation invariant, with generators which are well localized in both space and frequency domains.

This paper is presented in three chapters: In this chapter, we will introduce basic notations and definitions that are frequently used in this note as well as the property of finitely generated SIS with additional invariance. In Chapter II, we will discuss the time frequency localization of generators of finitely generated SISs with additional invariance. Chapter III is devoted to the construction of PSISs with additional invariance nearest to a set of given functions. Most of the results in this paper are from [ASW11, AKTW11, TW11].

I.1 Fourier Analysis

We now begin with the review of Fourier analysis. Let us first look at the definition of Fourier series.

Definition I.1.1. If f is an integrable function on the interval $[a,b]$ of length L, the n-th Fourier coefficient of f is defined by

$$a_n = \widehat{f}(n) = \frac{1}{L}\int_a^b f(x)e^{-2\pi inx/L}dx, \quad n \in \mathbb{Z}.$$

The Fourier series of f is given by

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi inx/L}.$$

Similarly,

Definition I.1.2. The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined to be

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx, \quad \text{a.e.}\xi \in \mathbb{R}.$$

The Fourier transform can be extended to be a unitary operator on $L^2(\mathbb{R})$. For any subset $V \subset L^2(\mathbb{R})$, we denote $\widehat{V} = \{\widehat{f} : f \in V\}$.

Under suitable conditions, e.g. $f \in L^2(\mathbb{R})$ or f is in the Schwartz space, f can be reconstructed from $\widehat{f}$ using the inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi$$

Definition I.1.3. The convolution of two functions $f, g \in L^1(\mathbb{R})$ is defined to be

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

It satisfies

$$||f * g||_1 \leq ||f||_1 ||g||_1.$$

More generally, from Tonelli's theorem, we have that

$$||f * g||_p \leq ||f||_1 ||g||_p, \text{ for } 1 \ \leq p \leq \infty.$$

We will use the following properties of the Fourier transform throughout this paper:

(1) For any real number a, if $h(x) = f(x+a)$, then $\widehat{h}(\xi) = e^{2\pi i a\xi}\widehat{f}(\xi)$.

(2) For any real number ξ_0, if $h(x) = e^{2\pi i x\xi_0}f(x)$, then $\widehat{h}(\xi) = \widehat{f}(\xi - \xi_0)$.

(3) $\widehat{f * g} = \widehat{f}\widehat{g}$.

Proposition I.1.4. *If $f(x)$ is in $L^1(\mathbb{R})$, then $\widehat{f}(\xi)$ is uniformly continuous and decays to zero as $|\xi| \to \infty$, i.e. $\widehat{f} \in C_0(\mathbb{R})$.*

And we will frequently use the following results:

Theorem I.1.5 (Plancherel's)**.** *If $f(x) \in L^2(\mathbb{R})$, then $\widehat{f}(\xi)$ is also in $L^2(\mathbb{R})$ and*

$$\int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Theorem I.1.6 (Parseval's)**.** *If $f(x)$ and $g(x)$ are both in $L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx.$$

For more detail about Fourier analysis and its applications, see [SS03] and [W01].

I.2 Schwartz Spaces

Definition I.2.1. The Schwartz space $\mathcal{S}$, or the space of rapidly decreasing functions, is the set of all infinitely differentiable functions f so that f and all its derivatives $f^{'}, f^{''}, \dots, f^{(\ell)}, \dots$, are rapidly decreasing, in the sense that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty \text{ for all } k, \ell \geq 0.$$

Note that the Schwartz space is a topological vector space under the topology induced by the above seminorms. We can show that the Schwartz space is closed under differentiation and multiplication by polynomials.

The Schwartz space is very important in Fourier analysis. Actually, we have the following theorem.

Theorem I.2.2. *The Fourier transform is a continuous bijection of* $\mathcal{S}$.

The main ingredient of the proof of the theorem is the following properties of Fourier transform of function f in the Schwartz space $\mathcal{S}$:

- $\widehat{f^{(\ell)}}(\xi) = (2\pi i \xi)^{\ell} \widehat{f}(\xi)$
- If $h(x) = (-2\pi i x)^k f(x)$, then $\widehat{h}(\xi) = (\widehat{f})^{(k)}(\xi)$

A simple but important example of a function in $\mathcal{S}$ is the Gaussian function

$$f(x) = e^{-x^2},$$

which plays a key role in Fourier analysis as well as in probability theory.

After normalizing the Gaussian function, we have that

Theorem I.2.3. *If* $f(x) = e^{-\pi x^2}$, *then* $\widehat{f}(\xi) = f(\xi)$.

Proof. Define

$$F(\xi) = \widehat{f}(\xi) = \int_{\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx,$$

then

$$F(0) = \int_{\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

And from the fact that $f^{'}(x) = -2\pi x f(x)$ and the properties above,

$$\begin{aligned} F^{'}(\xi) &= \int_{\infty}^{\infty} -2\pi i x f(x) e^{-2\pi i x \xi} dx \\ &= i \int_{\infty}^{\infty} f^{'}(x) e^{-2\pi i x \xi} dx \\ &= i(2\pi i \xi) \widehat{f}(\xi) = -2\pi \xi F(\xi) \end{aligned}$$

By solving an ordinary differential equation, we will have $F(\xi) = e^{-\pi \xi^2}$. □

I.3 Frame Theory

Frames can be considered as the generalized notion of orthonormal bases. However, the frame elements in general are neither orthogonal to each other nor linearly independent. We now start with Riesz bases.

Definition I.3.1. A Riesz basis $\{\phi_k\}$ for a Hilbert space $\mathcal{H}$ is a family of the form $\{\phi_k = Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.

We have the following characterization of Riesz bases in terms of bases with extra conditions:

Proposition I.3.2. *A sequence $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ if and only if it is a basis for $\mathcal{H}$ and*

$$0 < A||c||_2 \leq ||\sum_k c_k \phi_k|| \leq B||c||_2 < \infty,$$

for all $c \in \ell^2$, where A and B are absolute constants.

Now we are ready to see the definition of a frame.

Definition I.3.3. A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist positive constants A and B such that

$$A||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B||f||^2, \text{for all } f \in \mathcal{H}.$$

The constants A and B are called frame bounds.

If $\{f_k\}_{k=1}^{\infty}$ is a frame of $\mathcal{H}$, the operator

$$T : \ell^2(\mathbb{N}) \to \mathcal{H}, T(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k f_k$$

is called the pre-frame operator and its adjoint operator is given by

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}), T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}.$$

The frame operator $S : \mathcal{H} \to \mathcal{H}$ given by

$$Sf = TT^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$$

is a positive invertible operator satisfying $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$ and $\frac{1}{B} I_{\mathcal{H}} \leq S^{-1} \leq \frac{1}{A} I_{\mathcal{H}}$.

For a given frame $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ and the frame operator S, the collection $\{\tilde{f}_k\}_{k=1}^{\infty} = \{S^{-1} f_k\}_{k=1}^{\infty}$ also forms a frame in $\mathcal{H}$. We call this frame the canonical dual frame of $\{f_k\}_{k=1}^{\infty}$, and we have

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle \tilde{f}_k = \sum_{k=1}^{\infty} \langle f, \tilde{f}_k \rangle f_k.$$

Please see [Chr03] for more review about frame theory.

I.4 Finitely Generated Shift-invariant Spaces

Now we are ready to have the formal definition of a shift-invariant space.

Definition I.4.1. A subspace V of $L^2(\mathbb{R})$ is a shift-invariant space (SIS) if it is invariant under integer translations, i.e.

$$f \in V \text{ implies } f(\cdot + k) \in V, \text{ for all } k \in \mathbb{Z}.$$

From now on, we only focus on finitely generated shift invariant spaces which are defined to be of the form

$$V(\Phi) = \{\sum_{k\in\mathbb{Z}} C^T(k)\Phi(\cdot + k) : C \in (\ell^2)^r\}$$

for some vector function $\Phi = (\phi_1, \cdots, \phi_r)^T$, where each $\phi_i \in L^2(\mathbb{R})$ and $C = (c_1, \cdots, c_r)^T$, where each $c_i = \{c_i(k)\} \in \ell^2$. In other words, every function $f \in V(\Phi)$ is of the form $\sum_{i=1}^r \sum_{k\in\mathbb{Z}} c_i(k)\phi_i(\cdot + k)$. In this case we say that the space $V(\Phi)$ is finitely generated by $\Phi = \{\phi_1, \cdots, \phi_r\}$.

Note that in general, the only assumption that we will require on Φ is that the Gramian matrix

$$G_\Phi(\omega) = \sum_{k\in\mathbb{Z}} \widehat{\Phi}(\omega + k)\overline{\widehat{\Phi}(\omega + k)}^T$$

is bounded, i.e.

$$G_\Phi(\omega) \le MI, \text{ a.e. } \omega \in \mathbb{R}, \tag{I.4.1}$$

where M is a positive constant. Recall that the Gramian matrix is a semi-positive definite Hermitian matrix. An equivalent condition on the matrix G_Φ

is that its components $(G_\Phi)_{i,j}$ belong to L^∞. Then we have that

$$\|\sum_{i=1}^{r}\sum_{k\in\mathbb{Z}} c_i(k)\phi_i(\cdot+k)\|_2^2 \leq (\sup_{\xi\in\mathbb{R}}\sum_{i,j}|(G_\Phi)_{i,j}(\xi)|)(\sum_{i=1}^{r}\|c_i\|_2^2) < \infty$$

which gives us that $V(\Phi)$ is a well-defined linear subspace of $L^2(\mathbb{R})$ under the condition (I.4.1). But that by no means implies that $V(\Phi)$ is a closed subspace of $L^2(\mathbb{R})$. We have the following characterization:

Theorem I.4.2 ([AST05]). *Let Φ be such that (I.4.1) holds. Then $V(\Phi)$ is closed in $L^2(\mathbb{R})$ iff there exists a positive constant $A > 0$ such that*

$$AG_\Phi(\xi) \leq G_\Phi^2(\xi) \text{ a.e. } \xi \in \mathbb{R}.$$

And also,

Theorem I.4.3 ([Bow00]). *Φ generates a Riesz basis for $V(\Phi)$ iff*

$$mI \leq G_\Phi(\xi) \leq MI$$

a.e. $\xi \in \mathbb{R}$ for some positive constants m and M.

We have the following more general result:

Theorem I.4.4 ([Bow00]). *Φ generates a frame for $V(\Phi)$ with frame constants A and B iff*

$$AG \leq G^2 \leq BG$$

If $V(\phi)$ is a principal shift invariant space (PSIS) generated by a single function ϕ, then by the definition, we have that

$$V(\phi) = \{\sum_{k=-\infty}^{\infty} c_k\phi(\cdot+k) : \{c_k\}_{k=-\infty}^{\infty} \in \ell^2\}.$$

From the discussion above, we know that $V(\phi)$ is a well-defined linear subspace of $L^2(\mathbb{R})$ if we assume that the overlap function

$$G_\phi(\xi) = \sum_j |\widehat{\phi}(\xi + k)|^2$$

has an upper bound, i.e. there exists a constant B such that $G_\phi(\xi) \leq B$ a.e. ξ. Furthermore, if we assume that $A \leq G_\phi(\xi)$ a.e. ξ, for some positive constant A, then $V(\phi)$ is closed subspace of $L^2(\mathbb{R})$ and ϕ generates a Riesz basis for $V(\phi)$. In that case, we can equivalently define $V(\phi)$ to be

$$V(\phi) = \overline{\text{span}}\{\phi(\cdot + k) : k \in \mathbb{R}\} \tag{I.4.2}$$

where the closure is taken with respect to the L^2-norm. From now on, we always assume that every finitely generated SIS $V(\Phi)$ we talk about in this note is a closed subspace of $L^2(\mathbb{R})$ and Φ generates a frame, in other words, we assume that there exist positive constants A and B such that

$$AG_\Phi \leq G_\Phi^2(\xi) \leq BG_\Phi, \text{ a.e. } \xi \in \mathbb{R}.$$

In particular, for principal SIS $V(\phi)$, we assume that ϕ generates a frame, i.e.

$$A\sum_j |\widehat{\phi}(\xi + k)|^2 \leq (\sum_j |\widehat{\phi}(\xi + k)|^2)^2 \leq B\sum_j |\widehat{\phi}(\xi + k)|^2, \text{ a.e. } \xi \in \mathbb{R} \tag{I.4.3}$$

for positive constants A and B.

We have the following well-known result,

Proposition I.4.5. *If Φ generates an orthonormal basis for the SIS $V(\Phi)$, then we must have*

$$G_\Phi(\omega) = I, \text{ a.e. } \omega \in \mathbb{R} \tag{I.4.4}$$

where I is the $r \times r$ identity matrix.

Proof. Since Φ generates a orthonormal basis, for any $i \neq j$ and any $k_i, k_j \in \mathbb{Z}$, from Parseval's theorem, we must have that

$$\begin{aligned}
0 &= \int_{\infty}^{\infty} \phi_i(x+k_i)\overline{\phi_j(x+k_j)}dx \\
&= \int_{\infty}^{\infty} \widehat{\phi}_i(\xi)e^{2\pi i k_i \xi}\overline{\widehat{\phi_j}(\xi)}e^{-2\pi i k_j \xi}d\xi \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 \widehat{\phi}_i(\xi+k)e^{2\pi i k_i(\xi+k)}\overline{\widehat{\phi_j}(\xi+k)}e^{-2\pi i k_j(\xi+k)}d\xi \\
&= \int_0^1 \sum_{j\in\mathbb{Z}} \widehat{\phi}_i(\xi+k)\overline{\widehat{\phi_j}(\xi+k)}e^{2\pi i(k_i-k_j)\xi}d\xi
\end{aligned}$$

The sum and integral above can be interchanged by Fubini's theorem.

Consider $G_\Phi(\xi)_{i,j} = \sum_{k\in\mathbb{Z}} \widehat{\phi}_i(\xi+k)\overline{\widehat{\phi_j}(\xi+k)}$. It is a 1-periodic function and satisfies

$$\int_0^1 G_\Phi(\xi)_{i,j}e^{2\pi i(k_i-k_j)\xi}d\xi = 0$$

for all k_i and k_j. Then all the Fourier coefficients are zero, which means if $i \neq j$, we have $G_\Phi(\xi)_{i,j} = 0$ for almost everywhere $\xi \in \mathbb{R}$. For any i, from the normality, using similar computations, we have that for $k = 0$

$$1 = \int_{\infty}^{\infty} \phi_i(x)\overline{\phi_i(x)}dx = \int_0^1 \sum_{l\in\mathbb{Z}} \widehat{\phi}_i(\xi+l)\overline{\widehat{\phi}_i(\xi+l)}e^{-2\pi i k\xi}d\xi,$$

and for any $k \neq 0$

$$\begin{aligned}
0 &= \int_{\infty}^{\infty} \phi_i(x)\overline{\phi_i(x+k)}dx \\
&= \int_{\infty}^{\infty} \widehat{\phi}_i(\xi)\overline{\widehat{\phi}_i(\xi)}e^{-2\pi ik\xi}d\xi \\
&= \sum_{l\in\mathbb{Z}} \int_0^1 \widehat{\phi}_i(\xi+l)\overline{\widehat{\phi}_i(\xi+l)}e^{-2\pi ik(\xi+l)}d\xi \\
&= \int_0^1 \sum_{l\in\mathbb{Z}} \widehat{\phi}_i(\xi+l)\overline{\widehat{\phi}_i(\xi+l)}e^{-2\pi ik\xi}d\xi
\end{aligned}$$

So we have $G_\Phi(\xi)_{i,i} = 1$ for almost every $\xi \in \mathbb{R}$. That shows that $G_\Phi(\omega) = I$, a.e. $\omega \in \mathbb{R}$ $\square$

Corollary I.4.6. *If ϕ generates an orthonormal basis for the PSIS $V(\phi)$, then we have that*

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi+k)|^2 = 1, \ \ a.e.\ \omega \in \mathbb{R}.$$

I.5 Shift-invariant spaces with $\frac{1}{n}\mathbb{Z}$-invariance

We say V is $\frac{1}{n}\mathbb{Z}$-invariant if V is invariant under translations by multiples of $1/n$, i.e.

$$f \in V \text{ implies } f(\cdot + \frac{k}{n}) \in V, \text{ for all } k \in \mathbb{Z}.$$

We say a subspace is translation-invariant if it is invariant under all real numbers, i.e.

$$f \in V \text{ implies } f(\cdot + \alpha) \in V, \text{ for all } \alpha \in \mathbb{R}.$$

Obviously, any translation-invariant space is shift-invariant and $\frac{1}{n}\mathbb{Z}$-invariant and any $\frac{1}{n}\mathbb{Z}$-invariant space is shift-invariant, but the inverses are not true in general.

Given a SIS V, let G be the set of all parameters θ such that V is invariant under integer multiple of θ. We must have that $\mathbb{Z} \subset G$ and if $\theta \in G$, then $l + k\theta \in G$ for all integers l and k. In particular, if θ is irrational, then G is dense in $\mathbb{R}$, since V is closed, we can show that V is translation invariant.

It is shown in [ACHKM10] that there are only two possibilities:

- V is translation-invariant, or
- there exists an $n \in \mathbb{N}$ such that V is $\frac{1}{n}\mathbb{Z}$-invariant, but not $\frac{1}{m}\mathbb{Z}$-invariant with $m > n$.

Definition I.5.1. Given a SIS V, we say that V has invariance order n if n is the maximum positive integer such that V is $\frac{1}{n}\mathbb{Z}$-invariant. If this maximum does not exist, we say that V has invariance order ∞; in this case V is translation-invariant.

Given a positive integer n, we divide the real line into n subset. For $k = 0, \cdots, n-1$, set $I_k = [k, k+1)$, and define

$$B_k = \bigcup_{j \in \mathbb{Z}} (I_k + nj).$$

Note that B_k implicitly depends on the choice of n.

Suppose V is a SIS in $L^2(\mathbb{R})$, we define the following spaces:

$$U_k = \{f \in L^2(\mathbb{R}) : \widehat{f} = \widehat{g}\chi_{B_k} \text{ for some } g \in V\}, \qquad k = 0, \ldots, n-1. \tag{I.5.1}$$

Since the sets B_k are disjoint, we have that U_k are mutually orthogonal.

If $f \in V$, we use f^k to denote the function

$$\widehat{f^k} = \widehat{f}\chi_{B_k},$$

and P_k to denote the orthogonal projection onto U_k,

$$f^k = P_k f.$$

The following results are proven in [ACHKM10]

Theorem I.5.2. *If V is a SIS, then for each $k = 0, \dots, n-1$, the subspace U_k is a SIS that is also $\frac{1}{n}\mathbb{Z}$-invariant.*

Proposition I.5.3. *Suppose V is a SIS, then V is also $\frac{1}{n}\mathbb{Z}$-invariant iff $U_k \subset V$ for all $k = 0, \dots, n-1$. In that case, we have that V is the orthogonal direct sum*

$$V = U_0 \dot{\oplus} \cdots \dot{\oplus} U_{n-1}.$$

This proposition states that if a SIS is also $\frac{1}{n}\mathbb{Z}$-invariant then every function in it can be uniquely written as a sum of projections in the frequency domain.

Definition I.5.4. Given $f \in L^2(\mathbb{R})$ and $\omega \in [0,1)$, the *fiber* $\widehat{f}_\omega$ of f at ω is the sequence

$$\widehat{f}_\omega = \{\widehat{f}(\omega + k)_{k \in \mathbb{Z}}\}.$$

It is easy to check that the fiber $\widehat{f}_\omega$ belongs to $\ell^2(\mathbb{Z})$ for almost every $\omega \in [0,1)$. Similarly, given a closed subspace V of $L^2(\mathbb{R})$ and $\omega \in [0,1)$, the *fiber space* of V at ω is

$$\mathcal{J}_V(\omega) = \overline{\operatorname{span}}\{\widehat{f}_\omega : f \in V \text{ and } \widehat{f}_\omega \in \ell^2(\mathbb{Z})\},$$

where the closure is taken in the norm of $\ell^2(\mathbb{Z})$.

If $V(\Phi)$ is a finitely generated SIS with $\Phi = \{\phi_1, \ldots, \phi_r\}$ and $f \in V(\Phi)$, there exist $\mathbb{Z}$-periodic functions $a_1, \ldots, a_r$ such that

$$\widehat{f}(\xi) = \sum_{i=1}^{r} a_i(\xi)\widehat{\phi}(\xi), \text{a.e. } \xi,$$

which implies that the fiber space $\mathcal{J}_{V(\Phi)}(\omega)$ is generated by the fibers of the generators at ω. That is, for a.e. ω we have that

$$\mathcal{J}_{V(\Phi)}(\omega) = \text{span}\{(\widehat{\phi}_i)_\omega : i = 1, \ldots, r\}.$$

Now we define the $r \times \mathbb{Z}$ matrix $\mathcal{F}_\Phi(\omega)$,

$$(\mathcal{F}_\Phi(\omega))_{i,j} = \widehat{\phi}_i(w + j), \text{ for } 1 \leq i \leq r, j \in \mathbb{Z}.$$

From the definition of Grammian matrix, we have that $G_\Phi(\omega) = \mathcal{F}_\Phi(\omega) \times \mathcal{F}_\Phi^*(\omega)$, where $\mathcal{F}_\Phi^*(\omega)$ is the adjoint matrix of $\mathcal{F}_\Phi(\omega)$. Then we must have that

$$\dim(\mathcal{J}_{V(\Phi)}(\omega)) = \text{rank } [\mathcal{F}_\Phi(\omega)] = \text{rank } [G_\Phi(\omega)], \tag{I.5.2}$$

for a.e. ω.

In a similar manner, the SIS U_k is generated by $\Phi^k = P_k\Phi = \{\phi_1^k, \ldots, \phi_r^k\}$, where $\phi_i^k = P_k\phi_i$. The fiber spaces $\mathcal{J}_{U_k}(\omega)$ satisfy

$$\mathcal{J}_{U_k}(\omega) = \text{span}\{(\widehat{\phi}_i^{\,k})_\omega : i = 1, \ldots, r\}.$$

for a.e. ω.

We denote by G_{Φ^k} the Grammian matrix associated with the generators of U_k.

From above, we must have

$$\dim(\mathcal{J}_{U_k}(\omega)) = \text{rank } [G_{\Phi^k}(\omega)], \tag{I.5.3}$$

for a.e. ω and $k = 0, \dots, n$.

Using the notation above, we have the following characterization,

Theorem I.5.5 ([ACHKM10]). *If $V(\Phi)$ is finitely generated by $\Phi = (\phi_1, \cdots, \phi_r)^T$, then the following statements are equivalent:*

(a) V is $\frac{1}{n}\mathbb{Z}$-invariant.

(b) For almost every $\omega \in [0, 1)$,

$$\dim\left(\mathcal{J}_V(\omega)\right) = \sum_{k=0}^{n-1} \dim\left(\mathcal{J}_{U_k}(\omega)\right).$$

(c) For almost every $\omega \in [0, 1)$,

$$\text{rank } [G_\Phi(\omega)] = \sum_{k=0}^{n-1} \text{rank } [G_{\Phi^k}(\omega)].$$

The proof of this theorem follows directly from (I.5.2) and (I.5.3) and the orthogonality of the U_k.

CHAPTER II

UNCERTAINTY PRINCIPLES

In many applications, it is always desired to construct "good" generating functions in the sense that the functions have good localization in both time and frequency domains, i.e. the generating functions and their Fourier transforms both have fast decay. It is of interest in some applications for a SIS to be translation invariant like the Paley-Wiener space. However, the generating function $\sin \pi x / \pi x$ has slow decay. So we want to construct a SIS which is also $\frac{1}{n}\mathbb{Z}$-invariant such that the generators have fast decay, but also have good frequency decay.

In general, there are obstructions for the time-frequency localization. For example, the classical uncertainty principle tells us that a function and its fourier transform cannot both decay as rapidly as we want.

Theorem II.0.6. *For any function $f \in L^2(\mathbb{R})$, we have*

$$||f||_2^2 \leq 4\pi ||xf(x)||_2 ||\xi \widehat{f}(\xi)||_2,$$

and the equality holds only if

$$f(x) = e^{-sx^2}$$

for $s > 0$.

See [BF94] for the proof of a more general version of the classical uncertainty principle.

The time-frequency localization deteriorates if we impose more conditions. In fact, for sequence of functions, it is even impossible for

$$|f_n(x)| \leq c(1+|x|)^p, \ |\widehat{f}_n(\xi)| \leq c(1+|\xi|)^p$$

to hold, when $p > \frac{1}{2}$, for all f_n in an infinite orthonormal set [BBS92], where c is a absolute positive constant which dose not depend on n.

In fact, the time-frequency non-localization for a basis of $L^2(\mathbb{R})$ is an active subject of research. For example, if the Gabor system $\{E_m T_n f\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imx} f(x+n)\}_{m,n\in\mathbb{Z}}$ of a function f is a Riesz basis for $L^2(\mathbb{R})$, we will have the following Balian-Low theorem:

Theorem II.0.7. *Let $f \in L^2(\mathbb{R})$. If $\{E_m T_n f\}$ is a Riesz basis for $L^2(\mathbb{R})$, then*

$$\Big(\int_{-\infty}^{\infty} |xf(x)|^2 dx\Big)\Big(\int_{-\infty}^{\infty} |\xi\widehat{f}(\xi)|^2 d\xi\Big) = \infty$$

The Balian-Low theorem implies that if function f generates a Gabor Riesz basis, then it is impossible that g and $\widehat{g}$ satisfy

$$|f(x)| < \frac{c}{x^r}, \ \ |\widehat{f}(\xi)| < \frac{c}{\xi^r}$$

simultaneously with $r > 3/2$.

See [HW96] for the proof of The Balian-Low theorem and [HP08] and the references therein for related work.

In this chapter, we want to consider similar questions but in SIS which are also $\frac{1}{n}\mathbb{Z}$-invariant. The questions can be formulated as follows:

Question II.0.8.

1. *If functions $\phi_1, \dots, \phi_r$ and their integer shifts generate a Riesz basis (or even a frame) for the shift-invariant space $V(\Phi)$, what can we say about*

their time-freqency localizations?

2. *Furthermore, if $V(\Phi)$ is also $\frac{1}{n}\mathbb{Z}$-invariant for some $n > 1$, how fast can ϕ_i decay in both time and frequency domains simultaneously?*

Note that the generators only generate a subspace of $L^2(\mathbb{R})$ and we don't consider the modulation structures of the functions. So the Balian-Low theorem doesn't apply here. In fact, it is possible to construct an orthonormal generator ϕ for the shift invariant space $V(\phi)$ such that ϕ and $\widehat{\phi}$ are both in the Schwartz space. However, we show that if $V(\phi)$ has additional invariance then we get a severe time-freqency obstruction.

First, we will consider the easy case when the SIS is a PSIS, that is, generated by a single function. Then we will consider the more general finitely generated SIS. Although, the PSIS result is really a special case for general finitely generated SIS result, we will consider them separately. Also, to make things smoother, we will show the results first and prove them later.

II.1 Principal Shift-invariant Space case

In general, if we do not require any additional invariance other than integer shift, we can construct a PSIS to have an orthonormal generator with exponential decay. Consider the Schwartz $\mathcal{S}$ space of rapidly decreasing functions. Let $g(x) = e^{-\pi x^2}$ and then $\widehat{g}(\xi) = e^{-\pi\xi^2}$, hence g and $\widehat{g}$ are both in $\mathcal{S}$. Consider the overlap function $G_g(\xi) = \sum_j |\widehat{g}(\xi + j)|^2 = \sum_j e^{-\pi(\xi+j)^2}$. We also know that $\widehat{g}(\xi) = e^{-\pi\xi^2}$ is never zero. In particular, there exists positive constant c such that $\widehat{g}(\xi) \geq c > 0$ for all $\xi \in [0,1]$. Then we have $G_g(\xi) \geq |\widehat{g}(\xi)|^2 \geq c^2 > 0$ for all $\xi \in [0,1]$. Since $G_g(\xi)$ is a 1-periodic function, $G_g(\xi) \geq c^2 > 0$ for all ξ.

Now it makes sense to consider the function $\phi(x)$ such that $\widehat{\phi}(\xi) = \widehat{g}(\xi)/(G_g(\xi))^{\frac{1}{2}}$. And it follows that the overlap function $G_\phi(\xi) = \sum_j |\widehat{\phi}(\xi + j)|^2 = \sum_j |\widehat{g}(\xi +$

$j)|^2/G_g(\xi) = 1$. Thus, $\{\phi(x+k)\}_k$ is an orthonormal basis for the PSIS $V(\phi)$, and ϕ decays exponentially in both the time domain and frequency domain. However, $V(\phi)$ is neither translation invariant nor $\frac{1}{n}\mathbb{Z}$-invariant for some $n > 1$. This leads us to the consider next question.

As we can see from the PW space, if a PSIS is translation invariant, there may be some obstructions in terms of the time-frequency localization of the generator. Actually, this slow spatial-decay property for the generator of principal shift-invariant spaces $V(\phi)$ that are also translation-invariant is not unique to the space of bandlimited functions PW. In fact, we can show that the generator ϕ of any translation-invariant principal shift-invariant space $V(\phi)$ is not integrable.

Theorem II.1.1. *Let $\phi \in L^2$ and $\{\phi(\cdot - k)|\ k \in \mathbb{Z}\}$ be a Riesz basis for its generating space $V(\phi)$. If $V(\phi)$ is translation-invariant, then $\phi \notin L^1$.*

The slow spatial-decay of the generators of shift-invariant spaces that are also translation-invariant is a disadvantage for the numerical implementation of some analysis and processing algorithms. As we discussed before, this is the reason we are considering the PSIS which is $\frac{1}{n}\mathbb{Z}$-invariant.

We try to circumvent some of the problems by seeking PSIS $V(\phi)$ that are close to being translation invariant, with a generator ϕ which is well localized in both space and frequency domains, i.e., ϕ and $\widehat{\phi}$ are well localized. Specifically, we ask whether we can find a shift-invariant space $V(\phi)$ such that $V(\phi)$ is also $\frac{1}{n}\mathbb{Z}$ invariant for some $2 \leq n \in \mathbb{N}$, and such that ϕ and $\widehat{\phi}$ are well localized. It turns out that it is possible to construct functions ϕ that are well-localized in time and frequency domains, that generate shift-invariant spaces $V(\phi)$ that are also $\frac{1}{n}\mathbb{Z}$ invariant. But in light of the Balian-Low Theorem and uncertainty principle, we should expect there must be some other obstructions. It turns out that we can obtain the following surprising result:

Theorem II.1.2. *If $\phi \in L^2$ has the property that $\{\phi(\cdot - k)|\ k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V(\phi)$ and $V(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for some $n \geq 2$, then for any $\epsilon > 0$, we have*

$$\int_{\mathbb{R}} |\phi(x)|^2 |x|^{1+\epsilon} dx = +\infty. \tag{II.1.1}$$

Remark 1.

(i) Theorem II.1.2 is a Balian-Low type result. If we choose $\epsilon = 1$ in (II.1.1) of Theorem II.1.2, we get $\int_{\mathbb{R}} |x\phi(x)|^2 dx = +\infty$. It should be noted that in the Balian-Low Theorem $\int_{-\infty}^{\infty} |xg(x)|^2 dx$ can be finite, while in the case of Theorem II.1.2 $\int_{\mathbb{R}} |x\phi(x)|^2 dx$ is always infinite. For the case $\Delta_p = \int_{\mathbb{R}} |\phi(x)|^2 |x|^p dx$, the theorem above should be comparable to the $(1, \infty)$ version of the Balian-Low Theorem ([BCPS06], [Gau09]).

(ii) If we do not require other invariances besides integer shifts, then we can find $V(\phi)$ such that $\{\phi(\cdot - k) :\ k \in \mathbb{Z}\}$ is an orthonormal basis for V and such that ϕ decays exponentially in both time and frequency. In particular for such a ϕ it is obvious that $\Big(\int_{-\infty}^{\infty} |x|^{\alpha} |g(x)|^2 dx\Big)\Big(\int_{-\infty}^{\infty} |\xi|^{\beta} |\widehat{g}(\xi)|^2 d\xi\Big) < \infty$, where $\alpha, \beta > 0$ are any positive real numbers.

There is also a decay restriction in the Fourier domain. Specifically, the Fourier transform of an integrable generator ϕ of a principal shift-invariant space which is $\frac{1}{n}\mathbb{Z}$-invariant for some integer $n \geq 2$ cannot decay faster than $|\xi|^{-1/2-\epsilon}$ for any $\epsilon > 0$.

Theorem II.1.3. *Let $2 \leq n \in \mathbb{N}$. Let $\phi \in L^1 \cap L^2$ have the property that $\{\phi(\cdot - k)|\ k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V(\phi)$ and $V(\phi)$ is*

$\frac{1}{n}\mathbb{Z}$*-invariant, then for any* $\epsilon > 0$*,*

$$\sup_{\xi\in\mathbb{R}} |\widehat{\phi}(\xi)||\xi|^{1/2+\epsilon} = +\infty. \tag{II.1.2}$$

We conclude from Theorem II.1.3 that there is an obstruction to pointwise frequency (non)-localization property.

Remark 2. The conclusion of Theorem II.1.3 remains valid if we weaken the condition that $\phi \in L^1 \cap L^2$ to $\phi \in L^2$ and $\widehat{\phi}$ is continuous.

Now, we show the optimality of the results of Theorems II.1.2 and II.1.3.

The optimality of Theorem II.1.2 is obvious since the $\phi = sinc$ function generates a translation invariant space and $\int_{\mathbb{R}} |\phi(x)|^2|x|^{1-\epsilon}dx < \infty$ for any $0 < \epsilon < 1$. The following result shows that (II.1.2) in Theorem II.1.3 is sharp and that for any $2 \leq n \in \mathbb{N}$ there exists a generator $\phi \in L^1 \cap L^2$ (that depends on n) for $V(\phi)$ such that $\widehat{\phi}$ decays like $|\xi|^{-1/2}$. This is done by constructing time-frequency localized generators ϕ that achieve the desired properties:

Theorem II.1.4. *For each integer* $n \geq 2$*, there exists a function* $\phi \in L^1 \cap L^2$ *(and hence* $\widehat{\phi}$ *is continuous), which depends on* n*, such that* $\{\phi(\cdot - k)|\ k \in \mathbb{Z}\}$ *is an orthonormal basis for its generating space* $V(\phi)$*,* $V(\phi)$ *is* $\frac{1}{n}\mathbb{Z}$*-invariant, and*

$$\int_{\mathbb{R}} |\phi(x)|^2(1 + |x|)^{1-\epsilon}dx < \infty, \tag{II.1.3}$$

$$\sup_{\xi\in\mathbb{R}} |\widehat{\phi}(\xi)||\xi|^{1/2} < +\infty. \tag{II.1.4}$$

Remark 3.

(i) Note that by giving up the translation invariance and only allowing $1/n$ invariance as in Theorem II.1.4, we are able to have an L^1 generator, while this is not possible for translation invariance as shown in Theorem II.1.1.

(ii) Note that Theorem II.1.4 shows the optimality of both Theorems II.1.2 and II.1.3 simultaneously.

We turn our attention to the integral measure of time-frequency localization for generators of $\frac{1}{n}\mathbb{Z}$-invariant spaces. Unlike what was proven for the translation-invariant case in Theorem 1.1, we prove that by sacrificing a little frequency localization, it is possible for generators of such spaces to be in L^1, even when satisfying the optimality condition (II.1.3).

Theorem II.1.5. *For any $2 \leq n \in \mathbb{N}$, $\epsilon > 0$, $\gamma \geq 0$, $\delta > 0$, $1 \leq q < \infty$ with $1 + \delta - q/2 < 1/(2\gamma)$, there exists $\phi \in L^2$ (that depends on $\epsilon, \delta, q, \gamma, n$) such that $\{\phi(\cdot - k) |\ k \in \mathbb{Z}\}$ is an orthonormal basis for its generating space $V(\phi)$, $V(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant and ϕ satisfies the following conditions:*

1. $\int_{\mathbb{R}} |\phi(x)|^2 (1 + |x|)^{1-\epsilon} dx < \infty$,
2. $\int_{\mathbb{R}} |\phi(x)| (1 + |x|)^{\gamma} dx < \infty$,
3. $\int_{\mathbb{R}} |\widehat{\phi}(\xi)|^q (1 + |\xi|)^{\delta} d\xi < \infty$.

Remark 4.

(i) Note that the orthonormal generator $\phi = sinc$ for the Paley-Wiener space PW satisfies the first and third localization properties in Theorem II.1.5. However, the *sinc* function does not satisfies the second time localization inequality. In fact no function ϕ generating a shift-invariant space $V(\phi)$ that is also translation invariant can satisfy the second inequality of Theorem II.1.5, as is shown in Theorem II.1.1. Thus by relaxing translation invariance to $\frac{1}{n}\mathbb{Z}$ invariance we are able to get better time localization in the sense of the second localization inequality above. For this, however, we needed to trade off some frequency localization by allowing infinite support in frequency.

(ii) We do not know what happens for the case $\epsilon = 0$.

(iii) Using Lemmas II.2.4, II.2.5 and II.2.6, Theorem II.1.5 can be shown to be valid for other norms and other weights.

II.2 Proofs

II.2.1 Proof of Theorem II.1.1

To prove Theorem II.1.1, we first look at the following proposition which is a special case of the result in Theorem I.5.5.

Proposition II.2.1. *Let $\phi \in L^2$ with the property that $\{\phi(\cdot - k) |\ k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V(\phi)$. Then $V(\phi)$ is translation-invariant if and only if for almost all $\xi \in \mathbb{R}$,*

$$\widehat{\phi}(\xi)\widehat{\phi}(\xi + k) = 0 \text{ for all } 0 \neq k \in \mathbb{Z}.$$

Now we start to prove Theorem II.1.1.

Proof of Theorem II.1.1. Suppose on the contrary that there exists a principal shift-invariant space $V(\phi)$ on the real line such that $V(\phi)$ is translation-invariant and the generator ϕ is integrable. Let

$$\mathcal{O} := \{\xi \in \mathbb{R} |\ \widehat{\phi}(\xi) \neq 0\}.$$

Since $\phi \in L^1$ by assumption, $\widehat{\phi}$ is continuous, and hence $\mathcal{O}$ is an open set. From Proposition II.2.1, it follows that the Lebesgue measure of the set $(\mathcal{O}+j)\cap(\mathcal{O}+k)$ is zero for all $j \neq k \in \mathbb{Z}$. This together with the fact that $\mathcal{O}$ is an open set gives that

$$(\mathcal{O} + j) \cap (\mathcal{O} + k) = \emptyset \quad \text{for all } j \neq k \in \mathbb{Z}. \tag{II.2.1}$$

Recall that $\mathbb{R}$ is connected and that any connected set is not a union of nonempty disjoint open sets. Thus $\{\mathcal{O}+k | k \in \mathbb{Z}\}$ is not an open covering of the real line, i.e., $\mathbb{R}\backslash(\cup_{k\in\mathbb{Z}}(\mathcal{O}+k)) \neq \emptyset$, which in turn implies the existence of a real number $\xi_0 \in \mathbb{R}$ with the property that

$$\widehat{\phi}(\xi_0 + k) = 0 \quad \text{for all } k \in \mathbb{Z}. \tag{II.2.2}$$

As $\widehat{\phi}$ is uniformly continuous by the assumption that $\phi \in L^1$, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\widehat{\phi}(\xi + k) - \widehat{\phi}(\xi_0 + k)| < \epsilon \quad \text{for all } |\xi - \xi_0| < \delta \text{ and } k \in \mathbb{Z}. \tag{II.2.3}$$

By (II.2.1), for any $\xi \in \mathbb{R}$ there exists an integer $l(\xi)$ such that

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + k)|^2 = |\widehat{\phi}(\xi + l(\xi))|^2. \tag{II.2.4}$$

Combining (II.2.2), (II.2.3) and (II.2.4) yields

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + k)|^2 < \epsilon^2 \quad \text{whenever } |\xi - \xi_0| < \delta. \tag{II.2.5}$$

Since $\epsilon > 0$ can be chosen to be arbitrarily small, the last inequality contradicts the Riesz basis property that there exists $m > 0$ such that $m \le \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi+k)|^2$ for almost all $\xi \in \mathbb{R}$. $\square$

II.2.2 Proof of Theorem II.1.2

We first have the following proposition which is also a special case of Theorem I.5.5.

Proposition II.2.2. *Let $n \geq 2$ be an integer, and $\phi \in L^2$ with the property that $\{\phi(\cdot - k) |\ k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V(\phi)$. Then $V(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant if and only if for almost all $\xi \in \mathbb{R}$, one and only one of the following vectors*

$$\Phi_m(\xi) := (\cdots, \widehat{\phi}(\xi + m - n), \widehat{\phi}(\xi + m), \widehat{\phi}(\xi + m + n), \cdots), \quad 0 \leq m \leq n - 1, \tag{II.2.6}$$

is nonzero.

Proof of Theorem II.1.2. Suppose on the contrary that

$$\int_{\mathbb{R}} |\phi(x)|^2 (1 + |x|)^{1+\epsilon} dx < \infty. \tag{II.2.7}$$

Then $\phi \in L^1$, which implies that $\widehat{\phi}$ is a uniformly continuous function. Let $\mathcal{O}_m = \{\xi \in \mathbb{R} |\ \Phi_m(\xi) \neq 0\}, 0 \leq m \leq n - 1$, where Φ_m is defined as in (II.2.6). Since

$$\mathcal{O}_m = \bigcup_{k \in \mathbb{Z}} \{\xi \in \mathbb{R} |\ \widehat{\phi}(\xi + m + kn) \neq 0\},$$

then $\mathcal{O}_m, 0 \leq m \leq n - 1$ are open sets, and

$$\mathcal{O}_m + m = \mathcal{O}_0 \text{ and } \mathcal{O}_m + nk = \mathcal{O}_m \quad \text{for all } 0 \leq m \leq n - 1 \text{ and } k \in \mathbb{Z}. \tag{II.2.8}$$

Moreover, the intersection between the sets $\mathcal{O}_m$ with different m have zero Lebesgue measure (hence are empty sets). Therefore $\{\mathcal{O}_m | 0 \leq m \leq n - 1\}$ is not an open covering of the real line $\mathbb{R}$, which implies that the existence of a real number $\xi_0 \in \mathbb{R}$ with the property that

$$\widehat{\phi}(\xi_0 + k) = 0 \quad \text{for all } k \in \mathbb{Z}. \tag{II.2.9}$$

Let $N \geq 1$ be a sufficiently large integer, $\delta = N^{-1-\epsilon/2}$, and h be a smooth function supported on $[-2,2]$ and satisfy $0 \leq h \leq 1$ and $h(x) = 1$ when $x \in [-1,1]$. Define $\phi_N(x) = h(x/N)\phi(x)$. Then we obtain that

$$\begin{aligned}
& \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}|(\widehat{\phi}-\widehat{\phi}_N)(\xi_0+\xi+k)|^2 d\xi\Big)^{1/2} \\
\leq\ & \Big(\frac{1}{2\delta}\int_{\mathbb{R}}|(\widehat{\phi}-\widehat{\phi}_N)(\xi)|^2 d\xi\Big)^{1/2} = \Big(\frac{1}{2\delta}\int_{\mathbb{R}}|\phi(x)-\phi_N(x)|^2 dx\Big)^{1/2} \\
\leq\ & N^{-\epsilon/4}\Big(\int_{\mathbb{R}}|\phi(x)|^2(1+|x|)^{1+\epsilon}dx\Big)^{1/2}, \qquad \text{(II.2.10)}
\end{aligned}$$

and

$$\begin{aligned}
& \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}|\widehat{\phi}_N(\xi_0+\xi+k)-\widehat{\phi}_N(\xi_0+k)|^2 d\xi\Big)^{1/2} \\
=\ & \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}\Big|\int_0^{\xi}\widehat{\phi}_N'(\xi_0+\xi'+k)d\xi'\Big|^2 d\xi\Big)^{1/2} \\
\leq\ & \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}\xi\int_0^{\xi}\sum_{k\in\mathbb{Z}}\Big|\widehat{\phi}_N'(\xi_0+\xi'+k)\Big|^2 d\xi' d\xi\Big)^{1/2} \\
\leq\ & \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}\xi\int_0^{\xi}\sum_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}}N^2|\widehat{h}'(N\eta)||\widehat{\phi}(\xi_0+\xi'+k-\eta)|d\eta\Big|^2 d\xi' d\xi\Big)^{1/2} \\
\leq\ & \Big(\frac{N^3}{2\delta}\|\widehat{h}'\|_1\int_{-\delta}^{\delta}\xi\int_0^{\xi}\int_{\mathbb{R}}|\widehat{h}'(N\eta)|\Big(\sum_{k\in\mathbb{Z}}|\widehat{\phi}(\xi_0+\xi'+k-\eta)|^2\Big)d\eta d\xi' d\xi\Big)^{1/2} \\
\leq\ & N^{-\epsilon/2}\|\widehat{h}'\|_1\Big(\operatorname{ess\,sup}_{\xi\in\mathbb{R}}\sum_{k\in\mathbb{Z}}|\widehat{\phi}(\xi+k)|^2\Big)^{1/2}, \qquad \text{(II.2.11)}
\end{aligned}$$

where $\widehat{\phi}_N(\xi) = N\int_{\mathbb{R}}\widehat{h}(N\eta)\widehat{\phi}(\xi-\eta)d\eta$ is used to obtain the second inequality, while the third inequality is obtained by letting $|\widehat{h}'(N\eta)| = |\widehat{h}'(N\eta)|^{1/2}|\widehat{h}'(N\eta)|^{1/2}$

and using Hölder inequality. Also we have that

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} |\widehat{\phi}_N(\xi_0+k)|^2 &= \sum_{k\in\mathbb{Z}} \Big| \int_{\mathbb{R}} e^{-2\pi i(\xi_0+k)x}\phi(x)(1-h(x/N))dx\Big|^2 \\
&\le \int_0^1 \Big(\sum_{l\in\mathbb{Z}} |\phi(x+l)||1-h((x+l)/N)|\Big)^2 dx \\
&\le \int_0^1 \Big(\sum_{l\in\mathbb{Z}} |\phi(x+l)|^2(1+|x+l|)^{1+\epsilon}\Big) \\
&\qquad \times \Big(\sum_{l\in\mathbb{Z}} (1-h((x+l)/N))^2(1+|x+l|)^{-1-\epsilon}\Big)dx \\
&\le 2\Big(\sum_{l=N}^{\infty} |l|^{-1-\epsilon}\Big) \times \Big(\int_{\mathbb{R}} |\phi(x)|^2(1+|x|)^{1+\epsilon}dx\Big)
\end{aligned} \tag{II.2.12}
$$

where the first equality follows from (II.2.9). Combining (II.2.10), (II.2.11) and (II.2.12) with the Riesz basis condition gives

$$
\begin{aligned}
m &\le \operatorname{ess\,inf}_{\xi\in\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi+k)|^2\Big)^{1/2} \\
&\le \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta} \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi_0+\xi+k)|^2 d\xi\Big)^{1/2} \\
&\le \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta} \sum_{k\in\mathbb{Z}} |\widehat{\phi}_N(\xi_0+\xi+k)-\widehat{\phi}_N(\xi_0+k)|^2 d\xi\Big)^{1/2} \\
&\quad + \Big(\sum_{k\in\mathbb{Z}} |\widehat{\phi}_N(\xi_0+k)|^2\Big)^{1/2} + \Big(\frac{1}{2\delta}\int_{-\delta}^{\delta} \sum_{k\in\mathbb{Z}} |(\widehat{\phi}-\widehat{\phi}_N)(\xi_0+\xi+k)|^2 d\xi\Big)^{1/2} \\
&\le CN^{-\epsilon/4} \to 0 \text{ as } N\to\infty,
\end{aligned} \tag{II.2.13}
$$

which is a contradiction. □

II.2.3 Proof of Theorem II.1.3

Proof. Note that $\phi \in L^1$ implies that $\widehat{\phi}$ is uniformly continuous. Now, suppose on the contrary that

$$|\widehat{\phi}(\xi)| \leq C(1+|\xi|)^{-1/2-\epsilon} \tag{II.2.14}$$

for some positive constants C and $\epsilon > 0$. This together with the continuity of the function $\widehat{\phi}$ implies that $G_\phi(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi+k)|^2$ is a continuous function. Therefore there exists a positive constant m such that

$$G_\phi(\xi) \geq m \quad \text{for all } \xi \in \mathbb{R} \tag{II.2.15}$$

from the continuity of the function G_ϕ. Using the argument in the proof of Theorem II.1.2, we can find a real number $\xi_0 \in \mathbb{R}$ such that $\widehat{\phi}(\xi_0 + k) = 0$ for all $k \in \mathbb{Z}$, which implies that $G_\phi(\xi_0) = 0$. This contradicts (II.2.15). $\square$

II.2.4 Proof of Theorem II.1.4

To prove Theorems II.1.4 and II.1.5, we construct a family of principal shift-invariant spaces on the real line which are $\frac{1}{n}\mathbb{Z}$-invariant for a given integer $n \geq 2$. Let g be an infinitely-differentiable function that satisfies $g(x) = 0$ when $x \leq 0$, $g(x) = 1$ when $x \geq 1$, and $(g(x))^2 + (g(1-x))^2 = 1$ when $0 \leq x \leq 1$. For positive numbers $\alpha, \beta > 0$ and a natural number $n \geq 2$, define $\psi_{\alpha,\beta,n}$ with the help of the Fourier transform by

$$\begin{aligned}\widehat{\psi_{\alpha,\beta,n}}(\xi) &= h_0(\xi) + \sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} (\beta_j)^{-1/2} h_j(\xi - n(\gamma_j + l)) \\ &\quad + \sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} (\beta_j)^{-1/2} h_j(-\xi - n(\gamma_j + l)), \end{aligned} \tag{II.2.16}$$

where $\beta_j = \lceil 2^{j\beta} \rceil$ (the smallest integer larger than or equal to $2^{j\beta}$), $\gamma_j = \sum_{k=0}^{j-1} \beta_k$, $g_0(x) = g(x+1)g(-x+1)$, $g_1(x) = g(x+1)g(-2^{\alpha}x+1)$, and

$$h_j(\xi) = \begin{cases} g_0(2\xi/(1-2^{-\alpha})) & \text{if } j = 0, \\ g_1(2^{j\alpha}(2\xi - 1 + 2^{-j\alpha})/(2^{\alpha}-1)) & \text{if } j \geq 1. \end{cases} \tag{II.2.17}$$

The functions $\widehat{\psi_{\alpha,\beta,n}}(\xi)$ with $\alpha = 1, \beta = 2$ and $n = 2$ and $h_i(\xi), 0 \leq i \leq 3$, with $\alpha = 1$ are plotted in Figure II.1.

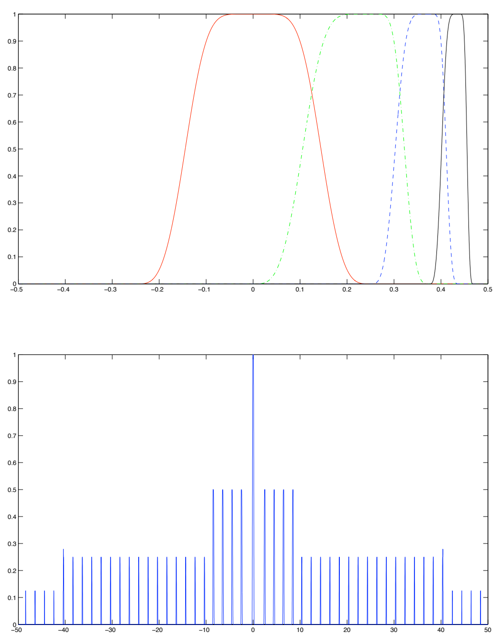

Figure II.1: The functions $h_i, 0 \leq i \leq 3$ with $\alpha = 1$ on the top, and the function $\widehat{\psi_{\alpha,\beta,n}}$ with $\alpha = 1, \beta = 2$ and $n = 2$ on the bottom.

Lemma II.2.3. *For $\alpha, \beta > 0$ and an integer $n \geq 2$, let $\psi_{\alpha,\beta,n}$ be defined as in* (II.2.16)*. Then $\psi_{\alpha,\beta,n}$ is an orthonormal generator of its generating space $V(\psi_{\alpha,\beta,n})$ and the principal shift-invariant space $V(\psi_{\alpha,\beta,n})$ is $\frac{1}{n}\mathbb{Z}$-invariant.*

Proof. As each h_j, for $j \geq 0$, is supported in $(-1/2, 1/2)$ by construction,

$$
\begin{aligned}
|\widehat{\psi_{\alpha,\beta,n}}(\xi)|^2 &= |h_0(\xi)|^2 + \sum_{j=1}^{\infty} \sum_{l=0}^{\beta_j - 1} (\beta_j)^{-1} |h_j(\xi - n(\gamma_j + l))|^2 \\
&\quad + \sum_{j=1}^{\infty} \sum_{l=0}^{\beta_j - 1} (\beta_j)^{-1} |h_j(-\xi - n(\gamma_j + l))|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} |\widehat{\psi_{\alpha,\beta,n}}(\xi + k)|^2 &= \sum_{k\in\mathbb{Z}} |h_0(\xi + k)|^2 + \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \left(|h_j(\xi + k)|^2 + |h_j(-\xi + k)|^2\right) \\
&= |h_0(\xi)|^2 + \sum_{j=1}^{\infty} |h_j(\xi)|^2 + \sum_{j=1}^{\infty} |h_j(-\xi)|^2 \qquad \text{(II.2.18)}
\end{aligned}
$$

for any $\xi \in (-1/2, 1/2)$. Set

$$
H(\xi) := |h_0(\xi)|^2 + \sum_{j=1}^{\infty} |h_j(\xi)|^2 + \sum_{j=1}^{\infty} |h_j(-\xi)|^2.
$$

Then $H(\xi)$ is a symmetric function supported on $(-1/2, 1/2)$ and for any $\xi \in [1 - 2^{-j\alpha}, 1 - 2^{-(j+1)\alpha}]/2$ with $j \geq 0$,

$$
\begin{aligned}
H(\xi) &= |h_j(\xi)|^2 + |h_{j+1}(\xi)|^2 \\
&= |g(-2^{(j+1)\alpha}(2\xi - 1 + 2^{-j\alpha})/(2^\alpha - 1) + 1)|^2 \\
&\quad + |g(2^{(j+1)\alpha}(2\xi - 1 + 2^{-(j+1)\alpha})/(2^\alpha - 1) + 1)|^2 \\
&= 1 \qquad \text{(II.2.19)}
\end{aligned}
$$

by the construction of the functions g and $h_j, j \geq 0$. Therefore $H(\xi) = 1$ for all $\xi \in (-1/2, 1/2)$, which together with (II.2.18) implies that

$$
\sum_{k\in\mathbb{Z}} |\widehat{\psi_{\alpha,\beta,n}}(\xi + k)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}\backslash(1/2 + \mathbb{Z}). \qquad \text{(II.2.20)}
$$

Then $\psi_{\alpha,\beta,n}$ is an orthonormal generator for its generating space $V(\psi_{\alpha,\beta,n})$ by (II.2.20).

By (II.2.16), $\widehat{\psi_{\alpha,\beta,n}}$ is supported on $(-1/2, 1/2) + n\mathbb{Z}$. Then $V(\psi_{\alpha,\beta,n})$ is $\frac{1}{n}\mathbb{Z}$-invariant by (II.2.20) and Proposition III.2.4. $\square$

We are now ready to prove Theorem II.1.4.

Proof of Theorem II.1.4. Let $\psi_{\alpha,\beta,n}$ be as in (II.2.16) for $\alpha, \beta > 0$, and set $\phi = \psi_{\alpha,\beta,n}$. Then by Lemma II.2.3 it suffices to prove (II.1.4) for the function ϕ just defined. From (II.2.16) it follows that

$$
\begin{aligned}
& |\widehat{\phi}(\xi)||\xi|^{1/2} = |\widehat{\psi_{\alpha,\beta,n}}(\xi)||\xi|^{1/2} \\
\leq \quad & \sup\Big\{|h_0(\xi)||\xi|^{1/2}, \sup_{j\geq 1, 0\leq l\leq \beta_j-1} \beta_j^{-1/2}|h_j(\xi - n(\gamma_j + l))||\xi|^{1/2}, \\
& \sup_{j\geq 1, 0\leq l\leq \beta_j-1} \beta_j^{-1/2}|h_j(-\xi - n(\gamma_j + l))||\xi|^{1/2}\Big\}
\end{aligned}
$$

Note that, from its definition, $h_j(\xi - n(\gamma_j + l))$ has support in $[n(\gamma_j + l), n(\gamma_j + l) + 1]$ and has maximal value 1. Thus the term $|h_j(\xi - n(\gamma_j + l))||\xi|^{1/2}$ can be bounded above by $\big(n(\gamma_j + l) + 1\big)^{1/2}$ for all ξ and $0 \leq l \leq \beta_j - 1$. Thus, it follows from the last inequality and the relation $\gamma_j + \beta_j = \gamma_{j+1}$ that

$$
|\widehat{\phi}(\xi)||\xi|^{1/2} \leq 1 + C\sup_{j\geq 1}(\beta_j)^{-1/2}(\gamma_{j+1})^{1/2} < \infty, \tag{II.2.21}
$$

where C is a positive constant. Hence (II.1.4) holds. In particular, we can show that

$$
0 < \limsup_{|\xi|\to\infty} |\widehat{\phi}(\xi)||\xi|^{1/2} < \infty. \tag{II.2.22}
$$

This proves the pointwise frequency localization of the theorem. The time localization inequality is a direct consequence of Lemma II.2.4 below. The fact that ϕ is also in L^1 follows from Lemma II.2.5 choosing $p = 1$, $\gamma = 0$. $\square$

II.2.5 Proof of Theorem II.1.5

Theorem II.1.5 is an immediate consequence of the following three lemmas:

Lemma II.2.4. *Let $\epsilon \in (0,1)$, $\alpha, \beta > 0$, n be an integer with $n \geq 2$, and $\psi_{\alpha,\beta,n}$ be defined as in* (II.2.16)*. Then*

$$\int_{\mathbb{R}} |\psi_{\alpha,\beta,n}(x)|^2 |x|^{1-\epsilon} dx < \infty. \tag{II.2.23}$$

Lemma II.2.5. *Let $\gamma \geq 0$, $1 \leq p < 2$, n be an integer with $n \geq 2$, and $\psi_{\alpha,\beta,n}$ be defined as in* (II.2.16) *for positive numbers $\alpha, \beta > 0$ with $\beta(1/p - 1/2) + \alpha(p - 1 - \gamma)/p > 0$. Then*

$$\int_{\mathbb{R}} |\psi_{\alpha,\beta,n}(x)|^p (1 + |x|)^\gamma dx < \infty. \tag{II.2.24}$$

Lemma II.2.6. *Let $\delta > 0$, $1 \leq q < \infty$, n be an integer with $n \geq 2$, and $\psi_{\alpha,\beta,n}$ be defined as in* (II.2.16) *for positive numbers $\alpha, \beta > 0$ with $\alpha > \beta(1 + \delta - q/2)$. Then*

$$\int_{\mathbb{R}} |\widehat{\psi_{\alpha,\beta,n}}(\xi)|^q (1 + |\xi|)^\delta d\xi < \infty. \tag{II.2.25}$$

Proof of Lemma II.2.4. Taking the inverse Fourier transform of both sides of (II.2.16) yields

$$\begin{aligned}
\psi_{\alpha,\beta,n}(x) \quad = \quad & \frac{1 - 2^{-\alpha}}{2} g_0^\vee \big(\frac{1 - 2^{-\alpha}}{2} x\big) + \frac{2^\alpha - 1}{2} \sum_{j=1}^{\infty} (\beta_j)^{-1/2} 2^{-j\alpha} g_1^\vee \big(\frac{2^\alpha - 1}{2^{j\alpha+1}} x\big) \\
& \times e^{\pi i x (1 - 2^{-j\alpha})} \Big(\sum_{l=0}^{\beta_j - 1} e^{2\pi i x n (\gamma_j + l)} \Big) + \frac{2^\alpha - 1}{2} \sum_{j=1}^{\infty} (\beta_j)^{-1/2} 2^{-j\alpha} \\
& \times g_1^\vee \big(- \frac{2^\alpha - 1}{2^{j\alpha+1}} x\big) \times e^{-\pi i x (1 - 2^{-j\alpha})} \Big(\sum_{l=0}^{\beta_j - 1} e^{-2\pi i x n (\gamma_j + l)} \Big),
\end{aligned} \tag{II.2.26}$$

where $g_0^\vee$ and $g_1^\vee$ are the inverse Fourier transforms of the functions g_0 and g_1 respectively. Since both g_0 and g_1 are compactly supported and infinitely differentiable, their inverse Fourier transforms $g_0^\vee$ and $g_1^\vee$ have polynomial decay at infinity. In particular,

$$|g_0^\vee(x)| + |g_1^\vee(x)| \le C(1+|x|)^{-2},\ x \in \mathbb{R}$$

for some positive constant C. Hence

$$\begin{aligned}
&\Big(\int_{\mathbb{R}} |\psi_{\alpha,\beta,n}(x)|^2(1+|x|)^{1-\epsilon}dx\Big)^{1/2} \le \Big(\frac{1-2^{-\alpha}}{2}\Big)\Big(\int_{\mathbb{R}} |g_0^\vee(x)|^2(1+|x|)^{1-\epsilon}dx\Big)^{1/2} \\
&\qquad +(2^\alpha-1)\sum_{j=1}^\infty(\beta_j)^{-1/2}2^{-j\alpha}\Big(\int_{\mathbb{R}}\Big|g_1^\vee\big(\frac{2^\alpha-1}{2^{j\alpha+1}}x\big)\Big|^2\Big(\frac{\sin\beta_j n\pi x}{\sin n\pi x}\Big)^2(1+|x|)^{1-\epsilon}dx\Big)^{1/2} \\
\le\ & C + C\sum_{j=1}^\infty 2^{-j(\beta+\alpha+\alpha\epsilon)/2}\Big(\int_{\mathbb{R}}(1+2^{-j\alpha}|x|)^{-2}\Big(\frac{\sin\beta_j\pi x}{\sin\pi x}\Big)^2dx\Big)^{1/2} \\
=\ & C + C\sum_{j=1}^\infty 2^{-j(\beta+\alpha+\alpha\epsilon)/2}\Big(\int_{-1/2}^{1/2}\Big(\sum_{l\in\mathbb{Z}}(1+2^{-j\alpha}|x+l|)^{-2}\Big)\Big(\frac{\sin\beta_j\pi x}{\sin\pi x}\Big)^2dx\Big)^{1/2} \\
\le\ & C + C\sum_{j=1}^\infty 2^{-j(\beta/2+\alpha\epsilon/2)}\Big(\int_{-1/2}^{1/2}\Big(\frac{\sin\beta_j\pi x}{\sin\pi x}\Big)^2dx\Big)^{1/2} \\
\le\ & C + C\sum_{j=1}^\infty 2^{-j(\beta+\alpha\epsilon)/2}\Big(\int_{-1/2}^{1/2}\big(\min(\beta_j,\frac{1}{2|x|})\big)^2dx\Big)^{1/2} \\
\le\ & C + C\sum_{j=1}^\infty 2^{-j\alpha\epsilon/2} < \infty, \qquad\qquad \text{(II.2.27)}
\end{aligned}$$

where C is a positive constant which could be different at different occurrences.

□

Proof of Lemma II.2.5. Similar to the argument in Lemma II.2.4 we have

$$\begin{aligned}
& \Big(\int_{\mathbb{R}} |\psi_{\alpha,\beta,n}(x)|^p (1+|x|)^\gamma dx\Big)^{1/p} \\
\leq\ & C + C\sum_{j=1}^{\infty} 2^{-j(\beta/2+\alpha(1-(1+\gamma)/p))}\Big(\int_{-1/2}^{1/2}\Big(\frac{\sin\beta_j\pi x}{\sin\pi x}\Big)^p dx\Big)^{1/p} \\
\leq\ & \begin{cases} C + C\sum_{j=1}^{\infty} 2^{-j(\beta(1/p-1/2)+\alpha(p-1-\gamma)/p)} & \text{if } 1<p<2 \\ C + C\sum_{j=1}^{\infty} j2^{-j(\beta/2-\gamma\alpha)} & \text{if } p=1 \end{cases} \\
<\ & \infty,
\end{aligned}$$

from which the lemma follows. □

Proof of Lemma II.2.6. By (II.2.16), we have

$$\begin{aligned}
& \int_{\mathbb{R}} |\widehat{\psi_{\alpha,\beta,n}}(\xi)|^q (1+|\xi|)^\delta d\xi \\
=\ & \int_{\mathbb{R}} |h_0(\xi)|^q (1+|\xi|)^\delta d\xi + \sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} \beta_j^{-q/2}\int_{\mathbb{R}} |h_j(\xi - n(\gamma_j+l))|^q (1+|\xi|)^\delta d\xi \\
& + \sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} \beta_j^{-q/2}\int_{\mathbb{R}} |h_j(-\xi - n(\gamma_j+l))|^q (1+|\xi|)^\delta d\xi \\
\leq\ & C + C\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} 2^{-\beta j(q/2-\delta)}\int_{\mathbb{R}} |h_j(\xi - n(\gamma_j+l))|^q d\xi \\
& + C\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} 2^{-\beta j(q/2-\delta)}\int_{\mathbb{R}} |h_j(-\xi - n(\gamma_j+l))|^q d\xi \\
\leq\ & C + C\sum_{j\geq 1} 2^{j\beta(\delta-q/2+1)-\alpha j} < \infty, \qquad \text{(II.2.28)}
\end{aligned}$$

where C is a positive constant which could be different at different occurrences. Hence the lemma is established. □

II.3 Finitely Generated Shift-invariant Space case

In this section, we generalize some of the results from the previous section. We generalize the results in two ways. First, we consider finitely generated shift-invariant spaces, second, we only require the generators and their integer shifts form a frame of the generating space.

Recall that if a shift-invariant space $V(\Phi)$ is generated by $\Phi = (\phi_1, \ldots, \phi_r)^T$ with $\phi_1, \ldots, \phi_r$ in $L^2(\mathbb{R})$ and $\{\phi_i(\cdot - k) |\ k \in \mathbb{Z}, i = 1, \ldots, r\}$ forms a frame for $V(\Phi)$, we have that

$$AG(\xi) \leq G^2(\xi) \leq BG(\xi) \tag{II.3.1}$$

for almost every $\xi \in \mathbb{R}$ ([Bow00]), where $G(\xi) = G_\Phi(\xi)$ is the Grammian matrix for $\Phi = (\phi_1, \ldots, \phi_r)^T$.

Here we have to make a natural assumption that $\mathrm{Length}(V(\Phi)) = r$ where for any finitely generated shift-invariant space V, $\mathrm{Length}(V)$ is the cardinality of the smallest generating set for V, that is

$$\mathrm{Length}(V) \stackrel{\text{def}}{=} \min\{\#\Phi : V = V(\Phi)\}.$$

Under this assumption, using Theorem 2.3 in [ACHM07], we can easily show that there exists at least a point $\xi \in \mathbb{R}$ such that

$$\det G(\xi) \neq 0.$$

We now show that if a finitely generated shift-invariant space is translation invariant, then at least one of its frame generators cannot be in L^1. In an equivalent form, we have the following theorem.

Theorem II.3.1. *Suppose that functions $\phi_1, \ldots, \phi_r$ in $L^2 \cap L^1$ satisfy that*

$\{\phi_i(\cdot - k)|\ k \in \mathbb{Z}, i = 1, \ldots, r\}$ forms a frame for the generating space $V(\Phi)$. Then $V(\Phi)$ cannot be translation-invariant.

We also have the following Balian-Low type obstructions, the following theorem can be considered as generalization of Theorem II.1.2,

Theorem II.3.2. *Suppose $\phi_i \in L^2$ are such that $\{\phi_i(\cdot - k)|\ k \in \mathbb{Z}, i = 1, \ldots, r\}$ is a frame for $V(\Phi)$ which is $\frac{1}{n}\mathbb{Z}$-invariant for some $n \nmid r$. Then there exits $i_0 \in \{1, \ldots, r\}$ such that*

$$\int_{\mathbb{R}} |\phi_{i_0}(x)|^2 |x|^{1+\epsilon} dx = +\infty$$

for all $\epsilon > 0$.

There is also a pointwise decay restriction in the Fourier domain. Specifically, the following theorem can be compared with Theorem II.1.3,

Theorem II.3.3. *Suppose that functions $\phi_1, \ldots, \phi_r$ in L^2 satisfy that $\widehat{\phi}_i$ is continuous for $i = 1, \ldots, r$ and $\{\phi_i(\cdot - k)|\ k \in \mathbb{Z}, i = 1, \ldots, r\}$ is a frame for the generating space $V(\Phi)$ which is $\frac{1}{n}\mathbb{Z}$-invariant for some $n \nmid r$, then there exits $i_0 \in \{1, \ldots, r\}$ such that $\xi^{1/2+\epsilon}\widehat{\phi}_{i_0}(\xi) \notin L^\infty$ for any $\epsilon > 0$, i.e.,*

$$\sup_{\xi \in \mathbb{R}} |\widehat{\phi}_{i_0}(\xi)||\xi|^{1/2+\epsilon} = +\infty.$$

Note that the condition $n \nmid r$ in the above two theorems is very easy to satisfy since we are hoping to have shift-invariant spaces close to being translation invariant. In other words, n is usually very large, in which case the condition is automatically satisfied.

II.4 Proofs

We first have the following proposition which is directly from Theorem I.5.5

Proposition II.4.1 ([ACHKM10]). *If $V(\Phi)$ is finitely generated by $\Phi = (\phi_1, \cdots, \phi_r)^T$ and $V(\Phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant, then for almost every $\omega \in [0,1)$,*

$$\text{rank } G(\omega) = \sum_{l=0}^{n-1} \text{rank } G_l(\omega), \tag{II.4.1}$$

where $G_l(\omega)_{ij} = \sum_{k\in\mathbb{Z}} \widehat{\phi}_i^l(\omega+k)\,\overline{\widehat{\phi}_j^l(\omega+k)}$ and $\widehat{\phi}_i^l = \widehat{\phi}_i \chi_{B_l}$.

Here and later on we will use a slightly different version of Proposition II.4.1,

Corollary II.4.2. *If $V(\Phi)$ is finitely generated by $\Phi = (\phi_1, \cdots, \phi_r)^T$ and $V(\Phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant, then for almost every $\omega \in \mathbb{R}$,*

$$\text{rank } G(\omega) = \sum_{l=0}^{n-1} \text{rank } A_l(\omega), \tag{II.4.2}$$

where $A_l(\omega)_{ij} = \sum_{k\in\mathbb{Z}} \widehat{\phi}_i(\omega+kn+l)\,\overline{\widehat{\phi}_j(\omega+kn+l)}$.

Proof. From the definition of the matrices $G_l(\omega)$ and $A_l(\omega)$, it is easy to show that for $\omega \in [0,1)$, we have that

$$G_l(\omega) = A_l(\omega).$$

In particular, we have (II.4.2) is true for almost every $\omega \in [0,1)$. Note that $A_l(\omega+1) = A_{l+1}(\omega)$, for all $l = 0, 1, \ldots, n-1$. Then $\sum_{l=0}^{n-1} \text{rank } [A_l(\omega)]$ is a periodic function with period 1. It is also easy to see that rank $[G(\omega)]$ is a periodic function with period 1. So we must have that (II.4.2) is true for almost every $\omega \in \mathbb{R}$. $\square$

Now we are ready to see an important lemma of this section.

Lemma II.4.3. *Suppose that functions $\phi_1, \ldots, \phi_r$ in L^2 satisfy that $\widehat{\phi_i}$ is continuous for $i = 1, \ldots, r$ and the generating space $V(\Phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for some $n \nmid r$, then there exists $\xi \in \mathbb{R}$ such that $\det(G(\xi)) = 0$.*

Proof. Suppose on the contrary that for all $\xi \in \mathbb{R}$, $\det(G(\xi)) \neq 0$, that is rank $G(\xi) = r$. Since $V(\Phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant, from Corollary II.4.2 we have that

$$r = \text{rank } G(\xi) = \sum_{l=0}^{n-1} \text{rank } A_l(\xi) \tag{II.4.3}$$

for almost every $\xi \in \mathbb{R}$. Now we want to show that this is true for all $\xi \in \mathbb{R}$. Note that $G(\xi) = \sum_{l=0}^{n-1} A_l(\xi)$, which implies that

$$\text{rank } G(\xi) \leq \sum_{l=0}^{n-1} \text{rank } A_l(\xi),$$

for all $\xi \in \mathbb{R}$. This shows that if (II.4.3) is not true for some $\xi_0 \in \mathbb{R}$, we must have that

$$r = \text{rank } G(\xi_0) < \sum_{l=0}^{n-1} \text{rank } A_l(\xi_0).$$

Since $A_l(\xi) = \sum_{s=0}^{\infty} A_l^s(\xi)$, where $A_l^s(\xi)$, defined as $A_l^s(\xi)_{ij} = \sum_{|k|=s} \widehat{\phi}_i(\omega + kn + l)\, \overline{\widehat{\phi}_j(\omega + kn + l)}$, are all positive semidefinite matrices. So there must be a $p(l, \xi)$ such that

$$\text{rank } A_l(\xi) = \text{rank } \sum_{s=0}^{p(l,\xi)} A_l^s(\xi).$$

Since $\widehat{\phi}_i$ is continuous, $\sum_{s=0}^{p(l,\xi)} A_l^s(\xi)$ is a continuous matrix whose rank can only increase locally, which implies

$$\text{rank } A_l(\eta) \geq \text{rank } \sum_{s=0}^{p(l,\xi)} A_l^s(\eta) \geq \text{rank } \sum_{s=0}^{p(l,\xi)} A_l^s(\xi) = \text{rank } A_l(\xi),$$

for all $|\eta-\xi| \le \epsilon$ for some small ϵ. This proves that rank $A_l(\xi)$ can only increase locally. Then there must exist some $\epsilon_0 > 0$ such that

$$r < \sum_{l=0}^{n-1} \text{rank } A_l(\xi_0) \le \sum_{l=0}^{n-1} \text{rank } A_l(\eta),$$

for all $|\eta - \xi_0| < \epsilon_0$, this contradicts that (II.4.3) is true for almost every $\xi \in \mathbb{R}$. Then (II.4.3) is true for all $\xi \in \mathbb{R}$.

Since rank $A_l(\xi)$ can only increase locally and their sum is always constant, we must have that rank $A_l(\xi)$ is locally constant in $\mathbb{R}$. But $\mathbb{R}$ is connected, and so rank $A_l(\xi)$ has to be a constant function in $\mathbb{R}$. We denote rank $A_l(\xi) = r_l$, so $r = \sum_{l=0}^{n-1} r_l$. From the definition of $A_{\Phi^l}(\xi)$, for any l_1 and l_2, we have that $r_{l_1} = A_{l_1}(\xi) = A_{l_1}(\xi + l_2 - l_1) = A_{l_2}(\xi) = r_{l_2}$. This is a contradiction to the assumption that $n \nmid r$. So we have $\xi \in \mathbb{R}$ such that $\det[G(\xi)] = 0$. $\square$

Now we are ready to prove Theorem II.3.1.

Proof of Theorem II.3.1. Suppose the contrary that $V(\Phi)$ is translation invariant, in particular, it is $\frac{1}{n}\mathbb{Z}$-invariant for any $n \ge 1$. Since $\phi_i \in L^1$, we must have that $\widehat{\phi}_i$ is continuous for $i = 1, \dots, r$.

Define $E = \{\xi | \det(G(\xi)) \neq 0\}$, by Lemma II.4.3, $E \neq \mathbb{R}$. For any $\xi \in E$, rank $G(\xi) = r$. Since the rank of $G(\xi)$ is non-decreasing locally as showed in the proof of Lemma II.4.3. It is easy to see that E is an open set in $\mathbb{R}$. So E has to be the union of disjoint open intervals.

Choose $\xi_0 \in \overline{E} \backslash E$ to be the left end point of one of the intervals whose union is E, then $\det(G(\xi_0)) = 0$. There must exist a nonzero vector $\vec{a}$ such that $G(\xi_0)\vec{a} = 0$. Since $G(\xi) = F(\xi)F^*(\xi)$ where $F(\xi)$ is a $r \times \mathbb{Z}$ matrix with $F(\xi)_{ik} = \widehat{\phi}_i(\xi + k)$, we have $F^*(\xi_0)\vec{a} = 0$. Then we can show that

$$\sum_{i=1}^{r} a_i \widehat{\phi}_i(\xi_0 + k) = 0 \quad \text{for all } k \in \mathbb{Z}.$$

Define $f(x) = \sum_{i=1}^{r} a_i \phi_i(x)$. Then $\widehat{f}(\xi_0 + k) = 0$ for all the integer k. Also we must have that $f \in L^1$, which implies that $\widehat{f}$ is uniformly continuous. Then for any $\epsilon > 0$, we can choose $\delta > 0$ small enough, such that $(\xi_0, \xi_0 + \delta) \subset E$ and that

$$|\widehat{f}(\xi + k)| = |\widehat{f}(\xi + k) - \widehat{f}(\xi_0 + k)| < \epsilon \quad \text{for all } \xi \in (\xi_0, \xi_0 + \delta) \text{ and } k \in \mathbb{Z}.$$

Note that for any $\xi \in \mathbb{R}$, we have

$$\text{rank } F(\xi) = \text{rank } G(\xi) \leq r.$$

By (II.4.1) and the fact that $V(\Phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for any $n \geq 1$, we have that for any $1 \leq i \leq r$ and $\xi \in \mathbb{R}$

$$\widehat{\phi}_i(\xi + k) \neq 0,$$

for at most r integers k. Since f is a linear combination of ϕ_i, we must have that

$$\widehat{f}(\xi + k) \neq 0,$$

for at most r^2 integers k. Then for $\xi \in (\xi_0, \xi_0 + \delta)$,

$$\sum_{k \in \mathbb{Z}} |f(\xi + k)|^2 < r^2 \epsilon^2,$$

which implies that

$$\lim_{\delta \to 0^+} \frac{1}{\delta} \int_{\xi_0}^{\xi_0+\delta} \sum_{k \in \mathbb{Z}} |\widehat{f}(\xi + k)|^2 d\xi = 0.$$

But

$$\sum_{k \in \mathbb{Z}} |\widehat{f}(\xi + k)|^2 d\xi = \|F^*(\xi)\vec{a}\|_{\ell^2}^2 = \langle G(\xi)\vec{a}, \vec{a}\rangle \geq A\|\vec{a}\|^2$$

for $\xi \in (\xi_0, \xi_0 + \delta)$, which is a contradiction.

$\square$

Before proving Theorem II.3.2, we first introduce the following lemma whose proof can be found in the proof of Theorem II.1.2,

Lemma II.4.4 ([ASW11])**.** *If $f \in L^2$ satisfies that*

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{1+\epsilon} dx < \infty, \quad \text{for some } \epsilon > 0,$$

and that there exists a constant C such that

$$\sum_{k \in \mathbb{Z}} |\widehat{f}(\xi + k)|^2 \leq C \quad \text{for almost every } \xi \in \mathbb{R}. \tag{II.4.4}$$

and that there exists ξ_0 is such that

$$\widehat{f}(\xi_0 + k) = 0 \quad \text{for all } k \in \mathbb{Z},$$

then we must have

$$\lim_{\delta \to 0^+} \frac{1}{\delta} \int_0^{\delta} \sum_{k \in \mathbb{Z}} |\widehat{f}(\xi_0 + \xi + k)|^2 d\xi = 0.$$

Proof of Theorem II.3.2. Suppose the contrary that there exits $\epsilon > 0$ such that

$$\int_{\mathbb{R}} |\phi_i(x)|^2 |x|^{1+\epsilon} dx < \infty, \tag{II.4.5}$$

for every i, then we must have that $\widehat{\phi}_i$ is continuous for every i.

Define $E = \{\xi | \det(G(\xi)) \neq 0\}$, by Lemma II.4.3, $E \neq \mathbb{R}$. From the same argument in the proof of Theorem II.3.1, we can choose $\xi_0 \in \overline{E} \setminus E$ to be the left end point of one of the intervals whose union is E, then $\det(G(\xi_0)) = 0$. With the same notation, there must exist a nonzero vector $\vec{a}$ such that $F^*(\xi_0)\vec{a} = 0$. Define a function $f(x) = \sum_{i=1}^r a_i \phi_i(x)$, then $\widehat{f}(\xi_0 + k) = 0$ for all integers k. Also from (II.4.5), we will have that

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{1+\epsilon} dx < \infty.$$

Also from (II.3.1), one can easily show that (II.4.4) is true for all ϕ_i. In particular it is true for f. Then by Lemma II.4.4,

$$\begin{aligned} 0 &= \lim_{\delta \to 0^+} \frac{1}{\delta} \int_0^\delta \sum_{k \in \mathbb{Z}} |\widehat{f}(\xi_0 + \xi + k)|^2 d\xi \\ &= \lim_{\delta \to 0^+} \frac{1}{\delta} \int_0^\delta \|F^*(\xi_0 + \xi)\vec{a}\|_{\ell^2}^2 d\xi \\ &= \lim_{\delta \to 0^+} \frac{1}{\delta} \int_0^\delta \langle G(\xi_0 + \xi)\vec{a}, \vec{a}\rangle d\xi \end{aligned} \tag{II.4.6}$$

This is a contradiction for the same reason as shown in Theorem II.3.1. $\square$

The following is the proof for Theorem II.3.3

Proof of Theorem II.3.3. Suppose the contrary that there exists $\epsilon > 0$ such that $\xi^{1/2+\epsilon}\widehat{\phi}_i(\xi) \in L^\infty$ for all ϕ_i. Then $G(\xi)$ will be a continuous matrix function. This, together with the fact that $\{\phi_i(\cdot - k)|\ k \in \mathbb{Z}, i = 1, \ldots, r\}$ is a frame for

$V(\Phi)$, will give us that

$$AG(\xi) \leq G^2(\xi), \quad \text{for all } \xi \in \mathbb{R}. \tag{II.4.7}$$

But $V(\Phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for some $n \nmid r$. Define $E = \{\xi | \det(G(\xi)) \neq 0\}$. By Lemma II.4.3, $E \neq \mathbb{R}$. It is easy to see that E is an open set in $\mathbb{R}$ since $\det G(\xi)$ is a continuous function. So E has to be the union of disjoint open intervals. From the same argument in the proof of Theorem II.3.1, we can choose ξ_0 and $\delta > 0$ such that $\det G(\xi_0) = 0$ and $(\xi_0, \xi_0 + \delta) \subset E$.

For any $\xi \in (\xi_0, \xi_0 + \delta)$ we have $\det G(\xi) \neq 0$. Then from (II.4.7), we have that $A \leq G(\xi)$ which implies that $\det G(\xi) \geq A^r$. But this contradicts that $\det G(\xi_0) = 0$ by the continuity of $\det G(\xi)$.

□

CHAPTER III

OPTIMAL SHIFT-INVARIANT SPACES

III.1 Background

In many signal and image processing applications, it is assumed that the functions representing the signals belong to some PSIS. For example, the signals are often assumed to be bandlimited or to belong to a PSIS generated by splines, see e.g., [AG01]. These assumptions are useful in applications. However, the choice of the particular PSIS typically is not deduced from a set of data or observations of the underlying class of signals. Thus, it is natural to search for a shift-invariant space that is nearest to a set of some observed data, for example, in the sense of the least squares. Existence of such spaces is guaranteed by the following theorem which is a special case of a more general result in [ACHM07]:

Theorem III.1.1 ([ACHM07]). *Let $\mathcal{F} = \{f_1, \dots, f_m\} \subset L^2(\mathbb{R})$ be a set of functions in $L^2(\mathbb{R})$, then there exists $V = V(\phi) \in \mathcal{V}_1$ such that*

$$\sum_{i=1}^{m} \|f_i - P_V f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_{V'} f_i\|^2, \text{for all } V' \in \mathcal{V}_1,$$

where P_V and $P_{V'}$ are orthogonal projections on V and V' respectively.

From now on, we will denote by $\mathcal{V}$ the class of all PSIS with an orthonormal generator which are also translation invariant. Similarly, by $\mathcal{V}_n$ we denote the class of all PSI spaces with an orthonormal generator which are also $\frac{1}{n}\mathbb{Z}$-invariant.

Although the theorem above assures us of the existence of a principal shift-invariant space nearest to a set of data, it does not provide an answer, if we require the PSIS to have additional invariance. For this reason, we consider the following two problems:

Problem 1. *Given a set of functions* $\mathcal{F} = \{f_1, \dots, f_m\} \subset L^2(\mathbb{R})$, *we want to find a principal shift-invariant space* $V(\phi)$ *which satisfies*

$$V(\phi) = \operatorname{argmin}_{V' \in \mathcal{V}} \sum_{i=1}^{m} \|f_i - P_{V'} f_i\|^2,$$

where $P_{V'}$ *is the orthogonal projection onto* V'.

Problem 2. *Given a set of functions* $\mathcal{F} = \{f_1, \dots, f_m\} \subset L^2(\mathbb{R})$ *and an integer* $n \geq 2$, *we want to find a principal shift-invariant* $V(\phi)$ *which satisfies*

$$V(\phi) = \operatorname{argmin}_{V' \in \mathcal{V}_n} \sum_{i=1}^{m} \|f_i - P_{V'} f_i\|^2,$$

where $P_{V'}$ *is the orthogonal projection onto* V'.

Following [ACHM07], we say that if Problem 1 (resp. Problem 2) has a solution for any finite set of data functions $\mathcal{F}$, then $\mathcal{V}$ (resp. $\mathcal{V}_n$), satisfies the Minimum Subspace Approximation Property (MSAP).

Definition III.1.2. A set $\mathcal{C}$ of closed subspaces of a separable Hilbert space $\mathcal{H}$ containing the zero subspace has MSAP if for every finite subset $\mathcal{F} \subset \mathcal{H}$ there exists an element $V \in \mathcal{C}$ such that $V = \operatorname{argmin}_{V' \in \mathcal{C}} \sum_{f \in \mathcal{F}} \|f - P_{V'} f\|^2_{\mathcal{H}}$.

Necessary and sufficient conditions for MSAP were obtained in [AT10]. Specifically, let us identify any subspace $V \in \mathcal{C}$ with the orthogonal projection $Q = Q_V$ whose kernel is exactly V (i.e., $Q = I - P_V$). The set $\mathcal{C}$ is then identified with a set of projections $\{Q \in \Pi(\mathcal{H}) : \ker Q \in \mathcal{C}\}$ where $\Pi(\mathcal{H})$ is the set of all orthogonal projections on $\mathcal{H}$. Defining $\mathcal{C}^+ := \mathcal{C} + \mathcal{P}^+(\mathcal{H})$, where $\mathcal{P}^+(\mathcal{H}) \subset B(\mathcal{H})$ is the set of all positive semidefinite operators, we have the following sufficient topological condition for MSAP.

Theorem III.1.3 ([AT10])**.** $(\mathcal{C} = \overline{\mathcal{C}}) \Rightarrow (\mathcal{C}^+ = \overline{\mathcal{C}^+}) \Rightarrow$ *($\mathcal{C}$ satisfies MSAP); where the closures are taken with respect to the weak operator topology.*

For finite dimensional spaces the last implication is, in fact, an equivalence. The first implication is, however, a strict one. For infinite dimensional spaces, both implications are strict.

The characterization in the infinite dimensional case is in terms of contact half spaces containing $\mathcal{C}^+$. Specifically, a *half space* in $B(\mathcal{H})$ is a set

$$H_{\Phi,a} = \{A \in B(\mathcal{H}) : \Phi(A) \geq a\},$$

where $a \in \mathbb{R}$ and Φ is a bounded $\mathbb{R}$-linear functional on $\mathcal{B}(\mathcal{H})$. A contact half space to $\mathcal{C}^+$ is a half space containing $\mathcal{C}^+$ such that its boundary has non-empty intersection with $\mathcal{C}^+$. The set of all contact half spaces is denoted by $\eta(\mathcal{C}^+)$.

Theorem III.1.4 ([AT10])**.** *Let $\mathcal{C}$ be a set of projectors in $\mathcal{B}(\mathcal{H})$. Then $\mathcal{C}$ has MSAP if and only if*

$$\eta(\mathcal{C}^+) = \eta(\overline{\mathcal{C}^+}).$$

In other words, $\mathcal{C}$ has MSAP if and only if given $a \in \mathbb{R}$, a bounded $\mathbb{R}$-linear functional Φ on $B(\mathcal{H})$, and an operator $A' \in \overline{\mathcal{C}^+}$ satisfying $\Phi(A') = a$, there is $A \in \mathcal{C}^+$ such that $\Phi(A) = a$.

Note that the characterizations in both finite and infinite dimensional cases give necessary and sufficient conditions for existence of a best approximating subspace $V \in \mathcal{C}$. However, the conditions may be difficult to check and the theorem does not give a way to construct V. Thus, one of the goals of this paper is to use a constructive method to prove that $\mathcal{V}$ and $\mathcal{V}_n$ satisfy MSAP. But first, we show that $\mathcal{V}^+$ is not closed under the weak operator topology, where $\mathcal{V}^+$ is defined the same way as $\mathcal{C}^+$. Thus, we give a new example that shows that the second implication of Theorem III.1.3 is not an equivalence. Consider the sets

$$E_k = \left\{ \bigcup_{i=0}^{2^{k-1}-1} \left[-1 + \frac{i}{2^{k-1}}, -1 + \frac{1}{2^k} + \frac{i}{2^{k-1}} \right) \right\} \cup \left\{ \bigcup_{i=0}^{2^{k-1}-1} \left[\frac{i}{2^{k-1}}, \frac{1}{2^k} + \frac{i}{2^{k-1}} \right) \right\}$$

for all $k \in \mathbb{Z}^+$. Note that every E_k is a *tiling set*, that is it has measure 1 and a *packing property*, i.e., $\bigcup_{k\in\mathbb{Z}} E_k = \mathbb{R}$, and $E_k \cap (E_k + j)$ has measure 0 for all $j \in \mathbb{Z}\backslash\{0\}$. If we define ϕ_k via $\widehat{\phi}_k = \chi_{E_k}$, we have that $I - P_{V(\phi_k)} \in \mathcal{V} \subset \mathcal{V}^+$. It is not difficult to show that $I - P_{V(\phi_k)}$ converges weakly to $I - P_{V(\psi)}$ where $\widehat{\psi} = \frac{1}{2}\chi_{[-1,1)}$. Since for any $V(\phi) \in \mathcal{V}$, we must have $\widehat{\phi} = \chi_E$ for some tiling set E (in particular the measure of E is 1). There is, however, no non-negative self adjoint operator A such that $P_{V(\psi)} = P_{V(\phi)} - A$, that is $I - P_{V(\psi)} = I - P_{V(\phi)} + A$, and, hence, $I - P_{V(\psi)} \notin \mathcal{V}^+$. Therefore, $\mathcal{V}^+$ is not closed in the weak operator topology. The same example also shows that $\mathcal{V}_n^+$ is not closed for any integer $n \geq 2$.

III.2 Constructions

III.2.1 Translation invariant PSI spaces.

We first consider the simple case in which we only have one data function f. As we mentioned in the introduction, it is easy to show that if $V(\phi) \in \mathcal{V}$, we must have that $\widehat{\phi} = \chi_E$, where E is a tiling set. In such a way each $V(\phi) \in \mathcal{V}$ is characterized by the tiling set E. The following theorem ensures existence of an optimal subspace for one function f.

Theorem III.2.1. *For any data function $f \in L^2(\mathbb{R})$, there exists $V(\phi) \in \mathcal{V}$ such that*

$$\|f - P_{V(\phi)}f\|^2 \leq \|f - P_{V'}f\|^2, \text{ for all } V' \in \mathcal{V}.$$

Before proving this theorem, first note that

$$\|f - P_{V(\phi)}f\|^2 = \|f\|^2 - \|P_{V(\phi)}f\|^2,$$

and

$$\|P_{V(\phi)}f\| = \|\widehat{P_{V(\phi)}f}\| = \|P_E\widehat{f}\| = \|\widehat{f}\chi_E\|,$$

where $\widehat{\phi} = \chi_E$. Now if we can maximize $\|P_E\widehat{f}\|$ with a measurable tiling set E, then $V(\phi)$ defined by $\widehat{\phi} = \chi_E$ will be an optimal subspace whose existence is asserted by Theorem III.2.1. In the proof below we construct such a tiling set.

Proof. (Theorem III.2.1). Since $f \in L^2(\mathbb{R})$, we have $\widehat{f} \in L^2(\mathbb{R})$ and $\{\widehat{f}(\xi + k)\}_{k\in\mathbb{Z}} \in \ell^2$ for almost every $\xi \in \mathbb{R}$. Then the function

$$M\widehat{f}(\xi) = \max_k\{|\widehat{f}(\xi + k)|\},$$

is well defined for almost every $\xi \in \mathbb{R}$, meaurable and 1-periodic.

Consider the set $S_0 = \{\xi \in D(M\widehat{f})|\ M\widehat{f}(\xi) = |\widehat{f}(\xi)|\}$, where $D(M\widehat{f})$ is the domain of $M\widehat{f}$. Since $\widehat{f}$ and $M\widehat{f}$ are measurable functions, S_0 is a measurable set. Our goal is to construct a (measurable) tiling set $E \subseteq S_0$. Indeed, such a set would maximize $\|P_{E'}\widehat{f}\|$ among all tiling sets E', because then

$$\|P_E\widehat{f}\|^2 = \int_E |\widehat{f}(\xi)|^2 d\xi = \int_E M\widehat{f}(\xi)^2 d\xi = \int_{E'} M\widehat{f}(\xi)^2 d\xi \geq \int_{E'} |\widehat{f}(\xi)|^2 d\xi = \|P_{E'}\widehat{f}\|^2.$$

Clearly, to construct E we may consider for each $\xi \in D(M\widehat{f}) \cap [0, 1)$ the non-empty set $(\xi + \mathbb{Z}) \cap S_0$ and pick one point from each of these sets. We only need to make sure that the result is Lebesgue measurable. We do it in the following way.

First, for each integer $l \geq 0$, we define the sets L_l via

$$L_l = \begin{cases} [\frac{l}{2}, \frac{l}{2}+1), & \text{if } l \text{ is even;} \\ \\ [-\frac{l+1}{2}, -\frac{l-1}{2}), & \text{if } l \text{ is odd.} \end{cases}$$

Next, using recursion, we let $E_l = S_l \cap L_l$ and $S_{l+1} = S_l \setminus \bigcup_{k\in\mathbb{Z}} (E_l + k)$, $l \geq 0$, where $E_l + k$ is the set E_l translated by k. Finally, we define $E = \bigcup_{l\geq 0} E_l$. It is easily seen that E_l and S_l are all measurable sets, and so E is measurable. We claim that E is the desired tiling set.

Observe that by construction $E \subseteq S_0$, and, for each $\xi \in D(M\widehat{f}) \cap [0,1)$, the set $(x+\mathbb{Z}) \cap E$ is a singleton. Hence, E does, indeed, have the desired properties and the theorem is proved. $\square$

It turns out that the case of several data functions is a simple corollary of the above theorem.

Corollary III.2.2. *For data functions $f_1, \ldots, f_m \in L^2(\mathbb{R})$, there exists $V(\phi) \in \mathcal{V}$ such that*

$$\sum_{i=1}^{m} \|f_i - P_{V(\phi)} f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_{V(\psi)} f_i\|^2, \text{ for all } V(\psi) \in \mathcal{V}.$$

Proof. Note that for any $V(\psi) \in \mathcal{V}$, if $\widehat{\psi} = \chi_{E'}$ we have

$$\begin{aligned}\sum_{i=1}^{m} \|f_i - P_{V(\psi)} f_i\|^2 &= \sum_{i=1}^{m} (\|f_i\|^2 - \|P_{V(\psi)} f_i\|^2) \\ &= \sum_{i=1}^{m} \|f_i\|^2 - \sum_{i=1}^{m} \|\widehat{f}_i \chi_{E'}\|^2 \\ &= \sum_{i=1}^{m} \|f_i\|^2 - \int_{E'} \sum_{i=1}^{m} |\widehat{f}_i|^2 dx \\ &= \sum_{i=1}^{m} \|f_i\|^2 - \|P_{V(\psi)} f\|^2,\end{aligned}$$

where f is such that $\widehat{f} = \sqrt{\sum_{i=1}^{m} |\widehat{f}_i|^2}$. It remains to apply Theorem III.2.1 to this function f. $\square$

III.2.2 $\frac{1}{n}\mathbb{Z}$-invariant PSI spaces.

It is impossible to construct a translation invariant PSI space such that the generator is in L^1 ([ASW11]). Hence, it is natural to relax one of the conditions and study $\frac{1}{n}\mathbb{Z}$-invariant spaces for some integer n. This leads us to consider Problem 2 in the introduction, and we have

Theorem III.2.3. *For data functions $f_1, \ldots, f_m \in L^2(\mathbb{R})$, there exists $V(\phi) \in \mathcal{V}_n$ such that*

$$\sum_{i=1}^{m} \|f_i - P_{V(\phi)} f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_{V(\psi)} f_i\|^2 \text{ for all } V(\psi) \in \mathcal{V}_n.$$

As in the previous section we observe that proving the above theorem is equivalent to finding an eligible function ϕ that maximizes the expression

$$\sum_{i=1}^{m} \|P_{V(\phi)} f_i\|^2. \tag{III.2.1}$$

To prove Theorem III.2.3, we will use the following characterization of $\frac{1}{n}\mathbb{Z}$-invariance, which is a special case of Proposition II.2.1.

Proposition III.2.4. *Let $n \geq 2$ be an integer, and $\phi \in L^2(\mathbb{R})$ be an orthonormal generator for the space $V(\phi)$. Then $V(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant if and only if for almost every $\xi \in [0,1)$, one of the following vectors*

$$\Phi_j(\xi) := (\cdots, \widehat{\phi}(\xi + j - n), \widehat{\phi}(\xi + j), \widehat{\phi}(\xi + j + n), \cdots), \quad j = 0, 1, \ldots n-1, \tag{III.2.2}$$

is a unit vector in ℓ^2 and the others are zero vectors.

For almost every $\xi \in [0,1)$ we let $j(\xi) \in \{0, 1, \ldots, n-1\}$ be such that $\Phi_{j(\xi)}(\xi)$ is the unique unit vector in the above proposition.

We also need the following Lemma:

Lemma III.2.5. *Suppose that $h_j(x)$, $j = 0, \ldots, n-1$ are measurable functions on $\mathbb{R}$ and $e^j(x)$, $j = 0, \ldots, n-1$ are real valued measurable functions on $[0,1)$. For $x \in [0,1)$, define $s(x)$ to be the smallest index such that $e^{s(x)}(x) \leq e^l(x)$ for all $l \neq s(x)$, and let $S(x)$, be the periodization on $\mathbb{R}$ of $s(x)$. Then the function $h(x) := h_{S(x)}(x)$ is also measurable.*

Proof. Define $E_0 = \{x \in [0,1) | e^0(x) \leq e^i(x), i \neq 0\}$, and define E_j for $j = 1, \ldots, n-1$ as

$$E_j = \{x \in [0,1) | e^j(x) \leq e^i(x), i \neq j\} \setminus \cup_{k<j} E_k.$$

Clearly, the sets E_j, $j = 0, \ldots, n-1$, are measurable. Then define $E^j, 0 \leq j \leq n-1$ to be

$$E^j = \cup_{k \in \mathbb{Z}} E_j + k,$$

which are also measurable. From its definition, $h(x)$ can be written as

$$h(x) = \sum_{j=0}^{n-1} h_j(x)\chi_{E^j}(x)$$

which is a clearly a measurable function. $\square$

Proof. (of Theorem III.2.3) We begin by noting that for any $V(\phi) \in \mathcal{V}_n$, and $\{\phi(\cdot + k), k \in \mathbb{Z}\}$ is an orthonormal basis for $V(\phi)$, we have

$$\begin{aligned}
\sum_{i=1}^{m} \|f_i - P_{V(\phi)}f_i\|^2 &= \sum_{i=1}^{m} \int_{\mathbb{R}} |f_i(x) - P_{V(\phi)}f_i(x)|^2 dx \\
&= \sum_{i=1}^{m} \int_{\mathbb{R}} |\widehat{f}_i(\xi) - \widehat{P_{V(\phi)}f_i}(\xi)|^2 d\xi \\
&= \sum_{i=1}^{m} \int_{\mathbb{R}} |\widehat{f}_i(\xi) - C_\phi^i(\xi)\widehat{\phi}(\xi)|^2 d\xi,
\end{aligned}$$

where $C_\phi^i(\xi) = \sum_{l\in\mathbb{Z}} \widehat{f}_i(\xi + l)\overline{\widehat{\phi}(\xi + l)}$ is a periodic function. Rewriting the integral and using Fubini's theorem, we get

$$\begin{aligned}
\sum_{i=1}^{m} \|f_i - P_{V(\phi)}f_i\|^2 &= \int_0^1 \sum_{i=1}^{m} \sum_{k=-\infty}^{\infty} |\widehat{f}_i(\xi + k) - C_\phi^i(\xi)\widehat{\phi}(\xi + k)|^2 d\xi \\
&= \int_0^1 \sum_{i=1}^{m} \sum_{j=0}^{n-1} \sum_{k=-\infty}^{\infty} |\widehat{f}_i(\xi + nk + j) - C_\phi^i(\xi)\widehat{\phi}(\xi + nk + j)|^2 d\xi \\
&= \int_0^1 \sum_{i=1}^{m} \sum_{j=0}^{n-1} \|\widehat{f}_i(\xi + n\cdot + j) - C_\phi^i(\xi)\widehat{\phi}(\xi + n\cdot + j)\|_{\ell^2}^2 d\xi
\end{aligned}$$

since for almost every $\xi \in [0, 1)$, we have $\{\widehat{f}_i(\xi + nk + j) - C_\phi^i(\xi)\widehat{\phi}(\xi + nk + j)\}_k \in \ell^2$.

Observe that, using the notation introduced after Proposition III.2.4 and denoting $\vec{F}_i(\xi + l)$ to be the ℓ^2 vector $\{\hat{f}_i(\xi + nk + l)\}_k$, the last expression is equal to

$$\int_0^1 \sum_{i=1}^{m} \|\vec{F}_i(\xi + j(\xi)) - P_{\Phi_{j(\xi)}(\xi)}\vec{F}_i(\xi + j(\xi))\|_{\ell^2}^2 + \sum_{i=1}^{m}\sum_{l\neq j(\xi)} \|\vec{F}_i(\xi + l)\|_{\ell^2}^2 d\xi, \tag{III.2.3}$$

where $j(\xi) \in \{0, 1, \ldots, n-1\}$ and $\Phi_{j(\xi)}(\xi)$ are defined after Proposition III.2.2. Hence, in order to construct $\phi \in L^2(\mathbb{R})$ such that $V(\phi) \in \mathcal{V}_n$ minimizes

$$\sum_{i=1}^{m} \|f_i - P_{V(\phi)} f_i\|^2,$$

it suffices to minimize the integrand in Equation III.2.3 for almost every $\xi \in [0, 1)$.

Since for fixed j, ξ, and f_i the expression $\sum_{i=1}^{m}\sum_{l\neq j} \|\vec{F}_i(\xi + l)\|_{\ell^2}^2$ is constant, minimizing the above expression becomes a problem of finding, for each ξ, the best approximation of vectors in ℓ^2 by a one dimensional space generated by $\Psi_j(\xi)$. This problem can be solved using the singular value decomposition (SVD).

Define $G_j(\xi)$ to be a $m \times m$ matrix, where $[G_j(\xi)]_{l_1,l_2} = \sum_k \widehat{f}_{l_1}(\xi + nk + j)\overline{\widehat{f}_{l_2}(\xi + nk + j)}$ for $j = 0, 1, \ldots, n-1$. Suppose the eigenvalues of $G_j(\xi)$ are $\lambda_{j,1}(\xi) \geq \lambda_{j,2}(\xi) \geq \cdots \geq \lambda_{j,m}(\xi) \geq 0$. Then the eigenvalues are measurable. Also there exists measurable a matrix function $U_j(\xi)$ such that $U_j(\xi)U_j^*(\xi) = I$ almost everywhere and we have the singular value decomposition

$$G_j(\xi) = U_j(\xi)\Lambda_j(\xi)U_j^*(\xi),$$

where $\Lambda_j(\xi) = \mathrm{diag}\big(\lambda_{j,1}(\xi), \ldots, \lambda_{j,m}(\xi)\big)$.(see [ACHM07])

Let us denote the vector formed by the first row of $U_j^*(\xi)$ as $U_j^1(\xi) = \big(y_j^1(\xi), \ldots, y_j^m(\xi)\big)^T$.

Then it is not difficult to check that $U_j^1(\xi)$ is a left-eigenvector of $G_j(\xi)$ with eigenvalue $\lambda_{j,1}(\xi)$. Define

$$q_j(\xi) = \sum_{l=1}^{m} y_j^l(\xi)\widehat{f}_l(\xi)\chi_{B_j},$$

where $B_j = \cup_{k\in\mathbb{Z}}[j, j+1) + nk$. Since U_j^1 is a measurable function of ξ, then q_j is also measurable.

Define $\Omega_j = \{\xi \in [0,1) | \sum_{k\in\mathbb{Z}} |q_j(\xi+k)|^2 = 0\}$, it is easy to see that Ω_j is a measurable set. Define another function

$$g_j(\xi) = \chi_{\Omega_j+[j,j+1)} + q_j(\xi).$$

.

Now $g_j(\xi)$ is a measurable function supported on B_j and $\|g_j(\xi+\cdot)\|_{\ell_2}^2 = \sum_{k\in\mathbb{Z}} |g_j(\xi+k)|^2 = \sum_{k\in\mathbb{Z}} |g_j(\xi+nk+j)|^2 \neq 0$ for almost every $\xi \in [0,1)$.

Now define $h_j(\xi) = \frac{g_j(\xi)}{\|g_j(\xi+\cdot)\|_{\ell_2}}$. One can check that $h_j(\xi)$ is a measurable function supported on B_j and for almost every $\xi \in [0,1)$

$$\sum_{k\in\mathbb{Z}} |h_j(\xi+k)|^2 = \sum_{k\in\mathbb{Z}} |h_j(\xi+nk+j)|^2 = \sum_{k\in\mathbb{Z}} \frac{|g_j(\xi+nk+j)|^2}{\|g_j(\xi+\cdot)\|_{\ell_2}^2} = 1, \quad \text{(III.2.4)}$$

Also, by the Eckart-Young theorem,

$$\sum_{i=1}^{m} \|\vec{F}_i(\xi+j) - P_{\vec{H}_j(\xi)}\vec{F}_i(\xi+j)\|_{\ell^2}^2 \leq \sum_{i=1}^{m} \|\vec{F}_i(\xi+j) - P_W\vec{F}_i(\xi+j)\|_{\ell^2}^2 \quad \text{(III.2.5)}$$

for all subspaces $W \subset \ell^2$ with $\dim W = 1$, where $\vec{H}_j(\xi)$ is the ℓ^2 vector $\{h_j(\xi+nk+j)\}_k$.

We also define a function for almost every $\xi \in [0,1)$,

$$\begin{aligned} e^j(\xi) &= \sum_{i=1}^{m} \|\vec{F}_i(\xi+j) - P_{\vec{H}_j(\xi)}\vec{F}_i(\xi+j)\|_{\ell^2}^2 \\ &+ \sum_{i=1}^{m}\sum_{l\neq j} \|\vec{F}_i(\xi+l)\|_{\ell^2}^2, \end{aligned}$$

Since h_j and $\widehat{f}_j$ are all measurable, $e^j(\xi)$ is also a measurable function. Suppose $s(\xi)$ is the smallest index such that $e^{s(\xi)}(\xi) \leq e^l(\xi)$ for all l. Then by Lemma III.2.5, we can construct a measurable function $h(\xi) = h_{S(\xi)}(\xi)$ where $S(\xi)$ is the periodization of $s(\xi)$. Note that $S(\xi) = S(\xi+k)$ for all $k \in \mathbb{Z}$. Then

$$\begin{aligned} \int_{\mathbb{R}} |h(\xi)|^2 d\xi &= \int_0^1 \sum_{k\in\mathbb{Z}} |h(\xi+k)|^2 d\xi \\ &= \int_0^1 \sum_{k\in\mathbb{Z}} |h_{S(\xi)}(\xi+k)|^2 d\xi \\ &= 1, \end{aligned}$$

which implies $h \in L^2$ and $\|h\|_{L^2} = 1$. Define $\phi(x)$ to be such that $\widehat{\phi} = h$.

For almost every $\xi \in [0,1)$ and $0 \leq l \leq n-1$

$$\begin{aligned} \|\Phi_l(\xi)\|_{\ell^2} &= \|\{\widehat{\phi}(\xi+nk+l)\}_k\|_{\ell^2} = \|\{h(\xi+nk+l)\}_k\|_{\ell^2} \\ &= \|\{h_{S(\xi)}(\xi+nk+l)\}_k\|_{\ell^2} \end{aligned}$$

since $h_{S(\xi)}(\xi)$ is supported on $B_{S(\xi)}$, there exits $l(\xi)$ such that $\|\Phi_{l(\xi)}(\xi)\|_{\ell^2} = 1$ and $\|\Phi_i(\xi)\|_{\ell^2} = 0$ for $i \neq l(\xi)$. By Proposition III.2.4, $V(\phi) \in \mathcal{V}_n$

Finally, we need to show that ϕ is indeed an optimal generator.

For any $V(\psi) \in \mathcal{V}_n$ and almost every $\xi \in [0,1)$, there exists $j_1 \in \{1,2,\ldots,n\}$

such that $\|\widehat{\psi}(\xi+n\cdot+j_1)\|_{\ell^2}=1$ and $\|\widehat{\psi}(\xi+n\cdot+j)\|_{\ell^2}=0$ for $j\neq j_1$. Then from Equation III.2.5, we have that

$$\sum_{i=1}^{m}\|\vec{F}_i(\xi+j_1)-P_{\vec{H}_{j_1(\xi)}}\vec{F}_i(\xi+j_1)\|_{\ell^2}^2 \leq \sum_{i=1}^{m}\|\vec{F}_i(\xi+j_1)-P_{\Psi_{j_1}(\xi)}\vec{F}_i(\xi+j_1)\|_{\ell^2}^2,$$

where $\Psi_{j_1}(\xi)$ is the ℓ^2 vector $\{\widehat{\psi}(\xi+nk+j_1)\}_k$, so

$$e^{j_1}(\xi)\leq\sum_{i=1}^{m}\|\vec{F}_i(\xi+j_1)-P_{\Psi_{j_1}(\xi)}\vec{F}_i(\xi+j_1)\|_{\ell^2}^2+\sum_{i=1}^{m}\sum_{l\neq j_1}\|\vec{F}_i(\xi+l)\|_{\ell^2}^2.$$

From the construction of $h(\xi)$, we have $\{h(\xi+nk+j)\}_k=\{h_{S(\xi)}(\xi+nk+j)\}_k$ and

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=0}^{n-1}&\|\vec{F}_i(\xi+j)-P_{\{h(\xi+nk+j)\}_k}\vec{F}_i(\xi+j)\|_{\ell^2}^2\\ &=\sum_{i=1}^{m}\|\vec{F}_i(\xi+S(\xi))-P_{\vec{H}_{S(\xi)}(\xi)}\vec{F}_i(\xi+S(\xi))\|_{\ell^2}^2\\ &+\sum_{i=1}^{m}\sum_{l\neq S(\xi)}\|\vec{F}_i(\xi+l)\|_{\ell^2}^2=e^{S(\xi)}(\xi)\leq e^{j_1}(\xi),\end{aligned}$$

where $\vec{H}_{S(\xi)}(\xi) = \{h_{S(\xi)}(\xi + nk + S(\xi))\}_k$. Then we have

$$\sum_{i=1}^{m} \sum_{j=0}^{n-1} \|\vec{F}_i(\xi + j) - P_{\{h(\xi+nk+j)\}_k} \vec{F}_i(\xi + j)\|_{\ell^2}^2$$

$$\leq \sum_{i=1}^{m} \sum_{j=0}^{n-1} \|\vec{F}_i(\xi + j) - P_{\Psi_j(\xi)} \vec{F}_i(\xi + j)\|_{\ell^2}^2$$

for a.e. $\xi \in [0, 1)$. So we must have that

$$\sum_{i=1}^{m} \|f_i - P_{V(\phi)} f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_{V(\psi)} f_i\|^2$$

and the theorem is proved. $\square$

BIBLIOGRAPHY

[ACHM07] A. Aldroubi, C. A. Cabrelli, D. Hardin, and U. M. Molter, *Optimal shift invariant spaces and their parseval frame generators*, Appl. Comput. Harmon. Anal., **23** (2007), 273–283.

[ACHKM10] A. Aldroubi, C. A. Cabrelli, C. Heil, K. Kornelson and U. M. Molter, *Invariance of a shift invariant space*, J. Fourier Anal. and Appl., **16** (2010), 60-75.

[AG01] A. Aldroubi and K. Gröchenig, *Non-uniform sampling in shift-invariant space*, SIAM Rev., **43** (2001), no. 4, 585–620.

[AKTW11] A. Aldroubi, I. Krishtal, R. Tessera and H. Wang, *Principal Shift-Invariant Spaces with Extra Invariance Nearest to Observed Data*, submitted, 2011.

[AST05] A. Aldroubi, Q. Sun and W.-S. Tang, *Convolution, average sampling, and Calderon resolution of the identity of shift-invariant spaces*, J. Fourier Anal. Appl., **11** (2005), 215–244.

[ASW11] A. Aldroubi, Q. Sun and H. Wang, *Uncertainty principles and Balian-Low type theorems in principal shift-invariant spaces*, Appl. Comput. Harmon. Anal., **30** (2011), 337–347.

[AT10] A. Aldroubi, and R. Tessera, *Sparse approximations and the minimum subspace approximation property*, preprint(2010).

[ACP09] M. Anastasio, C. Cabrelli and V. Paternostro, *Invariance of a shift-invariant space in several variables*, Preprint, 2009.

[BBS92] J. Benedetto, J.S. Bymes and H.S. Shapiro, *Wavelet auditory models and irregular sampling*, Monthly status report, February.(1992).

[BCGP03] J. Benedetto, W. Czaja, P. Gadzinski and A. Powell, *The Balian-Low Theorem and regularity of Gabor systems*, J. Geom. Anal., **13** (2003), 239–254.

[BCPS06] J. Benedetto, W. Czaja, A. Powell and J. Sterbenz , *An endpoint* $(1, \infty)$ *Balian-Low theorem*, Math. Res. Lett. **13** (2006), no. 2–3, 467–474.

[BF94] J. Benedetto and M. Frazier, *Wavelets: mathematics and applications*, CRC, 1994.

[BHW95] J. Benedetto, C. Heil, and D. Walnut, *Differentiation and the Balian-Low theorem*, J. Fourier Anal. Appl. **1**(1995), no. 4, 355–402.

[Bow00] M. Bownik, *The structure of shift-invariant subspaces of* $L^2(\mathbb{R}^n)$, J. Funct. Anal., **177**(2000), 282–309.

[Chr03] O. Christensen, *An introduction to frames and Riesz basis*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2003.

[CP07] W. Czaja and A. Powell, *Recent developments in the Balian-Low theorem*, Harmonic analysis and applications, 79–100, Appl. Numer. Harmon. Anal., Birkhäuser, 2006.

[CS03] C. K. Chui and Q. Sun, *Tight frame oversampling and its equivalence to shift-invariance of affine frame operators*, Proc. Amer. Math. Soc., **131** (2003), 1527–1538.

[CS07] C. K. Chui and Q. Sun, *Affine frame decompositions and shift-invariant spaces*, Appl. Comput. Harmon. Anal., **20** (2006), 74–107.

[DDR94] C. de Boor, R.A. DeVore, and A. Ron, *The structure of finitely generated shift-invariant spaces in* $L_2(R^d)$, J. Funct. Anal. **119** (1994), no. 1, 37–78.

[dBDR94] C. de Boor, R. DeVore, and A. Ron, *Approximation from shift-invariant subspaces of* $L_2(\mathbf{R}^d)$, Trans. Amer. Math. Soc., **341**(2) (1994), 787—806.

[Gau09] Z. Gautam, *A critical-exponent Balian-Low theorem.* Math. Res. Lett. **15** (2008), no. 3, 471–483.

[Grö01] K. Gröchenig, *Foundations of time-frequency analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2001.

[GH04] J.-P. Gabardo and D. Han, *Balian-Low phenomenon for subspace Gabor frames*, J. Math. Phys., **45** (2004), no. 8, 3362–3378.

[GHHK02] K. Gröchenig, D. Han, C. Heil, and G. Kutyniok, *The Balian-Low theorem for symplectic lattices in higher dimensions*, Appl. Comput. Harmon. Anal. **13** (2002), no. 2, 169–176.

[HL09] J. Hogan and J. Lakey, *Non-translation-invariance and the synchronization problem in wavelet sampling* Acta Appl. Math., **107** (2009), 373–398.

[HP06] C. Heil and A. Powell, *Gabor Schauder bases and the Balian-Low theorem*, J. Math. Phys., **47** (2006), no. 11,

[HP08] C. Heil and A. Powell, *Regularity for complete and minimal Gabor systems on a lattice*, Illinois J. Math. **53** (2009), no. 4, 1077-1094.

[HW96] E. Hernández and G. Weiss, *A first course on wavelets*, CRC Press, Boca Raton, FL, 1996.

[Leo07] C. Leonetti, *Reconstruction from error-affected sampled data in shift-invariant spaces*, Ph.D thesis, Vanderbilt University, 2007.

[UB00] M. Unser and T. Blu, *Fractional splines and wavelets*, SIAM Rev., **42** (2000), 43–67.

[SS03] E. Stein, and R. Shakarchi, *Fourier analysis, an introduction*, Princeton University Press, 2003.

[TW11] R. Tessera and H. Wang, *Uncertainty principles in finitely generated shift-Invariant spaces*, preprint, 2011.

[W01] D. Walnut, *An introduction to wavelet analysis*, Birkhäuser, 2001.

[Web00] E. Weber, *On the translation invariance of wavelet subspaces*, J. Fourier Anal. Appl., **6** (2000), 551–558.