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Force Distribution in Closed Kinematic Chains

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VIJAY R. KUMAR, MEMBER, IEEE, AND KENNETH J. WALDRON

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INTRODUCTION

THIS PAPER addresses the problem of force distribution in systems with closed kinematic chains involving multiple frictional contacts between an actively controlled structure and an object. Such systems are statically indeterminate and active coordination demands optimal solutions for force control [13]. One example of such a redundant system can be found in walking vehicles [6], [11], [15] in which the legs of the vehicle and the terrain form closed loops (see Fig. 1). A similar situation exists in multifingered grippers [1]–[3], [5], [7], [14], [17]. It has been shown that the redundancy in such systems can be resolved by linear programming techniques [5], [11] or by the application of the Moore-Penrose Generalized Inverse [6], [8]. The nature of the generalized inverse or the pseudo-inverse solution, which in turns leads to a decomposition of the force field, is explored in this paper for the special case in which the contact interaction is limited to a pure force (or zero pitch wrench [4]) through a contact center or contact point. This point-contact model is valid even for distributed contacts provided contact moments can be neglected and the contact center is known.

It is convenient to decompose the system of contact forces or the force field consisting of (only) the contact forces, into an *equilibrating force field* and an *interaction force field.* The *interaction force* between any two contact points is defined as the component of the difference of the contact forces along the line joining the two contact points. This condition may be mathematically expressed as

$$(\boldsymbol{F}_i - \boldsymbol{F}_j) \cdot (\boldsymbol{r}_i - \boldsymbol{r}_j) = 0 \tag{1}$$

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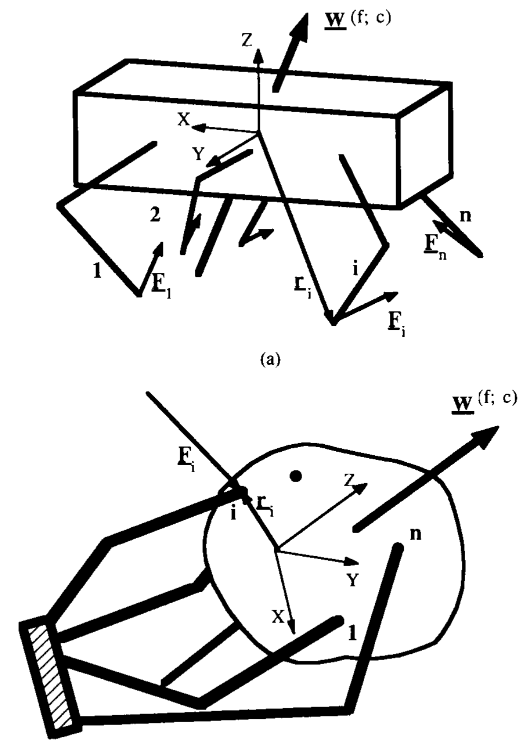

Fig. 1. Examples of closed-loop kinematic chains in an actively coordinated mechanisms. (a) A walking vehicle. (b) A multifingered robotic gripper. ($\boldsymbol{r}_i$ is the position vector and $\boldsymbol{F}_i$ is the contact wrench at the ith contact point; $\boldsymbol{w}$ is the load wrench (f and c are the associated force and couple), which is the resultant of the weight of the object, and inertial forces and moments. Any convenient object-fixed or vehicle-fixed reference frame can be used.)

where $\boldsymbol{F}_i$ and $\boldsymbol{F}_j$ are the contact forces, and $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ are the position vectors at the ith and jth contacts (in any convenient body-fixed or object-fixed reference frame), respectively. This is illustrated through examples for a two- and a three-contact case in Fig. 2. The equilibrating force field consists of *equilibrating forces,* which are the forces required to maintain equilibrium against an external load. Further, these forces have no interaction force components. Thus the interaction force field consists of forces which must have a zero net resultant. It includes force components which squeeze the body (in the case of multifingered grippers) or the terrain (in the case of walking vehicles).

It has been shown [8] that the pseudo-inverse solution for the force system belongs to the equilibrating force field. Further, the interaction force field was shown to be the set of forces belonging to the null space. This result is presented here

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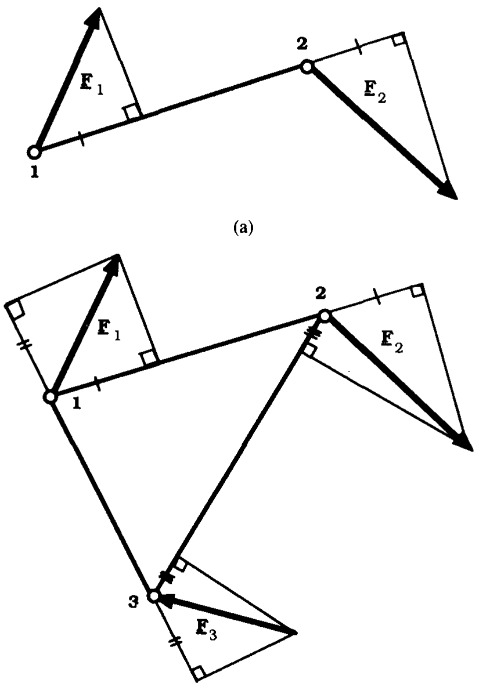

Fig. 2. The zero interaction force condition for (a) two and (b) three contacts (F_i is the contact force at the ith contact).

in the form of a theorem. The relationship between the generalized inverse solution and the decomposition of the force field is analyzed in greater detail, and the nature of the two force fields is explored. In particular, it is shown that the equilibrating force field is mathematically isomorphic to the velocity field in a rigid body. A computationally efficient, analytical method to obtain this solution is also presented.

The Pseudo-Inverse Solution

It has been assumed here that the contact interaction is such that moments cannot be transmitted, which means that there is a total of three force components at each contact. The equilibrium equations for the grasped object, or for the walking vehicle, may be written in the form

$$Gq = w \tag{2}$$

where w is the 6×1 external load vector consisting of the inertial forces and torques, and the weight of the object (vehicle body), q is the unknown $3n \times 1$ force vector, and n is the number of contact interactions. G is the $6 \times 3n$ coefficient matrix which is analogous to the Jacobian matrix encountered in the kinematics of serial chain manipulators. Each 6×1 column vector is a zero-pitch screw through a point of contact in the screw (in this case, line) coordinates (see Hunt [4] for a definition of screw coordinates). If S_{ix}, S_{iy}, and S_{iz} are the zero-pitch screw axes parallel to the x, y, and z axes (of any convenient coordinate system) passing through the ith contact point

$$G = \{S_{1x}\, S_{1y}\, S_{1z}\, S_{2x}\, S_{2y}\, S_{2z} \cdots S_{nx}\, S_{ny}\, S_{nz}\}.$$

Alternatively, the same expression may be written as

$$G = \begin{bmatrix} I_3 & I_3 & \cdots & I_3 \\ R_1 & R_2 & \cdots & R_n \end{bmatrix} \tag{3}$$

where I_3 is the 3×3 identity matrix and R_i is a skew-symmetric 3×3 matrix.

$$R_i = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

where (x_i, y_i, z_i) are the coordinates of the ith point of contact.

In general, the Moore–Penrose Generalized Inverse or the pseudo-inverse of G, G^+, seeks to find the minimum norm, least squares solution [12] for the force vector q. In this problem, if the screw system defined by the $3n$ zero-pitch wrenches is a sixth-order screw system or a six-system, w always belongs to the column space of G. It is assumed that this is the case here. The pseudo-inverse, then, is a right inverse which yields a minimum norm solution which must belong to the row space of G.

$$q = G^+ w. \tag{4}$$

Therefore, as the solution vector must belong to the row space of G

$$q = G^T c \tag{5}$$

where c is a 6×1 constant vector. If F_i is the force at the ith contact point and c_0 and c_1 are two 3×1 vectors, such that $c = \{c_0, c_1\}^T$, then

$$F_i = c_0 + R_i c_1 = c_0 - c_1 \times r_i. \tag{6}$$

It can be easily shown that this force system has no interaction forces [8]

$$\begin{aligned}(F_i - F_j) \cdot (r_i - r_j) &= [(c_0 - c_1 \times r_i) - (c_0 - c_1 \times r_j)] \cdot (r_i - r_j) \\ &= (r_i - r_j) \times c_1 \cdot (r_i - r_j) \\ &= 0.\end{aligned}$$

Thus the minimum norm condition implies that the solution vector must belong to the row space of G and hence to the equilibrating force field. As a corollary, all force vectors in the interaction force field must be represented in the null space. The following statement can now be made:

Theorem 1

If a body is subjected to multiple frictional contacts modeled by point contact, and if the system of zero-pitch contact wrenches span a six-dimensional space, the Moore–Penrose Generalized Inverse (or the pseudo-inverse) solution to the equilibrium equations yields a solution vector which lies in the equilibrating force field and has no interaction force components.

The Helicoidal Vector Field

Consider a system of ∞^2 coaxial helices, each of a constant pitch h. For an infinitesimal twist of a rigid body about the

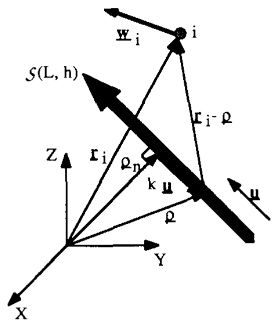

Fig. 3. The helicoidal vector field; Ψ. ($\mathcal{S}$ is the (screw) axis centreal to the vector field; h is the pitch, and L is the intensity of the field; $\boldsymbol{u}$ is a unit vector along $\mathcal{S}$; $\boldsymbol{\rho}$ is the position vector of an arbitrary point on the axis; $\boldsymbol{r}_i$ is the position vector, and $\boldsymbol{w}_i$ is the vector at the ith point.)

common axis of the helices, the velocity of any point on the body is tangential to that helix, which passes through the point, at that point. Such a system of ∞^3 tangents has been called a helicoidal velocity field by Hunt [4] and the common axis is the instantaneous screw axis.

A helicoidal vector field or an axial field $\boldsymbol{\Psi}$ is defined to be a system of ∞^3 vectors associated with an instantaneous screw axis $\mathcal{S}$. In Fig. 3, the vector at any point i is given by

$$\boldsymbol{w}_i = L\boldsymbol{u} \times (\boldsymbol{r}_i - \boldsymbol{\rho}) + hL\boldsymbol{u}. \tag{7}$$

If the force field is a helicoidal vector field, then $\boldsymbol{F}_i$ is of the same form as $\boldsymbol{w}_i$ in (7)

$$\boldsymbol{F}_i = L\boldsymbol{u} \times (\boldsymbol{r}_i - \boldsymbol{\rho}) + hL\boldsymbol{u}$$

and

$$(\boldsymbol{F}_i - \boldsymbol{F}_j) \cdot (\boldsymbol{r}_i - \boldsymbol{r}_j) = L\boldsymbol{u} \times (\boldsymbol{r}_i - \boldsymbol{r}_j) \cdot (\boldsymbol{r}_i - \boldsymbol{r}_j) = 0.$$

Hence, an important conclusion can be reached:

Theorem 2a

The force field given by a helicoidal vector field has no interaction force components and hence belongs to the equilibrating force field.

In fact, this could have been deduced by reducing (6) to the form in (7). This can be done because, it is always possible to find L, $\boldsymbol{u}$, $\boldsymbol{\rho}$, and h such that $\boldsymbol{F}_i$ can be described by (7).

It is more difficult to prove the converse of this theorem. One way of doing this is by writing the zero interaction force condition (1) for a hypothetical continuum (with contact points distributed continuously in three-dimensional space) in a differential form

$$\frac{dF_n}{dn} = 0 \tag{8}$$

where $\boldsymbol{n}$ is a unit vector representing a given direction and F_n is the component of force at any point along $\boldsymbol{n}$. In other words, the force component in a general direction (given by $\boldsymbol{n}$) does not vary along that direction. In vector notation, the following expression describes the same zero interaction force condition:

$$\nabla(\boldsymbol{F} \cdot \boldsymbol{n}) \cdot \boldsymbol{n} = 0. \tag{9}$$

In particular, the following relationships may be inferred from (9):

$$\begin{aligned} &\frac{\partial F_x}{\partial x} = 0 \qquad \frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x} = 0 \\ &\frac{\partial F_y}{\partial y} = 0 \qquad \frac{\partial F_x}{\partial z} + \frac{\partial F_z}{\partial x} = 0 \\ &\frac{\partial F_z}{\partial z} = 0 \qquad \frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y} = 0. \end{aligned} \tag{10}$$

It may be shown by integrating the partial differential equations in (10) that the vector field $\boldsymbol{F}$ must be of the form

$$\boldsymbol{F} = \boldsymbol{c}_0 + \boldsymbol{R}\,\boldsymbol{c}_1 \tag{11}$$

where $\boldsymbol{R}$ is the skew symmetric three-dimensional second-order tensor, which was defined earlier, and $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ are constant 3×1 vectors. Considering (6), (7), and (11), it is always possible to find h, L, $\boldsymbol{u}$, and $\boldsymbol{\rho}$ in (7) so that the expressions in (7) and (11) are identical. The zero interaction force condition thus requires that the forces belong to a system of ∞^3 tangents in *some* helicoidal vector field. This is stated in the form of a theorem as follows:

Theorem 2b

A force field satisfying the zero interaction force components must be a helicoidal vector field.

This is the converse of Theorem 2a. However, the assumption about the continuum ignores a number of singular cases which occur with a finite number of contact points. To prove that the zero interaction force condition implies a helicoidal vector field for a finite number of contact points is analogous to proving that the rigidity condition for points on the rigid body implies that the displacement of the points caused by an infinitesimal displacement of the body can be only described by a helicoidal vector field. A proof along these lines is included in the Appendix.

From Theorems 2a and 2b, the equivalence of the equilibrating force field and the helicoidal vector field is evident.

Theorem 3

The equilibrating force field ***is*** a helicoidal vector field.

Now it may be concluded that a system of wrenches arising through multiple frictional contacts which spans the six-dimensional space and satisfies the equilibrium equations, belongs to the equilibrating force field, if and only if it belongs to a helicoidal vector field, and if and only if it is the minimum norm solution to the equilibrium equations.

The reader familiar with kinematics will immediately recognize the analogy between the helicoidal system of forces and the velocity field for a rigid body. The zero interaction force condition is analogous to the rigidity condition, which requires the difference in velocities between any two points to have no components along the line joining the two points. This rigidity condition may be used to prove, in turn, a well-known result along the same lines, that the most general form of displacement of a rigid body can be instantaneously represented by a twist about a screw axis.

Equilibrating Forces

The best approach to finding the equilibrating force field or the minimum norm solution, is to find the axis, which is central to the helicoidal force field. Let F_i be given by (7). The equations of equilibrium (2) can be written in the form

$$\sum_i^n F_i = Q \tag{12}$$

and

$$\sum_i^n (r_i \times F_i) = T \tag{13}$$

where Q and T are the net external force and moment components of the load vector w. Substituting the expression in (7) in (12) and (13)

$$hLu + Lu \times \bar{r} - Lu \times \rho = \frac{1}{n} Q \tag{14}$$

and

$$\bar{r} \times hLu - \bar{r} \times (Lu \times \rho) + \frac{L}{n} \sum_i^n (r_i \times (u \times r_i)) = \frac{1}{n} T \tag{15}$$

where $\bar{r}$ is the position vector of the centroid. If I is the centroidal moment of inertia tensor, given by

$$I = -R_i R_i = \begin{bmatrix} \overline{y^2} + \overline{z^2} & -\overline{xy} & -\overline{xz} \\ -\overline{xy} & \overline{x^2} + \overline{z^2} & -\overline{yz} \\ -\overline{xz} & -\overline{yz} & \overline{x^2} + \overline{y^2} \end{bmatrix}$$

in which

$$\overline{x^2} = \frac{1}{n} \sum_i^n (x_i - \bar{x})^2 \qquad \overline{yz} = \frac{1}{n} \sum_i^n (y_i - \bar{y})(z_i - \bar{z})$$

$$\overline{y^2} = \frac{1}{n} \sum_i^n (y_i - \bar{y})^2 \qquad \overline{xz} = \frac{1}{n} \sum_i^n (x_i - \bar{x})(z_i - \bar{z})$$

$$\overline{z^2} = \frac{1}{n} \sum_i^n (z_i - \bar{z})^2 \qquad \overline{xy} = \frac{1}{n} \sum_i^n (x_i - \bar{x})(y_i - \bar{y})$$

where

$$\bar{x} = \frac{1}{n} \sum_i^n x_i \quad \bar{y} = \frac{1}{n} \sum_i^n y_i$$

and

$$\bar{z} = \frac{1}{n} \sum_i^n z_i$$

then (15) can be rewritten as

$$\bar{r} \times hLu - \bar{r} \times (Lu \times \rho) + L(I\,u + \bar{r} \times (u \times \bar{r})) = \frac{1}{n} T.$$

Substituting from (14) in this expression

$$\bar{r} \times \frac{1}{n} Q + LI\,u = \frac{1}{n} T. \tag{16}$$

Now, expressions for the parameters describing the field axis may be computed

$$u = \frac{I^{-1}}{nL} (T - \bar{r} \times Q) \tag{17}$$

$$L = \left| \frac{1}{n} I^{-1} (T - \bar{r} \times Q) \right| \tag{18}$$

$$h = \frac{1}{nL} (u \cdot Q). \tag{19}$$

Finally, if $\rho = \rho_n + ku$, as shown in Fig. 3, substituting into (14) and cross-multiplying by u yields

$$\rho_n = \frac{1}{nL} (u \times Q) - (u \cdot \bar{r}) u + \bar{r}. \tag{20}$$

In a centroidal reference frame, (16)–(19) may be written more compactly as

$$u = \frac{I^{-1}}{nL} T \tag{21}$$

$$L = \left| \frac{I^{-1} T}{n} \right| \tag{22}$$

$$h = \frac{1}{nL} (u \cdot Q) \tag{23}$$

$$\rho_n = \frac{1}{nL} (u \times Q). \tag{24}$$

Now the force field can be obtained from (7). From the point of view of programming, computing the force distribution, (17)–(20) followed by (7), involves a total of $12n + 87$ multiplications and $16n + 43$ additions. In addition, it is easy to program as singularities in the algorithm can be easily detected and alternative steps can be followed for such special cases.

Interaction Forces

The expressions for equilibrating forces were derived in the previous section. The interaction forces pose problems which are less tractable. The following discussion describes the nature of the interaction force field.

The interaction force field may be characterized using screw system theory [4]. If the interaction force field consists of n wrenches (which must have a zero resultant), the screws corresponding to the n wrenches must, in general, belong to a screw system of order $n - 1$ [16]. In special cases, they belong to a screw system of order less than $n - 1$. Further, since the contact wrenches are pure forces, the screw system must allow zero-pitch wrenches.

The interaction force field does not exist for the trivial case of a single contact. If the number of contact points is equal to 2, the interaction force field is represented by a first-order screw system. Further, the defining screw must be of zero pitch. Thus the interaction forces must lie along the line joining the two contact points and be equal and opposite. If the number of contacts is equal to 3, the interaction force field corresponds to a special two-system [4], which consists of coplanar zero-pitch screws whose axes are either parallel or concurrent. Such a system of coplanar, zero-pitch wrenches has been used for three-fingered grasps [7]. Clearly, there are ∞^2 choices for the point of concurrence, and another degree of freedom corresponding to the intensity of the field.

The problem of determining interaction forces becomes more intractable when the number of contacts exceeds 3. If the number of contacts is equal to 4, the zero-pitch screws must belong to a third-order screw system. The axes of the zero-pitch wrenches lie, in general, along the generators of a hyperboloid of revolution. In a special case, they may all be concurrent [7]; the wrenches constitute the special three-system which consists of the bundle of lines through the contact point. With 5 contact points, the zero-pitch wrenches are, in general, members of a fourth-order screw system. The wrench axes belong to a linear congruence and, in general, two lines (which may be imaginary) intersect all the axes. If the number of contact points equals 6, the axes are members of a linear complex and have one reciprocal screw whose axis coincides with the axis of the complex. Finally, if the number of contact points is greater than 6, in general, the interaction force field spans the six-space, and there is no restriction on the axes of the zero-pitch wrenches.

A more productive characterization for the cases of 4, 5, or 6 contact points is probably the use of pairs of equal and opposite forces acting along the lines joining the contact points. In general, the number of degrees of freedom of the interaction force field is $3n - 6$. Thus $3n - 6$ equal and opposite force pairs are needed to specify the field. Since there exist ${}^nC_2$ such pairs and $3n - 6 \leq {}^nC_2$ for $n > 3$, there are always sufficient lines joining contact points to do this.

While the equilibrating force solution minimizes the norm of the force vector, the interaction forces may be used to satisfy constraints or to suitably optimize the solution. For example, in a real-world situation, it must be ensured that the friction angle at each of the contact points is within an acceptable limit. In addition, in multilimbed systems, the interaction forces must often be minimized as they increase the isometric work which arises as a consequence of the active coordination.

Conclusions

The force distribution between multiple frictional contacts between an actively coordinated structure and a body is analyzed. The force allocation can be decomposed into an equilibrating force field, or a particular solution, and an interaction force field or a homogeneous solution. When the set of available contact wrenches span the six-dimensional space, the equilibrating force solution is shown to be identical to the solution derived from the Moore–Penrose Generalized Inverse or the pseudo-inverse. This solution corresponds to a helicoidal vector field which is similar to the velocity field of a rigid body. The zero interaction force condition, which is imposed on the force system to obtain the equilibrating forces, is analogous to the rigidity condition in kinematics. An elegant and computationally efficient solution for the equilibrating force field is derived. A physical interpretation of the interaction force field in terms of screw system theory is also presented.

While this decomposition leads to an efficient computation of the equilibrating force field, optimization of the contact conditions will, in most cases, require manipulation of the homogeneous solution. Two approaches to the characterization of the interaction force field have been presented here. Techniques for utilizing them to optimize contact conditions are a subject for future research.

Appendix

Proof of Theorem 2b

The assumption about the continuum earlier in the paper ignores a number of singular cases which occur with a finite number of contact points. It is proved here that the zero interaction force condition implies a helicoidal vector field for a finite number of contact points. The cases with 1 or 2, 3, and more than 3 contact points are considered separately.

A. One or Two Contact Points and Three or More Colinear Points

With one contact point, the force system can only resist a wrench which belongs to the special 3-system consisting of zero-pitch screws passing through the point. If the number of contact points is increased to two, the body is still free to rotate about a line joining the two contact points. A situation with three or more colinear contact points is a similar one. As such force systems do not span the six-space and hence do not completely constrain the object, the forces may not belong to an axial field. The theorems derived in this paper do not apply to these special cases.

B. Three Contact Points

Let $\boldsymbol{u}$ be a unit vector such that the three contact forces, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$, have equal components along $\boldsymbol{u}$. Such a unit vector satisfies the condition

$$(\boldsymbol{F}_1-\boldsymbol{F}_2)\cdot\boldsymbol{u}=(\boldsymbol{F}_1-\boldsymbol{F}_3)\cdot\boldsymbol{u}=0. \tag{25}$$

$\boldsymbol{u}$ can be uniquely determined if $\boldsymbol{F}_1 - \boldsymbol{F}_2$ and $\boldsymbol{F}_1 - \boldsymbol{F}_3$ are linearly independent. In other words, the two free vectors must be antiparallel. Let Π be a plane perpendicular to $\boldsymbol{u}$ as shown in Fig. 4. Let the forces be resolved parallel to and normal to $\boldsymbol{u}$ to yield components $\boldsymbol{F}_{ui}$ and $\boldsymbol{F}_{ni}$, respectively. For the moment, the following assumptions are made:

a) $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ are not all equal
b) $\boldsymbol{F}_1 - \boldsymbol{F}_2$ and $\boldsymbol{F}_1 - \boldsymbol{F}_3$ are not parallel
c) the vector projections of the contact forces on the plane Π (which is orthogonal to $\boldsymbol{u}$) are not all parallel
d) the projections of the contact points along $\boldsymbol{u}$ on Π are not colinear.

These special cases are discussed later.

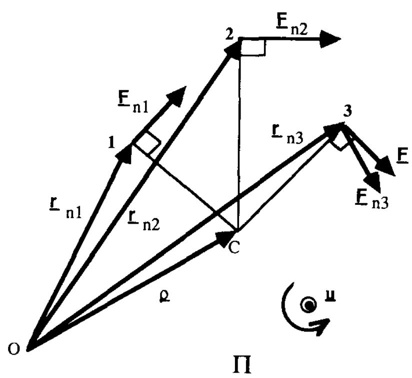

Fig. 4. Projection of the contact points and the contact forces onto Π (3 contact points). (u is the unit vector along the axis of the field given by (24); Π is the plane orthogonal to u; F_{ni} and r_{ni} are the projections of the ith contact force and the ith position vector onto Π).

Applying the zero interaction force condition (1) and recognizing that the components that are parallel to u (F_{ui}) are equal

$$(F_{ni}-F_{nj}) \cdot (r_i-r_j)=0.$$

If the position vectors are projected along u onto Π so that the ith vector yields r_{ni} at the ith contact point, the above equation involves only r_{ni} and r_{nj}

$$(F_{ni}-F_{nj}) \cdot (r_{ni}-r_{nj})=0. \tag{26}$$

All three force components cannot be parallel, because that would contradict the assumption in c). Let 1 and 2 designate two contact points such that F_{n1} and F_{n2} are not parallel. Let ρ be the position vector of the intersection to the perpendiculars to F_{n1} and F_{n2} at contact points 1 and 2 as shown in Fig. 4. Scalar constants λ_1 and λ_2 may be defined such that

$$\rho=r_{n1}+\lambda_1 u\times F_{n1}=r_{n2}+\lambda_2 u\times F_{n2}. \tag{27}$$

Cross multiplying by $r_{n1} - r_{n2}$

$$\lambda_1[(r_{n1}-r_{n2}) \cdot F_{n1}]u=\lambda_2[(r_{n1}-r_{n2}) \cdot F_{n2}]u. \tag{28}$$

The case in which the projections of the contact points are colinear is treated separately (see assumption d)). Therefore, if this case is excluded, r_{n1} cannot equal $r_{n2}\cdot F_{n1}$ and F_{n2} cannot both be perpendicular to $r_{n1} - r_{n2}$ as they are not parallel (according to hypothesis). Similarly, F_{n1} and F_{n2} cannot both be equal to zero. Otherwise, from (26), F_{n3} must be zero which again implies that all F_i are equal (which is excluded by a)). Thus one of the two following possibilities can be allowed:

$$\lambda_1=\lambda_2$$

or

$$\lambda_2=0 \quad \text{and} \quad (r_{n1}-r_{n2}) \cdot F_{n1}=0.$$

Consider the more general case where $\lambda_1 = \lambda_2 = \lambda$. λ cannot equal zero as this would mean that the projections are colinear ($r_{n1} = r_{n2}$). Let F be a force at contact point 3 given by

$$\rho=r_{n3}+\lambda u\times F$$

or

$$F=\lambda^{-1}(\rho-r_{n3})\times u. \tag{29}$$

Then

$$F \cdot (r_{n2}-r_{n3})=\lambda^{-1}(r_{n2}+\lambda u\times F_{n2}-r_{n3})\times u \cdot (r_{n2}-r_{n3})$$
$$=F_{n2} \cdot (r_{n2}-r_{n3})$$
$$=F_{n3} \cdot (r_{n2}-r_{n3})$$

from (26). Similarly,

$$F \cdot (r_{n1}-r_{n3})=F_{n3} \cdot (r_{n1}-r_{n3}).$$

As F and F_{n3} have equal components along two linearly independent directions, F must equal F_{n3}. Thus the normals to all three forces intersect at C and the ratio of the distance of the contact point from C to the magnitude of the contact force is λ.

If an axis is drawn through C parallel to u, the ith contact force may be expressed as

$$F_i=F_u+\lambda^{-1}(\rho-r_i)\times u \tag{30}$$

where ρ is now the position vector of any point on the axis and r_{ni} is replaced by r_i in (29). If $1/\lambda$ is denoted by L and $|F_u|/L = h$

$$F_i=hLu+Lu\times(r_i-\rho)$$

which is (7). Thus the axis is the axis of an axial field and the vector F_i belongs to a helicoidal field.

If $\lambda_2 = 0$ (and choosing λ_2 instead of λ_1 to be zero does not decrease the generality of the proof) then $r_{n2} = \rho$. Further, by applying (26) to points 1 and 2, as the case in which the projection of the points is colinear is argued separately, F_{n2} must be equal to zero. This implies that F_{n1} must be perpendicular to ($r_{n1} - r_{n2}$), F_{n3} to ($r_{n3} - r_{n2}$). It may be argued that F_{n1} and F_{n3} cannot be parallel (as it is assumed here that all three components are not parallel) and must be therefore related by an equation similar to (2). A similar conclusion may be reached once more, except now, the axis passes through a contact point (point 2).

Now, the special cases which were excluded earlier must be analyzed.

a) If F_1, F_2, and F_3 *are* all equal, the vector field is a helicoidal field with an infinite pitch central axis along F_1.

b) If $F_1 - F_2$ and $F_1 - F_3$ *are* parallel

$$(F_1-F_2)=\alpha(F_1-F_3)$$

where α is a scalar. Using (1)

$$(F_1-F_2) \cdot (r_1-r_2)=0$$

or

$$\alpha(F_1-F_3) \cdot (r_1-r_2)=0.$$

Also, by (1)

$$(F_1-F_3) \cdot (r_1-r_3)=0.$$

Thus ($F_1 - F_3$), and similarly ($F_1 - F_2$), must lie perpendicular to the plane of the contact points (the case in which all three forces are equal has already been considered). Thus $\boldsymbol{u}$ may lie along any line on a plane which is parallel to the plane containing the contact points. The projections of the contact points onto a plane perpendicular to $\boldsymbol{u}$ are colinear. In such a situation, the forces must lie on planes which are perpendicular to the plane containing the contact points.

In this case, let the x–y plane contain the contact points and the forces be resolved along the z-direction ($\boldsymbol{F}_{zi}$) and perpendicular to it ($\boldsymbol{F}_{pi}$). Clearly, since $\boldsymbol{F}_1 - \boldsymbol{F}_2$ and $\boldsymbol{F}_1 - \boldsymbol{F}_3$ must be perpendicular to the x–y plane, the $\boldsymbol{F}_{pi}$ are equal. The $\boldsymbol{F}_{zi}$ can always be described by a planar force distribution of the form

$$F_{zi} = Ax_i + By_i + C$$

where A, B, and C are constants which are functions of the locations of the contact points. If $Lu_x = B$, $Lu_y = -A$, $Lu_z = 0$, as $\boldsymbol{u}$ must lie on the x–y plane, and the perpendicular to the projection of the axis (which is parallel to $\boldsymbol{u}$) on the x–y plane, from the origin $\boldsymbol{\rho}_{np}$, is given by

$$L(\boldsymbol{\rho}_{np} \times \boldsymbol{u}) \cdot \boldsymbol{k} = C$$

then

$$\boldsymbol{F}_{zi} = (L\boldsymbol{u} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_{np}) \cdot \boldsymbol{k})\boldsymbol{k}.$$

Further, if hL is equal to $\boldsymbol{F}_{p1} \cdot \boldsymbol{u} (= \boldsymbol{F}_{p2} \cdot \boldsymbol{u} = \boldsymbol{F}_{p3} \cdot \boldsymbol{u})$, and $\boldsymbol{\rho}_{nz}$, the z-component of the position vector of any point on the axis, is defined so that

$$L\boldsymbol{u} \times \boldsymbol{\rho}_{nz} = -(\boldsymbol{F}_{pi} - (\boldsymbol{F}_{pi} \cdot \boldsymbol{u})\boldsymbol{u})$$

then

$$\boldsymbol{F}_i = hL\boldsymbol{u} - L\boldsymbol{u} \times (\boldsymbol{\rho}_{np} - \boldsymbol{r}_i) - L\boldsymbol{u} \times \boldsymbol{\rho}_{nz}$$

and if $\boldsymbol{\rho} = \boldsymbol{\rho}_{np} + \boldsymbol{\rho}_{nz}$

$$\boldsymbol{F}_i = hL\boldsymbol{u} + L\boldsymbol{u} \times (\boldsymbol{\rho} - \boldsymbol{r}_i)$$

which is, once more, (7). Thus even in the special case in which ($\boldsymbol{F}_1 - \boldsymbol{F}_3$) and ($\boldsymbol{F}_1 - \boldsymbol{F}_2$) are parallel, the force system is again a helicoidal vector field.

c) If the vector components of the contact forces on the plane Π (orthogonal to $\boldsymbol{u}$) *are* all parallel, then by (26), all three components are equal. This implies that $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ are equal—this case was considered above in the special case in which assumption a) is false.

d) If the projections of the contact points on Π along $\boldsymbol{u}$ *are* colinear, since all three contact forces must now have equal components along $\boldsymbol{u}$, and equal components along a line perpendicular to $\boldsymbol{u}$ (by applying (26)), ($\boldsymbol{F}_1 - \boldsymbol{F}_2$) and ($\boldsymbol{F}_1 - \boldsymbol{F}_3$) are parallel. This case is considered in b).

Thus in all cases, the zero interaction force condition requires a helicoidal force field for three contact point cases.

C. More than Three Contact Points

It has been shown that the zero interaction force condition implies a helicoidal force field for the 3 contact point case. Consider any set of 4 contact points. The contact forces at all 4 points satisfy the zero interaction force condition. Let the contact points 1, 2, and 3, be noncolinear (not losing any generality thereby) and $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ be given by (7). Now consider a general point q other than 1, 2, or 3. Let $\boldsymbol{F}$ denote a force at this point given by (7). By Theorem 2, $\boldsymbol{F}$ satisfies the zero interaction force condition and

$$\boldsymbol{F} \cdot (\boldsymbol{r}_3 - \boldsymbol{r}_q) = \boldsymbol{F}_3 \cdot (\boldsymbol{r}_3 - \boldsymbol{r}_q) = \boldsymbol{F}_q \cdot (\boldsymbol{r}_3 - \boldsymbol{r}_q).$$

Similarly

$$\boldsymbol{F} \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_q) = \boldsymbol{F}_q \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_q).$$

$$\boldsymbol{F} \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_q) = \boldsymbol{F}_q \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_q).$$

As the 4 points are, in general, noncoplanar, $\boldsymbol{F}$ and $\boldsymbol{F}_q$ have equal components along three linearly independent vectors and are, therefore, equal. Thus if the contact force at *any general fourth point* satisfies the zero interaction force condition, it also belongs to the same helicoidal force field.

This concludes the argument for a finite number of contact points. Thus when the force distribution completely constrains the body (object), the zero interaction force condition implies an axial or helicoidal vector field.

References

[1] J. M. Abel, W. Holzmann, and J. M. McCarthy, "On grasping objects with two articulated fingers," *IEEE J. Robotics Automat.*, vol. RA-1, no. 4, pp. 211-214, Dec. 1985.

[2] J. Hollerbach and S. Narasimhan, "Finger force computation without the grip jacobian," *IEEE Int. Conf. on Robotics and Automation* (San Francisco, CA, Apr. 7-10, 1986), pp. 871-875.

[3] W. Holzmann and J. M. McCarthy, "Computing the friction forces associated with a three fingered grasp," *IEEE J. Robotics Automat.*, vol. RA-1, no. 4, pp. 206-210, Dec. 1985.

[4] K. H. Hunt, *Kinematic Geometry of Mechanisms.* Oxford, UK: Clarendon Press, 1978.

[5] J. Kerr and B. Roth, "Analysis of multifingered hands," *Int. J. Robotics Res.*, vol. 4, no. 4, pp. 3-17, 1986.

[6] C. A. Klein, K. W. Olson, and D. R. Pugh, "Use of force and attitude sensors for locomotion of a legged vehicle over irregular terrain," *Int. J. Robotics Res.*, vol. 2, no. 2, pp. 3-17, 1983.

[7] V. Kumar and K. J. Waldron, "Sub-optimal algorithms for force distribution in multifingered grippers," in *Proc. IEEE Int. Conf. on Robotics and Automation* (Raleigh, NC, Mar. 31-Apr. 3, 1987), pp. 252-257.

[8] ——, "Actively coordinated vehicle systems," presented at the 1987 ASME Winter Annual Meeting, Boston, MA, Dec. 1987.

[9] ——, "Force distribution in closed kinematic chains," presented at the IEEE Int. Conf. on Robotics and Automation, Philadelphia, PA, Apr. 25-29, 1988.

[10] M. T. Mason, and J. K. Salisbury, *Robot Hands and the Mechanics of Manipulation.* Cambridge, MA: MIT Press, 1985.

[11] R. B. McGhee and D. E. Orin, "A mathematical programming approach to control of joint positions and torques in legged locomotion systems," in *Proc. ROMANSY-76 Symp.*" (Warsaw, Poland, Sept. 1976).

[12] B. Noble, "Methods for computing the Moore-Penrose Generalized Inverse, and related matters," in *Proc. Advanced Seminar* (sponsored by the Mathematics Research Center, University of Wisconsin, M. Z. Nashed, Ed.). New York, NY: Academic Press, 1976.

[13] J. K. Salisbury and J. J. Craig, "Analysis of multifingered hands: Force control and kinematic issues," *Int. J. Robotics Res.*, vol. 1, no. 1, pp. 4-17, 1982.

[14] J. K. Salibury, and B. Roth, "Kinematic and force analysis of articulated mechanical hands," *J. Mech., Transmiss. and Automat. in Des.*, vol. 105, pp. 35-41, 1983.

[15] K. J. Waldron, "Force and motion management in legged locomotion," *IEEE J. Robotics Automat.*, vol. RA-2, no. 4, pp. 214-220, Dec. 1986.

[16] K. J. Waldron and K. H. Hunt, "Series-parallel dualities in actively

coordinated mechanisms," presented at the 4th Int. Symp. on Robotics Research (Santa Cruz, CA, 1987).

[17] T. Yoshikawa and K. Nagai, "Manipulating and grasping forces in manipulation by multifingered grippers," in *Proc. IEEE Int. Conf. on Robotics and Automation* (Raleigh, NC, Mar. 31-Apr. 3, 1987), pp. 1998-2007.

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