Construction of Super Edge Magic Total Graphs

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Abstract

In this paper, we investigate the adjacency matrix of (a, d)-edge antimagic vertex graph and use this graph to construct other super edge magic total graphs with the same edge-weight set. Additionally, by combining known super edge magic total labeled graphs, we give a construction for a new super edge magic total graph.

Keywords: adjacency matrix, edge antimagic vertex labeling, super edge magic labeling.

1 Introduction

Let G be a finite simple undirected graph. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively, $v = |V(G)|$ and $e = |E(G)|$. For simplicity, we denote $V(G)$ by V and $E(G)$ by E.

A *labeling* of a graph G is a mapping that carries a set of graph elements into a set of numbers (usually positive integers), called *labels*. Kotzig and Rosa in 1970 introduced edge magic total labeling [5].

An (a, d)-*edge antimagic vertex* (EAV) labeling is a one-to-one mapping f from V onto the integers 1,2,...,v with the property that for every $xy \in E$, the edge-weight set $\{f(x)+f(y)|x, y \in V\}=\{a, a+d, a+2d, ..., a+(e-1)d\}$ for some positive integers a and d. A graph that has an (a, d)-edge antimagic vertex labeling is called an (a, d)-*edge antimagic vertex graph.* Since in this paper, we only consider the case of $d = 1$, then for simplicity we denote $(a, 1)$-EAV graph as EAV graph.

An *edge magic total* (EMT) labeling is a one-to-one mapping f from $V \cup E$ onto the integers 1,2,...,$v + e$ with the property that for every (x, y) in E, $f(x) + f(y) + f(xy) = k$ for some constant k. A graph that has an edge magic total labeling is called an *edge magic total graph.* An edge magic total labeling is called a *super edge magic total* (*SEMT*) *labeling* if $f(V) = \{1, 2, ..., v\}$ and a graph that has SEMT labeling is called a *SEMT graph.*

Research in SEMT labeling has been particularly popular during the last decade. For details, see the Gallian's dynamic survey [4]. There are many open problems, some of which will be listed in the conclusion of this paper.

Concerning SEMT graph, researchers usually concentrate on some specific class of families of graphs, such as trees, cycles, bipartite graphs, friendship graphs, wheels, generalised Petersen graphs. See [2, 3, 5, 6, 7, 10]. In this paper, we use the adjacency matrix of a known SEMT graph to construct other labeled graphs with the same edge-weights set. Additionally, we give a construction of new graphs by combining several graphs that have SEMT. Adjacency matrix methods have been used to generate a super $(a, 1)$-EAT graph in [11]. However, this is the first time that adjacency matrices are used to generate SEMT graphs.

2 Adjacency matrix

Let $G = (V(G), E(G))$ be a graph and f be an EAV labeling of G. Let $V = \{x_1, x_2, ..., x_v\}$ be the set of vertices in G with the labels $\{1, 2, ..., v\}$. Let A be an adjacency matrix of G, then the rows and columns of A can be labeled using 1,2,...,v. A is symmetric and every skew-diagonal (diagonal of A which is traversed in the "northeast" direction) line of matrix A has at most two "1" elements. The weights set $\{f(x) + f(y) : x, y \in V\}$ generates a consecutive integers $a, a+1, ..., a+e-1$ for some positive integer a. The weight $f(x) + f(y)$ is the same as the sum of labels of vertices on skew diagonal adjacency matrix that has "1" element.

A graph that has an EAV labeling and has the maximum possible number of edges is called *maximal EAV graph*. If G has a maximal EAV labeling then $a = 3$. Enomoto *et al.* [2] proved that the maximal number of edges in a SEMT graph is $2v - 3$.
Let $A = (a_{ij})$ be an adjacency matrix of a maximal EAV graph G. We can easily see that $|\{a_{1j} : a_{1j} \neq 0, j = 1, ..., v\}| = v - 1$ and $|\{a_{i1} : a_{i1} \neq 0, i = 1, ..., v\}| = v - 1$. Note that a_{vv} is counted twice. Thus the maximal width of the band of non-empty skew diagonal line is $2v - 3$.

Let A be the adjacency matrix of an EAV graph G of order v. If we move the element "1" of A along the skew-diagonal line, then this matrix is an adjacency matrix of an EAV graph that has the same weights set as A. Two graphs G and G^* are *EAV-equivalent* if G^* is obtained by the previous technique of moving the "1" element from G. Note that EAV-equivalent graphs are not necessarily isomorphic with respect to the graph structure and/or to the vertex labels. Figure 1 shows an example of generating a new maximal EAV graph from an old one. Graph G^* is obtained from graph G by moving the element "1" from position (1,4) to position (2,3) in the same skew-diagonal line.
Bača *et al.* [1] proved that if G has an EAV labeling then G has SEMT labeling. Thus, in this paper, we consider an adjacency matrix of an EAV graph.

Another known result for maximal SEMT labeling is given in the next section.

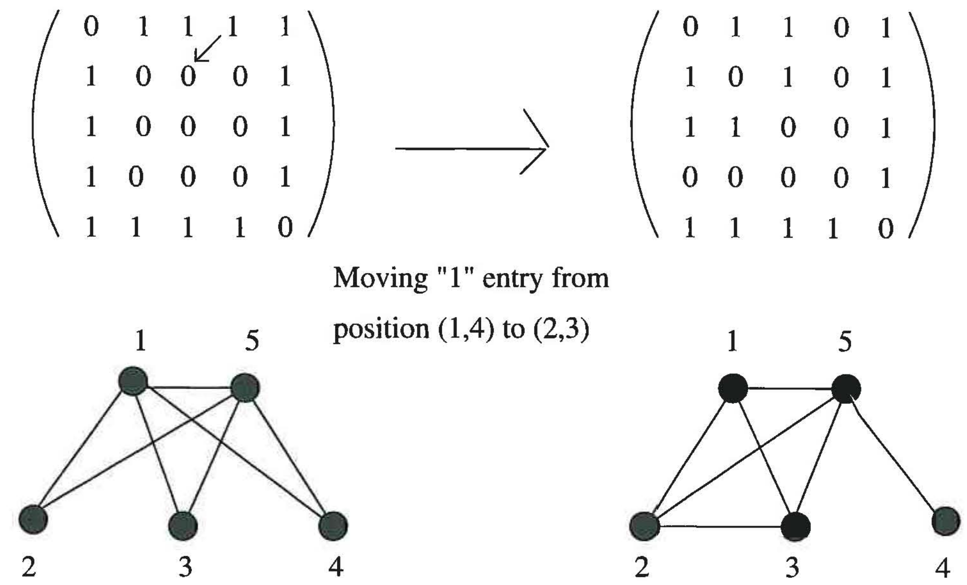

Figure 1: Generating a new EAV graph.

3 Maximal SEMT labeling

Figure 2 gives all maximal EAV-equivalent graphs with the one in Fig 1. Using the computer search we can find all possibilities of maximal EAV-equivalent graph from a given EAV graph with small order. Table 1 gives the result of the searching. Sugeng and Miller in [11] showed that the number of maximal EAV-equivalent (both connected and disconnected) graphs with size v is

- $(\frac{v-3}{2}!)^4(\frac{v-1}{2})^3$, for v odd,
- $(\frac{v-2}{2}!)^4\frac{v}{2}$, for v even.

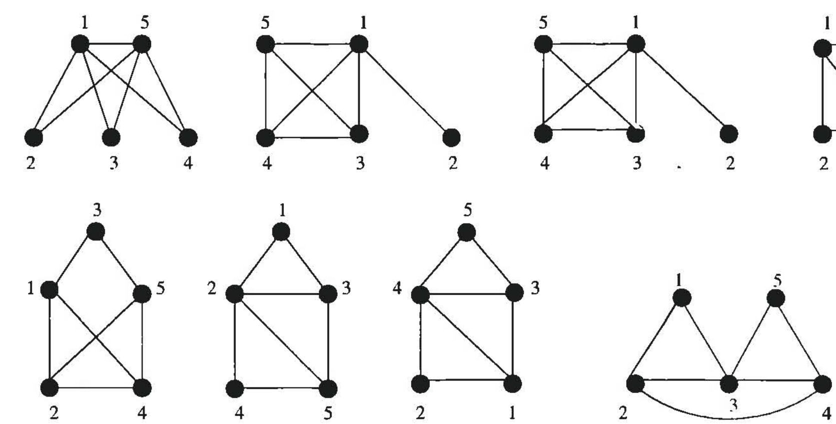

Figure 2: Maximal EAV-equivalent graphs on 5 vertices.

v	Connected EAV-equivalent	Disconnected EAV-equivalent
5	8	0
6	48	0
7	420	12
8	4896	288

Table 1: Maximal EAV-equivalent graphs on v vertices.

MacDougall and Wallis [9] studied SEMT maximal graphs. They called SEMT a *strong edge-magic-total* labeling. They proved the following propositions:

Proposition 1 *[9] Any SEMT labeling for a graph of order v can be obtained from any other by a sequence of single edge replacements.*

This proposition is the same as our technique of moving the "1" element along the skew-diagonal line of the adjacency matrix of a EAV graph.

Proposition 2 *[9] Every maximal SEMT graph of order v can be extended to one of order $v+1$.*

Proposition 3 can be generalised to the following theorem, giving a new SEMT graph from two known maximal SEMT graphs.

Theorem 1 *[9] Let G_1 and G_2 be any maximal SEMT graphs of order v and w, respectively. Then there are SEMT graphs of orders $v+w-2, v+w-1$, and $v+w$, each of which contains G_1 and G_2 as induced subgraphs.*

Considering the new maximal SEMT graph G with order $v+w$ like in the above theorem, then we have the following observation.

Observation 1 *If G_1 and G_2 are maximal SEMT graphs order v and w respectively, then we can construct a new maximal graph G with order $v+w$.*

Next, we give new results on maximal SEMT labeling of regular graph.

Proposition 3 *If an r-regular graph G is a maximal SEMT graph then the number of vertices v is equal to 2,3 or 6 and*

- *if $v=2$ then $r=1$, or*
- *if $v=3$ then $r=2$, or*
- *if $v=6$ then $r=3$.*

Proof. If G is an r-regular maximal SEMT labeling then $\frac{rv}{2}=2v-3$. It follows that $v|6$. Thus, v is equal to 2,3 or 6. □

The 1-regular graph with two vertices is K_2 and the 2-regular graph with three vertices is cycle C_3. It is known that K_2 and C_3 are SEMT graphs. Figure 3 gives EAV 3-regular graph on 6 vertices.

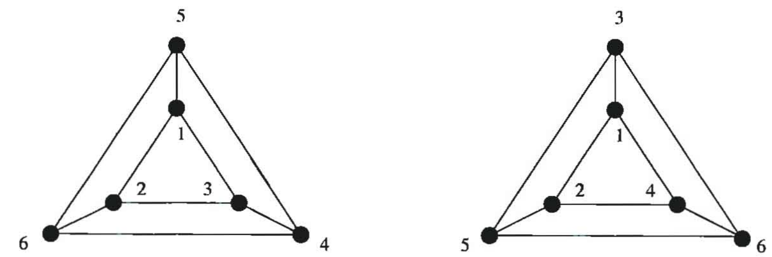

Figure 3: Maximal EAV 3-regular graph on 6 vertices.

4 Non-maximal SEMT graph

In this section, we show how the adjacency matrix of an EAV graph can be used for manipulating a given non-maximal SEMT graph.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 4: Expanding non-maximal SEMT graph on 5 vertices.

Theorem 2 *Any non-maximal SEMT graph can be extended to a maximal SEMT graph.*

Proof.

If G is a non-maximal SEMT graph of order v, then its adjacency matrix A has v rows and v columns but only $p < 2v-3$ non-empty skew-diagonal lines. Putting element "1" in $2v-3-p$ empty skew-diagonal lines, we obtain a maximal SEMT graph. □

Since the composition of edge in the graph has changed, then the edge labels for the new graph will also change. Figure 4 illustrates a maximal SEMT labeling extending a non-maximal SEMT graph of order 5. We can see that P_5 is not a maximal SEMT graph. It has only 4 edges. To extend P_5 to a maximal SEMT graph, we need 3 more edges.

Theorem 3 *Let G_1 and G_2 be any non-maximal SEMT graphs of order v and w respectively. Then there exists an SEMT graph of order $v+w$ which contains G_1 and G_2 as induced sub graphs. The minimum number of additional edges needed is $2v-1+\min\{wt(e_i) : e_i \in E(G_2)\} - \max\{wt(e_j) : e_j \in E(G_21)\}$.*

Proof.

Note that the weight of an edge xy under a labeling α is $wt(xy) = \alpha(x) + \alpha(y) + \alpha(xy)$. Let G_1 and G_2 be non-maximal SEMT graphs of order v and w respectively, and with number of edges e and f, respectively. Let $V(G_1) = \{x_1, ..., x_v\}$ and $V(G_2) = \{y_1, ..., y_w\}$. Label the vertices in G_1 and G_2 as

$$\alpha(x_i) = i, \text{ for } i = 1, ..., v.$$

$$\alpha(y_j) = v + j, \text{ for } j = 1, ..., w.$$

Let A and B be the adjacency matrices of G_1 and G_2, respectively. Create a new adjacency matrix C with order $(v+w) \times (v+w)$ such that

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Matrix C contains several empty skew-diagonal line bands in the middle. If we put "1" elements in every skew-diagonal line of the set of these empty skew-diagonal bands and make the matrix symmetric, then we obtain a EAV graph with $v+w$ vertices. Complete the edge labels then we have an SEMT graph G with order $v+w$. □

We already knew how to generate a bigger order SEMT graph from given SEMT graphs. On the other hand, we can also generate a smaller maximal (respectively, non-maximal) SEMT graph by deleting k vertices (and edges incident with those vertices) of a maximal (respectively, non-maximal) SEMT graph G to obtain a SEMT subgraph G'. However, we can only delete vertices that have the following properties:

- the k-largest labeled vertices, or
- the k-smallest labeled vertices, or
- the l-largest labeled vertices and the $(k-l)$-smallest labeled vertices.

Note that $l \leq k \leq v$. This requirement keeps the d-band set of the adjacency matrix of such graphs preserved to be a set of consecutive integers. The subgraph G has $v-k$ vertices. Note that, if we use either of the last two options, then we not only have to re-label the edges, but we also have to re-label the vertices by

- $\alpha^*(v_i) = \alpha(v_i) - k$ for the second option,
- $\alpha^*(v_i) = \alpha(v_i) - (k-l)$ for the third option.

Thus, we have the following observation.

Observation 2 *Every SEMT graph with order at least 3 contains a smaller SEMT subgraph.*

5 Conclusion

As mentioned in the introduction section, there are many results in SEMT labeling. However, many interesting problems remain unsolved. Here we list just a few.

- Are all trees SEMT graphs? (Conjecture from Enomoto *et al.* [2]).
- Can we use adjacency matrix to obtain all path-like trees? (Note that path-like tree is a tree that is derived from a path by moving some edges [8]).
- Can we find a relationship between SEMT labeling and other labeling using adjacency matrices?
- Can we use the algebraic properties of the adjacency matrix to find new properties of SEMT graph?
- Find SEMT labeling for various families of graphs.
- Find SEMT labeling by utilising properties of decomposition of graphs.

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