Properties of Consecutive Edge Magic Total Graphs

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Abstract

Let $G = (V, E)$ be a finite (non-empty) graph, where V and E are the sets of vertices and edges of G. An edge-magic total labeling is a bijection α from $V \cup E$ to the integers $1, 2, \ldots, n+e$, with the property that for every $xy \in E$, $\alpha(x)+\alpha(y)+\alpha(xy) = k$, for some constant k. Such a labeling is called an a-consecutive edge magic total labeling if $\alpha(V) = \{a+1, \ldots, a+n\}$ and a b-consecutive edge magic total if $\alpha(E) = \{b+1, b+2, \ldots, b+e\}$. In this paper we study the properties of a-consecutive and b-consecutive edge magic graphs.

Keywords: Graph, magic labeling, consecutive edge magic total labeling.

1 Introduction

All graphs considered are finite, simple and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $e = |E|$ and $n = |V| > 1$. A bijection

$$\alpha : V(G) \cup E(G) \to \{1, 2, \ldots, n+e\}$$

is called a *total labeling* for G and the associated weight $w_\alpha(xy)$ of an edge xy in G is $w_\alpha(xy) = \alpha(x) + \alpha(y) + \alpha(xy)$. In this paper we consider only total labelings, from now on by a labeling we shall always mean a total labeling. The total labeling α of G is *edge magic* if every edge has the same weight, and G is called an *edge magic graph* if an edge magic total labeling of G exists. If $\alpha(V) = \{1, ..., n\}$ then α is called *super edge magic labeling.* Magic labeling of graphs were introduced by Sedláček [2] in 1963, and since then there are many results in magic labeling, especially in edge magic labeling. For a recent dynamic survey of graph labellings see [1].

A bijection $\beta : V(G) \cup E(G) \to \{1, 2, ..., n+e\}$ is called an *a-consecutive edge magic labeling* of $G = G(V, E)$ if β is an edge magic labeling and $\beta(V) = \{a+1, ..., a+n\}$, $0 \le a \le e$. On the other hand, $\gamma : V(G) \cup E(G) \to \{1, 2, ..., n+e\}$ is called a *b-consecutive edge magic labeling* of $G = G(V, E)$ if γ is an edge magic labeling and $\gamma(E) = \{b+1, ..., b+e\}$, $0 \le b \le n$. A graph G that has a-consecutive (respectively, b-consecutive) edge magic labeling is called an *a-consecutive* (respectively, *b-consecutive*) *edge magic graph.*

Next we present known results of the dual labeling of super edge magic labeling.

Define $M = e + n$. Let $\gamma : V \cup E \to 1, 2, ..., M$ be a super edge magic labeling for a graph G. Define the labeling $\gamma' : V \cup E \to 1, 2, ..., M$ as follows.

$$\gamma'(x) = M + 1 - \gamma(x), x \in V,$$

$$\gamma'(xy) = M + 1 - \gamma(xy), xy \in E.$$

Then γ' is called the *dual* of γ.

Theorem 1.1 *[5] The dual of a super edge magic labeling for a graph G is also a super edge magic labeling.*

Similar results in dual labeling for a-consecutive and b-consecutive edge magic labeling will be presented in the next sections.

Let $V(G) = \{x_1, x_2, ..., x_n\}$ be the set of vertices in G with labels in $\{1, 2, ..., n + e\}$. A symmetric matrix $A = (a_{ij})$, $i, j = 1, ..., n$, is called an *adjacency matrix of* G if

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between } x_i \text{ and } x_j \\ 0 & \text{if there is no edge between } x_i \text{ and } x_j \end{cases}$$

A bijection $\alpha : V(G) \to \{1, 2, ..., n\}$ is called an (a, d)*-edge-antimagic vertex (EAV) labeling* of $G = G(V, E)$ if the set of the edge-weights of all edges in G is $\{a, a + d, ..., a + (e - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

In EAV labeling, we only give labels to the vertices of G. However, the evaluation is done for each edge in G. A graph that has EAV labeling can be represented by a special adjacency matrix.

If G is an EAV graph then the rows and columns of A can be labeled by 1,2,...,n. A is symmetric and every skew-diagonal (diagonal of A which is traversed in the "northeast" direction) line of matrix A has at most two "1" elements. The set $\{\alpha(x) + \alpha(y) : x, y \in V(G)\}$ generates a sequence of integers of difference d. Each entry "1" in a skew-diagonal line has a one-to-one correspondence to an element of the edge-weight set $\{\alpha(x) + \alpha(y) : x, y \in V(G)\}$. If $d = 1$ then the non-zero off diagonal lines form a band of consecutive integers. In this paper, EAV labeling always refers to an $(a, 1)$-EAV labeling.

2 Some properties of a-consecutive edge magic graphs

Let G be an a-consecutive edge magic graph and β be an a-consecutive edge magic labeling of G. Then $\beta(x) \in \{a+1, a+2, \ldots, a+n\}$, for every $x \in V(G)$, $0 \leq a \leq e$. Super edge magic labeling is a special case when $a = 0$. In this paper we consider a-consecutive edge magic labeling for $1 \leq a \leq n - 1$; For results in 0-consecutive edge magic labeling, that is, super edge magic labeling. For further results, see [1, 5].

Theorem 2.1 *The dual of an a-consecutive edge magic labeling for a graph G is an* $(e - a)$*-consecutive edge magic labeling.*

Proof: Let G be a graph that has a-consecutive edge magic labeling β with magic constant k.

Thus $\beta(V) = \{a+1, a+2, ..., a+n\}$. Let $M = e+n$. Define the labeling $\beta' : V \cup E \to 1, 2, ..., M$ as follows.

$$\beta'(x) = M + 1 - \beta(x), x \in V,$$

$$\beta'(xy) = M + 1 - \beta(xy), xy \in E.$$

We can see that $\beta'(V) = \{M + 1 - a - n, M - a - n + 2, ..., M - a\}$. Since the dual of an edge magic labeling is also an edge magic labeling, then β' is an $(e - a)$-consecutive edge magic labeling. □

Let us consider the adjacency matrix of a-consecutive edge magic graph. Since all labels of vertices in G are consecutive integers, then the adjacency matrix A of G consists of all elements in $\{a+1, a+2, ..., a+n\}$ in its rows and columns. The maximum number of edges in this graphs will be $2n-3$ [4, 3]. If $a \neq 0$ and $a \neq e$, and we know that $\beta(E) = \{1, ..., a\} \cup \{a+n+1, ..., n+e\}$, then there will be a gap in the set of edge-weights $\{\beta(x) + \beta(y) : x, y \in V(G)\}$. Thus, the labels divided into two blocks. The width of the gap must be n, the same as the gap in edge labels. Thus we can conclude that the maximum number of edges in G is $(2n - 3) - n = n - 3$. As a consequence, G cannot be connected. Thus we proved the following theorem.

Theorem 2.2 *If G has an a-consecutive edge magic labeling,* $a \neq 0$ *and* $a \neq e$*, then G is a disconnected graph.*

A natural guess is that G might be the union of 3 trees. However, the following results prove that this cannot be happen.

Corollary 1 *If* $a \neq 0$ *and* $a \neq e$ *then there is no a-consecutive edge magic labeling for* $3K_2$.

Proof:

Suppose that $3K_2$ has an a-consecutive edge magic labeling β, $a \neq 0$ and $a \neq e$, then

$$ek = \frac{(n + e)(n + e + 1)}{2} = \frac{(9.10)}{2} = 45$$

or $k = 15$. Since $n = 6$ and $e = 3$, the only possibilities for $0 < a < e$ are $a = 1$ or $a = 2$.

Case $a = 1$.

$$\beta(v) \in \{2, 3, 4, 5, 6, 7\} \text{ and } \beta(e) \in \{1, 8, 9\}, \quad \text{for } v \in V(K_2) \text{ and } e \in E(K_2).$$

Since $k = 15$, then $\beta(x) + \beta(y) \in \{6, 7, 14\}$, for $x, y \in V(G)$. However, the sum of the two largest labels of the vertices in G is less than 14, and so we cannot have the labeling.

Case $a = 2$.

$$\beta(v) \in \{3, 4, 5, 6, 7, 8\} \text{ and } \beta(e) \in \{1, 2, 9\}.$$

Since $k = 15$, then $\beta(x) + \beta(y) \in \{6, 13, 14\}$, for $x, y \in V(G)$. However, the sum of the two smallest labels of the vertices in G is more than 6, and so we cannot have the labeling.

Thus there is no a-consecutive edge magic labeling for $3K_2$ $\square$

Theorem 2.3 *If $a \neq 0$ and $a \neq e$, and G has an a-consecutive edge magic labeling β then G cannot be the union of three trees T_1, T_2 and T_3, where $|V(T_i)| > 2$, $i = 1, 2, 3$.*

Proof: Let G be an a-consecutive edge magic graph, $a \neq 0$ and $a \neq e$. Suppose that $G = T_1 \cup T_2 \cup T_3$, where T_1, T_2, T_3 are three arbitrary trees. Let $n_1 > 2$ (respectively, $n_2 > 2$, $n_3 > 2$) and e_1 (respectively, e_2, e_3) be the number of vertices and the number of edges in T_1 (respectively, T_2, T_3). Since G is an a-consecutive edge magic graph then there are two blocks of the edge labels. Consequently, the set of vertices in G also forms two disjoint subsets, say S_1 and S_2. Suppose that T_1 and T_2 are in the same block. Then

$$\begin{aligned} n_1 + n_2 &= (e_1 + 1) + (e_2 + 1) \\ &= e_1 + e_2 + 2. \end{aligned} \tag{1}$$

Thus

$$e_1 + e_2 = n_1 + n_2 - 2. \tag{2}$$

Since the number of edges must be the maximum possible, then

$$e_1 + e_2 = 2(n_1 + n_2) - 3. \tag{3}$$

From Equations (2) and (3), we obtain $(n_1 + n_2) = 1$ and this is a contradiction of $n_1 + n_2 > 1$. Similar results are obtained if T_2 and T_3 are in the same block. Thus G cannot be the disjoint union of three trees. $\square$

Figure 1 gives examples of a-consecutive edge magic labelings of graphs. Note that the maximal a-consecutive edge magic graphs means the graph has $n - 3$ edges.

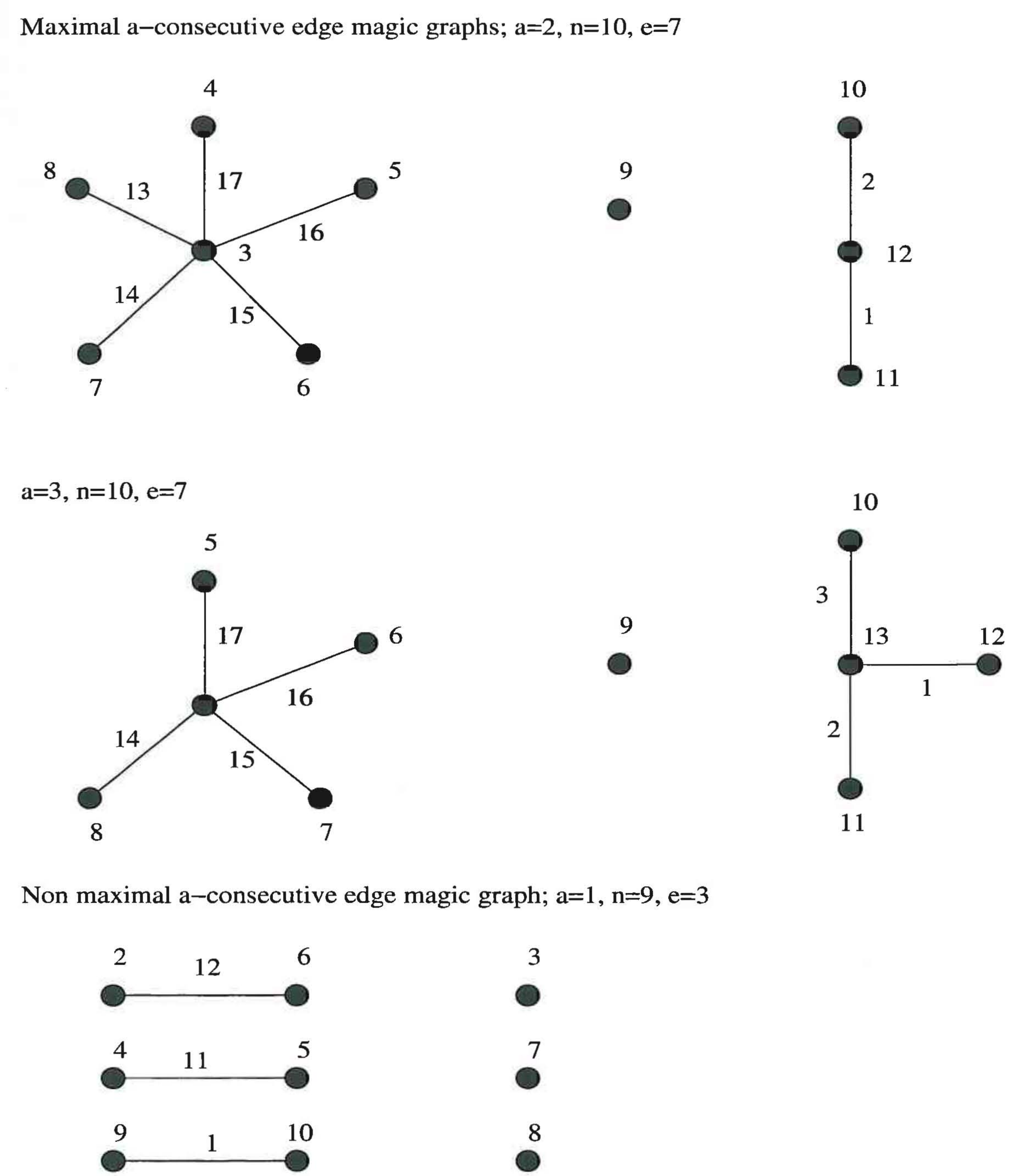

Figure 1: Examples of a-consecutive edge magic graphs.

If G is the union of three subgraphs then the only possibility is $G = T_1 \cup T_2 \cup T_3$, where T_1, T_2 and T_3 are trees. However, the previous theorem established that this is not possible. Thus if G has more than two connected components, then G has to have at least one isolated vertex.

If G has a-consecutive edge magic labeling, $a \neq 0$ and $a \neq e$, and the maximum number of edges then in every case we need at least one isolated vertex. By counting the maximum number of edges and comparing this to the number of vertices in the graph, we derive the number of isolated vertices that are needed in the following observation.

Observation 1 *Let G be an a-consecutive edge magic graph, $a \neq 0$ and $a \neq e$. If G consists of*

- *two trees then the number of isolated vertices is one.*
- *one tree and one unicyclic graph then the number of isolated vertices is two.*
- *two unicyclic graphs then the number of isolated vertices is three.*
- *otherwise the number of isolated vertices is at least three.*

Theorem 2.4 *There is an a-consecutive edge magic graph for every a and n.*

Proof: Let G be the union of two stars, S_1 and S_2, and one isolated vertex x. Let $v_{1i}, i = 1, ..., t_1$, $t_1 = e - a$, denote the leaves of S_1 and let $v_{2j}, j = 1, ..., t_2$, $t_2 = a$, denote the leaves of S_2. Label the vertices of G as follows.

$$\beta(v) = \begin{cases} n-1 & \text{if } v = x \\ a+1 & \text{if } v \text{ is a center of } S_1 \\ a+1+i & \text{if } v = v_{1i}, i = 1, ..., t_1 \\ a+n & \text{if } v \text{ is a center of } S_2 \\ a+n-j & \text{if } v = v_{2j}, j = 1, ..., t_2 \end{cases}$$

Complete the edge labels $\{1, 2, ..., a\} \cup \{a+n+1, ..., e+n\}$. Then we have an a-consecutive edge magic labeling for G. □

3 Some properties of b-consecutive edge magic graphs

Let G be a b-consecutive edge magic graph and let γ be a b-consecutive edge magic labeling of G. Then $\gamma(xy) \in \{b+1, b+2, \ldots, b+n\}$. The super edge magic labeling is a special case of b-consecutive edge magic labeling, when $b = n$. Since the vertex labels in a b-consecutive edge magic labeling, $1 \leq b \leq n-1$, do not form a set of consecutive integers, it follows that the row/column of the adjacency matrix of b-consecutive edge magic graph are labeled according to the vertex labels of G not consecutively as $1, 2, ..., n$.

Theorem 3.1 *Every b-consecutive edge magic graph has edge antimagic vertex labeling.*

Proof: Let G be a b-consecutive edge magic graph, then all the edge labels are the consecutive integers $\{b+1, ..., b+e\}$. If we define γ' as a restriction mapping of γ in V then we can see that γ' is an edge antimagic vertex labeling. □

Considering the dual labeling from Theorem 1, if a graph G has a b-consecutive edge magic labeling, a similar result in the dual also holds.

Theorem 3.2 *The dual of a b-consecutive edge magic labeling for a graph G is an $(n-b)$-consecutive edge magic labeling.*

Proof: Let G be a graph that has b-consecutive edge magic labeling with magic constant k.

Thus $\gamma(E) = \{b+1, b+2, ..., b+e\}$. Let $M = e+n$. Define the labeling $\gamma' : V \cup E \to 1, 2, ..., M$ as follows.

$$\gamma'(x) = M + 1 - \gamma(x), x \in V,$$
$$\gamma'(xy) = M + 1 - \gamma(xy), xy \in E.$$

We can see that $\gamma'(E) = \{M+1-b-e, M-b-e+2, ..., M-b\}$. Since the dual of an edge magic labeling is also an edge magic labeling, then γ' is an $(n-b)$-consecutive edge magic labeling. □

A *caterpillar* is a graph derived from a path by hanging any number of leaves from the vertices of the path. The caterpillar can be seen as a sequence of stars $S_1 \cup S_2 \cup \ldots \cup S_r$, where each S_i is a star with centre c_i and n_i leaves, $i = 1, 2, \ldots, r$, and the leaves of S_i include c_{i-1} and $c_{i+1}, i = 2, ..., r-1$. We denote the caterpillar as $S_{n_1, n_2, ..., n_r}$, where the vertex set is $V(S_{n_1,n_2,...,n_r}) = \{c_i : 1 \leq i \leq r\} \cup \bigcup_{i=2}^{r-1} \{x_i^j : 2 \leq j \leq n_i - 1\} \cup \{x_1^j : 1 \leq j \leq n_1 - 1\} \cup \{x_r^j : 2 \leq j \leq n_r\}$ and the edge set is $E(S_{n_1,n_2,...n_r}) = \{c_i c_{i+1} : 1 \leq i \leq r-1\} \cup \bigcup_{i=2}^{r-1} \{c_i x_i^j : 2 \leq j \leq n_i - 1\} \cup \{c_1 x_1^j : 1 \leq j \leq n_1 - 1\} \cup \{c_r x_r^j : 2 \leq j \leq n_r\}$.

Theorem 3.3 *There exists a b-consecutive edge magic graph for every b.*

Proof:
Let r be $b = \frac{r}{2}$, for r even; and $b = \frac{r+1}{2}$, for r odd. Let G be a caterpillar $S_{n_1,n_2,...,n_r}$, with centre $c_1, c_2, ..., c_r$, such that every centre c_i with i even has degree 2. Note that a star can be regarded as caterpillar S_{n_1,n_2}, with $n_2 = 1$. Let γ be a b-consecutive edge magic labeling for G. Label the odd centres as

$$\gamma'(c_i) = \frac{i+1}{2}, \text{ for } i \text{ odd.}$$

Let v_i^j be the j-th leaf of the centre i. Label the leaves of the odd centre by

$$\gamma'(v_i^j) = b + e + 1 + j + (\sum_{k=1}^{i-1} (n_k + 1))$$

If $i - 1 \leq 1$ then $\sum_{k=1,k\text{odd}}^{i-1} (n_k - 1) = 0$ and the even centre is treated as a leaf of the previous odd centre and is given the largest labels among the leaves.
Thus for every b, $b = \frac{r}{2}$, for r even; and $b = \frac{r+1}{2}$, for r odd, we have constructed a b-consecutive edge magic graph. □

We have an example of a b-consecutive edge magic labeling for every b. Figure 2 gives examples of labelings for some value of b. In general, we have

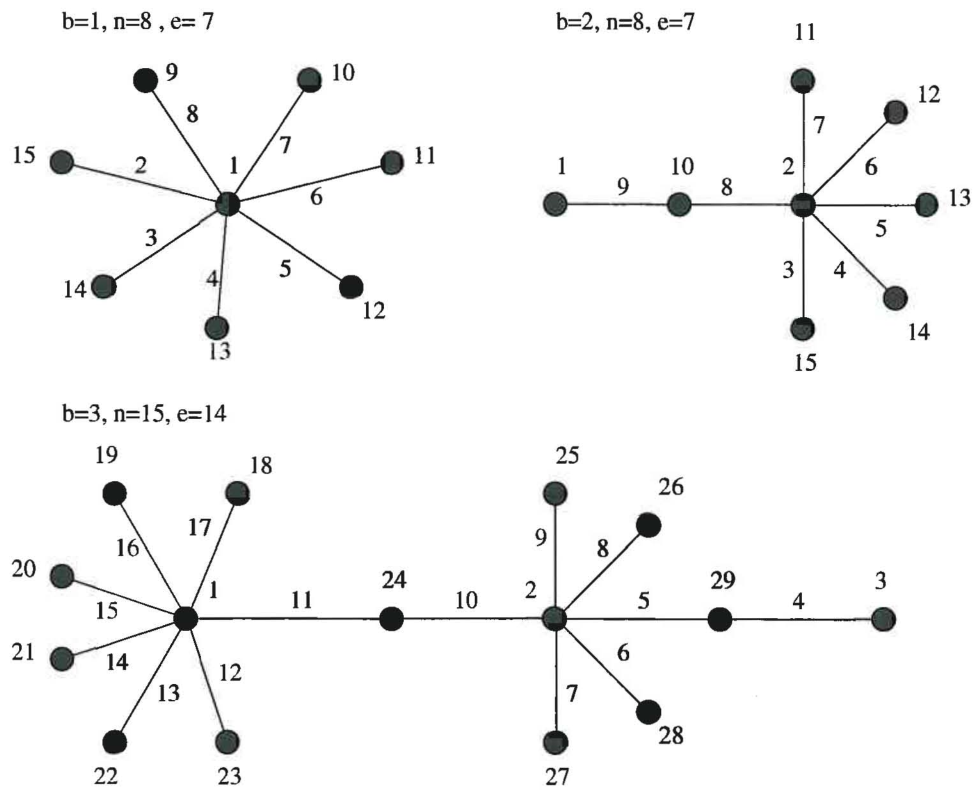

Figure 2: Examples of b-consecutive edge magic graphs.

Theorem 3.4 *If a connected graph G has a b-consecutive edge magic labeling, where $b \in \{1, ..., n-1\}$, then G is a tree.*

Proof: Suppose that G has a b-consecutive edge magic labeling γ. Then $\gamma(V) = V_1 \cup V_2$, where $V_1 = \{1, 2, ..., b\}$ and $V_2 = \{b+e+1, b+e+2, ..., n+e\}$. Let $b \in \{1, ..., n-1\}$. Let γ' be the restriction of γ under V. Thus γ' is a VAE labeling.

Let A be the adjacency matrix of G. Since the set of vertex labels is a union of two disjoint subset V_1 and V_2, then the adjacency matrix of G consists of four blocks as follows.

$$A = \begin{pmatrix} A_1* & A_2 \\ A_3 & A_4 \end{pmatrix}$$

Since A is a symmetric matrix it follows that A_3 is the transpose of A_2. The entries of A_1 represent all the edges between the vertices inside V_1, the entries of A_2 represent all the edges between vertices in V_1 and vertices in V_2, and the entries of A_4 represent all the edges between vertices inside V_2.

Suppose that A_1 is a nonzero submatrix. Then there is at least one edge xy between the vertices in V_1. $\gamma'(xy) = \gamma(x) + \gamma(y) \leq 2b - 1$. Let $x'y'$ be an edge between a vertex in V_1 and a vertex in V_2. Then $\gamma(x'y') = \gamma(x') + \gamma(y') \geq b + e + 2$. Since γ' is a VAE labeling, the edge-weights under γ' must be a set of consecutive integers. This means that $(b+e+2) < (2b-1)$ or $b > e + 3$. We know that $b \leq n - 1$, whence $e < n - 4$. This means G is disconnected. Similarly, if A_4 is nonzero.

Suppose that A_2 is a zero submatrix. Then A_1 and A_4 cannot be zero submatrices of A. Obviously, G will then be a disconnected graph.

If G is connected then A_1 and A_4 must be zero submatrices of A. Consider the submatrix A_2. The maximum edge-weight under γ' is $n + e + b$ and the minimum edge-weight is $b + e + 2$. Thus the maximum number of edges will be $(n + e + b) - (b + e + 2) - 1 = n - 1$. Then G is a tree. □

Corollary 2 *A double star S_{n_1,n_2} has a b-consecutive edge magic labeling for some $b \in \{1, 2, ..., n\}$ and*

- *If $b = 1$ then S_{n_1,n_2} is a star.*
- *If $b > 1$ then $b = n_2 + 1$.*

Proof:

Let $S_{n1,n2}$ be a double star with centres c_1 and c_2. Let n_1 be the number of leaves of c_1, excluding c_2, and let n_2 be the number of leaves of c_1, excluding c_1.

- $b = 1$.
  Label vertices and edges of $S_{n1,n2}$ as follows.

$$\gamma(v)=\begin{cases}1 & \text{if } v=c_1\\ 1+i & \text{if } v=v_i,\ v_i \text{ leaves of } c_2\\ n+e & \text{if } v=c_2\end{cases}$$

$$\gamma(c_1v_i)=i+1 \text{ for } i=1,...,n_1+1$$

Then γ is a b-consecutive edge magic labeling for $S_{n1,n2}$.

- $b>1$.
  Let $b=n_2+1$. Label vertices and edges of $S_{n1,n2}$ as follows.

$$\gamma(v)=\begin{cases}1 & \text{if } v=c_1\\ 1+i & \text{if } v=v_i,\ v_i \text{ leaves of } c_2\\ b+e+j & \text{if } v=v_j,\ v_j \text{ leaves of } c_1\\ n+e & \text{if } v=c_2\end{cases}$$

$$\gamma(c_1v_i)=i+1 \text{ for } i=1,...,n_1+1$$

  Complete labeling of all edges with elements of $\{b+1,...,b+e\}$, in such a way that γ is a b-consecutive edge magic labeling for S_{n1,n_2}.

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