GLOBAL OPTIMALITY CONDITIONS FOR MIXED INTEGER WEAKLY CONCAVE PROGRAMMING PROBLEMS

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Abstract. In this paper, some necessary and some sufficient global optimality conditions for a class of mixed integer programming problems whose objective functions are the difference of quadratic functions and convex functions are established. The numerical examples are also presented to show the significance of the global optimality conditions for this class of programming problems.

Keywords. Global optimality conditions, mixed integer programming problems, weakly concave programming problems.

AMS (MOS) subject classification: 41A65, 41A29, 90C30.

1 Introduction

Consider the following mixed integer programming problem:

$$(MIP) \qquad \begin{aligned} \min \quad & f(x) = \frac{1}{2}x^T Ax + d^T x - g(x) \\ \text{s.t.} \quad & x_i \in [u_i, v_i], i \in L, \\ & x_j \in \{p_j, p_j + 1, \dots, q_j\}, j \in M \end{aligned} \tag{1.1}$$

where A is a $n \times n$ symmetric matrix, $d \in \mathbb{R}^n$, $g(x) : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable convex function on $\mathbb{R}^n$, $L, M \subset \{1, \dots, n\}$ with $L \cap M = \emptyset$ and $L \cup M = \{1, \dots, n\}$, $u_i, v_i \in \mathbb{R}$ with $u_i < v_i, \forall\ i \in L$, and p_j, q_j are integers with $p_j < q_j, \forall j \in M$. We call problem (MIP) as a mixed integer weakly concave programming problem. Throughout the paper, we let

$$\mathcal{F} := \{x = (x_1, \dots, x_n)^T \mid x_i \in [u_i, v_i], i \in L; x_j \in \{p_j, p_j{+}1, \dots, q_j\}, j \in M\}.$$

There are a vast number of applications of mixed integer programming problems in many areas, such as engineering design, computational chemistry, computational biology and reliability networks as well as optimization of core reload patterns for nuclear reactors. For applications of mixed integer programming models in optimization, see [1], [7], [9], [10], [11], [16], [17].

Most existing approaches to solve the mixed integer programming problems belong to the branch-and-bound, decomposition or outer approximation methods. The discussion of the branch-and-bound methods can be found in [3], [15], [18]. The generalized Benders decomposition method was proposed in [8]. The outer approximation method can be seen in [5], [6]. It should be noted that mixed integer nonlinear programming problems are very difficult due to the nonlinearity and the mixture of continuous and discrete variables. One extremely tough task is how to verify whether the found solutions are global ones or not. Thus the global optimality conditions become the focus of many researches in recent years. Recently, some global optimality conditions characterizing global minimizer of quadratic minimization problem has been discussed in [2], [14], [19], [21]. Reference [4] also discussed some global optimality conditions for integer quadratic minimization problem when A is a positive semidefinite matrix. Especially [13] established conditions which ensure that a feasible point is a global minimizer of a quadratic minimization problem subject to box constraints or binary constraints by using a new approach which makes use of a global subdifferential. Reference [20] presented some global optimality conditions for quadratic programming problems with mixed variables. Reference [12] have derived some global optimality conditions for the minimization of the difference of quadratic and convex functions over box constraints or binary constraints.

In this paper, we will establish some necessary and sufficient global optimality conditions for the mixed integer weakly concave programming problems, which extend the results given in [12] and [20]. We also give some numerical examples to show the significance of the global optimality conditions.

The layout of the paper is as follows. Section 2 presents some necessary global optimality conditions for problem (MIP). The sufficient global optimality condition for (MIP) will be given in section 3.

2 Necessary global optimality conditions for (MIP)

In this section, we will derive the necessary global optimality conditions for problem (MIP). We use the following notations throughout this paper. The real line is denoted by $\mathbb{R}$ and the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. For vectors $x, y \in \mathbb{R}^n$, $x \geq y$ means that $x_i \geq y_i$, for $i = 1, \ldots, n$. The notation $A \succeq B$ means $A - B$ is a positive semidefinite and $A \preceq 0$ means $-A \succeq 0$. A diagonal matrix with diagonal elements

$\alpha_1, \dots, \alpha_n$ is denoted by $\mathrm{diag}(\alpha_1, \dots, \alpha_n)$. Let $A = (a_{ij})_{n\times n} \in S^n$, denote diag $(A) :=$ diag $(a_{11}, \dots, a_{nn})$ and diag $(\tilde{A}) :=$ diag $(\tilde{a}_{11}, \dots, \tilde{a}_{nn})$, where S^n is the set of $n \times n$ symmetric matrices, and

$$\tilde{a}_{ii} = \begin{cases} \min\{0, a_{ii}\}, & \forall i \in L, \\ a_{ii}, & \forall j \in M \end{cases}.$$

Let $\beta = (\beta_1, \dots, \beta_n)^T \in \mathbb{R}^n$, denote diag $(\beta) :=$ diag $(\beta_1, \dots, \beta_n)$. Let $\bar{x} \in \mathcal{F}$. For any $i \in L, j \in M$, denote

$$\tilde{\bar{x}}_i : = \begin{cases} -1, & \text{if } \bar{x}_i = u_i \\ 1, & \text{if } \bar{x}_i = v_i \\ \mathrm{sign}(d + A\bar{x} - \nabla g(\bar{x}))_i, & \text{if } u_i < \bar{x}_i < v_i \end{cases}, \quad (2.1)$$

$$\tilde{\bar{x}}_j : = \begin{cases} -1, & \text{if } \bar{x}_j = p_j \\ 1, & \text{if } \bar{x}_j = q_j \\ \mathrm{sign}(d + A\bar{x} - \nabla g(\bar{x}))_j, & \text{if } \bar{x}_j = p_j + 1, \dots, q_j - 1 \end{cases}, \quad (2.2)$$

$$b_{\bar{x}_i} : = \frac{2\tilde{\bar{x}}_i(d + A\bar{x} - \nabla g(\bar{x}))_i}{v_i - u_i} \quad (2.3)$$

$$b_{\bar{x}_j} : = 2\max\{\frac{\tilde{\bar{x}}_j(d + A\bar{x} - \nabla g(\bar{x}))_j}{1}, \frac{\tilde{\bar{x}}_j(d + A\bar{x} - \nabla g(\bar{x}))_j}{q_j - p_j}\}, \quad (2.4)$$

$$b_{\bar{x}} : = (b_{\bar{x}_1}, \dots, b_{\bar{x}_n})^T, \quad (2.5)$$

$$\mathcal{F}_i : = [u_i, v_i], , \quad (2.6)$$

$$\mathcal{F}_j : = \{p_j, p_j + 1, \dots, q_j\}, \quad (2.7)$$

where $\mathrm{sign}(d + A\bar{x} - \nabla g(\bar{x}))_k = \begin{cases} -1, & (d + A\bar{x} - \nabla g(\bar{x}))_k < 0 \\ 0, & (d + A\bar{x} - \nabla g(\bar{x}))_k = 0 \\ 1, & (d + A\bar{x} - \nabla g(\bar{x}))_k > 0 \end{cases}$.

Theorem 2.1. *Let $\bar{x} \in \mathcal{F}$, $A = (a_{ij})_{n\times n} \in S^n$. If $\bar{x}$ is a global minimizer of (MIP), then the following condition holds:*

$$[NC1] \qquad \mathrm{diag}(b_{\bar{x}}) \preceq \mathrm{diag}(\tilde{A}).$$

Proof. Let $\bar{x}$ be a global minimizer of problem (MIP). Then we have

$$\frac{1}{2}x^T Ax + d^T x - \frac{1}{2}\bar{x}^T A\bar{x} - d^T\bar{x} - g(x) + g(\bar{x}) \geq 0, \forall x \in \mathcal{F} \quad (2.8)$$

and we can get

$$\frac{1}{2}(x - \bar{x})^T A(x - \bar{x}) + (x - \bar{x})^T(d + A\bar{x} - \nabla g(\bar{x})) \geq 0, \forall x \in \mathcal{F}. \quad (2.9)$$

Let $x := (\bar{x}_1, \dots, \bar{x}_{k-1}, x_k, \bar{x}_{k+1}, \dots, \bar{x}_n)^T$, where $x \in \mathcal{F}, x_k \in \mathcal{F}_k$. Then by (2.9), we have

$$\frac{1}{2}(x_k - \bar{x}_k)^2 a_{kk} + (x_k - \bar{x}_k)(d + A\bar{x} - \nabla g(\bar{x}))_k \geq 0, \forall x_k \in \mathcal{F}_k, k = 1, \dots, n.$$

Now for any $i \in L, j \in M$ we consider the following cases:

1°. If $\bar{x}_i = u_i$, then

$$\begin{aligned}
& \frac{1}{2}(x_i-\bar{x}_i)^2 a_{ii} + (x_i-\bar{x}_i)(d+A\bar{x}-\nabla g(\bar{x}))_i \geq 0, \forall x_i \in \mathcal{F}_i \\
\Leftrightarrow\quad & \frac{1}{2}a_{ii}(x_i-\bar{x}_i) + (d+A\bar{x}-\nabla g(\bar{x}))_i \geq 0, \forall x_i \in (u_i, v_i] \\
\Leftrightarrow\quad & \begin{cases} (d+A\bar{x}-\nabla g(\bar{x}))_i \geq 0, & \text{if } a_{ii} \geq 0 \\ (d+A\bar{x}-\nabla g(\bar{x}))_i \geq -\frac{(v_i-u_i)a_{ii}}{2}, & \text{if } a_{ii} < 0 \end{cases} \\
\Leftrightarrow\quad & \tilde{\bar{x}}_i(d+A\bar{x}-\nabla g(\bar{x}))_i \leq \min\{0, \frac{(v_i-u_i)a_{ii}}{2}\}.
\end{aligned}$$

2°. If $\bar{x}_i = v_i$, then

$$\begin{aligned}
& \frac{1}{2}(x_i-\bar{x}_i)^2 a_{ii} + (x_i-\bar{x}_i)(d+A\bar{x}-\nabla g(\bar{x}))_i \geq 0, \forall x_i \in \mathcal{F}_i \\
\Leftrightarrow\quad & \frac{1}{2}a_{ii}(x_i-\bar{x}_i) + (d+A\bar{x}-\nabla g(\bar{x}))_i \leq 0, \forall x_i \in [u_i, v_i) \\
\Leftrightarrow\quad & \begin{cases} (d+A\bar{x}-\nabla g(\bar{x}))_i \leq 0, & \text{if } a_{ii} \geq 0 \\ (d+A\bar{x}-\nabla g(\bar{x}))_i \leq \frac{(v_i-u_i)a_{ii}}{2}, & \text{if } a_{ii} < 0 \end{cases} \\
\Leftrightarrow\quad & \tilde{\bar{x}}_i(d+A\bar{x}-\nabla g(\bar{x}))_i \leq \min\{0, \frac{(v_i-u_i)a_{ii}}{2}\}.
\end{aligned}$$

3°. If $u_i < \bar{x}_i < v_i$, then

$$\begin{aligned}
& \frac{1}{2}(x_i-\bar{x}_i)^2 a_{ii} + (x_i-\bar{x}_i)(d+A\bar{x}-\nabla g(\bar{x}))_i \geq 0, \forall x_i \in \mathcal{F}_i \\
\Leftrightarrow\quad & \begin{cases} \frac{1}{2}a_{ii}(x_i-\bar{x}_i) + (d+A\bar{x}-\nabla g(\bar{x}))_i \leq 0, & \forall x_i \in [u_i, \bar{x}_i) \\ \frac{1}{2}a_{ii}(x_i-\bar{x}_i) + (d+A\bar{x}-\nabla g(\bar{x}))_i \geq 0, & \forall x_i \in (\bar{x}_i v_i] \end{cases} \\
\Leftrightarrow\quad & (d+A\bar{x}-\nabla g(\bar{x}))_i = 0, a_{ii} \geq 0 \\
\Leftrightarrow\quad & \tilde{\bar{x}}_i(d+A\bar{x}-\nabla g(\bar{x}))_i \leq \min\{0, \frac{(v_i-u_i)a_{ii}}{2}\}.
\end{aligned}$$

4°. If $\bar{x}_j = p_j$, then

$$\begin{aligned}
& \frac{1}{2}(x_j-\bar{x}_j)^2 a_{jj} + (x_j-\bar{x}_j)(d+A\bar{x}-\nabla g(\bar{x}))_j \geq 0, \forall x_j \in \mathcal{F}_j \\
\Leftrightarrow\quad & \frac{1}{2}a_{jj}(x_j-\bar{x}_j) + (d+A\bar{x}-\nabla g(\bar{x}))_j \geq 0, \forall x_j \in \{p_j+1, \ldots, q_j\} \\
\Leftrightarrow\quad & \begin{cases} (d+A\bar{x}-\nabla g(\bar{x}))_j \geq -\frac{a_{jj}}{2}, & \text{if } a_{jj} \geq 0 \\ (d+A\bar{x}-\nabla g(\bar{x}))_j \geq -\frac{(q_j-p_j)a_{jj}}{2}, & \text{if } a_{jj} < 0 \end{cases} \\
\Leftrightarrow\quad & \tilde{\bar{x}}_j(d+A\bar{x}-\nabla g(\bar{x}))_j \leq \min\{\frac{a_{jj}}{2}, \frac{(q_j-p_j)a_{jj}}{2}\}.
\end{aligned}$$

5°. If $\bar{x}_j = q_j$, then

$$\begin{aligned} & \frac{1}{2}(x_j - \bar{x}_j)^2 a_{jj} + (x_j - \bar{x}_j)(d + A\bar{x} - \nabla g(\bar{x}))_j \geq 0, \forall x_j \in \mathcal{F}_j \\ \Leftrightarrow \quad & \frac{1}{2} a_{jj}(x_j - \bar{x}_j) + (d + A\bar{x} - \nabla g(\bar{x}))_j \leq 0, \forall x_j \in \{p_j, \ldots, q_j - 1\} \\ \Leftrightarrow \quad & \begin{cases} (d + A\bar{x} - \nabla g(\bar{x}))_j \leq \frac{a_{jj}}{2}, & \text{if } a_{jj} \geq 0 \\ (d + A\bar{x} - \nabla g(\bar{x}))_j \leq \frac{(q_j - p_j)a_{jj}}{2}, & \text{if } a_{jj} < 0 \end{cases} \\ \Leftrightarrow \quad & \tilde{\bar{x}}_j (d + A\bar{x} - \nabla g(\bar{x}))_j \leq \min\{\frac{a_{jj}}{2}, \frac{(q_j - p_j)a_{jj}}{2}\}. \end{aligned}$$

6°. If $\bar{x}_j \in \{p_j + 1, \ldots, q_j - 1\}$, then

$$\begin{aligned} & \frac{1}{2}(x_j - \bar{x}_j)^2 a_{jj} + (x_j - \bar{x}_j)(d + A\bar{x} - \nabla g(\bar{x}))_j \geq 0, \forall x_j \in \mathcal{F}_j \\ \Leftrightarrow \quad & \begin{cases} \frac{1}{2} a_{jj}(x_j - \bar{x}_j) + (d + A\bar{x} - \nabla g(\bar{x}))_j \geq 0, & \forall x_j \in \{p_j, \ldots, \bar{x}_j - 1\} \\ \frac{1}{2} a_{jj}(x_j - \bar{x}_j) + (d + A\bar{x} - \nabla g(\bar{x}))_j \leq 0, & \forall x_j \in \{\bar{x}_i + 1, \ldots, q_j\} \end{cases} \\ \Leftrightarrow \quad & -\frac{a_{jj}}{2} \leq (d + A\bar{x} - \nabla g(\bar{x}))_j \leq \frac{a_{jj}}{2}, \ a_{jj} > 0 \\ \Leftrightarrow \quad & \tilde{\bar{x}}_j (d + A\bar{x} - \nabla g(\bar{x}))_j \leq \min\{\frac{a_{jj}}{2}, \frac{(q_j - p_j)a_{jj}}{2}\}. \end{aligned}$$

Hence, if $\bar{x}$ is a global minimizer of (MIP), then condition $[NC1]$ holds. □

Example 2.1. *Consider the problem*

$$\begin{aligned} (EP1) \qquad & \min \quad f(x) := -{x_1}^2 + 4{x_2}^2 + x_1 x_2 - 2x_1 - \frac{1}{4}{x_4}^4 \\ & \text{s.t.} \quad x_1 \in [-1, 1] \\ & \qquad\ \ x_2 \in \{-1, 0, 1\}. \end{aligned}$$

Let

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 8 \end{pmatrix}, d = (-2, 0)^T, g(x) = \frac{1}{4}x_2^4,$$

then

$$f(x) = \frac{1}{2}x^T A x + d^T x - g(x).$$

For $\bar{x} = (\bar{x}_1, \bar{x}_2)^T \in \mathcal{F}$, *we have*

$$d + A\bar{x} = (-2 - 2\bar{x}_1 + \bar{x}_2, \bar{x}_1 + 8\bar{x}_2)^T, \tilde{a}_{11} = -2, \tilde{a}_{22} = 8,$$

$$\nabla g(\bar{x}) = (0, \bar{x}_2^3)^T, \nabla^2 g(\bar{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 3\bar{x}_2^2 \end{pmatrix}.$$

Hence the necessary condition $(NC1)$ *of Theorem 2.1 at* $\bar{x}$ *is*

$$\begin{cases} 2\tilde{\bar{x}}_1(-2 - 2\bar{x}_1 + \bar{x}_2)/2 \leq -2 \\ 2\max\{\tilde{\bar{x}}_2(\bar{x}_1 + 8\bar{x}_2 - 2\bar{x}_2^3), \tilde{\bar{x}}_2(\bar{x}_1 + 8\bar{x}_2 - 2\bar{x}_2^3)/2\} \leq 8 \end{cases} \tag{2.10}$$

We can verify that (2.10) holds at $\bar{x}$ if and only if $\bar{x} = (1,0)^T$. In fact $\bar{x} = (1,0)^T$ is the global minimizer of $(EP1)$, since $\bar{x} = (1,0)^T$ also satisfies the sufficient global optimality condition $[SC3]$ given by Corollary 3.2 in Section 3, see Example 3.1.

Corollary 2.1. *Let $\bar{x} \in \mathcal{F}$, $A = (a_{ij})_{n\times n} \in S^n$. If $L = \emptyset$, and $\bar{x}$ is a global minimizer of (MIP), then the following condition holds:*

$$[NC2] \qquad \mathrm{diag}(b_{\bar{x}}) \preceq \mathrm{diag}(A).$$

Obviously, condition $[NC2]$ extends the condition $[BNC1]$ given in [12] for binary constraints $\mathcal{F} := \prod_{i=1}^n \{0,1\}$, i.e., $[BNC1]$ is just $[NC2]$ for the special case: $L = \emptyset, p_j = 0, q_j = 1, \forall j \in M$.

Corollary 2.2. *Let $\bar{x} \in \mathcal{F}$, $A = (a_{ij})_{n\times n} \in S^n$. If $M = \emptyset$, and $\bar{x}$ is a global minimizer of (MIP), then the following condition holds:*

$$[NC3] \quad 2\tilde{\bar{x}}_i(a + A\bar{x} - \nabla g(\bar{x}))_i - \tilde{a}_{ii}(v_i - u_i) \leq 0$$

Note that the condition $[NC3]$ is just the necessary condition $[NC1]$ given in [12] for box constraints $\mathcal{F} := \prod_{i=1}^n [u_i, v_i]$. Moreover, if $g(x) = 0$, then the condition $[NC1]$ given in this paper extends the necessary condition $[NC1]$ given in [20] for quadratic programming problems with mixed variables and also extends the necessary condition $[NC1]$ given in [14] for quadratic programming problems with binary constraints.

3 Sufficient global optimality conditions for (MIP)

In this section, we derive some sufficient global optimality conditions for problem (MIP). Let $\bar{\mathcal{F}} := \{x \in R^n \mid x_i \in [u_i, v_i], x_j \in [p_j, q_j], i \in L, j \in M\}$.

Theorem 3.1. (**Global Sufficient Condition for** (MIP)) Let $\bar{x} \in \mathcal{F}$. If

$$[SC1] \qquad \begin{cases} b_{\bar{x}_i} \leq 0, \forall i \in L \\ \mathrm{diag}(b_{\bar{x}}) \preceq A - \nabla^2 g(x), \forall x \in \bar{\mathcal{F}} \end{cases}$$

hold, then $\bar{x}$ is a global minimizer of (MIP).

Proof. Let $G := \mathrm{diag}(b_{\bar{x}})$ and $\bar{G} := \mathrm{diag}(\bar{b}_{\bar{x}_1}, \ldots, \bar{b}_{\bar{x}_n})$, where

$$\bar{b}_{\bar{x}_i} = \begin{cases} \min\{0, b_{\bar{x}_i}\}, i \in L \\ b_{\bar{x}_i}, i \in M \end{cases},$$

and let

$$\begin{aligned} l(x) &:= \frac{1}{2}x^T Gx + (d - \nabla g(\bar{x}) + (A - G)\bar{x})^T x, \\ \varphi(x) &:= f(x) - l(x), \end{aligned}$$

then $\nabla^2\varphi(x) = A - \nabla^2 g(x) - G$ is positive semidefinite for each $x \in \bar{\mathcal{F}}$ by the second part of the $[SC1]$. Hence, $\varphi(x)$ is convex on $\bar{\mathcal{F}}$. By $\nabla\varphi(\bar{x}) = 0$, we know that $\bar{x}$ is a global minimizer of $\varphi(x)$ on $\bar{\mathcal{F}}$, i.e.,

$$f(x) - f(\bar{x}) \geq l(x) - l(\bar{x}), \forall x \in \bar{\mathcal{F}}.$$

It is easy to verify that

$$l(x) - l(\bar{x}) = \frac{1}{2}\sum_{k=1}^{n}(x_k - \bar{x}_k)^2 b_{\bar{x}_k} + \sum_{k=1}^{n}(d - \nabla g(\bar{x}) + A\bar{x})_k(x_k - \bar{x}_k), \forall x \in \mathcal{F}.$$

Thus $l(x) - l(\bar{x}) \geq 0, \forall x \in \mathcal{F}$ is equivalent to

$$\frac{1}{2}(x_k - \bar{x}_k)^2 b_{\bar{x}_k} + (d - \nabla g(\bar{x}) + A\bar{x})_k(x_k - \bar{x}_k) \geq 0, \forall x_k \in \mathcal{F}_k, k = 1, \ldots, n.$$

By the proof of Theorem 2.1, we know that the above inequalities holds if and only if

$$\mathrm{diag}(b_{\bar{x}}) \preceq \bar{G},$$

i.e. $b_{\bar{x}_i} \leq 0, \forall i \in L$, which is exactly the first part of the $[SC1]$.

Therefore, if $[SC1]$ holds, then $\bar{x}$ is a global minimizer of (MIP). $\square$

Corollary 3.1. *Let $\bar{x} \in \mathcal{F}$, if $g(x) = 0$ and if*

$$[SC2] \qquad \left\{ \begin{array}{l} b_{\bar{x}_i} \leq 0, i \in L \\ \mathrm{diag}(b_{\bar{x}}) \preceq A \end{array} \right.$$

hold, then $\bar{x}$ is a global minimizer of (MIP).

Proof. It can be obtained directly from Theorem 3.1. $\square$

Note that condition $[SC2]$ extends the sufficient condition $[SC1]$ for the quadratic programming problems with mixed variables given in [20].

We denote the Hessian of g at x by $\nabla^2 g(x) = (g_{kt}(x))_{n\times n}$ and define for $k = 1, 2, \ldots, n$,

$\alpha_k = \min\{a_{kk} - g_{kk}(z) - \sum_{t\neq k,t=1}^{n} \mid a_{kt} - g_{kt}(z) \mid : z \in \bar{\mathcal{F}}\}$;

$Q = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$;

$\tilde{\alpha}_k = \min\{0, \alpha_k\}$, for $k \in L$ and $\tilde{\alpha}_k = \alpha_k$, for $k \in M$.

Corollary 3.2. *Let $\bar{x} \in \mathcal{F}$, $A = (a_{ij})_{n\times n} \in S^n$, if the diagonal matrix $\tilde{Q} := \mathrm{diag}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ satisfies*

$$[SC3] \qquad \mathrm{diag}(b_{\bar{x}}) \preceq \tilde{Q},$$

then $\bar{x}$ is a global minimizer of problem (MIP).

Proof. If $\operatorname{diag}(b_{\bar{x}}) \preceq \tilde{Q}$, then we have $b_{\bar{x}_i} \leq 0, \forall i \in L$ and $\operatorname{diag}(b_{\bar{x}}) \preceq Q$. By the definition of Q, we have that

$$\begin{aligned} & A - \nabla^2 g(z) - Q \\ = & \begin{bmatrix} a_{11} - g_{11}(z) - \alpha_1 & a_{12} - g_{12}(z) & \dots & a_{1n} - g_{1n}(z) \\ a_{21} - g_{21}(z) & a_{22} - g_{22}(z) - \alpha_2 & \dots & a_{2n} - g_{1n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} - g_{n1}(z) & a_{n2} - g_{n2}(z) & \dots & a_{nn} - g_{nn}(z) - \alpha_n \end{bmatrix}. \end{aligned}$$

By the definition of α_k we know that

$$a_{kk} - g_{kk}(z) - \alpha_k \geq \sum_{t=1, t\neq k}^{n} \mid a_{kt} - g_{kt}(z) \mid \geq 0$$

for any $z \in \bar{\mathcal{F}}$ and each $k = 1, 2, \dots, n$. Then for any $z \in \bar{\mathcal{F}}$ the matrix $A - \nabla^2 g(z) - Q$ is diagonally dominant with nonnegative diagonal elements. It follows $A - \nabla^2 g(x) - Q \succeq 0, \forall x \in \bar{\mathcal{F}}$. Hence $\operatorname{diag}(b_{\bar{x}}) \preceq A - \nabla^2 g(x), \forall x \in \bar{\mathcal{F}}$. By Theorem 3.1, we know that if $[SC3]$ holds, then $\bar{x}$ is a global minimizer of problem (MIP). □

Example 3.1. *Consider the problem* $(EP1)$ *in Example 2.1.*

$$\begin{aligned} (EP1) \qquad & \min \quad f(x) := -{x_1}^2 + 4{x_2}^2 + x_1 x_2 - 2x_1 - \frac{1}{4}{x_2}^4 \\ & \text{s.t.} \quad x_1 \in [-1, 1] \\ & \qquad\ x_2 \in \{-1, 0, 1\} \end{aligned}$$

We can easily verify that $(SC1)$ *holds at* $\bar{x} = (1, 0)^T$, *thus it is a global minimizer of* $(EP1)$. *In fact, we can get* $\alpha_1 = -3$, $\alpha_2 = 4$, *hence* $\tilde{\alpha}_1 = -3, \tilde{\alpha}_2 = 4$. *Then the sufficient condition* $(SC1)$ *of Theorem 3.1 at* $\bar{x}$ *is*

$$\begin{cases} 2\tilde{\bar{x}}_1(-2 - 2\bar{x}_1 + \bar{x}_2)/2 \leq -3 \\ 2\max\{\tilde{\bar{x}}_2(\bar{x}_1 + 8\bar{x}_2 - 2\bar{x}_2^3), \tilde{\bar{x}}_2(\bar{x}_1 + 8\bar{x}_2 - 2\bar{x}_2^3)/2\} \leq 4 \end{cases} \tag{3.1}$$

which holds at $\bar{x}$ *if and only if* $\bar{x} = (1, 0)^T$.

Corollary 3.3. *Let* $\bar{x} \in \mathcal{F}$. *If* $M = \emptyset$ *and for each* $k = 1, 2, \dots, n$

$$[SC4] \quad \tilde{\bar{x}}_k(a + A\bar{x} - \nabla g(\bar{x}))_k - \frac{1}{2}\tilde{\alpha}_k(v_k - u_k) \leq 0$$

holds, then $\bar{x}$ *is a global minimizer of problem* (MIP).

Note that the condition $[SC4]$ is just the condition $[SC1]$ given in [12] for box constraints.

Let $\lambda_k(x), k = 1, 2, \dots, n$, be the eigenvalues of $(A - \nabla^2 g(x))$. We now derive a sufficient condition for global optimality of (MIP) in terms of eigenvalues.

Corollary 3.4. *Let* $\bar{x} \in \mathcal{F}$, $\mu := \min_{x \in \bar{\mathcal{F}}} \min_{k=1,2,\ldots,n} \lambda_k(x)$ *and let*

$$\tilde{\mu}_i := \min\{0, \mu\}, \forall i \in L, \ \tilde{\mu}_j := \mu, \forall j \in M.$$

If

$$[SC5] \qquad b_{\bar{x}_k} \le \tilde{\mu}_k \text{ for each } k = 1, 2, \ldots, n,$$

then $\bar{x}$ *is a global minimizer of problem* (MIP).

Proof. Let $x \in \mathcal{F}$ and $l(x) = \frac{1}{2}x^T G x + (d - \nabla g(\bar{x}) + (A - G)\bar{x})^T x$, where $G = \mu I$ with I the $n \times n$ identity matrix. From the definition of μ, we know that the eigenvalues of $A - \nabla^2 g(x) - G$ are nonnegative for any $x \in \bar{\mathcal{F}}$, hence $A - \nabla^2 g(x) - G$ is positive semidefinite. Then as in the proof of Theorem 3.1, we have that $\bar{x}$ is a global minimizer of problem (MIP). $\square$

Example 3.2. *Consider the problem*

$$
\begin{aligned}
(EP2) \qquad \min \quad & f(x) := \frac{11}{2}x_1^2 + {x_2}^2 + \frac{5}{2}{x_3}^2 + 2x_1x_2 - 8x_1x_3 + 10x_2x_3 \\
& \quad + x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 \in [3, 9] \\
& x_2 \in [10, 16] \\
& x_3 \in \{0, 1, 2, 3\}.
\end{aligned}
$$

Let

$$A = \begin{pmatrix} 11 & 2 & -8 \\ 2 & 2 & 10 \\ -8 & 10 & 5 \end{pmatrix}, \qquad d = (1, 2, 3)^T,$$

then $f(x) = \frac{1}{2}x^T A x + d^T x$. *Note that* $g(x) = 0$ *in this example. For* $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^T \in \mathcal{F}$, *we have*

$$
\begin{aligned}
& a + A\bar{x} \\
= \ & (1 + 11\bar{x}_1 + 2\bar{x}_2 - 8\bar{x}_3, 2 + 2\bar{x}_1 + 2\bar{x}_2 + 10\bar{x}_3, 3 - 8\bar{x}_1 + 10\bar{x}_2 + 5\bar{x}_3)^T
\end{aligned}
$$

and $\tilde{\mu} = -9$. *It can be verified that* $[SC4]$ *holds at* $\bar{x} = (3, 10, 0)^T$, *thus it is a global minimizer of* $(EP2)$. *In fact the sufficient condition* $[SC3]$ *of Theorem 1 at* $\bar{x}$ *is*

$$
\begin{cases}
2\tilde{\bar{x}}_1(1 + 11\bar{x}_1 + 2\bar{x}_2 - 8\bar{x}_3)/6 \le -9 \\
2\tilde{\bar{x}}_2(2 + 2\bar{x}_1 + 2\bar{x}_2 + 10\bar{x}_3)/6 \le -9 \\
2\max\{\tilde{\bar{x}}_3(3 - 8\bar{x}_1 + 10\bar{x}_2 + 5\bar{x}_3), \tilde{\bar{x}}_3(3 - 8\bar{x}_1 + 10\bar{x}_2 + 5\bar{x}_3)/3\} \le -9
\end{cases}
$$

We can easily check that the above condition holds at $\bar{x}$ *if and only if* $\bar{x} = (3, 10, 0)^T$.

Remark 3.1. *Note that if* $g(x) = 0$ *and* A *is a diagonal matrix, then the necessary condition* $[NC1]$ *is equivalent to the sufficient condition* $[SC1]$.

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5 References

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