Impulsive Synchronization of State Delayed Discrete Complex Networks with Switching Topology

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Abstract. In this paper, global exponential synchronization of a class of discrete delayed complex networks with switching topology is investigated by using Lyapunov-Ruzimiki method. The impulsive scheme is designed to work at the time instant of switching occurrence. A time-varying delay dependent criterion for impulsive synchronization is given to ensure the delayed discrete complex networks switching topology tending to a synchronous state. Furthermore, a numerical simulation is given to illustrate the effectiveness of main results.

Keywords: Complex networks, impulsive synchronization.

1 Introduction

It has long been understood that many physical, social, biological, and technological networks are modeled by a graph with non-trivial topological features. In this model, every node is an individual element of the whole system with certain pattern of connections, in which connections between each pair of nodes are neither entirely regular nor entirely random[1],[2],[3].Secure communication[4],[5],parallel image processing[6] and chemical reaction implemented by coupled chaotic systems have been an active research field during the last two decades. As a consequence, theory and methods for synchronization of different families of complex networks have been extensively studied by many researchers(such as,[7]–[10]) and references therein). The improvement on different regimes of synchronization of discrete complex networks are abstracted from papers authored by [9]. Some general cases of synchronization of complex networks with switching topology can be found in the literatures of [14]. Adaptive synchronization, impulsive synchronization scheme and pining control synchronization have been considered by authors in[14]-[16]. Impulsive control has been successfully used to stabilize and synchronize dynamical systems, for examples, [11]-[13]. And impulsive control technique could be an efficient method when a discrete change behavior is needed. The adjustment interest rate could agree with that. In this paper, we proposed an impulsive

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synchronization scheme for a state delayed discrete complex networks with switching topology. For this control scheme, we consider that the impulsive control signal is designed to be input into all of nodes.

The paper is organized as follows. Section 2 presents some mathematical preliminaries needed in this work, and a generalized mathematical model for delayed discrete complex networks with switching topology. The main theorem for global synchronization of this type of discrete complex networks are then given in Section 3. In Section 4, a small-world networks with 3 sub-networks involving 30 nodes is constructed to illustrate the effectiveness of our result. Section 5 concludes the paper.

2 Preliminary

First, we need to introduce some notations and definitions for the sake of exploring our main results. Let$\| \bullet \|$ denote the Euclidean norm; $\mathbb{R}^n$ denotes the n–dimensional Euclidean space,the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, and, for certain positive integer τ, we let $\mathbb{Z}_{-\tau} = \{-\tau, -\tau + 1, \ldots, 0\}$. The family of N linearly coupled discrete complex networks, consisting of time delay with respect to its system state and the switched topology, can be described by

$$x_i(k+1) = Ax_i(k) + Bf(x_i(k)) + Df(x_i(k-\tau(k))) + I(k) \\ + \sum_{j=1}^{N} c_{ij,\sigma(k)} \Gamma x_j(k-\tau(k)), \qquad i = 1, 2, ..., N, \qquad k \in \mathbb{N} \tag{1}$$

$$x_{ik_0} = \phi(\theta), \qquad \theta \in \mathbb{Z}_{-\tau}, \tag{2}$$

where $x_i(k) = (x_{i,1}, x_{i,2}, ..., x_{i,n}) \in \mathbb{R}^n$ represents the state vector of the i–th node at every instant of time k and n denotes the number of nodes affiliated to each sub-networks. $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ are known real matrices. $\boldsymbol{f(x_i(k))} = (\boldsymbol{f_1(x_{i,1}(k))}, \boldsymbol{f_2(x_{i,2}(k))}, ..., \boldsymbol{f_n(x_{i,n}(k))})^T$ and $\boldsymbol{f(\bullet)} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth nonlinear vector-valued functions. $\boldsymbol{I(k)} = (I_i(k), I_2(k), ..., I_n(k))^T$ is a n-dimensional vector from external input. $\boldsymbol{S}$ is a finite index set of r elements: $\boldsymbol{S} = \{s_1, s_2, ..., s_r\}$. Let the switching function be denoted by $\sigma(k) : \mathbb{N} \longrightarrow \boldsymbol{S}$, which is the switching signal from sudden changing of system dynamic without jumps in the state $\boldsymbol{x}$ at any switching instant. Specifically, we consider that it is a piecewise constant function and continuous from the right, indicating certain active subsystem regime, at every instant of time k the index $\sigma(k) = s_k \in \boldsymbol{S}$; meanwhile, let the switching instants of σ be denoted by $k_{m,x}(m = 1, 2, ...)$ and let $k_{0,x} := 0$ (without chattering). $C_{s_k} = (c_{ij,s_k}) \in \mathbb{Z}^{N\times N}$ represents the outer coupling configuration symmetric matrix defined as follows: for each active subsystem regime s_k , if there is a connection from node j to node i ($j \neq i$), then $c_{ij,s_k} = c_{ji,s_k} > 0$; otherwise $c_{ij,s_k} = c_{ji,s_k} = 0$. Assume that

$$c_{ii,s_k} = -\sum_{j=1,j\neq i}^{N} c_{ij,s_k} = -\sum_{j=1,j\neq i}^{N} c_{ji,s_k}, \qquad i \in N, \qquad s_k \in \boldsymbol{S}. \tag{3}$$

The notation represents $\Gamma \in \mathbb{R}^{n\times n}$ the diagonal inner coupling matrix between two connected nodes. $\tau(k)$ is a time-varying delay with respect to each instant of time k and satisfies $\tau(k) \in \mathbb{Z}_{-\tau}$. $\phi(\bullet) : \mathbb{Z}_{-\tau} \longrightarrow \mathbb{R}^{n\times N}$ is continuous everywhere except at a finite number of points. The norm of $\phi(\bullet)$ is defined by $\|\phi(\theta)\|_\tau = \sup_{\theta\in \mathbb{Z}_{-\tau}}\{\|\phi(\theta)\|\}$. We assume that at each active subsystem regime, the existence and uniqueness of a solution of system (1) for every initial condition and piecewise/continuous input can be guaranteed. In order to design an impulsive control scheme to synchronize system (1), we consider the evolutionary state is abruptly jumping at every impulsive instant of time k_u from its open-loop state , which can be formularized by

$$\Delta x_i(k_{m,u}) = J_u x_i^*(k_{m,u}), \qquad m = 1, 2, ...\mathbb{N} \tag{4}$$

where $x_i^*(k_{m,u})$ stands for the primal state at time instant $k_{m,u}$ without impulsive jump. As usual, every impulsive instant of time $k_{l,u}$ satisfies $0 = k_{0,u} < k_{1,u} < k_{2,u} < \cdots < k_{m,u} < k_{m+1,u} < \cdots$ and $\lim_{m\to\infty} k_{m,u} = \infty$;$J_u : \mathbb{R}^n \to \mathbb{R}^n (m = 1, 2, ...)$ represents the impulsive jump strength. Therefore, at every impulsive instant of time $k_{m,u}$, the coupled states $x_i(k)-x_j(k)$ between connected node i and j can be described by

$$x_i(k_{m,u}) - x_j(k_{m,u}) = x_i^*(k_{m,u}) - x_j^*(k_{m,u}) + J_u[x_i^*(k_{m,u}) - x_j^*(k_{m,u})]. \tag{5}$$

Intuitively, a family of impulsive controller can be designed as

$$U_i(k, x_i(k)) = \sum_{u=1}^{\infty} \delta(k - k_{m,u}) J_u(x_i^*(k_{m,u})), \qquad m = 1, 2, ...\mathbb{N}, \tag{6}$$

where $U_i(k, x_i(k))$ represents a class of impulsive controller at each instant of time $k_{m,u}$; $\delta(\bullet)$ denotes the Dirac discrete-time function.

Assumption 1. *For each nonlinear function* $\boldsymbol{f_i(\bullet)}$ *(i=1,2,...,n),suppose that it is globally Lipschitz continues function and satisfies*

$$\|f_i(x_1) - f_i(x_2)\| \le \hat{l}_i\|x_1 - x_2\|, \quad i = 1, 2, ..., n, for\, any\, x_1, x_2 \in \mathbb{R}, \tag{7}$$

where $\hat{l}_i$ *is certain positive constant.*

Definition 1. *The system of the impulsive controlled discrete complex networks (7) is said to be globally exponentially synchronized, if for any initial condition* $\phi(\bullet) : \mathbb{Z}_{-\tau} \to \mathbb{R}^{n\times N}$, *and there exist two positive constants* λ *and* $M_0 \ge 1$ *such that*

$$\|x_i(k) - x_j(k)\| \le M_0 e^{-\lambda(k-k_0)}, \quad 1 \le i \le j \le N \tag{8}$$

holds for all $k > k_0$.

Lemma 1. *Let* $\boldsymbol{W} = (w_{ij})_{N\times N}$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$,$\boldsymbol{x} = (x_1, x_2, ..., x_N)^T$ *and* $\boldsymbol{y} = (y_1, y_2, ..., y_N)^T$ *with* x_k,$y_k \in \mathbb{R}^n$ *(k=1,2,...,N). If* $\boldsymbol{W} = \boldsymbol{W}^T$ *and each row sum of* $\boldsymbol{W}$ *is zero, then*

$$\boldsymbol{x^T}(\boldsymbol{W} \otimes \boldsymbol{P})\boldsymbol{y} = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} w_{ij}(x_i - x_j)^T P(y_i - y_j). \tag{9}$$

3 Main Results

When the impulsive controller can be functioning simultaneously at the state of discrete complex networks' switching signal, the equivalent impulsive controlled system is rewritten by using the matrix Kronecker product

$$\begin{aligned} x(k+1) &= (I_N \otimes A)x(k) + (I_N \otimes B)F(x(k)) + (I_N \otimes D)F(x(k-\tau)) \\ &\quad + I(k) + (C_{\sigma(k)} \otimes \Gamma)x(k-\tau), \quad k \neq k_{m,u} \end{aligned} \tag{10}$$

$$x(k_{m,u}) = [I_N \otimes (I_N + J_u(k_{m,u}))]x(k_{m,u}-1), \tag{11}$$

for any k,$m \in \mathbb{N}$.

Theorem 2. *Under* ***Assumption 1.*** *the impulsive controlled complex network(12) is exponentially synchronized if there exists certain positive integer m_τ, positive scalars $\varepsilon_{\sigma(k)}$,$p_{\sigma(k)}$, $q_{\sigma(k)}$ and positive-definite matrices $P_{\sigma(k)} \in \mathbb{R}^{n\times n}$,$Q_{l,\sigma(k)} \in \mathbb{R}^{n\times n}$ (l=1,2,...6) such that*

(i) Given $\mu \geq 1$ and $P_{\sigma(k_{m,x})} \leq \mu P_{\sigma(k_{m+1,x})}$, for any $k \in [k_{m,x}, k_{m+1,x}-1]$ in corresponding sub-state $\sigma(k_{m,x})$,

$$p_{\sigma(k_{m,x})} - \left[\frac{\lambda_{max}(\Pi_{\sigma(k_{m,x})})}{\lambda_{min}(P^{-1}_{\sigma(k_{m,x})})} + \mu q_{\sigma(k_{m,x})} \frac{\lambda_{max}(\Omega_{\sigma(k_{m,x})})}{\lambda_{min}(P^{-1}_{\sigma(k_{m-m_\tau,x})})}\right] \geq 0, \tag{12}$$

where

$$\begin{aligned} \Pi_{\sigma(k_{m,x})} &= A^T P_{\sigma(k_{m,x})} A + L^T B^T P_{\sigma(k_{m,x})} BL + A^T Q_{1,\sigma(k_{m,x})} A \\ &\quad + L^T B^T P^T_{\sigma(k_{m,x})} Q_{1,\sigma(k_{m,x})} P_{\sigma(k_{m,x})} BL + A^T Q^{-1}_{2,\sigma(k_{m,x})} A \\ &\quad - NC_{\sigma(k_{m,x})} A^T Q^{-1}_{3,\sigma(k_{m,x})} ANC_{\sigma(k_{m,x})} + L^T B^T Q_{4,\sigma(k_{m,x})} BL \\ &\quad - L^T NC_{\sigma(k_{m,x})} B^T Q_{5,\sigma(k_{m,x})} BNC_{\sigma(k_{m,x})} L, \\ \Omega_{\sigma(k_{m,x})} &= L^T D^T P_{\sigma(k_{m,x})} DL - NC^2_{\sigma(k_{m,x})} \Gamma^T P_{\sigma(k_{m,x})} \Gamma \\ &\quad + L^T D^T P^T_{\sigma(k_{m,x})} Q_{2,\sigma(k_{m,x})} P_{\sigma(k_{m,x})} DL \\ &\quad - \Gamma^T P^T_{\sigma(k_{m,x})} Q_{3,\sigma(k_{m,x})} P_{\sigma(k_{m,x})} \Gamma \\ &\quad + L^T D^T P^T_{\sigma(k_{m,x})} Q_{4,\sigma(k_{m,x})} P_{\sigma(k_{m,x})} DL \\ &\quad - \Gamma^T P^T_{\sigma(k_{m,x})} Q_{5,\sigma(k_{m,x})} P_{\sigma(k_{m,x})} \Gamma \\ &\quad - L^T NC_{\sigma(k_{m,x})} D^T Q^{-1}_{6,\sigma(k_{m,x})} DNC_{\sigma(k_{m,x})} L \\ &\quad - \Gamma^T P^T_{\sigma(k_{m,x})} Q_{6,\sigma(k_{m,x})} P_{\sigma(k_{m,x})} \Gamma. \end{aligned}$$

(ii) $\mu \lambda^2_{max}(1 + J_u(k_{m,x})) < e^{\varepsilon_{\sigma(k_{m,x})}(k_{m+1,x}-k_{m,x})}$.

(iii) $q_{\sigma(k_{m,x})} \geq e^{\varepsilon_{\sigma(k_{m,x})}(k_{m+1,x}-k_{m,x}+1)+\sum_{i=0}^{m_\tau-1} \varepsilon_{k_{m-i,x}}(k_{m+1-i,x}-k_{m-i,x})}$, *where* $m_\tau = \lceil \frac{\tau}{\inf\{k_{m,x}-k_{m-1,x}\}} \rceil$.

Proof. Consider the following Lyapunov function:

$$V(k) = x^T(k)(W \otimes P_{\sigma(k)})x(k), \tag{13}$$

for any $k \in [k_{m,x}, k_{m+1,x} - 1]$, m=1,2,...
where

$$W = \begin{bmatrix} N-1 & -1 & ... & -1 \\ -1 & N-1 & ... & -1 \\ ... & ... & ... & ... \\ -1 & -1 & ... & N-1 \end{bmatrix}$$

.
One observes that for the case $k \in \mathbb{Z}_\tau$,

$$\begin{aligned} V(\theta) &= x^T(\theta)(W \otimes P_{\sigma(0)})x(\theta) \\ &= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (x_i(\theta) - x_j(\theta))^T P_{\sigma(0)}(x_i(\theta) - x_j(\theta)) \\ &= \varphi(\|\phi(\theta)\|_\tau^2). \end{aligned} \tag{14}$$

Choose $M \geq 1$, such that

$$\begin{aligned} \varphi(\|\phi(\theta)\|_\tau^2) &\leq M\varphi(\|\phi(\theta)\|_\tau^2)e^{-\lambda(k_{1,x}-k_{0,x})}e^{-\varepsilon_{\sigma(k_{0,x})}(k_{1,x}-k_{0,x})} \\ &< q_{\sigma(k_{0,x})}\varphi(\|\phi(\theta)\|_\tau^2). \end{aligned} \tag{15}$$

By claiming that

$$V(k) \leq M\varphi(\|\phi(\theta)\|_\tau^2)e^{-\lambda(k_{m,x}-k_{0,x})}, k \in [k_{m-1,x}, k_{m,x} - 1], \, m \in N. \tag{16}$$

And by virtue of mathematical induction, the claim (16) is true for each $k \in \mathbb{N}$. In view of (16) and **Definition 1.**, it can be obtained that

$$V(k) \leq M\varphi(\|\phi(\theta)\|_\tau^2)e^{-\lambda(k-k_{0,x})}, \; k \in [k_{m-1,x}, k_{m,x} - 1], \, m \in \mathbb{N}. \tag{17}$$

For any $k \in \mathbb{N}$,

$$\begin{aligned} &\min\left\{\lambda_{\min}(P_{\sigma(k)})\right\} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \|x_i(k) - x_j(k)\|^2 \\ &\leq \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (x_i(k) - x_j(k))^T P_{\sigma(k)}(x_i(k) - x_j(k)) \\ &\leq M\varphi(\|\phi(\theta)\|_\tau^2)e^{-\lambda(k-k_{0,x})}. \end{aligned} \tag{18}$$

Therefor, for any $k \in \mathbb{N}$,

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \|x_i(k) - x_j(k)\|^2 \leq \min\left\{\lambda_{\min}^{-1}(P_{\sigma(k)})\right\} M\varphi(\|\phi(\theta)\|_\tau^2)e^{-\lambda(k-k_{0,x})}, \tag{19}$$

which implies

$$\|x_i(k) - x_j(k)\| \leq M_0 e^{-\lambda(k-k_{0,x})}, \qquad 1 \leq i \leq j \leq N. \tag{20}$$

Therefore, the discrete complex networks (1) is globally exponentially synchronized under impulsive control. The proof is thus completed. □

Remark 1. We consider a multiple Lyapunov function for each sub-networks with arbitrarily fast switching signal in our theorem, which results in a less conservation criterion.

Remark 2. In the switched Lyapunov function, $p_{\sigma(k)}$ gives an upper bound on the estimation of divergence rate for each running sub-networks. By condition (ii) of **Theorem 1.**, the impulsive control gain is designed to compensate divergence from system itself and deteriorating effect from arbitrarily fast switching. If some certain sub-networks could be self-synchronizing, the impulsive control gain only need to compensate deteriorating effect.

4 Example and Numerical Simulations

This section presents a typical example to illustrate our result. Let us consider a 2-dimensional discrete chaotic neural networks is given as the isolated node of a small world network with 30 nodes,

$$x(k+1) = Ax(k) + Bf(x(k)) + Df(x(k-\tau(k))) + I(k), \tag{21}$$

where $x(k) = (x_1(k), x_2(k))^T$,$f(x(k)) = (tanh(x_1(k)), tanh(x_2(k)))^T$,$I(k) = (0,0)^(T)$,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -0.11 \\ -5 & 3.2 \end{bmatrix}, D = \begin{bmatrix} -1.6 & -0.1 \\ -0.18 & -2.4 \end{bmatrix},$$

and $\tau(k) = \frac{e^k}{1+e^(k)}$.Obviously, Lipschitz constants can be 1 here.Consider a small-world model involved with three different subsystem. The trajectory of each single node of this small-world model has random initial values in the interval [0.3,3] and [-3,-0.3], respectively. Given a switching signal $\sigma(t)$ in Fig1.(a), we have the state response of the switched complex networks, see Fig.1(b). From **Theorem 1**, for each sub-network, we have $J_1 = \begin{bmatrix} -0.6667 & 0 \\ 0 & -0.667 \end{bmatrix}$. $J_2 = \begin{bmatrix} -0.4079 & 0 \\ 0 & -0.4079 \end{bmatrix}$. $J_3 = \begin{bmatrix} -1.1576 & 0 \\ 0 & -1.1576 \end{bmatrix}$.

It is shown that all of nodes in each sub-networks could not reach into a synchronous state without a control. Indeed, the switched signal plays a role of deterioration accelerator to diverge the synchronous state,shown in Fig.1(b). Once the feasible impulsive controller is placed on discrete complex networks with topology switching, such complex networks would be synchronized, see Fig.1(c).

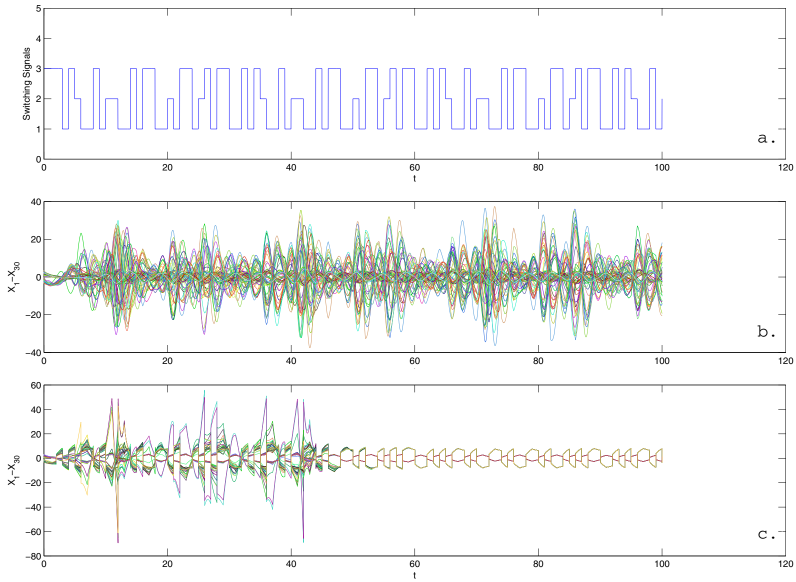

Fig. 1. (a) The switching signal σ(t); (b)The state responses of the switched system; (c)The synchronized state under impulsive control

5 Conclusion

In this paper, we have investigated impulsive synchronization control of a discrete delayed complex networks with switching topology by using Lyapunov Ruzimiki method. A time-varying delay dependent criteria for exponential synchronization is presented guarantee the switched discrete complex networks tending to be a synchronous manifold. It is worthwhile to see time-varying delay can take any value, even larger than any dwell time of a sub-networks.Futhermore, a numerical example with 3 sub-networks are presented by using the impulsive control technique.

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