Canonical Duality Theory and Algorithm for Solving Challenging Problems in Network Optimisation

Ning Ruan[1,2] and David Yang Gao[1]

[1] School of Sciences, Information Technology and Engineering, University of Ballarat, Ballarat, VIC 3353, Australia

[2] Department of Mathematics and Statistics, Curtin University, Perth, WA 6845, Australia

n.ruan@ballarat.edu.au

Abstract. This paper presents a canonical dual approach for solving a general nonconvex problem in network optimization. Three challenging problems, sensor network location, traveling salesman problem, and scheduling problem are listed to illustrate the applications of the proposed method. It is shown that by the canonical duality, these nonconvex and integer optimization problems are equivalent to unified concave maximization problem over a convex set and hence can be solved efficiently by existing optimization techniques.

Keywords: Global Optimization, Canonical Duality, Wireless Network, Traveling Salesman Problem, Scheduling Problem.

1 Introduction

Let us consider the following nonconvex (primal) optimization problem that arises in a wide range of applications:

$$(\mathcal{P}): \ \min \left\{ P(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + W(\mathbf{x}) \ : \ \mathbf{x} \in \mathcal{X}_a \right\}, \tag{1}$$

where $Q = \{q_{ij}\} \in \mathbb{R}^{n\times n}$ is a given symmetric matrix, $\mathbf{f} \in \mathbb{R}^n$ is a given vector, $\mathcal{X}_a \subset \mathbb{R}^n$ is a convex open set, and $W(\mathbf{x})$ is a nonconvex function. Note that in the context of constrained optimization problems, the function $W(\mathbf{x})$ could be simply defined as a (nonsmooth) indicator function of a feasible space $\mathcal{X}_c$:

$$W(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{X}_c \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

If $\mathcal{X}_a = \mathbb{R}^n$ and $\mathcal{X}_c = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{A}\mathbf{x} \le \mathbf{b},\ \ \mathbf{l} \le \mathbf{x} \le \mathbf{u}\}$, where $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a matrix, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{l}, \mathbf{u} \in \mathbb{R}^n$ are given vectors, then Problem $(\mathcal{P})$ reduces to a linearly constrained quadratic program:

$$(\mathcal{P}_q): \quad \min \left\{ P(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} \ : \ \ \mathbf{A}\mathbf{x} \le \mathbf{b},\ \mathbf{l} \le \mathbf{x} \le \mathbf{u},\ \mathbf{x} \in \mathbb{R}^n \right\}. \tag{3}$$

T. Huang et al. (Eds.): ICONIP 2012, Part III, LNCS 7665, pp. 702–709, 2012.

It is well-known that even this most simple problem is NP-hard if Q is indefinite and considerable efforts have been devoted to solve this type of problems.

The key idea of the canonical dual transformation is to choose a certain geometrically reasonable measure (operator) $\boldsymbol{\varepsilon} = \Lambda(\mathbf{x}) : \mathcal{X}_a \subset \mathbb{R}^n \to \mathcal{E}_a \subset \mathbb{R}^m$ such that the nonconvex functional $W(\mathbf{x})$ can be recast by adopting the canonical form $W(\mathbf{x}) = V(\Lambda(\mathbf{x}))$. Thus, the primal problem $(\mathcal{P})$ can be written in the following canonical form:

$$\min \{ P(\mathbf{x}) = V(\Lambda(\mathbf{x})) - U(\mathbf{x}) \ : \ \ \mathbf{x} \in \mathcal{X}_a \}, \tag{4}$$

where $U(\mathbf{x}) = -\frac{1}{2}\mathbf{x}^T Q \mathbf{x} + \mathbf{f}^T \mathbf{x}$. For the given canonical function $V(\boldsymbol{\varepsilon})$, its Legendre conjugate $V^*(\boldsymbol{\varsigma})$ can be defined uniquely by the Legendre transformation, and the following canonical duality relations hold:

$$\boldsymbol{\varsigma} = \nabla V(\boldsymbol{\varepsilon}) \ \ \Leftrightarrow \ \ \boldsymbol{\varepsilon} = \nabla V^*(\boldsymbol{\varsigma}) \ \ \Leftrightarrow \ \ V(\boldsymbol{\varepsilon}) + V^*(\boldsymbol{\varsigma}) = \boldsymbol{\varepsilon}^T \boldsymbol{\varsigma}. \tag{5}$$

In finite deformation mechanics, the one-to-one canonical duality relation $\boldsymbol{\varsigma} = \nabla V(\boldsymbol{\varepsilon})$ is called the canonical constitutive law [1]. By this canonical duality, the nonconvex term $W(\mathbf{x}) = V(\Lambda(\mathbf{x}))$ in the problem $(\mathcal{P})$ can be replaced by $\Lambda(\mathbf{x})^T \boldsymbol{\varsigma} - V^*(\boldsymbol{\varsigma})$ such that the nonconvex function $P(\mathbf{x})$ is reformulated as

$$\Xi(\mathbf{x}, \boldsymbol{\varsigma}) = \Lambda(\mathbf{x})^T \boldsymbol{\varsigma} - V^*(\boldsymbol{\varsigma}) - U(\mathbf{x}), \tag{6}$$

which is the so-called *total complementary function* introduced by Gao and Strang in nonconvex mechanics [1]. By using this total complementary function, the canonical dual function can be formulated as

$$P^d(\boldsymbol{\varsigma}) = \text{sta}\{ \Xi(\mathbf{x}, \boldsymbol{\varsigma}) \ : \ \ \mathbf{x} \in \mathcal{X}_a \} = U^\Lambda(\boldsymbol{\varsigma}) - V^*(\boldsymbol{\varsigma}), \tag{7}$$

where $U^\Lambda(\boldsymbol{\varsigma}) = \text{sta}\{\Lambda(\mathbf{x})^T \boldsymbol{\varsigma} - U(\mathbf{x}) \ : \ \ \mathbf{x} \in \mathcal{X}_a\}$ is the so-called Λ-conjugate of U, which is defined on the dual feasible space $\mathcal{S}_a$.

2 Challenging Problems and Applications

2.1 Wireless Network Localization

Consider the following general nonlinear programming problem arising from Euclidean distance geometry (see [3]):

$$(\mathcal{P}) \quad \min \left\{ P(\mathbf{X}) = \sum_{(i,j)\in\mathcal{S}} \frac{1}{2} w_{ij} \left(\frac{1}{2} \|\mathbf{x}_i - \mathbf{x}_j\|^2 - \mu_{ij} \right)^2 + \frac{1}{2} \langle \mathbf{X}, \mathbf{A}\mathbf{X} \rangle - \langle \mathbf{X}, \mathbf{T} \rangle \, | \mathbf{X} \in \mathcal{X}_a \right\},$$

where the decision variable $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n] = \{x_i^\alpha\}_{i,\alpha} \in \mathbb{R}^{r \times n}$ is a matrix (two-point tensor) with each column $\mathbf{x}_i \in \mathbb{R}^r$ as a position of each sensor such that

$$\|\mathbf{x}_i - \mathbf{x}_j\| = \left(\sum_{\alpha=1}^{r} (x_i^\alpha - x_j^\alpha)^2 \right)^{\frac{1}{2}}$$

denotes the Euclidian distance between $\mathbf{x}_i$ and $\mathbf{x}_j$, $(i,j) \in \mathcal{S} = \{1, 2, \cdots, n\}$; $\mathcal{X}_a \subset \mathbb{R}^{d\times n}$ is a feasible set; $\mathbf{T} = \{T^i_\alpha\} \in \mathcal{X}^* = \mathbb{R}^{n\times d}$ is a given matrix; $w_{ij} \geq 0$ and $\mu_{ij} \geq 0 \quad (\forall i, j \in \mathcal{S})$ are given weights and parameters for each pair $(\mathbf{x}_i, \mathbf{x}_j)$, respectively; $\mathbf{A} = \{A^{i,\beta}_{\alpha,j}\}$ is a fourth-order symmetric tensor, and $\mathbf{AX} = \{\sum_{j=1}^n \sum_{\beta=1}^r \mathbf{A}^{i,\beta}_{\alpha,j} x^\beta_j\}_{i,\alpha}$, the bilinear form $\langle \mathbf{X}, \mathbf{T}\rangle : \mathcal{X}_a \times \mathcal{X}^* \to \mathbb{R}$ is defined as

$$\langle \mathbf{X}, \mathbf{T}\rangle = tr(\mathbf{XT}) = \sum_{i=1}^n \sum_{\alpha=1}^d X^\alpha_i T^i_\alpha.$$

Canonical Geometric Measure and Dual Problem Since

$$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \in \mathbb{R}^{r\times n},$$

we have the identity

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = (\mathbf{x}_i - \mathbf{x}_j)^T(\mathbf{x}_i - \mathbf{x}_j) = (\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{X}^T \mathbf{X}(\mathbf{e}_i - \mathbf{e}_j),$$

where $\mathbf{e}_i$ is the i-th standard unit vector in $\mathbb{R}^n$. Introducing a linear (difference) operator $\mathbf{D} : \mathcal{X}_a \to \mathbb{R}^{r\times n\times n}$ such that

$$\mathbf{DX} = \{\mathbf{X}(\mathbf{e}_i - \mathbf{e}_j)\} = \{\mathbf{x}_i - \mathbf{x}_j\},$$

the *canonical strain measure* $\boldsymbol{\xi}$ can be defined as

$$\boldsymbol{\xi} = \{\xi_{ij}\} = \Lambda(\mathbf{X}) = \frac{1}{2}(\mathbf{DX})^T(\mathbf{DX}) = \frac{1}{2}\left\{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{X}^T \mathbf{X}(\mathbf{e}_i - \mathbf{e}_j)\right\},$$

where Λ is the so-called *geometrical nonlinear operator* from $\mathcal{X}_a \subset \mathbb{R}^{r\times n}$ into

$$\mathcal{V}_a = \{\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times n} |\ \boldsymbol{\xi} = \boldsymbol{\xi}^T, \ \ \boldsymbol{\xi} \succeq 0, \ \xi_{ii} = 0, \ i = 1, \cdots, n\}.$$

Clearly, $\xi_{ij} = \frac{1}{2}\|\mathbf{x}_i - \mathbf{x}_j\|^2$, which is corresponding to the Cauchy-Riemann strain tensor in finite deformation theory. By introducing a quadratic function $V : \mathcal{V}_a \to \mathbb{R}$,

$$V(\boldsymbol{\xi}) = \frac{1}{2}\sum_{i,j} w_{ij}(\xi_{ij} - \mu_{ij})^2 = \frac{1}{2}\langle(\boldsymbol{\xi} - \boldsymbol{\mu}); \mathbf{W} \circ (\boldsymbol{\xi} - \boldsymbol{\mu})\rangle,$$

where $\mathbf{W} = \{w_{ij}\}$, $\boldsymbol{\mu} = \{\mu_{ij}\}$, $\mathbf{W} \circ \boldsymbol{\mu} = \{w_{ij}\mu_{ij}\}$ represents the Hadamard product of two matrices, and $\langle *; *\rangle$ denotes the bilinear operator of two matrices. The primal problem $(\mathcal{P})$ can now be reformulated in the canonical form:

$$(\mathcal{P}) : \ \ \min\left\{\Pi(\mathbf{X}) = V(\Lambda(\mathbf{X})) + \frac{1}{2}\langle \mathbf{X}, \mathbf{AX}\rangle - \langle \mathbf{X}, \mathbf{T}\rangle : \ \ \mathbf{X} \in \mathcal{X}_a\right\}.$$

By the canonical dual transformation, the canonical dual problem can be proposed as follows:

$$(\mathcal{P}^d) : \ \ \mathrm{sta}\left\{P^d(\boldsymbol{\varsigma}) = -\frac{1}{2}\langle \mathbf{G}^+(\boldsymbol{\varsigma})\mathbf{T}, \mathbf{T}\rangle - \frac{1}{2}\langle \boldsymbol{\varsigma}; \mathbf{W}^{-1} \circ \boldsymbol{\varsigma}\rangle - \langle \boldsymbol{\mu}; \boldsymbol{\varsigma}\rangle \ \mid \ \boldsymbol{\varsigma} \in \mathcal{S}_a\right\},$$

where, $\mathbf{G}(\boldsymbol{\varsigma}) = \mathbf{A}+\mathbf{D}^T\boldsymbol{\varsigma}\mathbf{D}$ with $\mathbf{D}^T\boldsymbol{\varsigma} = (\mathbf{e}_i^T - \mathbf{e}_j^T)\boldsymbol{\varsigma}$, $\mathbf{G}^+$ represents the generalized inverse of $\mathbf{G}$, the dual feasible space $\mathcal{S}_a$ is a subset of $\mathbb{R}^{n\times n}$ such that for a given $\mathbf{T}$, the matrix equation $\mathbf{G}(\boldsymbol{\varsigma})\,\mathbf{X} = \mathbf{T}$ is solvable on $\mathcal{S}_a$.

Theorem 1 (Complementary-Dual Principle). *The problem* $(\mathcal{P}^d)$ *is a canonical dual of the primal problem* $(\mathcal{P})$ *in the sense that if* $\bar{\boldsymbol{\varsigma}}$ *is a critical point of* $(\mathcal{P}^d)$*, then*

$$\bar{\mathbf{X}} = \mathbf{G}^+(\bar{\boldsymbol{\varsigma}})\mathbf{T} \tag{8}$$

is a critical point of $(\mathcal{P})$ *and*

$$P(\bar{\mathbf{X}}) = P^d(\bar{\boldsymbol{\varsigma}}).$$

In order to identify extremality of the analytical solution (8), we need to introduce a useful feasible space

$$\mathcal{S}_a^+ = \{\boldsymbol{\varsigma} \in \mathcal{S}_a \mid \ \mathbf{G}(\boldsymbol{\varsigma}) \succ 0\}.$$

Theorem 2. *Suppose that* $\bar{\boldsymbol{\varsigma}} \in \mathcal{S}_a^+$ *is a critical point of the canonical dual function* $P^d(\bar{\boldsymbol{\varsigma}})$ *and* $\bar{\mathbf{X}} = \mathbf{G}^+(\bar{\boldsymbol{\varsigma}})\mathbf{T}$*. Then,* $\bar{\mathbf{X}}$ *is a global minimizer of* $P(\mathbf{X})$ *on* $\mathbb{R}^{r\times n}$ *if and only if* $\bar{\boldsymbol{\varsigma}}$ *is a global maximizer of* $P^d(\boldsymbol{\varsigma})$ *on* $\mathcal{S}_a^+$*, i.e.,*

$$P(\bar{\mathbf{X}}) = \min_{\mathbf{X}\in\mathbb{R}^{r\times n}} P(\mathbf{X}) \Leftrightarrow \max_{\boldsymbol{\varsigma}\in\mathcal{S}_a^+} P^d(\boldsymbol{\varsigma}) = P^d(\bar{\boldsymbol{\varsigma}}). \tag{9}$$

This theory shows that if the canonical dual function $P^d(\boldsymbol{\varsigma})$ has a critical point in $\mathcal{S}_a^+$, then the nonconvex primal problem $(\mathcal{P})$ is equivalent to a concave maximization problem $(\mathcal{P}^d)$ over a convex space $\mathcal{S}_a^+$, which can be solved easily by well-developed optimization methods.

2.2 Traveling Salesman Problem

Consider the well-known Traveling salesman problem (TSP), which we need to determine the shortest closed path passing through a set of n cities, with each city visited exactly once. Suppose $\mathcal{N} = \{1, 2, \cdots, n\}$ is the set of TSP cities, and the distance between city i and city j is given by d_{ij}. Assume

$$d_{ii} = 0,\ d_{ij} = d_{ji},\ \forall i, j \in \mathcal{N}.$$

Define a Boolean decision variable x_{ij} according to

$$x_{ij} = \begin{cases} 1 & \text{if city } i \text{ is in the } jth \text{ position,} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

To make sure the round trip, we assume

$$x_{i0} = x_{in},\ x_{i1} = x_{i(n+1)},\ \forall i, j \in \mathcal{N}.$$

Let $\mathbf{X} = \{x_{ij}\} \in \mathbb{R}^{n\times n}$, the Traveling salesman problem can be represented by following quadratic programming problem [8]:

$$
\begin{aligned}
(\mathcal{P}) \quad & \text{Minimize} \;\; P(\mathbf{X}) = \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} x_{ij} d_{ik} \cdot (x_{k(j+1)} + x_{k(j-1)}) \\
& \text{subject to} \;\; \sum_{j=1}^{n} x_{ij} = 1, \; \forall i \in \mathcal{N}, \quad \sum_{i=1}^{n} x_{ij} = 1, \; \forall j \in \mathcal{N}, \\
& \qquad\qquad\quad x_{ij} \in \{0,1\}, \; \forall i,j \in \mathcal{N}.
\end{aligned}
$$

Canonical Dual Problem. Let

$$G(\boldsymbol{\mu}) = \mathbf{A} + 2\text{Diag}\,(\boldsymbol{\mu}),$$

$$F(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\mu}) = \boldsymbol{\mu} - \mathbf{C}^T\boldsymbol{\sigma} - D^T\boldsymbol{\tau}.$$

By the canonical dual transformation [1], the canonical dual problem can be stated as follows:

$$
\begin{aligned}
(\mathcal{P}^d) \quad & \text{Maximize} \;\; P^d(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\mu}) = -\frac{1}{2} F(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\mu})^T G^{\dagger}(\boldsymbol{\mu}) F(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\mu}) - \boldsymbol{\sigma}^T\mathbf{e} - \boldsymbol{\tau}^T\mathbf{e} \\
& \text{subject to} \;\; \boldsymbol{\sigma} \neq 0, \; \boldsymbol{\tau} \neq 0, \boldsymbol{\mu} \neq 0, \\
& \qquad\qquad\quad \boldsymbol{\sigma} \in \mathbb{R}^n, \boldsymbol{\tau} \in \mathbb{R}^n, \boldsymbol{\mu} \in \mathbb{R}^{nn},
\end{aligned}
$$

where, $\mathbf{A} = \{a_{st}\} \in \mathbb{R}^{nn\times nn}$ is a block matrix, which satisfies

$$
a_{st} = \begin{cases}
d_{ik}, & \text{if } s = (i-1)N + j \text{ and } t = (k-1)N + (j-1), \; \forall i,k,j \in \mathcal{N}, \\
d_{ik}, & \text{if } s = (i-1)N + j \text{ and } t = (k-1)N + (j+1), \; \forall i,k,j \in \mathcal{N}, \\
d_{ki}, & \text{if } s = (k-1)N + (j-1) \text{ and } t = (i-1)N + j, \; \forall i,k,j \in \mathcal{N}, \\
d_{ki}, & \text{if } s = (k-1)N + (j+1) \text{ and } t = (i-1)N + j, \; \forall i,k,j \in \mathcal{N}, \\
0, & \text{otherwise},
\end{cases}
$$

$$
\mathbf{C} = \begin{bmatrix}
1 \cdots 1 \; 0 \cdots 0 \cdots 0 \cdots 0 \\
0 \cdots 0 \; 1 \cdots 1 \cdots 0 \cdots 0 \\
\vdots \; \ddots \; \vdots \; \vdots \; \ddots \; \vdots \; \ddots \; \vdots \; \ddots \; \vdots \\
0 \cdots 0 \; 0 \cdots 0 \cdots 1 \cdots 1
\end{bmatrix} \in \mathbb{R}^{n\times nn},
$$

$$
D = \begin{bmatrix}
1 \; 0 \cdots 0 \cdots\cdots 1 \; 0 \cdots 0 \\
0 \; 1 \cdots 0 \cdots\cdots 0 \; 1 \cdots 0 \\
\vdots \; \vdots \; \ddots \; \vdots \;\; \vdots \quad \vdots \;\; \vdots \; \vdots \; \ddots \; \vdots \\
0 \; 0 \cdots 1 \cdots\cdots 0 \; 0 \cdots 1
\end{bmatrix} \in \mathbb{R}^{n\times nn},
$$

$$\mathbf{e} = [1,\cdots,1,\cdots,1,\cdots,1]^T \in \mathbb{R}^n.$$

Theorem 3 (Complementary-Dual Principle). *Problem* $(\mathcal{P}^d)$ *is a canonical dual of Problem* $(\mathcal{P})$ *in the sense that if* $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\mu}})$ *is a KKT solution of Problem* $(\mathcal{P}^d)$*, then the vector* $\bar{\mathbf{X}} = \{x_{ij}\} \in \mathbb{R}^{n\times n}$ *defined by*

$$x_{ij} = y_{(i-1)n+j},\ \forall i,j \in \mathcal{N},\ \textit{and}\ \bar{\mathbf{y}} = G^{\dagger}(\bar{\boldsymbol{\mu}})F(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\mu}}) \in \mathbb{R}^{nn} \tag{11}$$

is a KKT solution of Problem $(\mathcal{P})$ *and* $P(\bar{\mathbf{X}}) = P^d(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\mu}})$.

To continue, let the feasible space $\mathcal{X}$ of problem $(\mathcal{P})$ and the dual feasible space $\mathcal{Z}$ be defined by

$$\mathcal{X} = \left\{ \mathbf{X} \in \mathbb{R}^{n\times n} : \sum_{j=1}^{n} x_{ij} = 1,\ \sum_{i=1}^{n} x_{ij} = 1,\ x_{ij} \in \{0,1\},\ \forall i,j \in \mathcal{N} \right\}$$
$$\mathcal{Z} = \{(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{nn} : \boldsymbol{\sigma} \neq 0,\ \boldsymbol{\tau} \neq 0, \boldsymbol{\mu} \neq 0\},$$
$$\mathcal{Z}_a^+ = \{(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathcal{Z} : G(\boldsymbol{\mu}) \succ 0\}.$$

We have the following theorem.

Theorem 4. *Assume that* $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\mu}})$ *is a KKT point of* $P^d(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu})$ *and* $\bar{X}$ *defined by (11). If* $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\mu}}) \in \mathcal{Z}_a^+$*, then* $\bar{X}$ *is a global minimizer of* $P(\mathbf{X})$ *and* $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\mu}})$ *is a global maximizer of* $P^d(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu})$ *with*

$$P(\bar{X}) = \min_{\mathbf{X}\in\mathcal{X}} P(\mathbf{X}) = \max_{(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\mu})\in\mathcal{Z}_a{}^+} P^d(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu}) = P^d(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\mu}}) \tag{12}$$

2.3 Scheduling Problem in Supply Chain

In project scheduling, a set of resource-constrained jobs has to be scheduled so as to minimize a given objective resources. The scheduling problem has a variety of applications in manufacturing, production planning, project management, and elsewhere.

We consider the problem to minimize the total cost of a schedule when the jobs are subject to temporal constraints only (i.e., there are no resource constraints). A common way to model scheduling problems as integer linear programs is to use time indexed variables. Let

$$x_{jt} = \begin{cases} 1 \text{ if job } j \text{ starts at time } t, \\ 0 \ \ \text{otherwise,} \end{cases}$$

where, $j \in J = 0, \cdots, n$. Jobs 0 and n are assumed to be artificial jobs indicating the project start and the project completion, respectively, d_{ij} be the integral length of a time lag (i,j) between two jobs $i,j \in J$, and let $L \subseteq J \times J$ be the set of all given time lags, T be the deadline of the project, and $t = 0, \cdots, T$, p_i be the processing time of activity i, the precedence relation $(i,j) \in L$ if activity j cannot start before activity i completes. Finally, let w_{jt} be the net present value of activity j when starting at time t. This leads to the following integer linear program:

$$(\mathcal{P}) \quad \text{Minimize} \;\; P(\mathbf{x}) = \sum_{j=0}^{n} \sum_{t=0}^{T} w_{jt} x_{jt} \tag{13}$$

$$\text{subject to} \;\; \sum_{t=0}^{T} x_{jt} = 1, \; j \in J, \;\; \sum_{t=0}^{T} t(x_{jt} - x_{it}) \geq d_{ij}, \; (i,j) \in L, \tag{14}$$

$$x_{jt} \in \{0,1\}, \; j \in J, \; t = 0, \cdots, T. \tag{15}$$

Canonical Dual Problem. Let

$$\begin{aligned} \mathbf{X} &= [x_{00}, \cdots, x_{0T}, \cdots, x_{n0}, \cdots, x_{nT}]^T, \\ \mathbf{W} &= [w_{00}, \cdots, w_{0T}, \cdots, w_{n0}, \cdots, w_{nT}]^T, \\ \mathbf{D} &= [d_{00}, \cdots, d_{0n}, \cdots, d_{n0}, \cdots, d_{nn}]^T, \; d_{ij} = 0 \;\text{ if }\; i \geq j \end{aligned}$$

and

$$\mathbf{B} = \begin{bmatrix} 1 \cdots 1 & 0 \cdots 0 & \cdots & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 & \cdots & 0 \cdots 0 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \ddots & \vdots \ddots \vdots \\ 0 \cdots 0 & 0 \cdots 0 & \cdots & 1 \cdots 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times[(T+1)\times(n+1)]},$$

$$\mathbf{A} = \begin{bmatrix} 0 \cdots T & 0 \cdots -T & \cdots\cdots & 0 \cdots 0 & 0 \cdots \;\; 0 \\ \vdots \ddots \vdots & \vdots \ddots \;\; \vdots & \ddots \;\; \ddots & \vdots \ddots \vdots & \vdots \ddots \;\; \vdots \\ 0 \cdots T & 0 \cdots \;\; 0 & \cdots\cdots & 0 \cdots 0 & 0 \cdots -T \\ \vdots \ddots \vdots & \vdots \ddots \;\; \vdots & \ddots \;\; \ddots & \vdots \ddots \vdots & \vdots \ddots \;\; \vdots \\ 0 \cdots 0 & 0 \cdots \;\; 0 & \cdots\cdots & 0 \cdots T & 0 \cdots -T \end{bmatrix} \in \mathbb{R}^{[(n+1)\times(n+1)]\times[(T+1)\times(n+1)]},$$

By the canonical dual theory [1], the canonical dual problem can be stated as follows:

$$\begin{aligned} (\mathcal{P}^d) \quad & \text{Maximize} \;\; P^d(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu}) = -\frac{1}{2} F(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu})^T \mathbf{G}^+(\boldsymbol{\mu}) F(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu}) - \boldsymbol{\sigma}^T \mathbf{e} + \boldsymbol{\tau}^T \mathbf{e} \\ & \text{subject to} \;\; \boldsymbol{\sigma} > 0, \; \boldsymbol{\tau} \geq 0, \boldsymbol{\mu} > 0, \\ & \qquad \boldsymbol{\sigma} \in \mathbb{R}^{n+1}, \boldsymbol{\tau} \in \mathbb{R}^{(n+1)\times(n+1)}, \boldsymbol{\mu} \in \mathbb{R}^{(T+1)\times(n+1)}, \end{aligned}$$

where,

$$\mathbf{G}(\boldsymbol{\mu}) = 2\text{Diag}\;(\boldsymbol{\mu}), \; F(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\mu}) = \boldsymbol{\mu} - \mathbf{W} - \mathbf{B}^T \boldsymbol{\sigma} - \mathbf{A}^T \boldsymbol{\tau}.$$

And we have complementary-dual principle and optimization criterion similar to Theorem 3 and Theorem 4.

3 Conclusions

We have presented simple applications of the canonical duality theory for three challenging problems. A general analytical solution is obtained by the complementary-dual principle. Results show that by using the canonical dual transformation, the nonconvex primal problem and integer programming problem can be converted to a unified concave maximization dual problem, which can be solved by well-developed convex minimization techniques. The idea and the method presented in this article can be used and generalized to solve much more difficult problems in global optimization, network communication, and scientific computations (see [2, 4–7]). The development of techniques is essential to extrapolate the complexities of the real world.

Acknowledgments. This paper was partially supported by a grant (AFOSR FA9550-10-1-0487) from the US Air Force Office of Scientific Research. Dr. Ning Ruan was supported by a funding from the Australian Government under the Collaborative Research Networks (CRN) program.

References

1. Gao, D.Y.: Duality Principles in Nonconvex Systems: Theory, Methods and Applications. Kluwer Academic Publishers, Dordrecht (2000)
2. Gao, D.Y., Ruan, N.: Solutions to Quadratic Minimization Problems with Box and Integer Constraints. J. Global Optim. 47, 463–484 (2010)
3. Gao, D.Y., Ruan, N., Pardalos, P.M.: Canonical Dual Solutions to Sum of Fourth-Order Polynomials Minimization Problems with Applications to Sensor Network Localization. In: Boginski, V.L., Commonder, C.W., Pardalos, P.M., Ye, Y.Y. (eds.) Sensors: Theory, Algorithms and Applications, vol. 61, pp. 37–54. Springer (2012)
4. Gao, D.Y., Ruan, N., Sherali, H.D.: Solutions and Optimality Criteria for Nonconvex Constrained Global Optimization Problems. J. Global Optim. 45, 473–497 (2009)
5. Gao, D.Y., Watson, L.T., Easterling, D.R., Thacker, W.I., Billups, S.C.: Solving the Canonical Dual of Box- and Integer-Constrained Nonconvex Quadratic Programs via a Deterministic Direct Search Algorithm. Optim. Method Softw. (2011), doi:10.1080/10556788.2011.641125
6. Gao, D.Y., Wu, C.Z.: On the Triality Theory for a Quartic Polynomial Optimization Problem. J. Ind. Manag. Optim. 8, 229–242 (2012)
7. Ruan, N., Gao, D.C., Jiao, Y.: Canonical Dual Least Square Method for Solving General Nonlinear Systems of Equations. Comput. Optim. Appl. 47, 335–347 (2010)
8. Smith, K.: An Argument for Abandoning the Traveling Salesman Problem as a Neural-Network Benchmark. IEEE Trans. Neural Networks 7, 1542–1544 (1996)