About regularity properties in variational analysis and applications in optimization

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Abstract

Regularity properties lie at the core of variational analysis because of their importance for stability analysis of optimization and variational problems, constraint qualifications, qualification conditions in coderivative and subdifferential calculus and convergence analysis of numerical algorithms. The thesis is devoted to investigation of several research questions related to regularity properties in variational analysis and their applications in convergence analysis and optimization.

Following the works by Kruger, we examine several useful regularity properties of collections of sets in both linear and Hölder-type settings and establish their characterizations and relationships to regularity properties of set-valued mappings.

Following the recent publications by Lewis, Luke, Malick (2009), Drusvyatskiy, Ioffe, Lewis (2014) and some others, we study application of the uniform regularity and related properties of collections of sets to alternating projections for solving nonconvex feasibility problems and compare existing results on this topic.

Motivated by Ioffe (2000) and his subsequent publications, we use the classical iteration scheme going back to Banach, Schauder, Lyusternik and Graves to establish criteria for regularity properties of set-valued mappings and compare this approach with the one based on the Ekeland variational principle.

Finally, following the recent works by Khanh et al. on stability analysis for optimization related problems, we investigate calmness of set-valued solution mappings of variational problems.

Statement of authorship

This thesis contains no work extracted in whole or in part from a thesis, dissertation or research paper previously presented for another degree or diploma except where explicit reference is made. No other person's work has been relied upon or used without due acknowledgment in the main text and bibliography of the thesis.

The content of this thesis, which is presented as a thesis incorporating published papers, consists of six published/accepted or submitted papers and one paper in draft form. The PhD candidate declares that he is the main author of these seven papers and that his contribution to each paper is fifty percent or more.

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Preface

Investigations of regularity properties of collections of sets have led to many fundamental ideas and important applications in variational analysis, optimization, optimal control and related topics. The notion of extremal behavior of collections of sets can be traced back to the pioneering work by Dubovitskii and Milyutin [31]. The concept of (local) extremality of collections of finitely many sets was first introduced and intensively examined in the 1980s by Kruger and Mordukhovich [53, 61, 62]. Its necessary condition in terms of Fréchet and limiting normals currently known as the *extremal principle* has been recognized as one of the cornerstones of variational analysis [77]. The extremal principle can be viewed as an extension of the classical *convex separation theorem* to the nonconvex setting and hence plays a fundamental role in many applications of nonconvex calculus, optimization and related topics. As shown by Kruger [54, 55], the conclusion of the extremal principle holds true under weaker than local extremality assumptions which can be interpreted as *stationarity* or *approximate stationarity* of the collection of sets. In Asplund spaces, the last property and the conclusion of the extremal principle are equivalent. This equivalence is now known as the *extended extremal principle*.

Along with extremality and stationarity, regularity properties of collections of sets have also attracted considerable attention of researchers in recent decades; cf. e.g., [13, 16, 17, 54, 55, 56, 57, 58, 88]. *Regularity* and *uniform regularity* properties of collections of sets first introduced and systematically investigated by Kruger [56] are the negations of stationarity and approximate stationarity properties, respectively. A dual version of the second property is also known as *alliedness* [82] or *transversality* [9, 18, 30, 36]. Several relaxations of the uniform regularity property of collections of sets have come to life recently, motivated by the needs of convergence analysis of algorithms: *relative transversality, inherent transversality* and *intrinsic transversality* [18, 30].

Earlier another important regularity property of collections of sets was introduced by Bauschke and Borwein [14, 15]. It is usually referred to as *linear regularity* although it is also in use under several other names. This concept is closely related to the (strong) *conical hull intersection property* (CHIP), the *Jameson's property* and the *metric inequality*. It has attracted a remarkable attention of researchers due to its importance in convergence analysis and approximation theory and its close connections to many important ideas in variational analysis and optimization theory; cf. e.g., [11, 13, 16, 17, 23, 39, 40, 52, 71, 72, 74, 80, 82, 90, 93].

Although the linear and uniform regularity properties of collections of sets were originally introduced to serve different purposes in very different contexts, they turn out to be very closely related. The uniform regularity of a collection of sets at a point in their intersection is in a sense equivalent to the linear regularity condition being satisfied uniformly at all nearby points. These two properties along with their interesting interpretations in terms of set-valued mappings are the main objects of this study.

Regularity properties of collections of sets play an important role in variational analysis, optimization and approximation theory, particularly as constraint qualifications in establishing optimality conditions, qualification conditions in coderivative and subdifferential calculus, and in analyzing convergence of numerical algorithms. From the applications point of view, this study mainly focuses on numerical issues such as linear convergence of projection methods for finding a common point of a collection of sets.

Introduced in the 1990s for computational purposes, the linear regularity property was used as a sufficient condition when establishing linear convergence of sequences generated by the cyclic projection algorithm for finding the projection of a point on the intersection of a collection of closed and convex sets. The application of the uniform regularity property to convergence analysis is a relatively new phenomenon starting in 2009 with the work by Lewis et al. [70] which extended the result by Bauschke and Borwein to collections of nonconvex sets. This nonconvex convergence criterion paved the way to using projection methods for solving nonconvex feasibility problems frequently appearing in conjunction with constrained optimization problems.

There are important relationships between regularity properties of collections of sets and those of set-valued mappings. The uniform regularity can be interpreted as the direct analogue of the metric regularity of set-valued mappings. The last property has been at the center of many important aspects of variational analysis and optimization theory, particularly convergence analysis of numerical algorithms for solving generalized equations, optimization and approximation problems; cf. e.g., [8, 9, 26, 28, 29, 40, 51, 59, 60, 70, 75, 76, 78, 86]. Correspondingly, the local version of linear regularity can be interpreted as the *metric subregularity* of set-valued mappings, a prominent relaxed version of metric regularity [28]. Its outstanding role in optimization and variational analysis in relation to the *calmness* property, *error bounds, weak sharp minima, slopes*, and subdifferential calculus has been verified through a vast number of publications; cf. e.g., [6, 7, 22, 27, 29, 33, 43, 44, 64, 65, 67, 77, 87]. For completeness, another regularity property of collections of sets known as *semiregularity* [64, 65] corresponds, in the same manner, to the *metric semiregularity* of set-valued mappings [58, 64, 65].

Wide range of applications of regularity properties of collections of sets suggests further investigation of different aspects related to variational analysis and optimization theory. Qualitative and quantitative characterizations of the properties in terms of both primal space and dual space elements are important. In many situations, particularly in numerical analysis, quantitative estimates characterizing the properties provide the "radius of effectiveness" within which the algorithm will perform effectively. A big variety of existing regularity properties and their characterizations require a kind of classification scheme. Formulating connections between them and other important concepts in variational analysis and optimization including slopes, error bounds and weak sharp minima is also of interest. In another perspective, in situations when conventional linear estimates are not satisfied, regularity properties of collections of sets can be extended and examined in the Hölder-type setting, or even more general nonlinear settings.

The breakthrough result by Lewis et al. [70] reveals a number of issues for investigation. Some effort has been contributed towards weakening the assumptions and improving convergence rates to increase the effectiveness of the method [18, 30]. The approach initiated in [70] not only works effectively for the averaged and alternating projection methods, but also seems to be applicable to the Douglas-Rachford iteration scheme [83]. Convergence analysis for the last method is, in general, difficult due to its complexity and leaves much room for further investigation. Considering projection methods in infinite dimensional (e.g., Hilbert) spaces may also have a potential because the mentioned approach still works well for appropriate inexact projection methods. Using approximate algorithms allows one to consider situations when exact projections do not exist. This can be considered as one of the main difficulties when dealing with projection methods in infinite dimensional spaces.

Regularity properties of set-valued mappings can be investigated in conjunction with their counterparts for collections of sets. Lying at the core of variational analysis, they were initially studied in the framework of the stability theory of solutions to *generalized equations* (initiated by Robinson [84, 85] in the 1970s). They have found numerous applications when studying stability of optimization and variational problems, constraint qualifications, qualification conditions in coderivative and subdifferential calculus and convergence rates of numerical algorithms; cf. books and surveys [1, 10, 12, 21, 29, 40, 42, 44, 51, 77, 82, 86] and the references therein.

This study is devoted to investigation of several specific research questions regarding regularity properties in variational analysis and their application in optimization, particularly convergence analysis and stability analysis. Firstly, continuing the initial works by Kruger [56, 57, 58], we examine the three mentioned earlier regularity properties of collections of sets in both the linear and Hölder-type settings and find out the relationships to their counterparts in terms of set-valued mappings. Secondly, following the approaches initiated by Lewis et al. [70] and Drusvyatskiy et al. [30], we apply the uniform regularity property and its relaxations to the method of alternating projections and compare existing results on this topic. Thirdly, we use the classical iteration scheme going back to Banach, Schauder, Lyusternik and Graves to establish criteria for the metric regularity property of set-valued mappings and its extensions and compare the results with those obtained with the help of the *Ekeland variational principle*. Finally, an application of regularity properties of set-valued mappings in analyzing stability of solutions to variational problems is studied.

In Chapter 1 [65], we systematically investigate the three properties of semiregularity (weak regularity), subregularity (local linear regularity) and uniform regularity (strong regularity) of collections of sets in the linear setting. Following the lines of [56, 57, 58], we provide quantitative and qualitative characterizations of these properties in terms of both primal space and dual space elements. Complementing a group of results formulated in the mentioned articles, we establish several new characterizations and criteria for the properties. To the best of our knowledge, most of the primal and dual space characterizations of the subregularity property and a few primal space characterizations of the two other properties are established in this chapter for the first time. We also discuss their relationships with the corresponding regularity properties of set-valued mappings. Equivalences regarding the subregularity property are established here for the first time as direct analogues of those regarding the two other properties originally formulated by Kruger [58, Theorem 7]. Some quantitative estimates in that article are also improved. In comparison with the proofs of the mentioned results in [58] where some auxiliary set-valued mappings were used, we provide direct proofs of the results.

In Chapter 2 [64] which continues Chapter 1, we extend the investigation of the three regularity properties of collections of sets to the Hölder-type setting. This is motivated by the link between the collections of sets and set-valued mappings and the importance of Hölder-type extensions of regularity properties of set-valued mappings in variational analysis both in theory and in relatively new application in establishing convergence rates of numerical algorithms; cf. [6, 22, 32, 33, 34, 38, 67, 73, 87]. We introduce the Hölder-type extensions of the properties of collections of sets and provide appropriate examples of collections of sets where conventional regularity conditions are not satisfied, but one can still identify their Hölder-type analogues. Quantitative and qualitative characterizations of these new extended properties in terms of both primal space and dual space elements are provided along the lines of [65]. We formulated their close connections with the corresponding Hölder-type regularity properties of set-valued mappings.

In Chapter 3 [63], we examine the uniform regularity property of collections of sets in Hilbert spaces and apply the obtained results to analyzing linear convergence of the cyclic projection method for solving nonconvex feasibility problems. This is motivated by the recent developments in the employment of this property in convergence analysis of projection algorithms in Lewis et al. [70], Attouch et al. [9], Luke [75, 76], and Hesse and Luke [36].

The quantitative characterizations of the uniform regularity property in terms of Fréchet normal vectors in normed linear spaces developed in [56, 57, 58, 59] in Hilbert spaces admit simpler equivalent representations. For the most important for applications case of two sets, we provide two more dual space constants characterizing the property. One of them is a slight modification of the constant introduced by Lewis et al. [70] when formulating convergence rates of averaged and alternating projection methods in Euclidean spaces. We establish the exact relationships amongst these constants including the one introduced in [70]. Thanks to these relationships, each constant can be used for characterizing convergence rates of projection algorithms in Euclidean spaces. Following the approach initiated by Lewis et al. [70], we make an attempt to use these dual constants to establish a linear convergence result for the cyclic projection algorithm for solving nonconvex feasibility problems of finite collections of sets in Hilbert spaces. In the case of two sets, our convergence criterion goes back to the one formulated in [70].

In Chapter 4 [66], we continue examining the uniform regularity property of collections of sets and its relaxed versions and applying them to numerical algorithms. Unlike Chapter 3 where only the approach initiated by Lewis et al. [70] is studied, here the technique of analyzing linear convergence of the alternating projections initiated by Drusvyatskiy et al. [30] is also discussed. Comparisons of the regularity assumptions employed and the convergence rates obtained in the two approaches are provided when appropriate. We discuss relationships between the two relaxations of the uniform regularity property due to Bauschke et al. [19] and Drusvyatskiy et al. [30]: *intrinsic transversality* and *inherent transversality* (the terminology is taken from [30]). Examples demonstrating the independence of the two properties are provided. As a consequence, this shows the independence of the two groups of convergence results established by the mentioned two groups of researchers. We demonstrate that the approaches initiated in [30, 70] for establishing linear convergence of alternating projections are also applicable for certain "inexact" versions of the algorithm. These extended methods, which are motivated by [70] and first considered in this study, allow some appropriate inexactness when finding a projection of a point on a set. They cover the classical method of alternating projections as a special case. Following the lines of [30, 70], we formulate convergence criteria for the extended algorithms which go back to the original ones established in the mentioned articles as a special case when the method is exact.

In Chapter 5 [49], we investigate the *metric regularity* property - the most recognized and widely used property of set-valued mappings; cf. [12, 20, 21, 29, 40, 51, 77, 81, 82, 86] and the direct counterpart of the uniform regularity property of collections of sets [57, 58] discussed in the previous chapters. We propose and investigate a general regularity model for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces. This model not only covers the case of a family of set-valued mappings when formulating regularity criteria, but also can be of interest by itself. In this chapter, we focus on analogues of the properties of metric regularity and *openness* of set-valued mappings. Through formulating regularity criteria for extended set-valued mappings, which reduce to those for the metric regularity in the conventional setting, we demonstrate that the approach based on iteration procedures going back to Banach, Schauder, Lyusternik and Graves still possesses potential. In particular, we modify the *induction theorem* formulated by Khanh [46], which was used as the main tool when proving the other results in [46, 47, 48], and show that it can serve as a substitution of the Ekeland variational principle when establishing regularity and openness criteria. Criteria for both global and local versions of the metric regularity and openness properties for the conventional setting are derived as consequences of those for the extended setting. Motivated by Ioffe [44], we investigate regularity properties in a general nonlinear model involving certain gauge functions. Results for linear and Hölder-type regularity models can be easily obtained by considering gauge functions of the corresponding types.

In Chapter 6 [50] which continues Chapter 5, we investigate regularity properties of set-valued mappings between metric spaces. We demonstrate that the general regularity theory for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$ developed in Chapter 5 (article [49]) can be translated into the conventional setting to obtain criteria for the *metric subregularity* property of set-valued mappings. The need for investigating this important relaxation of the metric regularity property apparently comes from its outstanding role in optimization and variational analysis in connection with the calmness property, error bounds, weak sharp minima, slopes and subdifferential calculus which has been verified in a vast number of publications; cf. e.g., [6, 7, 22, 27, 29, 33, 43, 44, 64, 65, 67, 77, 87]. We also extend the regularity theory for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$ suggested in [49] by considering its subregularity variant. This property is a direct counterpart of the metric subregularity property of set-valued mappings in the conventional setting. Following the lines of [22, 40, 41, 44, 46, 47, 48, 49, 81], we first formulate the results for the most general model which involves a certain gauge function. The criteria for the linear and Hölder-type regularity models, which are of special interest for applications including convergence analysis of computational methods (cf., e.g., [34, 37, 68, 73, 89, 92]) follow by considering the gauge function of the corresponding types. It is also emphasized that in certain settings, for example, the one of Hölder-type of order k ($k > 1$) captured in [79], metric subregularity properties can no longer be captured via the understanding of regularity ones.

In Chapter 7 [6], we examine regularity properties of set-valued mappings in relation to stability analysis issues. Following the recent works by Anh and Khanh [2, 3, 4, 5] on stability analysis for optimization related problems, we investigate calmness of set-valued solution mappings of variational problems. In particular, we investigate the so-called $(l.\alpha)$-*Hölder calmness* of solutions to parametric equilibrium problems. When $\alpha = 1$, this is a kind of calmness property which is in general stronger than the property of the same name usually used in variational analysis, e.g., [24, 25, 35, 45, 69, 91]. As applications we investigate conditions for Hölder calmness of solutions to optimization problems and well-posedness in the Hölder sense. The last subject is intimately related to the stability analysis and plays a very important role in studying optimization and variational problems.

Overall, this PhD thesis contributes to several important aspects of variational analysis and optimization in both theory and applied perspectives. Its main contribution to the literature of variational analysis is the extensions and developments of the regularity theory of collections of sets (Chapters 1 and 2). Several regularity properties including the uniform regularity, subregularity and semiregularity of collections of sets are systematically discussed and represented in a coherent manner. Several new qualitative and quantitative characterizations in terms of primal space and dual space elements of these properties are formulated. Interesting and important mutual relationships between regularity properties of set-valued mappings and those of collections of sets are strengthened and refined. These regularity properties of collections of sets are also extended and considered in the more general Hölder-type setting. Their characterizations and relationships to the Hölder-type regularity properties of set-valued mappings are formulated. Besides, this study contributes to the topic of convergence analysis of numerical algorithms (Chapters 3 and 4). The role of the regularity properties, especially, the uniform regularity and its restricted versions in establishing linear convergence of projection methods is investigated and discussed. This provides an understanding of the nature of projection algorithms, especially, the developments on methods of approximate alternating projections. Some new understanding of the regularity theory of set-valued mappings is also achieved throughout the project (Chapters 5,6 and 7). Several criteria or metric regularity are formulated by using classical iteration schemes instead of the Ekeland variational principle. Applications of regularity properties in stability analysis of solutions to equilibrium problems are investigated.

All the results presented in this thesis, unless otherwise specified, are the results of the author's independent research carried out under the supervision of Assoc. Prof. Alexander Kruger, Prof. Phan Quoc Khanh and Assoc. Prof. Adil Bagirov at Federation University Australia. Chapter 7 is based on the joint article involving Assoc. Prof. Lam Quoc Anh as a contributing author.

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2. L. Q. Anh, A. Y. Kruger, and N. H. Thao. On Hölder calmness of solution mappings in parametric equilibrium problems. *TOP*, 22(1):331–342, 2014.

3. P. Q. Khanh, A. Y. Kruger, and N. H. Thao. An induction theorem and nonlinear regularity models. arXiv:1410.3032v1, pages 1–20, 2014. Submitted to *SIAM Journal on Optimization.*

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This study is presented as a PhD thesis incorporating published papers. The bibliography used in each chapter/article will be listed at the end of that chapter/article.

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Chapter 1

Quantitative characterizations of regularity properties of collections of sets

Several primal and dual quantitative characterizations of regularity properties of collections of sets in normed linear spaces are discussed. Relationships between regularity properties of collections of sets and those of set-valued mappings are provided.

1.1 Introduction

Regularity properties of collections of sets play an important role in variational analysis and optimization, particularly as constraint qualifications in establishing optimality conditions and coderivative/subdifferential calculus and in analyzing convergence of numerical algorithms.

The concept of linear regularity was introduced in [6, 7] as a key condition in establishing linear convergence rates of sequences generated by the cyclic projection algorithm for finding the projection of a point on the intersection of a collection of closed convex sets. This property has proved to be an important qualification condition in the convergence analysis, optimality conditions, and subdifferential calculus; cf., e.g., [5, 8, 9, 10, 19, 36, 42, 48, 51].

Recently, when investigating the extremality, stationarity and regularity properties of collections of sets systematically, several other kinds of regularity properties have been considered in [27, 28, 29, 30, 31, 32]. They have proved to be useful in convergence analysis [3, 17, 32, 34, 39, 40] and are closely related to certain stationarity properties involved in extensions of the extremal principle [25, 26, 29, 30, 41].

In this study, we aim at providing primal and dual quantitative characterizations of several regularity properties of collections of sets. We also discuss their relationships with the corresponding regularity properties of set-valued mappings.

After introducing in the next section some basic notation, we discuss in Section 1.3 three primal space local regularity properties of collections of sets, namely, *semiregularity*, *subregularity*, and *uniform regularity* as well as their quantitative characterizations. The main result of this section – Theorem 1 – gives equivalent metric characterizations of the three mentioned regularity properties. Section 1.4 is dedicated to dual characterizations of the regularity properties. In Theorem 2 (i), we give a sufficient condition of subregularity in terms of Fréchet normals. In Section 1.5, we present relationships between regularity properties of collections of sets and the corresponding regularity properties of set-valued mappings.

1.2 Notation

Our basic notation is standard; cf. [41, 45]. For a normed linear space X, its topological dual is denoted X^*, while $\langle\cdot,\cdot\rangle$ denotes the bilinear form defining the pairing between the two spaces. The closed unit ball in a normed space is denoted $\mathbb{B}$, $B_\delta(x)$ stands for the closed ball with radius δ and centre x. Products of normed spaces will be considered with the maximum type norms, if not specified otherwise.

The Fréchet normal cone to a set $\Omega \subset X$ at $x \in \Omega$ and the Fréchet subdifferential of a function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ at a point x with $f(x) < \infty$ are defined, respectively, by

$$N_\Omega(x) := \left\{ x^* \in X^* : \limsup_{u\to x,\, u\in\Omega\setminus\{x\}} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le 0 \right\},$$

$$\partial f(x) := \left\{ x^* \in X^* : \liminf_{u\to x,\, u\neq x} \frac{f(u) - f(x) - \langle x^*, u - x\rangle}{\|u - x\|} \ge 0 \right\}.$$

For a given set $\Omega \subset X$, the distance function associated with Ω is defined by

$$d(x, \Omega) := \inf_{\omega \in \Omega} \|x - \omega\|, \ \forall x \in X.$$

In the sequel, $\mathbf{\Omega}$ stands for a collection of m $(m \geq 2)$ sets $\Omega_1, \ldots, \Omega_m$ in a normed linear space X, and we assume the existence of a point $\bar{x} \in \bigcap_{i=1}^m \Omega_i$.

1.3 Regularity properties of collections of sets

In this section, we discuss local primal space regularity properties of finite collections of sets and their primal space characterizations.

1.3.1 Definitions

The next definition introduces several regularity properties of $\mathbf{\Omega}$ at $\bar{x}$.

Definition 1. (i) $\mathbf{\Omega}$ is semiregular at $\bar{x}$ iff there exist positive numbers α and δ such that

$$\bigcap_{i=1}^{m} (\Omega_i - x_i) \bigcap B_\rho(\bar{x}) \neq \emptyset \tag{1.1}$$

for all $\rho \in (0, \delta)$ and all $x_i \in X$ $(i = 1, \ldots, m)$ such that $\max\limits_{1 \leq i \leq m} \|x_i\| \leq \alpha\rho$.

(ii) $\mathbf{\Omega}$ is subregular at $\bar{x}$ iff there exist positive numbers α and δ such that

$$\bigcap_{i=1}^{m} (\Omega_i + (\alpha\rho)\mathbb{B}) \bigcap B_\delta(\bar{x}) \subseteq \left(\bigcap_{i=1}^{m} \Omega_i \right) + \rho\mathbb{B} \tag{1.2}$$

for all $\rho \in (0, \delta)$.

(iii) $\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ iff there exist positive numbers α and δ such that

$$\bigcap_{i=1}^{m} (\Omega_i - \omega_i - x_i) \bigcap (\rho\mathbb{B}) \neq \emptyset \tag{1.3}$$

for all $\rho \in (0, \delta)$, $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$, and all $x_i \in X$ $(i = 1, \ldots, m)$ such that $\max\limits_{1 \leq i \leq m} \|x_i\| \leq \alpha\rho$.

Remark 1. Among the three regularity properties in Definition 1, the third one is the strongest. Indeed, condition (1.1) corresponds to taking $\omega_i = \bar{x}$ in (1.3). To compare properties (ii) and (iii), it is sufficient to notice that condition (1.2) is equivalent to the following one: for any $x \in B_\delta(\bar{x})$, $\omega_i \in \Omega_i$, $x_i \in X$ $(i = 1, \ldots, m)$ such that $\max\limits_{1\le i\le m} \|x_i\| \le \alpha\rho$, and $\omega_i + x_i = x$ $(i = 1, \ldots, m)$, it holds

$$\bigcap_{i=1}^{m}(\Omega_i - x)\bigcap(\rho\mathbb{B}) \neq \emptyset.$$

This corresponds to taking $\omega_i + x_i = x$ $(i = 1, \ldots, m)$ in (1.3) (with $x \in X$) and possibly choosing a smaller $\delta > 0$. Hence, (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii).

Remark 2. When $\bar{x} \in \operatorname{int} \bigcap_{i=1}^{m} \Omega_i$, all the properties in Definition 1 hold true automatically.

Remark 3.[1] When $\Omega_1 = \Omega_2 = \ldots = \Omega_m$, property (ii) in Definition 1 is trivially satisfied (with $\alpha = 1$).

The regularity properties in Definition 1 can be equivalently defined using the following nonnegative constants which provide quantitative characterizations of these properties:

$$\theta[\mathbf{\Omega}](\bar{x}) := \liminf_{\rho\downarrow 0} \frac{\theta_\rho[\mathbf{\Omega}](\bar{x})}{\rho}, \tag{1.4}$$

$$\zeta[\mathbf{\Omega}](\bar{x}) := \lim_{\delta\downarrow 0} \inf_{0<\rho<\delta} \frac{\zeta_{\rho,\delta}[\mathbf{\Omega}](\bar{x})}{\rho}, \tag{1.5}$$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\omega_i \xrightarrow{\Omega_i} \bar{x}, \rho\downarrow 0} \frac{\theta_\rho[\Omega_1 - \omega_1, \ldots, \Omega_m - \omega_m](0)}{\rho}, \tag{1.6}$$

where, for $\rho > 0$ and $\delta > 0$,

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \sup\left\{ r \ge 0 : \bigcap_{i=1}^{m}(\Omega_i - x_i)\bigcap B_\rho(\bar{x}) \neq \emptyset,\ \forall x_i \in r\mathbb{B} \right\}, \tag{1.7}$$

$$\zeta_{\rho,\delta}[\mathbf{\Omega}](\bar{x}) := \sup\left\{ r \ge 0 : \bigcap_{i=1}^{m}(\Omega_i + r\mathbb{B})\bigcap B_\delta(\bar{x}) \subseteq \bigcap_{i=1}^{m}\Omega_i + \rho\mathbb{B} \right\}. \tag{1.8}$$

The next proposition follows immediately from the definitions.

Proposition 1. *(i) $\mathbf{\Omega}$ is semiregular at $\bar{x}$ if and only if $\theta[\mathbf{\Omega}](\bar{x}) > 0$. Moreover, $\theta[\mathbf{\Omega}](\bar{x})$ is the exact upper bound of all numbers α such that (1.1) is satisfied.*

[1] Observed by a reviewer.

(ii) $\mathbf{\Omega}$ *is subregular at* $\bar{x}$ *if and only if* $\zeta[\mathbf{\Omega}](\bar{x}) > 0$*. Moreover,* $\zeta[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* α *such that* (1.2) *is satisfied.*

(iii) $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if* $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$*. Moreover,* $\hat{\theta}[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* α *such that* (1.3) *is satisfied.*

Remark 4. Properties (i) and (iii) in Definition 1 were discussed in [28] (where they were called *regularity* and *strong regularity*, respectively) and [29] (properties $(\mathrm{R})_S$ and $(\mathrm{UR})_S$) and [30] (*regularity* and *uniform regularity*). The current terminology used in parts (i) and (ii) of Definition 1 comes from the standard terminology used for the corresponding regularity properties of set-valued mappings; cf. Section 1.5.

Constants (1.4), (1.6), and (1.7) can be traced back to [21, 22, 23, 24, 25, 26, 27]. Property (ii) in Definition 1 and constants (1.5) and (1.8) are new.

Remark 5. If finite, constants $\zeta[\mathbf{\Omega}](\bar{x})$ and $\hat{\theta}[\mathbf{\Omega}](\bar{x})$ always take values in $[0,1]$, while constant $\theta[\mathbf{\Omega}](\bar{x})$ can be strictly greater than one (cf. Example 4 below). In view of Remark 1, it is not difficult to check that $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \leq \min\{\theta[\mathbf{\Omega}](\bar{x}), \zeta[\mathbf{\Omega}](\bar{x})\}$.

The equivalent representation of constant (1.7) given in the next proposition can be useful.

Proposition 2. *For any* $\rho > 0$*,*

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) = \sup\left\{ r \geq 0 : \; r\mathbb{B}^m \subseteq \bigcup_{x \in B_\rho(\bar{x})} \prod_{i=1}^m (\Omega_i - x) \right\}, \tag{1.9}$$

where $\prod_{i=1}^m (\Omega_i - x) = (\Omega_1 - x) \times \ldots \times (\Omega_m - x)$ *and* $\mathbb{B}^m = \prod_{i=1}^m \mathbb{B}$.

Proof. It is sufficient to observe that condition

$$\bigcap_{i=1}^m (\Omega_i - x_i) \bigcap B_\rho(\bar{x}) \neq \emptyset$$

in (1.7) is equivalent to the existence of $x \in B_\rho(\bar{x})$ such that $x_i \in \Omega_i - x$ for all $i = 1, \ldots, m$. This holds true for all $x_i \in r\mathbb{B}$ if and only if

$$r\mathbb{B}^m \subseteq \bigcup_{x \in B_\rho(\bar{x})} \prod_{i=1}^m (\Omega_i - x).$$

□

From Propositions 1 and 2, we immediately obtain equivalent representations of semiregularity and uniform regularity.

Corollary 1. *(i)* $\mathbf{\Omega}$ *is semiregular at* $\bar{x}$ *if and only if there exist positive numbers* α *and* δ *such that*

$$(\alpha\rho)\mathbb{B}^m \subseteq \bigcup_{x\in B_\rho(\bar{x})} \prod_{i=1}^{m} (\Omega_i - x) \tag{1.10}$$

for all $\rho \in (0,\delta)$*. Moreover,* $\theta[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* α *such that* (1.10) *is satisfied.*

(ii) $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if there exist positive numbers* α *and* δ *such that*

$$(\alpha\rho)\mathbb{B}^m \subseteq \bigcap_{\substack{\omega_i\in\Omega_i\cap B_\delta(\bar{x})\\(i=1,\ldots,m)}} \bigcup_{x\in\rho\mathbb{B}} \prod_{i=1}^{m} (\Omega_i - \omega_i - x) \tag{1.11}$$

for all $\rho \in (0,\delta)$*. Moreover,* $\hat{\theta}[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* α *such that* (1.11) *is satisfied.*

Remark 6. The definition of subregularity in Definition 1 (ii) is already of inclusion type in the setting of the original space X. There is no need to consider the product space X^m.

1.3.2 Examples

We next present examples illustrating that properties (i) and (ii) in Definition 1 are in general independent and none of these two properties implies property (iii) in Definition 1.

Example 1. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 = \Omega_2 := \mathbb{R} \times \{0\}$$

and the point $\bar{x} = (0,0) \in \Omega_1 \cap \Omega_2$. The collection $\{\Omega_1, \Omega_2\}$ is subregular at $\bar{x}$, while it is not semiregular at this point.

Proof. In view of Remark 3, $\{\Omega_1, \Omega_2\}$ is subregular at $\bar{x}$. Observe also that $(\Omega_1 - (0, -\varepsilon)) \cap$

$(\Omega_2 - (0,\varepsilon)) = \emptyset$ for any $\varepsilon > 0$. Hence, by (1.7) and (1.4), $\{\Omega_1, \Omega_2\}$ is not semiregular at $\bar{x}$. ◻

Example 2. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 := \left\{(u,v) \in \mathbb{R}^2 : \ u \le 0 \text{ or } v \ge u^2\right\}, \ \Omega_2 := \left\{(u,v) \in \mathbb{R}^2 : \ u \le 0 \text{ or } v \le 0\right\}$$

and the point $\bar{x} = (0,0) \in \Omega_1 \cap \Omega_2$. The collection $\{\Omega_1, \Omega_2\}$ is semiregular at $\bar{x}$, while it is not subregular at this point.

Proof. We first show that $\{\Omega_1, \Omega_2\}$ is semiregular at $\bar{x}$. For any number $\rho > 0$, we set $x_\rho := (-\rho, 0)$. Then $B_\rho(x_\rho) \subseteq \Omega_i$, i.e., $x_\rho + x_i \in \Omega_i$ for any $x_i \in \rho\mathbb{B}$ $(i = 1,2)$, and consequently

$$x_\rho \in (\Omega_1 - x_1) \cap (\Omega_2 - x_2) \cap B_\rho(\bar{x}), \ \forall x_i \in \rho\mathbb{B} \ (i = 1,2).$$

Hence, $\theta_\rho[\{\Omega_1, \Omega_2\}](\bar{x}) \ge \rho$ and $\theta[\{\Omega_1, \Omega_2\}](\bar{x}) \ge 1$. (One can show that these are actually equalities.) Thus, $\{\Omega_1, \Omega_2\}$ is semiregular at $\bar{x}$.

Suppose that inclusion (1.2) holds for some positive numbers α and δ and all $\rho \in (0,\delta)$. Set $\rho_n := \frac{1}{n}$ and $x_n := \left(\sqrt{\alpha\rho_n}, \alpha\rho_n\right)$. Then $x_n \in (\Omega_1 + (\alpha\rho_n)\mathbb{B}) \bigcap (\Omega_2 + (\alpha\rho_n)\mathbb{B})$, $d(x_n, \Omega_1 \bigcap \Omega_2) = \sqrt{\alpha\rho_n}$ and, for sufficiently large n, $\rho_n < \delta$ and $x_n \in B_\delta(\bar{x})$. It follows from (1.2) that $\sqrt{\alpha\rho_n} \le \rho_n$, and consequently $\alpha \le \rho_n$. This yields $\alpha \le 0$ which contradicts the assumptions. Hence, $\{\Omega_1, \Omega_2\}$ is not subregular at $\bar{x}$. ◻

Example 3. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 = \Omega_2 := \left\{(u,v) \in \mathbb{R}^2 : \ u \le 0 \text{ or } v = 0\right\}$$

and the point $\bar{x} = (0,0) \in \Omega_1 \cap \Omega_2$. The collection $\{\Omega_1, \Omega_2\}$ is both semiregular and subregular at $\bar{x}$, while it is not uniformly regular at this point.

Proof. In view of Remark 3, $\{\Omega_1, \Omega_2\}$ is subregular at $\bar{x}$. Using the arguments from the first part of Example 2, it is easy to check that the collection is semiregular at $\bar{x}$. We next show that $\{\Omega_1, \Omega_2\}$ is not uniformly regular at this point. Indeed, for any given numbers $\delta, \alpha > 0$, we find positive numbers $\rho < r < \delta$ and take

$$\omega_i = (r, 0) \in \Omega_i \cap B_\delta(\bar{x})\ (i = 1, 2), \quad a_1 = (0, \alpha\rho), \quad a_2 = (0, -\alpha\rho) \in \alpha\rho\mathbb{B}.$$

We have

$$(\Omega_1 - \omega_1 - a_1) \bigcap (\Omega_2 - \omega_2 - a_2) \bigcap (\rho\mathbb{B}) = \{(u, v) \in \mathbb{R}^2 :\ u \le -r\} \bigcap (\rho\mathbb{B}) = \emptyset.$$

□

The following example demonstrates that the constant $\theta[\mathbf{\Omega}](\bar{x})$ can take values greater than one.

Example 4. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 := \mathbb{R}^2, \Omega_2 := \left\{(u, v) \in \mathbb{R}^2 :\ u - \sqrt{3}v \ge 0 \text{ or } u + \sqrt{3}v \ge 0\right\}$$

and the point $\bar{x} = (0, 0) \in \Omega_1 \cap \Omega_2$. Then, $\theta[\mathbf{\Omega}](\bar{x}) = 2 > 1$.

Proof. By the structure of the sets, we have

$$\begin{aligned}
\theta_\rho[\mathbf{\Omega}](\bar{x}) &= \sup\{r \ge 0 :\ (\Omega_2 - x) \bigcap (\rho\mathbb{B}) \ne \emptyset,\ \forall x \in r\mathbb{B}\} \\
&= \sup\{r \ge 0 :\ d(0, \Omega_2 - x) \le \rho,\ \forall x \in r\mathbb{B}\} \\
&= \sup\{r \ge 0 :\ d(x, \Omega_2) \le \rho,\ \forall x \in r\mathbb{B}\} \\
&= \sup\{r \ge 0 :\ \max\{d(x, \Omega_2) :\ x \in r\mathbb{B}\} \le \rho\} \\
&= \sup\{r \ge 0 :\ \frac{r}{2} \le \rho\} = 2\rho.
\end{aligned}$$

The second last equality holds true since for any $r > 0$,

$$\max\{d(x, \Omega_2) :\ x \in r\mathbb{B}\} = d(x_r, \Omega_2) = \frac{r}{2},$$

where $x_r := (-r, 0)$.

Hence, by definition,

$$\theta[\mathbf{\Omega}](\bar{x}) = \liminf_{\rho \downarrow 0} \frac{\theta_\rho[\mathbf{\Omega}](\bar{x})}{\rho} = 2.$$

□

1.3.3 Metric characterizations

The regularity properties of collections of sets in Definition 1 can also be characterized in metric terms. The next proposition provides equivalent metric representations of constants (1.4) – (1.6).

Proposition 3.

$$\theta[\mathbf{\Omega}](\bar{x}) = \liminf_{\substack{x_i \to 0\ (1\le i\le m)\\ \bar{x}\notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1\le i\le m} \|x_i\|}{d\Big(\bar{x}, \bigcap_{i=1}^m (\Omega_i - x_i)\Big)}, \tag{1.12}$$

$$\zeta[\mathbf{\Omega}](\bar{x}) = \liminf_{\substack{x\to\bar{x}\\ x\notin \bigcap_{i=1}^m \Omega_i}} \frac{\max_{1\le i\le m} d(x,\Omega_i)}{d\Big(x, \bigcap_{i=1}^m \Omega_i\Big)} \tag{1.13}$$

$$= \liminf_{\substack{x\to\bar{x}\\ \omega_i \xrightarrow{\Omega_i} \bar{x}\ (1\le i\le m)\\ x\notin \bigcap_{i=1}^m \Omega_i}} \frac{\max_{1\le i\le m} \|\omega_i - x\|}{d\Big(x, \bigcap_{i=1}^m \Omega_i\Big)},$$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \liminf_{\substack{x\to\bar{x}\\ x_i\to 0\ (1\le i\le m)\\ x\notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1\le i\le m} d(x+x_i,\Omega_i)}{d\Big(x, \bigcap_{i=1}^m (\Omega_i - x_i)\Big)} \tag{1.14}$$

$$= \liminf_{\substack{x\to\bar{x}\\ x_i\to 0,\ \omega_i \xrightarrow{\Omega_i} \bar{x}\ (1\le i\le m)\\ x\notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1\le i\le m} \|x + x_i - \omega_i\|}{d\Big(x, \bigcap_{i=1}^m (\Omega_i - x_i)\Big)}.$$

Proof. Equality (1.12). Let ξ stand for the right-hand side of (1.12). Suppose that $\xi > 0$ and fix an arbitrary number $\gamma \in (0,\xi)$. Then there is a number $\delta > 0$ such that

$$\gamma d\left(\bar{x}, \bigcap_{i=1}^m (\Omega_i - x_i)\right) \le \max_{1\le i\le m} \|x_i\|,\ \forall x_i \in \delta\mathbb{B}\ (i = 1,\ldots,m). \tag{1.15}$$

Choose a number $\alpha \in (0,\gamma)$ and set $\delta' = \frac{\delta}{\alpha}$. Then, for any $\rho \in (0,\delta')$ and $x_i \in (\alpha\rho)\mathbb{B}$ $(i = 1,\ldots,m)$, it holds $\max_{1\le i\le m} \|x_i\| \le \alpha\rho \le \alpha\delta' = \delta$. Hence, (1.15) yields

$$d\left(\bar{x}, \bigcap_{i=1}^m (\Omega_i - x_i)\right) \le \frac{1}{\gamma} \max_{1\le i\le m} \|x_i\| \le \frac{\alpha}{\gamma}\rho < \rho.$$

This implies (1.1) and consequently $\theta[\mathbf{\Omega}](\bar{x}) \ge \alpha$. Taking into account that α can be arbitrarily close to ξ, we obtain $\theta[\mathbf{\Omega}](\bar{x}) \ge \xi$.

Conversely, suppose that $\theta[\mathbf{\Omega}](\bar{x}) > 0$ and fix an arbitrary number $\alpha \in (0, \theta[\mathbf{\Omega}](\bar{x}))$. Then there is a number $\delta > 0$ such that (1.1) is satisfied for all $\rho \in (0, \delta)$ and $x_i \in (\alpha\rho)\mathbb{B}$ $(i = 1, \ldots, m)$. Choose a positive $\delta' < \alpha\delta$. For any $x_i \in \delta'\mathbb{B}$ $(i = 1, \ldots, m)$, it holds $\max_{1\le i\le m} \|x_i\| < \alpha\delta$. Pick up a $\rho \in (0, \delta)$ such that $\max_{1\le i\le m} \|x_i\| = \alpha\rho$. Then (1.1) yields

$$\alpha d\left(\bar{x}, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \le \alpha\rho = \max_{1\le i\le m} \|x_i\|.$$

This implies $\xi \ge \alpha$. Since α can be arbitrarily close to $\theta[\mathbf{\Omega}](\bar{x})$, we deduce $\xi \ge \theta[\mathbf{\Omega}](\bar{x})$.

Equality (1.13). Let ξ stand for the right-hand side of (1.13). Suppose that $\xi > 0$ and fix an arbitrary number $\alpha \in (0, \xi)$. Then there is a number $\delta > 0$ such that

$$\alpha d\left(x, \bigcap_{i=1}^{m}\Omega_i\right) \le \max_{1\le i\le m} d(x, \Omega_i), \ \forall x \in B_\delta(\bar{x}).$$

If $x \in \bigcap_{i=1}^{m}(\Omega_i + (\alpha\rho)\mathbb{B}) \bigcap B_\delta(\bar{x})$ for some $\rho \in (0, \delta)$, then $\max_{1\le i\le m} d(x, \Omega_i) \le \alpha\rho$, and consequently $d\left(x, \bigcap_{i=1}^{m}\Omega_i\right) \le \rho$, i.e., $\zeta_{\rho,\delta}[\mathbf{\Omega}](\bar{x}) \ge \alpha\rho$. Hence, $\zeta[\mathbf{\Omega}](\bar{x}) \ge \alpha$. Since α can be arbitrarily close to ξ, we obtain $\zeta[\mathbf{\Omega}](\bar{x}) \ge \xi$.

Conversely, suppose that $\zeta[\mathbf{\Omega}](\bar{x}) > 0$ and fix any $\alpha \in (0, \zeta[\mathbf{\Omega}](\bar{x}))$. Then there is a number $\delta > 0$ such that (1.2) is satisfied for all $\rho \in (0, \delta)$. Choose a positive number $\delta' < \min\{\alpha\delta, \delta\}$. For any $x \in B_{\delta'}(\bar{x})$, it holds

$$\max_{1\le i\le m} d(x, \Omega_i) \le \|x - \bar{x}\| \le \delta' < \alpha\delta.$$

Choose a $\rho \in (0, \delta)$ such that $\max_{1\le i\le m} d(x, \Omega_i) = \alpha\rho$. Then, by (1.2),

$$\alpha d\left(x, \bigcap_{i=1}^{m}\Omega_i\right) \le \alpha\rho = \max_{1\le i\le m} d(x, \Omega_i).$$

Hence, $\alpha \le \xi$. By letting $\alpha \to \zeta[\mathbf{\Omega}](\bar{x})$, we obtain $\zeta[\mathbf{\Omega}](\bar{x}) \le \xi$.

Equality (1.14) has been proved in [27, Theorem 1].

□

Propositions 1 and 3 imply equivalent metric characterizations of the regularity properties of collections of sets.

Theorem 1. *(i)* $\mathbf{\Omega}$ *is semiregular at* $\bar{x}$ *if and only if there exist positive numbers* γ *and* δ *such that*

$$\gamma d\left(\bar{x}, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \leq \max_{1\leq i\leq m} \|x_i\|, \ \forall x_i \in \delta\mathbb{B}\ (i = 1, \ldots, m). \tag{1.16}$$

Moreover, $\theta[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* γ *such that* (1.16) *is satisfied.*

(ii) $\mathbf{\Omega}$ *is subregular at* $\bar{x}$ *if and only if there exist positive numbers* γ *and* δ *such that*

$$\gamma d\left(x, \bigcap_{i=1}^{m}\Omega_i\right) \leq \max_{1\leq i\leq m} d(x, \Omega_i), \ \forall x \in B_\delta(\bar{x}). \tag{1.17}$$

Moreover, $\zeta[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* γ *such that* (1.17) *is satisfied.*

(iii) $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if there exist positive numbers* γ *and* δ *such that*

$$\gamma d\left(x, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \leq \max_{1\leq i\leq m} d(x + x_i, \Omega_i) \tag{1.18}$$

for any $x \in B_\delta(\bar{x})$, $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, m)$. *Moreover,* $\hat{\theta}[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* γ *such that* (1.18) *is satisfied.*

Remark 7. Property (1.17) in the above theorem (also known as local *linear regularity*, *linear coherence*, or *metric inequality*) has been around for more than 20 years; cf. [4, 5, 6, 7, 8, 9, 10, 18, 19, 20, 35, 36, 37, 42, 44, 48, 51]. It has been used as a key condition when establishing linear convergence rates of sequences generated by cyclic projection algorithms and a qualification condition for subdifferential and normal cone calculus formulae. The stronger property (1.18) is sometimes referred to as *uniform metric inequality* [27, 28, 29]. Property (1.16) seems to be new.

1.4 Dual characterizations

This section discusses dual characterizations of regularity properties of a collection of sets $\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ at $\bar{x} \in \bigcap_{i=1}^{m}\Omega_i$. We are going to use the notation $\widehat{\Omega} := \Omega_1 \times \ldots \times \Omega_m \subset X^m$.

Recall that the (normalized) *duality mapping* [38, Definition 3.2.6] J between a normed space Y and its dual Y^* is defined as

$$J(y) := \{y^* \in \mathbb{S}_{Y^*} : \langle y^*, y\rangle = \|y\|\}, \ \forall y \in Y.$$

Note that $J(-y) = -J(y)$.

The following simple fact of convex analysis is well known (cf., e.g., [46, Corollary 2.4.16]).

Lemma 1. *Let $(Y, \|\cdot\|)$ be a normed space.*

(i) $\partial\|\cdot\|(y) = J(y)$ *for any* $y \neq 0$.

(ii) $\partial\|\cdot\|(0) = \mathbb{B}^*$.

Making use of the convention that the topology in X^m is defined by the maximum type norm, it is not difficult to establish a representation of the duality mapping on X^m.

Proposition 4. *For each $(x_1, \ldots, x_m) \in X^m$,*

$$J(x_1, \ldots, x_m) = \Bigg\{(x_1^*, \ldots, x_m^*) \in (X^*)^m : \sum_{i=1}^m \|x_i^*\| = 1; \ \textit{either } x_i^* = 0 \\ \textit{or } \left(\|x_i\| = \max_{1\le j\le m} \|x_j\|, \ x_i^* \in \|x_i^*\| J(x_i)\right) \ (i = 1, \ldots, m)\Bigg\}.$$

Proof. Let $\hat{x} := (x_1, \ldots, x_m) \in X^m$, $\hat{x}^* := (x_1^*, \ldots, x_m^*) \in (X^*)^m$. Then

$$\|\hat{x}\| = \max_{1\le i\le m} \|x_i\|, \quad \|\hat{x}^*\| = \sum_{i=1}^m \|x_i^*\|, \quad \langle \hat{x}^*, \hat{x}\rangle = \sum_{i=1}^m \langle x_i^*, x_i\rangle.$$

Suppose $\|\hat{x}^*\| = 1$, i.e., $\sum_{i=1}^m \|x_i^*\| = 1$. Then $\hat{x}^* \in J(\hat{x})$ if and only if $\sum_{i=1}^m \langle x_i^*, x_i\rangle = \|\hat{x}\|$. In its turn, the last equality holds true if and only if $\langle x_i^*, x_i\rangle = \|x_i^*\| \cdot \|\hat{x}\|$ for all $i = 1, \ldots, m$.

Indeed, if $\langle x_i^*, x_i\rangle = \|x_i^*\| \cdot \|\hat{x}\|$ for all $i = 1, \ldots, m$, then adding these m equalities, we obtain $\sum_{i=1}^m \langle x_i^*, x_i\rangle = \|\hat{x}\|$. Conversely, if $\langle x_i^*, x_i\rangle \neq \|x_i^*\| \cdot \|\hat{x}\|$, i.e., $\langle x_i^*, x_i\rangle < \|x_i^*\| \cdot \|\hat{x}\|$ for some $i \in \{1, \ldots, m\}$, then

$$\sum_{j=1}^m \langle x_j^*, x_j\rangle = \langle x_i^*, x_i\rangle + \sum_{j\neq i} \langle x_j^*, x_j\rangle < \|x_i^*\| \cdot \|\hat{x}\| + \|\hat{x}\| \sum_{j\neq i} \|x_j^*\| = \|\hat{x}\|.$$

Finally, $\langle x_i^*, x_i\rangle = \|x_i^*\| \cdot \|\hat{x}\|$ for some $i \in \{1, \ldots, m\}$ if and only if either $\|x_i\| = \|\hat{x}\|$ and $x_i^* \in \|x_i^*\| J(x_i)$ or $x_i^* = 0$. $\square$

In this section, along with the maximum type norm on $X^{m+1} = X \times X^m$, we are going to use another one depending on a parameter $\rho > 0$ and defined as follows:

$$\|(x, \hat{x})\|_\rho := \max\{\|x\|, \rho\,\|\hat{x}\|\}, \quad x \in X,\ \hat{x} \in X^m. \tag{1.19}$$

It is easy to check that the corresponding dual norm has the following representation:

$$\|(x^*, \hat{x}^*)\|_\rho = \|x^*\| + \rho^{-1}\|\hat{x}^*\|, \quad x^* \in X^*,\ \hat{x}^* \in (X^m)^*. \tag{1.20}$$

Note that if, in (1.19) and (1.20), $\hat{x} = (x_1, \ldots, x_m)$ and $\hat{x}^* = (x_1^*, \ldots, x_m^*)$ with $x_i \in X$ and $x_i^* \in X^*$ $(i = 1, 2, \ldots, m)$, then $\|\hat{x}\| = \max_{1\le i\le m} \|x_i\|$ and $\|\hat{x}^*\| = \sum_{i=1}^m \|x_i^*\|$.

The next few facts of subdifferential calculus are used in the proof of the main theorem below.

Lemma 2. *Let X be a normed space and $\varphi(u, \hat{u}) = \|(u - u_1, \ldots, u - u_m)\|$, $u \in X$, $\hat{u} := (u_1, \ldots, u_m) \in X^m$. Suppose $x \in X$, $\hat{x} := (x_1, \ldots, x_m) \in X^m$, and $\hat{v} := (x - x_1, \ldots, x - x_m) \neq 0$. Then*

$$\partial\varphi(x, \hat{x}) \subseteq \big\{(x^*, \hat{x}^* = (x_1^*, \ldots, x_m^*)) \in X^* \times (X^*)^m : \\ -\hat{x}^* \in J(\hat{v}),\ x^* = -(x_1^* + \ldots + x_m^*)\big\}.$$

Proof. Let $(x^*, \hat{x}^* = (x_1^*, \ldots, x_m^*)) \in \partial\varphi(x, \hat{x})$, i.e.,

$$\|(u - u_1, \ldots, u - u_m)\| - \|(x - x_1, \ldots, x - x_m)\| \ge \langle x^*, u - x\rangle + \sum_{i=1}^m \langle x_i^*, u_i - x_i\rangle$$

for any $u \in X$ and $\hat{u} := (u_1, \ldots, u_m) \in X^m$. In particular, with $u = x$ and $u_i = x_i - x_i'$ $(i = 1, \ldots, m)$ for an arbitrary $\hat{x}' := (x_1', \ldots, x_m') \in X^m$, we have

$$\|\hat{v} + \hat{x}'\| - \|\hat{v}\| \ge -\langle \hat{x}^*, \hat{x}'\rangle,$$

i.e., $-\hat{x}^* \in J(\hat{v})$. Similarly, with $u = x + x'$ and $u_i = x_i + x'$ ($i = 1, \ldots, m$) for an arbitrary $x' \in X$, we have

$$\left\langle x^* + \sum_{i=1}^{m} x_i^*, x' \right\rangle \leq 0,$$

and consequently $x^* + x_1^* + \ldots + x_m^* = 0$. □

Lemma 3. *Let X be a normed space and $\hat{\omega} := (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$. Then $N_{\widehat{\Omega}}(\hat{\omega}) = N_{\Omega_1}(\omega_1) \times \ldots \times N_{\Omega_m}(\omega_m)$.*

Proof. follows directly from the definition of the Fréchet normal cone. □

The proof of the main theorem of this section relies heavily on two fundamental results of variational analysis: the *Ekeland variational principle* (Ekeland [14]; cf., e.g., [25, Theorem 2.1], [41, Theorem 2.26]) and the *fuzzy* (*approximate*) *sum rule* (Fabian [15]; cf., e.g., [25, Rule 2.2], [41, Theorem 2.33]). Below we provide these results for completeness.

Lemma 4 (Ekeland variational principle)**.** *Suppose X is a complete metric space, and $f : X \to \mathbb{R}_\infty$ is lower semicontinuous and bounded from below, $\varepsilon > 0, \lambda > 0$. If*

$$f(v) < \inf_X f + \varepsilon,$$

then there exists $x \in X$ such that

(a) $d(x, v) < \lambda$,

(b) $f(x) \leq f(v)$,

(c) $f(u) + (\varepsilon/\lambda) d(u, x) \geq f(x)$ for all $u \in X$.

Lemma 5 (Fuzzy sum rule)**.** *Suppose X is Asplund, $f_1 : X \to \mathbb{R}$ is Lipschitz continuous and $f_2 : X \to \mathbb{R}_\infty$ is lower semicontinuous in a neighbourhood of $\bar{x}$ with $f_2(\bar{x}) < \infty$. Then, for any $\varepsilon > 0$, there exist $x_1, x_2 \in X$ with $\|x_i - \bar{x}\| < \varepsilon$, $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ ($i = 1, 2$) such that*

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(x_1) + \partial f_2(x_2) + \varepsilon \mathbb{B}^*.$$

The next theorem gives dual sufficient conditions for regularity of collections of sets.

Theorem 2. *Let X be an Asplund space and $\Omega_1, \ldots, \Omega_m$ be closed.*

(i) $\mathbf{\Omega}$ *is subregular at* $\bar{x}$ *if there exist positive numbers* α *and* δ *such that, for any* $\rho \in (0,\delta)$, $x \in B_\rho(\bar{x})$, $\omega_i \in \Omega_i \cap B_\rho(x)$ $(i = 1,\ldots,m)$ *with* $\omega_i \neq x$ *for some* $i \in \{1,\ldots,m\}$, *there is an* $\varepsilon > 0$ *such that, for any* $x' \in B_\varepsilon(x)$, $\omega_i' \in \Omega_i \cap B_\varepsilon(\omega_i)$, $x_i^* \in N_{\Omega_i}(\omega_i') + \rho\mathbb{B}^*$ $(i = 1,\ldots,m)$ *satisfying*

$$x_i^* = 0 \quad \text{if} \quad \|x' - \omega_i'\| < \max_{1\le j\le m} \|x' - \omega_j'\|,$$
$$\langle x_i^*, x' - \omega_i'\rangle \ge \|x_i^*\|(\|x' - \omega_i'\| - \varepsilon),$$
$$\sum_{i=1}^m \|x_i^*\| = 1, \tag{1.21}$$

it holds

$$\left\|\sum_{i=1}^m x_i^*\right\| > \alpha. \tag{1.22}$$

(ii) $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if there are positive numbers* α *and* δ *such that* (1.22) *holds true for all* $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ *and* $x_i^* \in N_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,m)$ *satisfying* (1.21).

The proof of Theorem 2 (i) consists of a series of propositions providing lower estimates for constant (1.13) and, thus, sufficient conditions for subregularity of $\mathbf{\Omega}$ which can be of independent interest. Observe that constant (1.13) can be rewritten as

$$\zeta[\mathbf{\Omega}](\bar{x}) = \liminf_{\substack{x\to\bar{x},\,\omega_i\to\bar{x}\ (1\le i\le m)\\ \hat{\omega}=(\omega_1,\ldots,\omega_m)\\ x\notin\bigcap_{i=1}^m \Omega_i}} \frac{f(x,\hat{\omega})}{d\left(x, \bigcap_{i=1}^m \Omega_i\right)} \tag{1.23}$$

with function $f : X^{m+1} \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ defined as

$$f(x,\hat{x}) = \max_{1\le i\le m} \|x - x_i\| + \delta_{\widehat{\Omega}}(\hat{x}), \quad x \in X,\ \hat{x} := (x_1,\ldots,x_m) \in X^m, \tag{1.24}$$

where $\delta_{\widehat{\Omega}}$ is the indicator function of $\widehat{\Omega}$: $\delta_{\widehat{\Omega}}(\hat{x}) = 0$ if $\hat{x} \in \widehat{\Omega}$ and $\delta_{\widehat{\Omega}}(\hat{x}) = +\infty$ otherwise.

Proposition 5. *Let* X *be a Banach space and* $\Omega_1, \ldots, \Omega_m$ *be closed.*

(i) $\hat{\zeta}[\mathbf{\Omega}](\bar{x}) \leq \zeta[\mathbf{\Omega}](\bar{x})$*, where*

$$\hat{\zeta}[\mathbf{\Omega}](\bar{x}) := \lim_{\rho \downarrow 0} \inf_{\substack{\|x-\bar{x}\|<\rho \\ \hat{\omega}=(\omega_1,\ldots,\omega_m)\in\widehat{\Omega} \\ 0<\max\limits_{1\leq i\leq m}\|x-\omega_i\|<\rho}} \zeta_\rho[\mathbf{\Omega}](x,\hat{\omega}) \tag{1.25}$$

and, for $x \in X$ *and* $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$*,*

$$\zeta_\rho[\mathbf{\Omega}](x,\hat{\omega}) := \limsup_{\substack{(u,\hat{v})\to(x,\hat{\omega}) \\ (u,\hat{v})\neq(x,\hat{\omega}) \\ \hat{v}=(v_1,\ldots,v_m)\in\widehat{\Omega}}} \frac{\left(\max\limits_{1\leq i\leq m}\|x-\omega_i\| - \max\limits_{1\leq i\leq m}\|u-v_i\|\right)_+}{\|(u,\hat{v})-(x,\hat{\omega})\|_\rho}. \tag{1.26}$$

(ii) If $\hat{\zeta}[\mathbf{\Omega}](\bar{x}) > 0$*, then* $\mathbf{\Omega}$ *is subregular at* $\bar{x}$*.*

Proof. (i) Let $\zeta[\mathbf{\Omega}](\bar{x}) < \alpha < \infty$. Choose a $\rho \in]0,1[$ and set

$$\eta := \min\left\{\frac{\rho}{2}, \frac{\rho}{\alpha}, \rho^{\frac{2}{\rho}}\right\}. \tag{1.27}$$

By (1.23), there are $x' \in B_\eta(\bar{x})$ and $\hat{\omega}' = (\omega'_1, \ldots, \omega'_m) \in \widehat{\Omega}$ such that

$$0 < f(x',\hat{\omega}') < \alpha d\left(x', \bigcap_{i=1}^m \Omega_i\right). \tag{1.28}$$

Denote $\varepsilon := f(x',\hat{\omega}')$ and $\mu := d\,(x', \bigcap_{i=1}^m \Omega_i)$. Then $\mu \leq \|x'-\bar{x}\| \leq \eta \leq \frac{\rho}{2} < 1$. Observe that f is lower semicontinuous. Applying to f Lemma 4 with ε as above and

$$\lambda := \mu(1-\mu^{\frac{\rho}{2-\rho}}), \tag{1.29}$$

we find points $x \in X$ and $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in X^m$ such that

$$\left\|(x,\hat{\omega})-(x',\hat{\omega}')\right\|_\rho < \lambda,\ f(x,\hat{\omega}) \leq f(x',\hat{\omega}'), \tag{1.30}$$

and

$$f(u,\hat{v}) + \frac{\varepsilon}{\lambda}\|(u,\hat{v})-(x,\hat{\omega})\|_\rho \geq f(x,\hat{\omega}), \tag{1.31}$$

for all $(u, \hat{v}) \in X \times X^m$. Thanks to (1.30), (1.29), (1.27), and (1.28), we have

$$\|x - x'\| < \lambda < \mu \leq \|x' - \bar{x}\|,$$

$$d\left(x, \bigcap_{i=1}^{m} \Omega_i\right) \geq d\left(x', \bigcap_{i=1}^{m} \Omega_i\right) - \|x - x'\| \geq \mu - \lambda = \mu^{\frac{2}{2-\rho}}, \tag{1.32}$$

$$\|x - \bar{x}\| \leq \|x - x'\| + \|x' - \bar{x}\| < 2\|x' - \bar{x}\| \leq 2\eta \leq \rho, \tag{1.33}$$

$$f(x, \hat{\omega}) \leq f(x', \hat{\omega}') < \alpha\mu \leq \alpha\eta \leq \rho. \tag{1.34}$$

It follows from (1.32), (1.33), and (1.34) that

$$\|x - \bar{x}\| < \rho,\ \hat{\omega} \in \widehat{\Omega},\ 0 < \max_{1 \leq i \leq m} \|x - \omega_i\| < \rho.$$

Observe that $\mu^{\frac{\rho}{2-\rho}} \leq \eta^{\frac{\rho}{2-\rho}} < \eta^{\frac{\rho}{2}} \leq \rho$, and consequently, by (1.28) and (1.29),

$$\frac{\varepsilon}{\lambda} < \frac{\alpha\mu}{\lambda} = \frac{\alpha}{1 - \mu^{\frac{\rho}{2-\rho}}} < \frac{\alpha}{1-\rho}.$$

Thanks to (1.31) and (1.24), we have

$$\max_{1 \leq i \leq m} \|x - \omega_i\| - \max_{1 \leq i \leq m} \|u - v_i\| \leq \frac{\alpha}{1-\rho} \|(u, \hat{v}) - (x, \hat{\omega})\|_\rho$$

for all $u \in X$ and $\hat{v} = (v_1, \ldots, v_m) \in \widehat{\Omega}$. It follows that $\zeta_\rho[\mathbf{\Omega}](x, \hat{\omega}) \leq \dfrac{\alpha}{1-\rho}$ and consequently

$$\inf_{\substack{\|x - \bar{x}\| < \rho \\ \hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega} \\ 0 < \max\limits_{1 \leq i \leq m} \|x - \omega_i\| < \rho}} \zeta_\rho[\mathbf{\Omega}](x, \hat{\omega}) \leq \frac{\alpha}{1-\rho}.$$

Taking limits in the last inequality as $\rho \downarrow 0$ and $\alpha \to \zeta[\mathbf{\Omega}](\bar{x})$ yields the claimed inequality.

(ii) follows from (i) and Proposition 1 (ii). □

Proposition 6. *Let X be an Asplund space and $\Omega_1, \ldots, \Omega_m$ be closed.*

(i) $\hat{\zeta}_1^*[\mathbf{\Omega}](\bar{x}) \le \hat{\zeta}[\mathbf{\Omega}](\bar{x})$*, where* $\hat{\zeta}[\mathbf{\Omega}](\bar{x})$ *is given by* (1.25)*,*

$$\hat{\zeta}_1^*[\mathbf{\Omega}](\bar{x}) := \lim_{\rho\downarrow 0} \inf_{\substack{\|x-\bar{x}\|<\rho \\ \hat{\omega}=(\omega_1,\ldots,\omega_m)\in\widehat{\Omega} \\ 0<\max\limits_{1\le i\le m}\|x-\omega_i\|<\rho}} \zeta_{\rho,1}^*[\mathbf{\Omega}](x,\hat{\omega}) \tag{1.35}$$

and, for $x \in X$ *and* $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$*,*

$$\zeta_{\rho,1}^*[\mathbf{\Omega}](x,\hat{\omega}) := \inf_{\substack{(x^*,\hat{y}^*)\in\partial f(x,\hat{\omega}) \\ \|\hat{y}^*\|<\rho}} \|x^*\| \tag{1.36}$$

(with the convention that the infimum over the empty set equals $+\infty$*).*

(ii) If $\hat{\zeta}_1^*[\mathbf{\Omega}](\bar{x}) > 0$*, then* $\mathbf{\Omega}$ *is subregular at* $\bar{x}$*.*

Proof. Let $\hat{\zeta}[\mathbf{\Omega}](\bar{x}) < \alpha < \infty$. Choose a $\beta \in (\hat{\zeta}[\mathbf{\Omega}](\bar{x}), \alpha)$ and an arbitrary $\rho > 0$. Set $\rho' = \min\{1, \alpha^{-1}\}\rho$. By (1.25) and (1.26), one can find points $x \in X$ and $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$ such that $\|x - \bar{x}\| < \rho'$, $0 < \max_{1\le i\le m} \|\omega_i - x\| < \rho'$, and

$$\max_{1\le i\le m} \|x-\omega_i\| - \max_{1\le i\le m} \|u - v_i\| \le \beta \left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_{\rho'}$$

for all $(u, \hat{v})$ with $\hat{v} = (v_1, \ldots, v_m) \in \widehat{\Omega}$ near $(x, \hat{\omega})$. In other words, $(x, \hat{\omega})$ is a local minimizer of the function

$$(u,\hat{v}) \mapsto \max_{1\le i\le m} \|u - v_i\| + \beta \left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_{\rho'}$$

subject to $\hat{v} = (v_1, \ldots, v_m) \in \widehat{\Omega}$. By definition (1.24), this means that $(x, \hat{\omega})$ minimizes locally the function

$$(u,\hat{v}) \mapsto f(u,\hat{v}) + \beta \left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_{\rho'},$$

and consequently its Fréchet subdifferential at $(x, \hat{\omega})$ contains zero. Take an

$$\varepsilon \in \left(0, \min\{\rho - \|x-\bar{x}\|, \rho - \max_{1\le i\le m} \|x-\omega_i\|, \alpha-\beta\}\right).$$

Applying Lemma 5 and Lemma 1 (ii), we can find points $x' \in X$, $\hat{\omega}' = (\omega_1', \ldots, \omega_m') \in \widehat{\Omega}$, and

$(x^*, \hat{y}^*) \in \partial f(x', \hat{\omega}')$ such that

$$\|x' - x\| < \varepsilon, \quad 0 < \max_{1\le i\le m} \|x' - \omega_i'\| \le \max_{1\le i\le m} \|x - \omega_i\| + \varepsilon,$$

$$\text{and} \quad \|(x^*, \hat{y}^*)\|_{\rho'} = \|x^*\| + \|\hat{y}^*\|/\rho' < \beta + \varepsilon.$$

It follows that

$$\|x' - \bar{x}\| < \rho, \;\; 0 < \max_{1\le i\le m} \|x' - \omega_i'\| < \rho, \;\; \|x^*\| < \alpha, \;\; \text{and} \;\; \|\hat{y}^*\| < \rho'\alpha \le \rho.$$

Hence, $\zeta^*_{\rho,1}[\boldsymbol{\Omega}](x', \hat{\omega}') < \alpha$, and consequently $\hat{\zeta}^*_1[\boldsymbol{\Omega}](\bar{x}) < \alpha$. By letting $\alpha \to \hat{\zeta}[\boldsymbol{\Omega}](\bar{x})$, we obtain the claimed inequality.

(ii) follows from (i) and Proposition 5 (ii). □

Proposition 7. *Let X be an Asplund space and $\Omega_1, \ldots, \Omega_m$ be closed.*

(i) $\hat{\zeta}^*_2[\boldsymbol{\Omega}](\bar{x}) \le \hat{\zeta}^*_1[\boldsymbol{\Omega}](\bar{x})$, *where $\hat{\zeta}^*_1[\boldsymbol{\Omega}](\bar{x})$ is given by* (1.35),

$$\hat{\zeta}^*_2[\boldsymbol{\Omega}](\bar{x}) := \lim_{\rho\downarrow 0} \inf_{\substack{\|x-\bar{x}\|<\rho \\ \hat{\omega}=(\omega_1,\ldots,\omega_m)\in\widehat{\Omega} \\ 0<\max_{1\le i\le m}\|x-\omega_i\|<\rho}} \lim_{\varepsilon\downarrow 0} \inf_{\substack{\|x'-x\|<\varepsilon \\ \hat{\omega}'\in\widehat{\Omega} \\ \|\hat{\omega}'-\hat{\omega}\|<\varepsilon}} \zeta^*_{\rho,\varepsilon,2}[\boldsymbol{\Omega}](x', \hat{\omega}') \tag{1.37}$$

and, for $x \in X$ and $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$ with $(x - \omega_1, \ldots, x - \omega_m) \ne 0$,

$$\zeta^*_{\rho,\varepsilon,2}[\boldsymbol{\Omega}](x, \hat{\omega}) := \inf\Bigg\{ \Bigg\| \sum_{i=1}^m x_i^* \Bigg\| : x_i^* \in N_{\Omega_i}(\omega_i) + \rho\mathbb{B}^* \quad (i = 1, \ldots, m), \\ x_i^* = 0 \;\; \text{if} \;\; \|x - \omega_i\| < \max_{1\le j\le m} \|x - \omega_j\|, \\ \langle x_i^*, x - \omega_i\rangle \ge \|x_i^*\|(\|x - \omega_i\| - \varepsilon), \\ \sum_{i=1}^m \|x_i^*\| = 1 \Bigg\}. \tag{1.38}$$

(ii) If $\hat{\zeta}^*_2[\boldsymbol{\Omega}](\bar{x}) > 0$, then $\boldsymbol{\Omega}$ is subregular at $\bar{x}$.

Proof. (i) Let $\rho > 0$, $x \in X$, $\hat{\omega} := (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$ with $\|x - \bar{x}\| < \rho$, $0 < \max_{1\le i\le m} \|x - \omega_i\| < \rho$, $(u^*, \hat{v}^*) \in \partial f(x, \hat{\omega})$, where f is given by (1.24), and $\|\hat{v}^*\| < \rho$.

Denote $\hat{v} := (x - \omega_1, \ldots, x - \omega_m)$. Then $0 < \|\hat{v}\| < \rho$. Observe that function f is the sum of two functions on X^{m+1}:

$$(x, \hat{x}) \mapsto \varphi(x, \hat{x}) := \|(x - x_1, \ldots, x - x_m)\| \quad \text{and} \quad (x, \hat{x}) \mapsto \delta_{\widehat{\Omega}}(\hat{x}),$$

where $\hat{x} := (x_1, \ldots, x_m)$ and $\delta_{\widehat{\Omega}}$ is the indicator function of $\widehat{\Omega}$. The first function is Lipschitz continuous while the second one is lower semicontinuous. One can apply Lemma 5. For any $\varepsilon > 0$, there exist points $x' \in X$, $\hat{x} := (x_1, \ldots, x_m) \in X^m$, $\hat{\omega}' := (\omega'_1, \ldots, \omega'_m) \in \widehat{\Omega}$, $(x^*, \hat{y}^*) \in \partial\varphi(x', \hat{x})$, and $\hat{\omega}^* \in N_{\widehat{\Omega}}(\hat{\omega}')$ such that

$$\|x' - x\| < \varepsilon, \quad \|\hat{x} - \hat{\omega}\| < \frac{\varepsilon}{4}, \quad \|\hat{\omega}' - \hat{\omega}\| < \frac{\varepsilon}{4},$$
$$\|(u^*, \hat{v}^*) - (x^*, \hat{y}^*) - (0, \hat{\omega}^*)\| < \varepsilon. \tag{1.39}$$

Taking a smaller ε if necessary, one can ensure that $\hat{v}' := (x' - \omega'_1, \ldots, x' - \omega'_m) \neq 0$, $\hat{v}'' := (x' - x_1, \ldots, x' - x_m) \neq 0$, $\|\hat{v}^*\| + \varepsilon < \rho$ and, for any $i = 1, \ldots, m$, $\|x' - x_i\| < \max_{1 \le j \le m} \|x' - x_j\|$ if and only if $\|x' - \omega'_i\| < \max_{1 \le j \le m} \left\|x' - \omega'_j\right\|$. By Lemma 2,

$$\hat{x}^* := -\hat{y}^* \in J(\hat{v}'') \quad \text{and} \quad x^* = x_1^* + \ldots + x_m^*,$$

where $\hat{x}^* = (x_1^*, \ldots, x_m^*)$. By Proposition 4,

$$\sum_{i=1}^m \|x_i^*\| = 1,$$
$$x_i^* = 0 \quad \text{if} \quad \left\|x' - \omega'_i\right\| < \max_{1 \le j \le m} \left\|x' - \omega'_j\right\|,$$

$$\langle x_i^*, x' - \omega'_i \rangle \ge \langle x_i^*, x' - x_i \rangle - \|x_i^*\| \, \|x_i - \omega'_i\| = \|x_i^*\|(\|x' - x_i\| - \|x_i - \omega'_i\|)$$
$$\ge \|x_i^*\|(\|x' - \omega'_i\| - 2\|x_i - \omega'_i\|) \ge \|x_i^*\|(\|x' - \omega'_i\| - \varepsilon) \quad (i = 1, \ldots, m).$$

Inequality (1.39) yields the estimates: $\|u^*\| > \|x^*\| - \varepsilon$, $\|\hat{x}^* - \hat{\omega}^*\| < \|\hat{v}^*\| + \varepsilon < \rho$, and consequently

$$\|u^*\| > \left\|\sum_{i=1}^m x_i^*\right\| - \varepsilon, \quad \hat{x}^* \in N_{\widehat{\Omega}}(\hat{\omega}') + \rho\mathbb{B}_m^*.$$

It follows from Lemma 3 and definitions (1.36) and (1.38) that

$$\zeta^*_{\rho,1}[\mathbf{\Omega}](x,\hat{\omega}) \geq \zeta^*_{\rho,\varepsilon,2}[\mathbf{\Omega}](x',\hat{\omega}') - \varepsilon.$$

The claimed inequality is a consequence of the last one and definitions (1.35) and (1.37).

(ii) follows from (i) and Proposition 6 (ii). □

Proof. of Theorem 2 (i) follows from Proposition 7 (ii) and definitions (1.37) and (1.38).

(ii) is a consequence of [29, Theorem 4]. □

Remark 8. One of the main tools in the proof of Theorem 2 is the fuzzy sum rule (Lemma 5) for Fréchet subdifferentials in Asplund spaces. The statements can be extended to general Banach spaces. For that, one has to replace Fréchet subdifferentials (and normal cones) with some other kind of subdifferentials satisfying a certain set of natural properties including the sum rule (not necessarily fuzzy) – cf. [30, p. 345].

If the sets $\Omega_1, \ldots \Omega_m$ are convex or the norm of X is Fréchet differentiable away from 0, then the fuzzy sum rule can be replaced in the proof by either the convex sum rule (Moreau–Rockafellar formula) or the simple (exact) differentiable rule (see, e.g., [25, Corollary 1.12.2]), respectively, to produce dual sufficient conditions for regularity of collections of sets in general Banach spaces in terms of either normals in the sense of convex analysis or Fréchet normals.

Remark 9. Since uniform regularity is a stronger property than subregularity (Remark 1), the criterion in part (ii) of Theorem 2 is also sufficient for the subregularity of the collection of sets in part (i).

The next example illustrates application of Theorem 2 (i) for detecting subregularity of collections of sets.

Example 5. Consider the collection $\{\Omega, \Omega\}$ of two copies of the set $\Omega := \mathbb{R} \times \{0\}$ in the real plane $\mathbb{R}^2$ with the Euclidean norm (cf. Example 1) and the point $\bar{x} = (0,0) \in \Omega$.

Obviously $N_\Omega(\omega) = \{0\} \times \mathbb{R}$ for any $\omega \in \Omega$. If $x_1^* := (a_1, b_1) \in N_\Omega(\omega_1') + \rho\mathbb{B}^*$ and $x_2^* := (a_2, b_2) \in N_\Omega(\omega_2') + \rho\mathbb{B}^*$ for some $\omega_1', \omega_2' \in \Omega$, then $|a_1| \leq \rho$ and $|a_2| \leq \rho$.

Take any positive numbers α and δ such that $\alpha^2 + 2\delta^2 < 1$ and any $\rho \in (0, \delta)$. Let $\omega_1, \omega_2 \in \Omega$, $x \in \mathbb{R}^2$, $\hat{v} := (\omega_1 - x, \omega_2 - x) \in \mathbb{R}^4 \setminus \{0\}$. Because of the definition of Ω, $\hat{v}$ has the following representation: $\hat{v} = (v_1, v, v_3, v)$.

If $v = 0$, then $\xi := v_1^2 + v_3^2 > 0$. Choose an $\varepsilon > 0$ such that

$$(\max\{|v_1| - \varepsilon, 0\})^2 + (\max\{|v_3| - \varepsilon, 0\})^2 > \xi/2 \quad \text{and} \quad 4\varepsilon^2/\xi < \alpha^2.$$

There are no pairs x_1^*, x_2^* satisfying the conditions of Theorem 2 (i). Indeed, if $\hat{v}' := (v_1', v_2', v_3', v_4') \in B_\varepsilon(\hat{v})$, then $|v_2'| \leq \varepsilon$, $|v_4'| \leq \varepsilon$, and $\|\hat{v}'\|^2 \geq |v_1'|^2 + |v_3'|^2 > \xi/2$. If $(x_1^*, x_2^*) \in J(\hat{v}')$, then $(x_1^*, x_2^*) = \hat{v}'/\|\hat{v}'\|$. Hence, $b_1^2 + b_2^2 \leq 2\varepsilon^2/\|\hat{v}'\|^2 < 4\varepsilon^2/\xi < \alpha^2$ and consequently $\|(x_1^*, x_2^*)\| < \alpha^2 + 2\delta^2 < 1$; a contradiction.

If $v \neq 0$, then we choose an $\varepsilon \in (0, |v|)$. If $\hat{v}' \in B_\varepsilon(\hat{v})$ and $(x_1^*, x_2^*) \in J(\hat{v}')$, then b_1 and b_2 have the same sign as v and $b_1^2 + b_2^2 \geq 1 - 2\delta^2$. Hence,

$$\|x_1^* + x_2^*\|^2 = (a_1 + a_2)^2 + (b_1 + b_2)^2 \geq (b_1 + b_2)^2 \geq b_1^2 + b_2^2 > \alpha^2.$$

By Theorem 2 (i), the collection $\{\Omega, \Omega\}$ is subregular at $\bar{x}$.

1.5 Regularity of set-valued mappings

In this section, we present relationships between regularity properties of collections of sets and the corresponding properties of set-valued mappings, which have been intensively investigated; cf., e.g., [11, 12, 13, 18, 29, 41, 43, 45].

Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F := \{(x, y) \in X \times Y : \ y \in F(x)\}$.

Definition 2. (i) F is *metrically semiregular* at $(\bar{x}, \bar{y})$ iff there exist positive numbers γ and δ such that

$$\gamma d\left(\bar{x}, F^{-1}(y)\right) \leq d(y, \bar{y}), \ \forall y \in B_\delta(\bar{y}). \tag{1.40}$$

The exact upper bound of all numbers γ such that (1.40) is satisfied will be denoted by $\theta[F](\bar{x}, \bar{y})$.

(ii) F is *metrically subregular* at $(\bar{x}, \bar{y})$ iff there exist positive numbers γ and δ such that

$$\gamma d\left(x, F^{-1}(\bar{y})\right) \leq d(\bar{y}, F(x)), \ \forall x \in B_\delta(\bar{x}). \tag{1.41}$$

The exact upper bound of all numbers γ such that (1.41) is satisfied will be denoted by $\zeta[F](\bar{x}, \bar{y})$.

(iii) F is *metrically regular* at $(\bar{x}, \bar{y})$ iff there exist positive numbers γ and δ such that

$$\gamma d\left(x, F^{-1}(y)\right) \leq d\left(y, F(x)\right), \ \forall (x, y) \in B_\delta(\bar{x}, \bar{y}). \tag{1.42}$$

The exact upper bound of all numbers γ such that (1.42) is satisfied will be denoted by $\hat{\theta}[F](\bar{x}, \bar{y})$.

Remark 10. Property (ii) and especially property (iii) in Definition 2 are very well known and widely used in variational analysis; see, e.g., [11, 12, 13, 18, 29, 41, 43, 45, 47, 49, 50]. Property (i) was introduced in [29]. In [1, 2], it is referred to as *metric hemiregularity.*

For a collection of sets $\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ in a normed linear space X, one can consider set-valued mapping $F : X \rightrightarrows X^m$ defined by (cf. [18, Proposition 5], [27, Theorem 3], [28, Proposition 8], [34, p. 491], [17, Proposition 33])

$$F(x) := (\Omega_1 - x) \times \ldots \times (\Omega_m - x), \ \forall x \in X.$$

It is easy to check that, for $x \in X$ and $u = (u_1, \ldots, u_m) \in X^m$, it holds

$$x \in \bigcap_{i=1}^m \Omega_i \iff 0 \in F(x), \quad F^{-1}(u) = \bigcap_{i=1}^m (\Omega_i - u_i).$$

The next proposition is a consequence of Theorem 1.

Proposition 8. *Consider* $\mathbf{\Omega}$ *and* F *as above and a point* $\bar{x} \in \bigcap_{i=1}^m \Omega_i$.

(i) $\mathbf{\Omega}$ *is semiregular at* $\bar{x}$ *if and only if* F *is metrically semiregular at* $(\bar{x}, 0)$. *Moreover,* $\theta[\mathbf{\Omega}](\bar{x}) = \theta[F](\bar{x}, 0)$.

(ii) $\mathbf{\Omega}$ *is subregular at* $\bar{x}$ *if and only if* F *is metrically subregular at* $(\bar{x}, 0)$. *Moreover,* $\zeta[\mathbf{\Omega}](\bar{x}) = \zeta[F](\bar{x}, 0)$.

(iii) $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if* F *is metrically regular at* $(\bar{x}, 0)$. *Moreover,* $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\theta}[F](\bar{x}, 0)$.

Remark 11. Assertion (iii) was proved in [28, Proposition 8] (see also [27, Theorem 3] and [34, p. 491]). The equivalence of subregularity of $\mathbf{\Omega}$ and metric subregularity of F has been established by Hesse and Luke in Proposition 33 (ii) of their recent preprint [16]. This proposition has not been included in the final version of their article which appeared in [17].

Conversely, regularity properties of set-valued mappings between normed linear spaces can be treated as realizations of the corresponding regularity properties of certain collections of two sets.

For a given set-valued mapping $F : X \rightrightarrows Y$ between normed linear spaces and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, one can consider the collection $\mathbf{\Omega}$ of two sets $\Omega_1 = \operatorname{gph} F$ and $\Omega_2 = X \times \{\bar{y}\}$ in $X \times Y$. It is obvious that $(\bar{x}, \bar{y}) \in \Omega_1 \cap \Omega_2$.

Theorem 3. *Consider F and $\mathbf{\Omega}$ as above.*

(i) F is metrically semiregular at $(\bar{x}, \bar{y})$ if and only if $\mathbf{\Omega}$ is semiregular at $(\bar{x}, \bar{y})$. Moreover,

$$\frac{\theta[F](\bar{x}, \bar{y})}{\theta[F](\bar{x}, \bar{y}) + 2} \leq \theta[\mathbf{\Omega}](\bar{x}, \bar{y}) \leq \theta[F](\bar{x}, \bar{y})/2. \tag{1.43}$$

(ii) F is metrically subregular at $(\bar{x}, \bar{y})$ if and only if $\mathbf{\Omega}$ is subregular at $(\bar{x}, \bar{y})$. Moreover,

$$\frac{\zeta[F](\bar{x}, \bar{y})}{\zeta[F](\bar{x}, \bar{y}) + 2} \leq \zeta[\mathbf{\Omega}](\bar{x}, \bar{y}) \leq \min\{\zeta[F](\bar{x}, \bar{y})/2, 1\}. \tag{1.44}$$

(iii) F is metrically regular at $(\bar{x}, \bar{y})$ if and only if $\mathbf{\Omega}$ is uniformly regular at $(\bar{x}, \bar{y})$. Moreover,

$$\frac{\hat{\theta}[F](\bar{x}, \bar{y})}{\hat{\theta}[F](\bar{x}, \bar{y}) + 2} \leq \hat{\theta}[\mathbf{\Omega}](\bar{x}, \bar{y}) \leq \min\{\hat{\theta}[F](\bar{x}, \bar{y})/2, 1\}. \tag{1.45}$$

Proof. (i) Suppose F is metrically semiregular at $(\bar{x}, \bar{y})$, i.e., $\theta[F](\bar{x}, \bar{y}) > 0$. Fix a $\gamma \in (0, \theta[F](\bar{x}, \bar{y}))$. Then there exists a number $\delta' > 0$ such that (1.40) is satisfied for all $y \in B_{\delta'}(\bar{y})$. Take any $\alpha > 0$ satisfying $2\alpha/\gamma + \alpha < 1$, and a $\delta := \frac{\delta'}{2\alpha}$. We are going to check that

$$(\Omega_1 - (u_1, v_1)) \bigcap (\Omega_2 - (u_2, v_2)) \bigcap B_\rho(\bar{x}, \bar{y}) \neq \emptyset \tag{1.46}$$

for all $\rho \in (0, \delta)$ and (u_1, v_1), $(u_2, v_2) \in (\alpha\rho)\mathbb{B}$. Indeed, take any $\rho \in (0, \delta)$ and

$(u_1, v_1), (u_2, v_2) \in (\alpha\rho)\mathbb{B}$. We need to find a point $(x, y) \in B_\rho(\bar{x}, \bar{y})$ satisfying

$$\begin{cases} (x, y) + (u_1, v_1) \in \operatorname{gph} F, \\ y = \bar{y} - v_2. \end{cases}$$

We set $y' := \bar{y} - v_2 + v_1$, so $y' \in B_{\delta'}(\bar{y})$ as $\|y' - \bar{y}\| = \|v_1 - v_2\| \le 2\alpha\rho < 2\alpha\delta = \delta'$. Then there is, by (1.40), an $x' \in F^{-1}(y')$ such that

$$\|\bar{x} - x'\| \le \frac{1}{\gamma}\|\bar{y} - y'\|.$$

Put $y := y' - v_1 = \bar{y} - v_2$ and $x := x' - u_1$. Then it holds

$$(x, y) + (u_1, v_1) = (x', y') \in \operatorname{gph} F, \quad \|y - \bar{y}\| = \|v_2\| \le \alpha\rho < \rho,$$

and

$$\begin{aligned} \|x - \bar{x}\| \le \|x - x'\| + \|x' - \bar{x}\| \le \|u_1\| + \frac{1}{\gamma}\|\bar{y} - y'\| \\ = \|u_1\| + \frac{1}{\gamma}\|v_1 - v_2\| \le (2\alpha/\gamma + \alpha)\rho < \rho. \end{aligned}$$

Hence, (1.46) is proved.

The above reasoning also yields the first inequality in (1.43).

To prove the inverse implication, we suppose $\mathbf{\Omega}$ is semiregular at $(\bar{x}, \bar{y})$, i.e., $\theta[\mathbf{\Omega}](\bar{x}, \bar{y}) > 0$. Fix an $\alpha \in (0, \theta[\mathbf{\Omega}](\bar{x}, \bar{y}))$. Then there exists a $\delta' > 0$ such that (1.46) holds true for all $\rho \in (0, \delta')$ and $(u_1, v_1), (u_2, v_2) \in (\alpha\rho)\mathbb{B}$. Set $\gamma := 2\alpha$ and $\delta < \alpha\delta'$. We are going to check that (1.40) is satisfied. Take any $y \in B_\delta(\bar{y})$, i.e., $\|y - \bar{y}\| \le \delta < \alpha\delta'$. Set $r \in (0, \delta')$ such that $\|y - \bar{y}\| = \alpha r$. Then, applying (1.46) for $(u_1, v_1) := \left(0, \frac{y - \bar{y}}{2}\right), (u_2, v_2) =: \left(0, \frac{\bar{y} - y}{2}\right) \in \left(\alpha\frac{r}{2}\right)\mathbb{B}$, we can find $(x_1, y_1) \in \operatorname{gph} F$ and $(x_2, \bar{y}) \in \Omega_2$ satisfying

$$(x_1, y_1) - (u_1, v_1) = (x_2, \bar{y}) - (u_2, v_2) \in B_{\frac{r}{2}}(\bar{x}, \bar{y}).$$

This implies that $y_1 = y$, $x_1 \in F^{-1}(y)$ and

$$\|x_1 - \bar{x}\| \leq \frac{r}{2} = \frac{1}{2\alpha}\|y - \bar{y}\| = \frac{1}{\gamma}\|y - \bar{y}\|.$$

Hence, (1.40) holds true.

The last reasoning also yields the second inequality in (1.43).

(ii) Suppose F is metrically subregular at $(\bar{x}, \bar{y})$, i.e., $\zeta[F](\bar{x}, \bar{y}) > 0$. Fix a $\gamma \in (0, \zeta[F](\bar{x}, \bar{y}))$. Then there exists $\delta' > 0$ such that (1.41) is satisfied for all $x \in B_{\delta'}(\bar{x})$. Take an $\alpha > 0$ satisfying $2\alpha/\gamma + \alpha < 1$, and a $\delta := \frac{\delta'}{\alpha+1}$. We are going to check that

$$(\Omega_1 + (\alpha\rho)\mathbb{B}) \bigcap (\Omega_2 + (\alpha\rho)\mathbb{B}) \bigcap B_\delta(\bar{x}, \bar{y}) \subseteq \Omega_1 \cap \Omega_2 + \rho\mathbb{B} \tag{1.47}$$

for all $\rho \in (0, \delta)$. Indeed, take any

$$(x, y) \in (\Omega_1 + (\alpha\rho)\mathbb{B}) \bigcap (\Omega_2 + (\alpha\rho)\mathbb{B}) \bigcap B_\delta(\bar{x}, \bar{y}).$$

Then $(x, y) = (x_1, y_1) + (u_1, v_1) = (x_2, \bar{y}) + (u_2, v_2)$ for some $(x_1, y_1) \in \operatorname{gph} F$, $x_2 \in X$, and (u_1, v_1), $(u_2, v_2) \in (\alpha\rho)\mathbb{B}$. Since

$$\|x_1 - \bar{x}\| \leq \|u_1\| + \|x - \bar{x}\| \leq \alpha\rho + \delta < (\alpha + 1)\delta = \delta',$$

by (1.41), there exists an $x' \in F^{-1}(\bar{y})$ such that $\|x_1 - x'\| \leq \frac{1}{\gamma} d(\bar{y}, F(x_1)) \leq \frac{1}{\gamma}\|\bar{y} - y_1\|$. Then

$$\begin{aligned} \big\|x_1 - x' + u_1\big\| \leq& \frac{1}{\gamma}\|\bar{y} - y_1\| + \|u_1\| = \frac{1}{\gamma}\|v_1 - v_2\| + \|u_1\| \\ \leq& \frac{2\alpha\rho}{\gamma} + \alpha\rho = \Big(\frac{2}{\gamma} + 1\Big)\alpha\rho < \rho, \\ \|v_2\| \leq& \alpha\rho < \rho. \end{aligned}$$

Hence, $(x, y) = (x', \bar{y}) + (x_1 - x' + u_1, v_2) \in \Omega_1 \cap \Omega_2 + \rho\mathbb{B}$.

The above reasoning also yields the first inequality in (1.44).

To prove the inverse implication, we suppose that $\boldsymbol{\Omega}$ is subregular at $(\bar{x}, \bar{y})$, i.e., $\zeta[\boldsymbol{\Omega}](\bar{x}, \bar{y}) > 0$. Fix an $\alpha \in (0, \zeta[\boldsymbol{\Omega}](\bar{x}, \bar{y}))$. Then there exists a $\delta' > 0$ such that (1.47) holds true for all $\rho \in (0, \delta')$. Set $\gamma := 2\alpha > 0$ and $\delta := \min\left\{\delta', \gamma\delta', \frac{2\delta'}{\gamma}\right\}$. We are going to check that (1.41) holds true. Take any $x \in B_\delta(\bar{x})$. Because $d(x, F^{-1}(\bar{y})) \le \|x - \bar{x}\| \le \delta$, it is sufficient to consider the case $0 < d(\bar{y}, F(x)) < \gamma\delta$. We choose a $y \in F(x)$ such that $d(\bar{y}, F(x)) \le \|y - \bar{y}\| := r < \gamma\delta$. Then

$$\left(x, \frac{y+\bar{y}}{2}\right) = (x, y) + \left(0, \frac{\bar{y}-y}{2}\right) = (x, \bar{y}) + \left(0, \frac{y-\bar{y}}{2}\right), \quad \left\|\frac{\bar{y}-y}{2}\right\| = \frac{r}{2} < \delta',$$

and consequently

$$\left(x, \frac{y+\bar{y}}{2}\right) \in \left(\Omega_1 + \frac{r}{2}\mathbb{B}\right) \bigcap \left(\Omega_2 + \frac{r}{2}\mathbb{B}\right) \bigcap B_{\delta'}(\bar{x}, \bar{y}). \tag{1.48}$$

Take $\rho := \frac{r}{2\alpha} < \delta \le \delta'$. Then $\frac{r}{2} = \alpha\rho$, and it follows from (1.47) and (1.48) that

$$\left(x, \frac{y+\bar{y}}{2}\right) \in \Omega_1 \cap \Omega_2 + \frac{r}{2\alpha}\mathbb{B} = F^{-1}(\bar{y}) \times \{\bar{y}\} + \frac{\|y-\bar{y}\|}{\gamma}\mathbb{B}.$$

Hence, there is an $x' \in F^{-1}(\bar{y})$ such that

$$\|x - x'\| \le \frac{1}{\gamma}\|y - \bar{y}\|.$$

Taking infimum in the last inequality over $x' \in F^{-1}(\bar{y})$ and $y \in F(x)$, we arrive at (1.41).

The last reasoning together with $\zeta[\mathbf{\Omega}](\bar{x}, \bar{y}) \le 1$, in view of (1.13), yields the second inequality in (1.44).

(iii) Suppose F is metrically regular at $(\bar{x}, \bar{y})$, i.e., $\hat{\theta}[F](\bar{x}, \bar{y}) > 0$. Fix a $\gamma \in (0, \hat{\theta}[F](\bar{x}, \bar{y}))$. Then there exists a $\delta' > 0$ such that (1.42) is satisfied for all $(x, y) \in B_{\delta'}(\bar{x}, \bar{y})$. Take an $\alpha > 0$ satisfying $2\alpha/\gamma + \alpha < 1$, and a $\delta := \frac{\delta'}{2\alpha+1}$. We are going to check that

$$(\Omega_1 - (x_1, y_1) - (u_1, v_1)) \bigcap (\Omega_2 - (x_2, \bar{y}) - (u_2, v_2)) \bigcap (\rho\mathbb{B}) \ne \emptyset \tag{1.49}$$

for all $\rho \in (0, \delta)$, $(x_1, y_1) \in \Omega_1 \cap B_\delta(\bar{x}, \bar{y}), x_2 \in B_\delta(\bar{x})$, and $(u_1, v_1), (u_2, v_2) \in (\alpha\rho)\mathbb{B}$. Take any such $\rho, (x_1, y_1), x_2, (u_1, v_1)$, and (u_2, v_2). We need to find $(a, b) \in \rho\mathbb{B}$ satisfying

$$\begin{cases} (x_1, y_1) + (u_1, v_1) + (a, b) \in \operatorname{gph} F, \\ b = -v_2. \end{cases}$$

We set $y' = y_1 - v_2 + v_1$, so $y' \in B_{\delta'}(\bar{y})$ as

$$\|y' - \bar{y}\| \leq \|y' - y_1\| + \|y_1 - \bar{y}\| \leq \|v_1 - v_2\| + \delta \leq 2\alpha\rho + \delta < (2\alpha + 1)\delta = \delta'.$$

Then, applying (1.42) for $(x_1, y') \in B_{\delta'}(\bar{x}, \bar{y})$, we find $x' \in F^{-1}(y')$ such that

$$\|x_1 - x'\| \leq \frac{1}{\gamma} d(y', F(x_1)) \leq \frac{1}{\gamma}\|y' - y_1\| = \frac{1}{\gamma}\|v_1 - v_2\| \leq \frac{2\alpha\rho}{\gamma}.$$

Put $a = x' - x_1 - u_1$ and $b = -v_2$. Then $\|a\| \leq \|x' - x_1\| + \|u_1\| \leq (2\alpha/\gamma + \alpha)\rho < \rho$, $\|b\| \leq \alpha\rho < \rho$, and it holds $(x_1, y_1) + (u_1, v_1) + (a, b) = (x', y') \in \operatorname{gph} F$.
Hence, (1.49) is proved.

The above reasoning also yields the first inequality in (1.45).

To prove the inverse implication, we suppose that $\boldsymbol{\Omega}$ is uniformly regular at $(\bar{x}, \bar{y})$, i.e., $\hat{\theta}[\boldsymbol{\Omega}](\bar{x}, \bar{y}) > 0$. Fix an $\alpha \in (0, \hat{\theta}[\boldsymbol{\Omega}](\bar{x}, \bar{y}))$. Then there exists a $\delta' > 0$ such that (1.49) holds true for all $\rho \in (0, \delta')$, $(x_1, y_1) \in \Omega_1 \cap B_{\delta'}(\bar{x}, \bar{y}), x_2 \in B_{\delta'}(\bar{x})$, and (u_1, v_1), $(u_2, v_2) \in (\alpha\rho)\mathbb{B}$. Set $\gamma := 2\alpha > 0$. Because $\theta[\boldsymbol{\Omega}](\bar{x}, \bar{y}) \geq \hat{\theta}[\boldsymbol{\Omega}](\bar{x}, \bar{y})$ (see Remark 5), assertion (i) implies that there exists a $\delta^* > 0$ such that (1.40) is satisfied for all $y \in B_{\delta^*}(\bar{y})$. Set

$$\delta := \min\left\{\delta^*, \frac{\delta'}{2\alpha + 2}, \frac{\alpha\delta'}{2\alpha + 1}\right\} > 0. \tag{1.50}$$

Now take any $(x, y) \in B_\delta(\bar{x}, \bar{y})$. We are going to check that (1.42) is satisfied. Because (1.40) implies

$$\gamma d(x, F^{-1}(y)) \leq \gamma\|x - \bar{x}\| + \gamma d(\bar{x}, F^{-1}(y)) \leq \gamma\delta + \|y - \bar{y}\| \leq (\gamma + 1)\delta,$$

it suffices to consider the case $d(y, F(x)) < (\gamma + 1)\delta \leq \alpha\delta'$. Choose a $y' \in F(x)$ such that

$$d(y, F(x)) \leq \|y - y'\| < (\gamma + 1)\delta,$$

and set $r \in (0, \delta')$ such that $\|y - y'\| = \alpha r < \alpha\delta'$. Then

$$\|y' - \bar{y}\| \le \|y' - y\| + \|y - \bar{y}\| < (2\alpha + 2)\delta \le \delta'$$

due to (1.50). Applying (1.49) with

$$(x_1, y_1) := (x, y') \in \operatorname{gph} F \cap B_{\delta'}(\bar{x}, \bar{y}), \quad (x_2, y_2) := (\bar{x}, \bar{y}),$$

$$(u_1, v_1) := \left(0, \frac{y - y'}{2}\right), \quad (u_2, v_2) =: \left(0, \frac{y' - y}{2}\right) \in \left(\alpha\frac{r}{2}\right)\mathbb{B},$$

we can find $(\tilde{x}, \tilde{y}) \in \operatorname{gph} F$ and $(z, \bar{y}) \in \Omega_2$ satisfying

$$(\tilde{x}, \tilde{y}) - (x_1, y_1) - (u_1, v_1) = (z, \bar{y}) - (x_2, \bar{y}) - (u_2, v_2) \in \frac{r}{2}\mathbb{B}.$$

This implies $\tilde{x} - x_1 \in \frac{r}{2}\mathbb{B}$ and $\tilde{y} = y_1 + v_1 - v_2 = y$, so $\tilde{x} \in F^{-1}(y)$. Then we obtain

$$d(x, F^{-1}(y)) \le \|x - \tilde{x}\| \le \frac{r}{2} = \frac{1}{2\alpha}\|y - y'\| = \frac{1}{\gamma}\|y - y'\|.$$

Taking infimum in the last inequality over $y' \in F(x)$, we arrive at (1.42).

The last reasoning together with $\hat{\theta}[\mathbf{\Omega}](\bar{x}, \bar{y}) \le 1$, in view of (1.14), yields the second inequality in (1.45).

□

Remark 12. The equivalences stated in Theorem 3 (i) and (iii) has been proved in [29, Theorem 7] by using some auxiliary set-valued mapping. The first inequalities in (1.43) and (1.45) improve the corresponding estimates given in the aforementioned reference because it is always true that

$$\frac{1}{2}\min\{\theta[F](\bar{x}, \bar{y})/2, 1\} \le \frac{\theta[F](\bar{x}, \bar{y})}{\theta[F](\bar{x}, \bar{y}) + 2},$$
$$\frac{1}{2}\min\{\hat{\theta}[F](\bar{x}, \bar{y})/2, 1\} \le \frac{\hat{\theta}[F](\bar{x}, \bar{y})}{\hat{\theta}[F](\bar{x}, \bar{y}) + 2}.$$

Statement (ii) in Theorem 3 seems to be new.

1.6 Conclusions

In this chapter, we continue investigating regularity properties of collections of sets in normed linear spaces.

We systematically examine three closely related primal space local regularity properties: *semiregularity*, *subregularity*, and *uniform regularity* and their quantitative characterizations. In Theorem 1, we establish equivalent metric characterizations of the three mentioned properties and demonstrate, in particular, the equivalence of subregularity and another important property, usually referred to as local *linear regularity*.

In Theorem 2 (i), in the Asplund space setting, we give a new dual space sufficient condition of subregularity in terms of Fréchet normals. The proof of this theorem consists of a series of propositions providing other (primal and dual space) sufficient conditions of subregularity which can be of independent interest.

We present also relationships between the mentioned regularity properties of collections of sets and the corresponding regularity properties of set-valued mappings which, in particular, explain the terminology adopted in this chapter.

The definitions and characterizations of the regularity properties of collections of sets discussed in this chapter can be extended to the more general Hölder type setting – cf. [33].

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Chapter 2

About $[q]$-regularity properties of collections of sets

We examine three primal space local Hölder type regularity properties of finite collections of sets, namely, $[q]$-*semiregularity*, $[q]$-*subregularity*, and *uniform* $[q]$-*regularity* as well as their quantitative characterizations. Equivalent metric characterizations of the three mentioned regularity properties as well as a sufficient condition of $[q]$-subregularity in terms of Fréchet normals are established. The relationships between $[q]$-regularity properties of collections of sets and the corresponding regularity properties of set-valued mappings are discussed.

2.1 Introduction

Regularity properties of collections of sets play an important role in variational analysis and optimization, particularly as constraint qualifications in establishing optimality conditions and coderivative/subdifferential calculus and in analyzing convergence of numerical algorithms.

The concept of *linear regularity* was first introduced in [7, 8] as a key condition in establishing linear convergence rates of sequences generated by the cyclic projection algorithm for finding a point in the intersection of a collection of closed convex sets. This property has proved to be an important qualification condition in the convergence analysis, optimality conditions, and subdifferential calculus, cf., [5, 6, 9, 10, 12, 26, 42, 43, 45, 61].

Recently, when investigating the extremality, stationarity and regularity properties of collections of sets systematically, several other kinds of regularity were introduced in [33] and have been further investigated in [34, 35, 36, 37, 38, 39, 52]. The *uniform regularity* is the negation of the *approximate stationarity* property of collections of sets which is the main ingredient in extensions of the *extremal principle* [31, 32, 49]. It has also proved to be useful in the convergence analysis [4, 38, 41, 47, 48].

The regularity properties of collections of sets are closely related to the well known regularity properties of set-valued mappings such as the *linear openness*, *covering*, *metric regularity*, *Aubin property*, and *calmness*. The Hölder extensions of these properties also play an important role in variational analysis both in theory and in establishing convergence rates of numerical algorithms, cf. [1, 11, 18, 19, 20, 22, 40, 44, 55].

In this chapter which continues the previous one, we attempt to extend regularity properties of collections of sets to the Hölder setting and establish their primal and dual space characterizations. We also discuss their relationships with the corresponding regularity properties of set-valued mappings.

In Section 2.2, we discuss three primal space local Hölder type regularity properties of finite collections of sets, namely, $[q]$-*semiregularity*, $[q]$-*subregularity*, and *uniform* $[q]$-*regularity* as well as their quantitative characterizations. The main result of this section – Theorem 4 – gives equivalent metric characterizations of the three mentioned regularity properties. We also give several examples illustrating these regularity properties. Section 2.3 is dedicated to dual characterizations of the regularity properties. In Theorem 5 (i), we give a sufficient condition of $[q]$-subregularity in terms of Fréchet normals. In Section 2.4, we present relationships between $[q]$-regularity properties of collections of sets and the corresponding regularity properties of set-valued mappings.

Our basic notation is standard, cf. [49, 54]. For a normed linear space X, its topological dual is denoted X^* while $\langle \cdot, \cdot \rangle$ denotes the bilinear form defining the pairing between the two spaces. The closed unit ball in a normed space is denoted $\mathbb{B}$. $B_\delta(x)$ stands for the closed ball with radius δ and center x. If not specified otherwise, products of normed spaces will be considered with the maximum type norms.

The Fréchet normal cone to a subset $\Omega \subset X$ at $x \in \Omega$ and the Fréchet subdifferential of a function $f : X \to \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$ at a point x with $f(x) < \infty$ are defined, respectively, by

$$N_\Omega(x) = \left\{ x^* \in X^* \mid \limsup_{u \to x,\, u \in \Omega \setminus \{x\}} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \le 0 \right\},$$

$$\partial f(x) = \left\{ x^* \in X^* \mid \liminf_{u \to x,\, u \neq x} \frac{f(u) - f(x) - \langle x^*, u - x \rangle}{\|u - x\|} \ge 0 \right\}.$$

For a given set Ω in X, its interior and boundary are denoted, respectively, $\operatorname{int}\Omega$ and $\operatorname{bd}\Omega$. The indicator and distance functions associated with Ω are defined, respectively, by

$$\delta_\Omega(x) = \begin{cases} 0, & \text{if } x \in \Omega, \\ \infty, & \text{if } x \in X \setminus \Omega, \end{cases}$$

$$d(x, \Omega) = \inf_{\omega \in \Omega} \|x - \omega\|, \ \forall x \in X.$$

2.2 $[q]$-regularity properties of collections of sets

In this section, we discuss local $[q]$-regularity properties of finite collections of sets and their primal space characterizations.

In the sequel, $\boldsymbol{\Omega}$ stands for a collection $\{\Omega_1, \ldots, \Omega_m\}$ of m $(m \ge 2)$ sets in a normed linear space X, $\bar{x} \in \bigcap_{i=1}^m \Omega_i$, and, if not specified otherwise, $q \in (0, 1]$.

2.2.1 Definitions

The next definition introduces several mutually related regularity properties of $\boldsymbol{\Omega}$ at $\bar{x}$.

Definition 3. (i) $\boldsymbol{\Omega}$ is $[q]$-semiregular at $\bar{x}$ if there exist positive numbers α and δ such that

$$\bigcap_{i=1}^m (\Omega_i - x_i) \bigcap B_\rho(\bar{x}) \neq \emptyset \tag{2.1}$$

for all $\rho \in (0, \delta)$ and all $x_i \in X$ $(i = 1, \ldots, m)$ such that $\max_{1 \le i \le m} \|x_i\| \le (\alpha\rho)^{\frac{1}{q}}$.

(ii) $\boldsymbol{\Omega}$ is $[q]$-subregular at $\bar{x}$ if there exist positive numbers α and δ such that

$$\bigcap_{i=1}^m \left(\Omega_i + (\alpha\rho)^{\frac{1}{q}} \mathbb{B} \right) \bigcap B_\delta(\bar{x}) \subseteq \left(\bigcap_{i=1}^m \Omega_i \right) + \rho\mathbb{B} \tag{2.2}$$

for all $\rho \in (0, \delta)$.

(iii) $\boldsymbol{\Omega}$ is uniformly $[q]$-regular at $\bar{x}$ if there exist positive numbers α and δ such that

$$\bigcap_{i=1}^{m}(\Omega_i - \omega_i - x_i)\bigcap(\rho\mathbb{B}) \neq \emptyset \tag{2.3}$$

for all $\rho \in (0, \delta)$, $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$, and all $x_i \in X$ $(i = 1, \ldots, m)$ such that $\max\limits_{1\le i\le m} \|x_i\| \le (\alpha\rho)^{\frac{1}{q}}$.

When $q = 1$, we will skip "[1]" in the name of the corresponding property and write simply "semiregular", "subregular", or "uniformly regular", cf. [39, Definition 3.1].

Remark 13. Among the three regularity properties in Definition 3, the third one is the strongest. Indeed, condition (2.1) corresponds to taking $\omega_i = \bar{x}$ in (2.3). To compare properties (ii) and (iii), it is sufficient to notice that condition (2.2) is equivalent to the following one: for any $x \in B_\delta(\bar{x})$, $\omega_i \in \Omega_i$, $x_i \in X$ $(i = 1, \ldots, m)$ such that $\max\limits_{1\le i\le m} \|x_i\| \le (\alpha\rho)^{\frac{1}{q}}$, and $\omega_i + x_i = x$ $(i = 1, \ldots, m)$, it holds

$$\bigcap_{i=1}^{m}(\Omega_i - x)\bigcap(\rho\mathbb{B}) \neq \emptyset.$$

This corresponds to taking $\omega_i + x_i = x$ $(i = 1, \ldots, m)$ in (2.3) (with $x \in X$) and possibly choosing a smaller $\delta > 0$. Hence, (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii).

Properties (i) and (ii) in Definition 3 are in general independent – see examples in Subsection 2.2.3.

Remark 14. The larger the order q is, the stronger the properties in Definition 3 are.

Remark 15. When $\bar{x} \in \operatorname{int} \bigcap_{i=1}^{m} \Omega_i$, all the properties in Definition 3 hold true automatically for any $q \in (0, \infty)$.

Remark 16. When $\Omega_1 = \Omega_2 = \ldots = \Omega_m$ and $q \in (0, 1]$, property (ii) in Definition 3 is trivially satisfied (with $\alpha = \delta = 1$).

Normally, it does not make sense to consider properties (ii) and (iii) in Definition 3 when $q > 1$. In the next proposition, we assume temporarily that all properties in Definition 3 are defined for all $q > 1$.

Proposition 9. *Let the sets Ω_i $(i = 1, \ldots, m)$ be closed and $q > 1$.*

(i) $\mathbf{\Omega}$ *is $[q]$-subregular at $\bar{x}$ $\quad\Leftrightarrow\quad$ $\mathbf{\Omega}$ is uniformly $[q]$-regular at $\bar{x}$ $\quad\Leftrightarrow\quad$ $\bar{x} \in \operatorname{int} \bigcap_{i=1}^m \Omega_i$.*

(ii) If $\bar{x} \in \operatorname{int} \bigcap_{i=1}^m \Omega_i$, then $\mathbf{\Omega}$ is $[q]$-semiregular at $\bar{x}$.

(iii) If $\mathbf{\Omega}$ is $[q]$-semiregular at $\bar{x}$ and the sets of primal proximal normals *[52, Definition 4.28] $N^P_{\Omega_i}(\bar{x}) := \{u \in X \mid \exists r > 0,\ d(\bar{x} + ru, \Omega_i) = r\|u\|\}$ are nontrivial for all $i = 1, \ldots, m$ such that $\bar{x} \in \operatorname{bd} \Omega_i$, then $\bar{x} \in \operatorname{int} \bigcap_{i=1}^m \Omega_i$.*

Proof. (i) The implications $\bar{x} \in \operatorname{int} \bigcap_{i=1}^m \Omega_i \quad\Rightarrow\quad \mathbf{\Omega}$ is uniformly $[q]$-regular at $\bar{x} \quad\Rightarrow\quad \mathbf{\Omega}$ is $[q]$-subregular at $\bar{x}$ are obvious. Next we show that $\mathbf{\Omega}$ is $[q]$-subregular at $\bar{x} \quad\Rightarrow\quad \bar{x} \in \operatorname{int} \bigcap_{i=1}^m \Omega_i$.

Suppose $\bar{x} \notin \operatorname{int} \bigcap_{i=1}^m \Omega_i$ while $\mathbf{\Omega}$ is $[q]$-subregular at $\bar{x}$, i.e., there exist numbers $\alpha > 0$ and $\delta > 0$ such that condition (2.2) holds true for all $\rho \in (0, \delta)$. Consider a sequence $x_k \to \bar{x}$ such that $r_k := d(x_k, \bigcap_{i=1}^m \Omega_i) > 0$ $(k = 1, 2, \ldots)$. Then

$$x_k \in \bigcap_{i=1}^m \Omega_i + r_k(1 + r_k)\mathbb{B} \subseteq \bigcap_{i=1}^m (\Omega_i + r_k(1 + r_k)\mathbb{B})$$

and $x_k \in B_\delta(\bar{x})$ for all sufficiently large k. Denote $\rho_k := \alpha^{-1}(r_k(1+r_k))^q$. Then $\rho_k < \delta$ for all sufficiently large k, and it follows from (2.2) that $x_k \in \bigcap_{i=1}^m \Omega_i + \rho_k\mathbb{B}$. Hence, $r_k \leq \rho_k$, and consequently $\alpha \leq r_k^{q-1}(1 + r_k)^q$. Letting $k \to \infty$, we arrive at a contradiction: $0 < \alpha \leq 0$.

(ii) is obvious.

(iii) Suppose $\bar{x} \notin \operatorname{int} \bigcap_{i=1}^m \Omega_i$ and there exist numbers $\alpha \geq 0$ and $\delta > 0$ such that condition (2.1) holds true for all $\rho \in (0, \delta)$ and all $x_i \in X$ $(i = 1, \ldots, m)$ such that $\max_{1 \leq i \leq m} \|x_i\| \leq (\alpha\rho)^{\frac{1}{q}}$. Then $\bar{x} \in \operatorname{bd} \Omega_j$ for some j. Choose a nonzero $u \in N^P_{\Omega_j}(\bar{x})$. Then there exists a number $r > 0$ such that $d(\bar{x} + tu, \Omega_j) = t\|u\|$ for all $t \in [0, r]$ [52, p. 284]. Denote $\rho_t := t\|u\|$ and $x_t := (\alpha\rho_t)^{\frac{1}{q}} \frac{u}{\|u\|}$. Then $\rho_t < \delta$ and $(\alpha\rho_t)^{\frac{1}{q}}/\|u\| < r$ for all sufficiently small t. Hence, $d(\bar{x}, \Omega_j - x_t) = d(\bar{x} + x_t, \Omega_j) = (\alpha\rho_t)^{\frac{1}{q}}$, and it follows from (2.1) that $(\alpha\rho_t)^{\frac{1}{q}} \leq \rho_t$, and consequently $0 \leq \alpha \leq \rho_t^{q-1}$. Letting $t \downarrow 0$, we conclude that $\alpha = 0$, i.e., $\mathbf{\Omega}$ is not $[q]$-semiregular at $\bar{x}$. ☐

Remark 17. Unlike $[q]$-subregularity and $[q]$-uniform regularity, when $\bar{x} \notin \operatorname{int} \bigcap_{i=1}^m \Omega_i$, the property of $[q]$-semiregularity can be fulfilled with $q > 1$ if the assumption of the existence of nontrivial primal proximal normals in Proposition 9 is not satisfied – see Example 9 below.

The regularity properties in Definition 3 can be equivalently defined using the following nonnegative constants which provide quantitative characterizations of these properties:

$$\theta^q[\mathbf{\Omega}](\bar{x}) := \liminf_{\rho\downarrow 0} \frac{(\theta_\rho[\mathbf{\Omega}](\bar{x}))^q}{\rho}, \tag{2.4}$$

$$\zeta^q[\mathbf{\Omega}](\bar{x}) := \lim_{\delta\downarrow 0} \inf_{0<\rho<\delta} \frac{(\zeta_{\rho,\delta}[\mathbf{\Omega}](\bar{x}))^q}{\rho}, \tag{2.5}$$

$$\hat{\theta}^q[\mathbf{\Omega}](\bar{x}) := \liminf_{\substack{\omega_i\to\bar{x},\,\omega_i\in\Omega_i\ (i=1,\ldots,m)\\ \rho\downarrow 0}} \frac{(\theta_\rho[\Omega_1-\omega_1,\ldots,\Omega_m-\omega_m](0))^q}{\rho}, \tag{2.6}$$

where, for $\rho>0$ and $\delta>0$,

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \sup\left\{r\ge 0 \mid \bigcap_{i=1}^m(\Omega_i-x_i)\bigcap B_\rho(\bar{x})\ne\emptyset,\ \forall x_i\in r\mathbb{B}\right\}, \tag{2.7}$$

$$\zeta_{\rho,\delta}[\mathbf{\Omega}](\bar{x}) := \sup\left\{r\ge 0 \mid \bigcap_{i=1}^m(\Omega_i+r\mathbb{B})\bigcap B_\delta(\bar{x})\subseteq\bigcap_{i=1}^m\Omega_i+\rho\mathbb{B}\right\}. \tag{2.8}$$

When $q=1$, we will not write superscript 1 in the denotations (2.4) – (2.6).

Using the equivalent representation of condition (2.2) in Remark 13, it is not difficult to check that $\hat{\theta}^q[\mathbf{\Omega}](\bar{x})\le\min\{\theta^q[\mathbf{\Omega}](\bar{x}),\zeta^q[\mathbf{\Omega}](\bar{x})\}$.

The next proposition follows immediately from the definitions.

Proposition 10. *(i) $\mathbf{\Omega}$ is $[q]$-semiregular at $\bar{x}$ if and only if $\theta^q[\mathbf{\Omega}](\bar{x})>0$. Moreover, $\theta^q[\mathbf{\Omega}](\bar{x})$ is the exact upper bound of all numbers α such that (2.1) is satisfied.*

(ii) $\mathbf{\Omega}$ is $[q]$-subregular at $\bar{x}$ if and only if $\zeta^q[\mathbf{\Omega}](\bar{x})>0$. Moreover, $\zeta^q[\mathbf{\Omega}](\bar{x})$ is the exact upper bound of all numbers α such that (2.2) is satisfied.

(iii) $\mathbf{\Omega}$ is uniformly $[q]$-regular at $\bar{x}$ if and only if $\hat{\theta}^q[\mathbf{\Omega}](\bar{x})>0$. Moreover, $\hat{\theta}^q[\mathbf{\Omega}](\bar{x})$ is the exact upper bound of all numbers α such that (2.3) is satisfied.

Remark 18. With $q=1$, properties (i) and (iii) in Definition 3 were discussed in [34] (see also [35, Properties $(\mathrm{R})_S$ and $(\mathrm{UR})_S$]), while property (ii) was introduced in [39]. Constants (2.4), (2.6), and (2.7) (with $q=1$) can be traced back to [29, 30, 31, 32, 33, 27, 28].

The equivalent representation of constant (2.7) given in the next proposition can be useful.

Proposition 11. *[39, Proposition 3.8] For any $\rho > 0$,*

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \sup\left\{ r \geq 0 \mid r\mathbb{B}^m \subseteq \bigcup_{x\in B_\rho(\bar{x})} \prod_{i=1}^{m} (\Omega_i - x) \right\}, \tag{2.9}$$

where $\prod_{i=1}^{m}(\Omega_i - x) = (\Omega_1 - x) \times \ldots \times (\Omega_m - x)$ and $\mathbb{B}^m = \prod_{i=1}^{m} \mathbb{B}$.

From Propositions 10 and 11, we immediately obtain equivalent representations of $[q]$-semiregularity and $[q]$-uniform regularity.

Corollary 2. *(i) $\mathbf{\Omega}$ is $[q]$-semiregular at $\bar{x}$ if and only if there exist positive numbers α and δ such that*

$$(\alpha\rho)^{\frac{1}{q}}\mathbb{B}^m \subseteq \bigcup_{x\in B_\rho(\bar{x})} \prod_{i=1}^{m} (\Omega_i - x) \tag{2.10}$$

for all $\rho \in (0, \delta)$. Moreover, $\theta^q[\mathbf{\Omega}](\bar{x})$ is the exact upper bound of all numbers α such that (2.10) is satisfied.

(ii) $\mathbf{\Omega}$ is uniformly $[q]$-regular at $\bar{x}$ if and only if there exist positive numbers α and δ such that

$$(\alpha\rho)^{\frac{1}{q}}\mathbb{B}^m \subseteq \bigcap_{\substack{\omega_i\in\Omega_i\cap B_\delta(\bar{x})\\ (i=1,\ldots,m)}} \bigcup_{x\in\rho\mathbb{B}} \prod_{i=1}^{m} (\Omega_i - \omega_i - x) \tag{2.11}$$

for all $\rho \in (0, \delta)$. Moreover, $\hat{\theta}^q[\mathbf{\Omega}](\bar{x})$ is the exact upper bound of all numbers α such that (2.11) is satisfied.

2.2.2 Metric characterizations

The $[q]$-regularity properties of collections of sets in Definition 3 can also be characterized in metric terms. The next proposition generalizing [39, Proposition 3.15] provides equivalent metric representations of constants (2.4) – (2.6).

Proposition 12.

$$\theta^q[\mathbf{\Omega}](\bar{x}) = \liminf_{\substack{x_i \to 0\ (i=1,\ldots,m) \\ \bar{x} \notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1\le i\le m} \|x_i\|^q}{d\Big(\bar{x}, \bigcap_{i=1}^m (\Omega_i - x_i)\Big)}, \tag{2.12}$$

$$\begin{aligned}\zeta^q[\mathbf{\Omega}](\bar{x}) &= \liminf_{\substack{x \to \bar{x} \\ x \notin \bigcap_{i=1}^m \Omega_i}} \frac{\max_{1\le i\le m} d^q(x, \Omega_i)}{d\Big(x, \bigcap_{i=1}^m \Omega_i\Big)} \\ &= \liminf_{\substack{x \to \bar{x} \\ \omega_i \to \bar{x},\, \omega_i \in \Omega_i\ (i=1,\ldots,m) \\ x \notin \bigcap_{i=1}^m \Omega_i}} \frac{\max_{1\le i\le m} \|\omega_i - x\|^q}{d\Big(x, \bigcap_{i=1}^m \Omega_i\Big)},\end{aligned} \tag{2.13}$$

$$\begin{aligned}\hat{\theta}^q[\mathbf{\Omega}](\bar{x}) &= \liminf_{\substack{x \to \bar{x} \\ x_i \to 0\ (i=1,\ldots,m) \\ x \notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1\le i\le m} d^q(x + x_i, \Omega_i)}{d\Big(x, \bigcap_{i=1}^m (\Omega_i - x_i)\Big)} \\ &= \liminf_{\substack{x \to \bar{x} \\ x_i \to 0,\, \omega_i \to \bar{x},\, \omega_i \in \Omega_i\ (i=1,\ldots,m) \\ x \notin \bigcap_{i=1}^m (\Omega_i - x_i)}} \frac{\max_{1\le i\le m} \|x + x_i - \omega_i\|^q}{d\Big(x, \bigcap_{i=1}^m (\Omega_i - x_i)\Big)}.\end{aligned} \tag{2.14}$$

Proof. Equality (2.12). Let ξ stand for the right-hand side of (2.12). Suppose that $\xi > 0$ and fix an arbitrary number $\gamma \in (0, \xi)$. Then there is a number $\delta > 0$ such that

$$\gamma d\left(\bar{x}, \bigcap_{i=1}^m (\Omega_i - x_i)\right) \le \max_{1\le i\le m} \|x_i\|^q, \ \forall x_i \in \delta\mathbb{B}\ (i = 1, \ldots, m). \tag{2.15}$$

Choose a number $\alpha \in (0, \gamma)$ and set $\delta' = \frac{\delta^q}{\alpha}$. Then, for any $\rho \in (0, \delta')$ and $x_i \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$ $(i = 1, \ldots, m)$, it holds $\max_{1\le i\le m} \|x_i\| \le (\alpha\rho)^{\frac{1}{q}} \le (\alpha\delta')^{\frac{1}{q}} = \delta$. Hence, (2.15) yields

$$d\left(\bar{x}, \bigcap_{i=1}^m (\Omega_i - x_i)\right) \le \frac{1}{\gamma} \max_{1\le i\le m} \|x_i\|^q \le \frac{\alpha}{\gamma}\rho < \rho.$$

This implies (2.1) and consequently $\theta^q[\mathbf{\Omega}](\bar{x}) \ge \alpha$. Taking into account that α can be arbitrarily close to ξ, we obtain $\theta^q[\mathbf{\Omega}](\bar{x}) \ge \xi$.

Conversely, suppose that $\theta^q[\mathbf{\Omega}](\bar{x}) > 0$ and fix an arbitrary number $\alpha \in (0, \theta^q[\mathbf{\Omega}](\bar{x}))$. Then there is a number $\delta > 0$ such that (2.1) is satisfied for all $\rho \in (0, \delta)$ and $x_i \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$ $(i = 1, \ldots, m)$. Choose a positive $\delta' < (\alpha\delta)^{\frac{1}{q}}$. For any $x_i \in \delta'\mathbb{B}$ $(i = 1, \ldots, m)$, it holds $\max_{1\le i\le m} \|x_i\| < (\alpha\delta)^{\frac{1}{q}}$. Pick up a $\rho \in (0, \delta)$ such that $\max_{1\le i\le m} \|x_i\| = (\alpha\rho)^{\frac{1}{q}}$. Then (2.1) yields

$$\alpha d\left(\bar{x}, \bigcap_{i=1}^m (\Omega_i - x_i)\right) \le \alpha\rho = \max_{1\le i\le m} \|x_i\|^q.$$

This implies $\xi \geq \alpha$. Since α can be arbitrarily close to $\theta^q[\mathbf{\Omega}](\bar{x})$, we deduce $\xi \geq \theta^q[\mathbf{\Omega}](\bar{x})$.

Equality (2.13). Let ξ stand for the right-hand side of (2.13). Suppose that $\xi > 0$ and fix an arbitrary number $\alpha \in (0, \xi)$. Then there is a number $\delta > 0$ such that

$$\alpha d\left(x, \bigcap_{i=1}^{m} \Omega_i\right) \leq \max_{1\leq i\leq m} d^q(x, \Omega_i), \ \forall x \in B_\delta(\bar{x}).$$

If $x \in \bigcap_{i=1}^m \left(\Omega_i + (\alpha\rho)^{\frac{1}{q}}\mathbb{B}\right) \bigcap B_\delta(\bar{x})$ for some $\rho \in (0, \delta)$, then $\max_{1\leq i\leq m} d^q(x, \Omega_i) \leq \alpha\rho$, and consequently $d\left(x, \bigcap_{i=1}^m \Omega_i\right) \leq \rho$, i.e., $\zeta_{\rho,\delta}[\mathbf{\Omega}](\bar{x}) \geq (\alpha\rho)^{\frac{1}{q}}$. Hence, $\zeta^q[\mathbf{\Omega}](\bar{x}) \geq \alpha$. Since α can be arbitrarily close to ξ, we obtain $\zeta^q[\mathbf{\Omega}](\bar{x}) \geq \xi$.

Conversely, suppose that $\zeta^q[\mathbf{\Omega}](\bar{x}) > 0$ and fix any $\alpha \in (0, \zeta^q[\mathbf{\Omega}](\bar{x}))$. Then there is a number $\delta > 0$ such that (2.2) is satisfied for all $\rho \in (0, \delta)$. Choose a positive number $\delta' < \min\{(\alpha\delta)^{\frac{1}{q}}, \delta\}$. For any $x \in B_{\delta'}(\bar{x})$, it holds

$$\max_{1\leq i\leq m} d(x, \Omega_i) \leq \|x - \bar{x}\| \leq \delta' < (\alpha\delta)^{\frac{1}{q}}.$$

Choose a $\rho \in (0, \delta)$ such that $\max_{1\leq i\leq m} d(x, \Omega_i) = (\alpha\rho)^{\frac{1}{q}}$. Then, by (2.2),

$$\alpha d\left(x, \bigcap_{i=1}^{m} \Omega_i\right) \leq \alpha\rho = \max_{1\leq i\leq m} d^q(x, \Omega_i).$$

Hence, $\alpha \leq \xi$. By letting $\alpha \to \zeta^q[\mathbf{\Omega}](\bar{x})$, we obtain $\zeta^q[\mathbf{\Omega}](\bar{x}) \leq \xi$.

Equality (2.14). Let ξ stand for the right-hand side of (2.14). Suppose that $\xi > 0$ and fix an arbitrary number $\gamma \in (0, \xi)$. Then there is a number $\delta > 0$ such that

$$\gamma d\left(x, \bigcap_{i=1}^{m} (\Omega_i - x_i)\right) \leq \max_{1\leq i\leq m} d^q(x + x_i, \Omega_i) \tag{2.16}$$

for any $x \in B_\delta(\bar{x})$ and $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, m)$. Fix any positive number $\alpha < \gamma$ and pick up a positive number δ' satisfying $\delta' + (\alpha\delta')^{\frac{1}{q}} \leq \delta$. Then, for any $\rho \in (0, \delta']$, $\omega_i \in \Omega_i \cap B_{\delta'}(\bar{x})$ and $a_i \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$ $(i = 1, \ldots, m)$, it holds

$$\|\omega_i - \bar{x} + a_i\| \leq \delta' + (\alpha\rho)^{\frac{1}{q}} \leq \delta' + (\alpha\delta')^{\frac{1}{q}} \leq \delta.$$

Applying (2.16) with $x = \bar{x}$ and $x_i = \omega_i - \bar{x} + a_i$, we get

$$d\left(0, \bigcap_{i=1}^{m}(\Omega_i - \omega_i - a_i)\right) \leq \gamma^{-1} \max_{1\leq i\leq m} d^q(\omega_i + a_i, \Omega_i)$$
$$\leq \gamma^{-1} \max_{1\leq i\leq m} \|a_i\|^q \leq \frac{\alpha}{\gamma}\rho < \rho.$$

Hence, (2.3) holds true and consequently $\hat{\theta}^q[\mathbf{\Omega}](\bar{x}) \geq \alpha$. Taking into account that α can be arbitrarily close to ξ, we obtain $\hat{\theta}^q[\mathbf{\Omega}](\bar{x}) \geq \xi$.

Conversely, suppose that $\hat{\theta}^q[\mathbf{\Omega}](\bar{x}) > 0$ and fix an arbitrary number $\alpha \in (0, \hat{\theta}^q[\mathbf{\Omega}](\bar{x}))$. Then there is some number $\delta > 0$ such that (2.3) is satisfied for all $\rho \in (0, \delta]$, $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $a_i \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$ $(i = 1, \ldots, m)$. We pick up some $\delta' > 0$ satisfying

$$(\delta'\alpha + (\delta')^q)^{\frac{1}{q}} + \frac{(\delta')^q}{\alpha} + 2\delta' < \delta. \tag{2.17}$$

Now, for $x \in B_{\delta'}(\bar{x})$ and $x_i \in \delta'\mathbb{B}$ $(i = 1, \ldots, m)$, we consider two cases.

Case 1. There exists some $j \in \{1, \ldots, m\}$ such that

$$d(x + x_j, \Omega_j) \geq (\delta'\alpha + (\delta')^q)^{\frac{1}{q}}.$$

Take $\rho = \frac{(\delta')^q}{\alpha} < \delta$, $\omega_i = \bar{x}$, $a_i = x_i$ $(i = 1, \ldots, m)$. Then $\|a_i\| \leq \delta' = (\alpha\rho)^{\frac{1}{q}}$. Applying (2.3), we find points

$$x'' \in \bigcap_{i=1}^{m}(\Omega_i - \bar{x} - x_i) \bigcap (\rho\mathbb{B})$$

and

$$x' := \bar{x} + x'' \in \bigcap_{i=1}^{m}(\Omega_i - x_i) \bigcap B_\rho(\bar{x}).$$

Hence,

$$d\left(x, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \leq \|x - x'\| \leq \|x - \bar{x}\| + \|x''\|$$
$$\leq \delta' + \rho = \frac{1}{\alpha}(\delta'\alpha + (\delta')^q)$$
$$\leq \frac{1}{\alpha} \max_{1\leq i\leq m} d^q(x + x_i, \Omega_i),$$

and consequently

$$\alpha d\left(x, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \le \max_{1\le i\le m} d^q(x + x_i, \Omega_i). \tag{2.18}$$

Case 2. $\max_{1\le i\le m} d(x + x_i, \Omega_i) < (\delta'\alpha + (\delta')^q)^{\frac{1}{q}}$.

Choose $\omega_i \in \Omega_i$ $(i = 1, \ldots, m)$ such that

$$\|x + x_i - \omega_i\| < (\delta'\alpha + (\delta')^q)^{\frac{1}{q}}.$$

Then, thanks to (7.3),

$$\|\omega_i - \bar{x}\| \le \|\omega_i - x - x_i\| + \|x_i\| + \|x - \bar{x}\| < (\delta'\alpha + (\delta')^q)^{\frac{1}{q}} + 2\delta' < \delta.$$

Setting

$$a_i := x + x_i - \omega_i \; (i = 1, \ldots, m), \;\; \rho := \frac{1}{\alpha} \max_{1\le i\le m} \|a_i\|^q,$$

we have

$$\rho < \frac{\delta'\alpha + (\delta')^q}{\alpha} < \delta, \;\; \|a_i\| \le (\alpha\rho)^{\frac{1}{q}} \; (i = 1, \ldots, m).$$

Applying (2.3) again, we find points

$$x'' \in \bigcap_{i=1}^{m}(\Omega_i - x - x_i) \bigcap (\rho\mathbb{B})$$

and

$$x' := x + x'' \in \bigcap_{i=1}^{m}(\Omega_i - x_i) \bigcap B_\rho(x).$$

Hence,

$$d\left(x, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \le \|x - x'\| \le \rho = \frac{1}{\alpha} \max_{1\le i\le m} \|x + x_i - \omega_i\|^q.$$

Taking infimum in the right-hand side of the last inequality over $\omega_i \in \Omega_i$ $(i = 1, \ldots, m)$, we again arrive at (2.18).

From (2.18) we conclude that $\alpha \le \xi$. Since α can be arbitrarily close to $\hat{\theta}^q[\mathbf{\Omega}](\bar{x})$, we deduce $\hat{\theta}^q[\mathbf{\Omega}](\bar{x}) \le \xi$.

The second equalities in the representations of $\zeta^q[\mathbf{\Omega}](\bar{x})$ and $\hat{\theta}^q[\mathbf{\Omega}](\bar{x})$ are straightforward.

□

Propositions 10 and 12 imply equivalent metric characterizations of the $[q]$-regularity properties of collections of sets.

Theorem 4. *(i)* $\mathbf{\Omega}$ *is* $[q]$*-semiregular at* $\bar{x}$ *if and only if it is* metrically $[q]$-semiregular *at* $\bar{x}$*, i.e., there exist positive numbers* γ *and* δ *such that*

$$\gamma d\left(\bar{x}, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \leq \max_{1\leq i\leq m} \|x_i\|^q, \ \forall x_i \in \delta\mathbb{B}\ (i=1,\ldots,m). \tag{2.19}$$

Moreover, $\theta^q[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* γ *such that* (2.19) *is satisfied.*

(ii) $\mathbf{\Omega}$ *is* $[q]$*-subregular at* $\bar{x}$ *if and only if it is* metrically $[q]$-subregular *at* $\bar{x}$*, i.e., there exist positive numbers* γ *and* δ *such that*

$$\gamma d\left(x, \bigcap_{i=1}^{m}\Omega_i\right) \leq \max_{1\leq i\leq m} d^q(x,\Omega_i), \ \forall x \in B_\delta(\bar{x}). \tag{2.20}$$

Moreover, $\zeta^q[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* γ *such that* 2.20 *is satisfied.*

(iii) $\mathbf{\Omega}$ *is uniformly* $[q]$*-regular at* $\bar{x}$ *if and only if it is* metrically uniformly $[q]$-regular *at* $\bar{x}$*, i.e., there exist positive numbers* γ *and* δ *such that*

$$\gamma d\left(x, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right) \leq \max_{1\leq i\leq m} d^q(x+x_i,\Omega_i) \tag{2.21}$$

for any $x \in B_\delta(\bar{x})$*,* $x_i \in \delta\mathbb{B}$ $(i=1,\ldots,m)$*. Moreover,* $\hat{\theta}^q[\mathbf{\Omega}](\bar{x})$ *is the exact upper bound of all numbers* γ *such that* (2.21) *is satisfied.*

Remark 19. With $q=1$, property (2.20) in the above theorem is known as the local *linear regularity*, *linear coherence*, or *metric inequality* [5, 6, 7, 8, 9, 10, 12, 23, 24, 26, 42, 43, 45, 50, 52, 58, 61]. It was used as the key condition when establishing linear convergence rates of sequences generated by cyclic projection algorithms and a qualification condition for subdifferential and normal cone calculus formulae. The stronger property (2.21) is sometimes referred to as *unform metric inequality* [33, 34, 35]. Property (2.19) with $q=1$ was investigated in [39].

2.2.3 Examples

In this subsection, we give several examples illustrating the discussed above regularity properties. We consider collections of two sets in $\mathbb{R}^2$ having a common point $\bar{x} = (0, 0)$. In the figures below (except Figure 2.4), the two sets are coloured cyan and yellow, respectively, while their intersection is coloured green.

Below we give two examples of collections of sets that do not satisfy certain q-regularity properties when $q = 1$, while the corresponding properties are fulfilled when $q = \frac{1}{2}$.

Example 6. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 := \left\{(u, v) \in \mathbb{R}^2 \mid v \geq 0\right\}, \quad \Omega_2 := \left\{(u, v) \in \mathbb{R}^2 \mid v \leq u^2\right\},$$

and the point $\bar{x} = (0, 0) \in \Omega_1 \cap \Omega_2$ (Figure 2.1). The collection $\{\Omega_1, \Omega_2\}$ is not semiregular at $\bar{x}$, while the $\left[\frac{1}{2}\right]$-semiregularity is satisfied at this point.

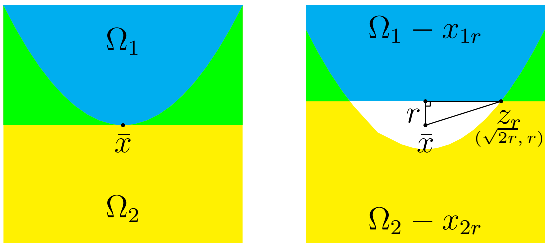

Figure 2.1: Semiregularity vs $[\frac{1}{2}]$-semiregularity

Proof. This example is taken from [35, Figure 8]. We first observe that, for any $r \in (0, 1)$ and all $x_1, x_2 \in r\mathbb{B}$, it holds

$$(\Omega_1 - x_1) \cap (\Omega_2 - x_2) \supseteq (\Omega_1 - x_{1r}) \cap (\Omega_2 - x_{2r}),$$

where $x_{1r} = (0, -r)$ and $x_{2r} = (0, r)$. Besides,

$$z_r := (\sqrt{2r}, r) \in (\Omega_1 - x_{1r}) \cap (\Omega_2 - x_{2r}),$$
$$d\left(\bar{x}, (\Omega_1 - x_{1r}) \cap (\Omega_2 - x_{2r})\right) = \|z_r\| = \sqrt{2r + r^2}.$$

Hence, by (2.7), for $\rho \in (0,1)$, we have

$$\theta_\rho[\{\Omega_1,\Omega_2\}](\bar{x}) = \sup\left\{r \geq 0 \mid \sqrt{2r+r^2} \leq \rho\right\} = \sqrt{1+\rho^2}-1,$$

and consequently, by (2.4),

$$\theta[\{\Omega_1,\Omega_2\}](\bar{x}) = \lim_{\rho\downarrow 0} \frac{\sqrt{1+\rho^2}-1}{\rho} = 0,$$
$$\theta^{\frac{1}{2}}[\{\Omega_1,\Omega_2\}](\bar{x}) = \lim_{\rho\downarrow 0} \frac{(\sqrt{1+\rho^2}-1)^{\frac{1}{2}}}{\rho} = \frac{1}{\sqrt{2}},$$

which means that $\{\Omega_1,\Omega_2\}$ is not semiregular at $\bar{x}$, while it is $\left[\frac{1}{2}\right]$-semiregular at this point.

One can easily show that $\theta_\rho[\{\Omega_1-\omega_1,\Omega_2-\omega_2\}](0) \geq \theta_\rho[\{\Omega_1,\Omega_2\}](\bar{x})$ for any $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, and consequently, by (2.6), $\hat{\theta}^{\frac{1}{2}}[\{\Omega_1,\Omega_2\}](\bar{x}) = \theta^{\frac{1}{2}}[\{\Omega_1,\Omega_2\}](\bar{x})$ and $\{\Omega_1,\Omega_2\}$ is even $\left[\frac{1}{2}\right]$-uniformly regular at $\bar{x}$.

Observe also that, for any $x \in \mathbb{R}^2$, $\max_{i=1,2} d(x,\Omega_i) = d(x,\Omega_1\cap\Omega_2)$, and consequently, by (2.13), $\zeta[\{\Omega_1,\Omega_2\}](\bar{x}) = 1$ and $\{\Omega_1,\Omega_2\}$ is subregular at $\bar{x}$. □

Example 7. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 := \left\{(x,x^2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\}, \quad \Omega_2 := \left\{(x,-x^2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\},$$

and the point $\bar{x} = (0,0) \in \Omega_1\cap\Omega_2$ (Figure 2.2). The collection $\{\Omega_1,\Omega_2\}$ is not subregular at $\bar{x}$, while the $\left[\frac{1}{2}\right]$-subregularity is satisfied at this point.

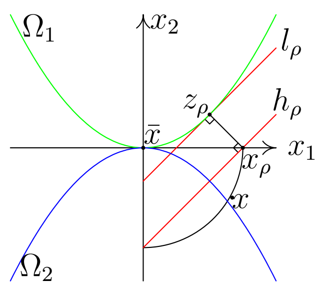

Figure 2.2: Subregularity vs $\left[\frac{1}{2}\right]$-subregularity

Proof. We first check that, for each number $\rho \in \left(0, \frac{1}{2}\right)$,

$$\min\left\{\max_{i=1,2} d(x, \Omega_i) \mid x \in \mathbb{R}^2, \|x\| = \rho\right\} = d(x_\rho, \Omega_1) = d(x_\rho, \Omega_2),$$

where $x_\rho := (\rho, 0)$. By the symmetry of the sets, it suffices to show that

$$\min\left\{d(x, \Omega_1) \mid x = (x_1, x_2) \in \mathbb{R}^2, \|x\| = \rho, x_1 \geq 0, x_2 \leq 0\right\} = d(x_\rho, \Omega_1). \tag{2.22}$$

Denote $z_\rho = (a, a^2) := P_{\Omega_1}(x_\rho)$ (the metric projection of x_ρ onto Ω_1). Then, with $f(x) = x^2$, we have $f'(z_\rho) \leq 1 = f'\left(\frac{1}{2}\right)$ for any $\rho \in \left(0, \frac{1}{2}\right)$. Thus, the lines h_ρ and l_ρ through x_ρ and z_ρ, respectively, with the slope $f'(z_\rho)$ separate the constraint set in (2.22) and Ω_1 and consequently, for any x in the constraint set in (2.22), it holds

$$d(x, \Omega_1) \geq d(x, l_\rho) \geq d(h_\rho, l_\rho) = d(x_\rho, \Omega_1),$$

which proves (2.22). One can easily check that $\rho = 2a^3 + a$ and $d(x_\rho, z_\rho) = \sqrt{4a^6 + a^4}$. Hence, by (2.13),

$$\begin{aligned}
\zeta[\{\Omega_1, \Omega_2\}](\bar{x}) &= \lim_{\rho \downarrow 0} \frac{d(x_\rho, z_\rho)}{\rho} = \lim_{a \downarrow 0} \frac{\sqrt{4a^6 + a^4}}{2a^3 + a} = 0, \\
\zeta^{\frac{1}{2}}[\{\Omega_1, \Omega_2\}](\bar{x}) &= \lim_{\rho \downarrow 0} \frac{d^{\frac{1}{2}}(x_\rho, z_\rho)}{\rho} = \lim_{a \downarrow 0} \frac{\sqrt[4]{4a^6 + a^4}}{2a^3 + a} = 1,
\end{aligned}$$

which means that $\{\Omega_1, \Omega_2\}$ is not subregular at $\bar{x}$, while it is $\left[\frac{1}{2}\right]$-subregular at this point.

Observe also that $(\Omega_1 - (0, -\varepsilon)) \cap (\Omega_2 - (0, \varepsilon)) = \emptyset$ for any $\varepsilon > 0$. Hence, by (2.7) and (2.4), $\{\Omega_1, \Omega_2\}$ is not $[q]$-semiregular at $\bar{x}$ for any $q > 0$. □

The above two examples show, in particular, that a collection of sets can be $[q]$-subregular at some point while not being $[q]$-semiregular at this point. In fact, these two regularity properties are independent. Next we give an example of a collection of sets that is semiregular at some point while it is not subregular at this point.

Example 8. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 := \left\{(u, v) \in \mathbb{R}^2 \mid u \leq 0 \text{ or } v \geq u^2\right\}, \; \Omega_2 := \left\{(u, v) \in \mathbb{R}^2 \mid u \leq 0 \text{ or } v \leq -u^2\right\},$$

and the point $\bar{x} = (0, 0) \in \Omega_1 \cap \Omega_2$ (Figure 2.3). The collection $\{\Omega_1, \Omega_2\}$ is semiregular at $\bar{x}$, while it is not subregular at this point.

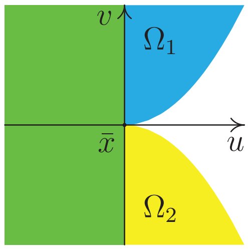

Figure 2.3: Subregularity vs Semiregularity

Proof. The proof of the absence of the subregularity in this example does not differ from that in Example 7. Next we show that $\{\Omega_1, \Omega_2\}$ is semiregular at $\bar{x}$. For any number $\rho > 0$, we set $x_\rho := (-\rho, 0)$. Then $B_\rho(x_\rho) \subseteq \Omega_i$, i.e., $x_\rho + x_i \in \Omega_i$ for any $x_i \in \rho\mathbb{B}$ $(i = 1, 2)$, and consequently

$$x_\rho \in (\Omega_1 - x_1) \cap (\Omega_2 - x_2) \cap B_\rho(\bar{x}), \ \forall x_i \in \rho\mathbb{B} \ (i = 1, 2).$$

Hence, $\theta_\rho[\{\Omega_1, \Omega_2\}](\bar{x}) \geq \rho$ and $\theta[\{\Omega_1, \Omega_2\}](\bar{x}) \geq 1$. (One can show that these are actually equalities.) Thus, $\{\Omega_1, \Omega_2\}$ is semiregular at $\bar{x}$. □

Example 9. In the real plane $\mathbb{R}^2$ with the Euclidean norm, consider two sets

$$\Omega_1 := \left\{(u, v) \in \mathbb{R}^2 \mid u \leq 0 \text{ or } |v| \geq u^2\right\}$$

(Figure 2.4) and $\Omega_2 := \mathbb{R}^2$, and the point $\bar{x} = (0, 0) \in \Omega_1 \cap \Omega_2$. The collection $\{\Omega_1, \Omega_2\}$ is q-semiregular at $\bar{x}$ for any $q \in (0, 1]$.

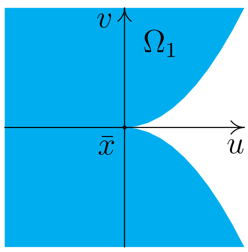

Figure 2.4: q-semiregularity

Proof. Obviously $\Omega_2 - x = \Omega_2 = \mathbb{R}^2$ for any $x \in \mathbb{R}^2$. Given a $\rho > 0$ and an $r \geq 0$, using the computations in Example 7, one can show that $(\Omega_1 - x) \bigcap B_\rho(\bar{x}) \neq \emptyset$ for all $x \in r\mathbb{B}$ if and only if $r \leq 2a^3 + a$ where a positive number a satisfies $4a^6 + a^4 = \rho^2$. Hence, $\theta_\rho[\{\Omega_1, \Omega_2\}](\bar{x}) = 2a^3 + a$ where $4a^6 + a^4 = \rho^2$ and consequently

$$\theta^q[\{\Omega_1, \Omega_2\}](\bar{x}) = \lim_{a \downarrow 0} \frac{(2a^3 + a)^q}{a^2\sqrt{4a^2 + 1}} = +\infty,$$

i.e., the collection $\{\Omega_1, \Omega_2\}$ is q-semiregular at $\bar{x}$ for any $q \in (0, 1]$.

Note that in fact the q-semiregularity condition is satisfied for any $q \leq 2$. □

2.3 Dual characterizations

This section discusses dual characterizations of $[q]$-regularity properties ($q \in (0, 1]$) of a collection of sets $\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ at $\bar{x} \in \bigcap_{i=1}^m \Omega_i$. We are going to use the notation $\widehat{\Omega} := \Omega_1 \times \ldots \times \Omega_m \subset X^m$.

Recall that the (normalized) *duality mapping* [46, Definition 3.2.6] J between a normed space Y and its dual Y^* is defined as

$$J(y) := \{y^* \in \mathbb{S}_{Y^*} \mid \langle y^*, y \rangle = \|y\|\}, \ \forall y \in Y.$$

Note that $J(-y) = -J(y)$.

The following simple fact of convex analysis is well known (cf., e.g., [56, Corollary 2.4.16]).

Lemma 6. *Let $(Y, \|\cdot\|)$ be a normed space.*

(i) $\partial\|\cdot\|(y) = J(y)$ for any $y \neq 0$.

(ii) $\partial\|\cdot\|(0) = \mathbb{B}^*$.

Making use of the convention that the topology in X^m is defined by the maximum type norm, it is not difficult to establish a representation of the duality mapping on X^m (cf. [39, Proposition 4.2]).

Proposition 13. *For each* $(x_1,\ldots,x_m)\in X^m$,

$$J(x_1,\ldots,x_m)=\Big\{(x_1^*,\ldots,x_m^*)\in(X^*)^m \mid \sum_{i=1}^m\|x_i^*\|=1;\ \textit{either } x_i^*=0 \\ \textit{or } \Big(\|x_i\|=\max_{1\le j\le m}\|x_j\|,\ x_i^*\in\|x_i^*\|J(x_i)\Big)\ (i=1,\ldots,m)\Big\}.$$

In this section, along with the maximum type norm on $X^{m+1}=X\times X^m$, we are going to use another one depending on a parameter $\rho>0$ and defined as follows:

$$\|(x,\hat{x})\|_\rho:=\max\{\|x\|,\rho\|\hat{x}\|\},\quad x\in X,\ \hat{x}\in X^m. \tag{2.23}$$

It is easy to check that the corresponding dual norm has the following representation:

$$\|(x^*,\hat{x}^*)\|_\rho=\|x^*\|+\rho^{-1}\|\hat{x}^*\|,\quad x^*\in X^*,\ \hat{x}^*\in(X^m)^*. \tag{2.24}$$

Note that if, in (2.23) and (2.24), $\hat{x}=(x_1,\ldots,x_m)$ and $\hat{x}^*=(x_1^*,\ldots,x_m^*)$ with $x_i\in X$ and $x_i^*\in X^*$ $(i=1,2,\ldots,m)$, then $\|\hat{x}\|=\max_{1\le i\le m}\|x_i\|$ and $\|\hat{x}^*\|=\sum_{i=1}^m\|x_i^*\|$.

The next few facts of subdifferential calculus are used in the proof of the main theorem below.

Lemma 7 ([39], Lemma 4.3)**.** *Let X be a normed space and $\varphi(u,\hat{u})=\|(u-u_1,\ldots,u-u_m)\|$ $(u\in X,\ \hat{u}:=(u_1,\ldots,u_m)\in X^m)$. Suppose $x\in X$, $\hat{x}:=(x_1,\ldots,x_m)\in X^m$, and $\hat{v}:=(x-x_1,\ldots,x-x_m)\neq 0$. Then*

$$\partial\varphi(x,\hat{x})\subseteq\big\{(x^*,\hat{x}^*)\in X^*\times(X^*)^m \mid -\hat{x}^*\in J(\hat{v}), \\ \hat{x}^*=(x_1^*,\ldots,x_m^*),\ x^*=-(x_1^*+\ldots+x_m^*)\big\}.$$

Lemma 8. *Let X be a normed space, $\varphi:X\to\mathbb{R}_\infty$, $q>0$, and $f(u):=(\varphi(u))^q$ $(u\in X)$. If $x\in X$ and $\varphi(x)\neq 0$, then $\partial f(x)=q(\varphi(x))^{q-1}\partial\varphi(x)$.*

Proof. follows from the standard chain rule for Fréchet subdifferentials, cf., e. g., [31, Corollary 1.14.1]. □

Lemma 9. *Let X be a normed space and $\hat{\omega} := (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$. Then $N_{\widehat{\Omega}}(\hat{\omega}) = N_{\Omega_1}(\omega_1) \times \ldots \times N_{\Omega_m}(\omega_m)$.*

Proof. follows directly from the definition of the Fréchet normal cone. □

The proof of the main theorem of this section relies heavily on two fundamental results of variational analysis: the *Ekeland variational principle* (Ekeland [16]; cf., e.g., [31, Theorem 2.1], [49, Theorem 2.26]) and the *fuzzy (approximate) sum rule* (Fabian [17]; cf., e.g., [31, Rule 2.2], [49, Theorem 2.33]). Below we provide these results for completeness.

Lemma 10 (Ekeland variational principle)**.** *Suppose X is a complete metric space, and $f : X \to \mathbb{R}_\infty$ is lower semicontinuous and bounded from below, $\varepsilon > 0, \lambda > 0$. If*

$$f(v) < \inf_X f + \varepsilon,$$

then there exists $x \in X$ such that

(a) $d(x, v) < \lambda$,

(b) $f(x) \leq f(v)$,

(c) $f(u) + (\varepsilon/\lambda)d(u, x) \geq f(x)$ for all $u \in X$.

Lemma 11 (Fuzzy sum rule)**.** *Suppose X is Asplund, $f_1 : X \to \mathbb{R}$ is Lipschitz continuous and $f_2 : X \to \mathbb{R}_\infty$ is lower semicontinuous in a neighborhood of $\bar{x}$ with $f_2(\bar{x}) < \infty$. Then, for any $\varepsilon > 0$, there exist $x_1, x_2 \in X$ with $\|x_i - \bar{x}\| < \varepsilon$, $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ $(i = 1, 2)$ such that*

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(x_1) + \partial f_2(x_2) + \varepsilon\mathbb{B}^*.$$

The next theorem gives dual sufficient conditions for $[q]$-regularity of collections of sets in Asplund spaces. Recall that a Banach space is called *Asplund* if any continuous convex function defined on a nonempty open convex set is Fréchet differentiable on a dense subset of its domain. Asplund spaces form a broad subclass of Banach spaces including, e. g., all spaces which admit Fréchet differentiable re-norms (in particular, Fréchet smooth spaces). Reflexive spaces are examples of Fréchet smooth spaces. Asplund property of a Banach space is necessary and sufficient for the fulfillment of some basic results involving Fréchet normals and subdifferentials (cf. [31, 49]). See [53] for various properties and characterizations of Asplund spaces.

Theorem 5. *Let X be an Asplund space and $\Omega_1, \ldots \Omega_m$ be closed.*

(i) $\boldsymbol{\Omega}$ *is $[q]$-subregular at $\bar{x}$ if there exist positive numbers α and δ such that, for any $\rho \in (0, \delta)$, $x \in B_\rho(\bar{x})$, $\omega_i \in \Omega_i \cap B_\rho(x)$ $(i = 1, \ldots, m)$ with $\omega_j \neq x$ for some $j \in \{1, \ldots, m\}$, there is an $\varepsilon > 0$ such that, for any $x' \in B_\varepsilon(x)$, $\hat{\omega}'_i \in \Omega_i \cap B_\varepsilon(\omega_i)$, $x_i^* \in N_{\Omega_i}(\omega'_i) + \rho\mathbb{B}^*$ $(i = 1, \ldots, m)$ satisfying $\hat{v} := (\omega'_1 - x', \ldots, \omega'_m - x') \neq 0$ and*

$$x_i^* = 0 \quad \text{if} \quad \|x' - \omega'_i\| < \max_{1 \le j \le m} \|x' - \omega'_j\|,$$
$$\langle x_i^*, x' - \omega'_i \rangle \ge \|x_i^*\|(\|x' - \omega'_i\| - \varepsilon),$$
$$\sum_{i=1}^m \|x_i^*\| = q\, \|\hat{v}\|^{q-1},$$

it holds

$$\left\| \sum_{i=1}^m x_i^* \right\| > \alpha. \tag{2.25}$$

(ii) $\boldsymbol{\Omega}$ *is uniformly $[q]$-regular at $\bar{x}$ if there are positive numbers α and δ such that (2.25) holds true for all $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i^* \in N_{\Omega_i}(\omega_i)$ $(i = 1, \ldots, m)$ satisfying $\sum_{i=1}^m \|x_i^*\| = 1$. The inverse implication holds true when $q = 1$.*

The proof of Theorem 5 (i) consists of a series of propositions providing lower estimates for constant (2.13) and, thus, sufficient conditions for $[q]$-subregularity of $\boldsymbol{\Omega}$ which can be of independent interest. Observe that constant (2.13) can be rewritten as

$$\zeta^q[\boldsymbol{\Omega}](\bar{x}) = \liminf_{\substack{x \to \bar{x},\, \omega_i \to \bar{x}\ (i=1,\ldots,m) \\ \hat{\omega} := (\omega_1, \ldots, \omega_m) \\ x \notin \bigcap_{i=1}^m \Omega_i}} \frac{f_q(x, \hat{\omega})}{d\left(x, \bigcap_{i=1}^m \Omega_i\right)} \tag{2.26}$$

with function $f_q : X^{m+1} \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ defined as

$$f_q(x, \hat{x}) = \max_{1 \le i \le m} \|x - x_i\|^q + \delta_{\widehat{\Omega}}(\hat{x}), \quad x \in X,\ \hat{x} := (x_1, \ldots, x_m) \in X^m, \tag{2.27}$$

where $\delta_{\widehat{\Omega}}$ is the indicator function of $\widehat{\Omega}$: $\delta_{\widehat{\Omega}}(\hat{x}) = 0$ if $\hat{x} \in \widehat{\Omega}$ and $\delta_{\widehat{\Omega}}(\hat{x}) = +\infty$ otherwise.

Proposition 14. *Let X be a Banach space and $\Omega_1, \ldots, \Omega_m$ be closed.*

(i) $\hat{\zeta}^q[\mathbf{\Omega}](\bar{x}) \le \zeta^q[\mathbf{\Omega}](\bar{x})$, *where*

$$\hat{\zeta}^q[\mathbf{\Omega}](\bar{x}) := \lim_{\rho \downarrow 0} \inf_{\substack{\|x-\bar{x}\|<\rho \\ \hat{\omega}=(\omega_1,\ldots,\omega_m)\in\widehat{\Omega} \\ 0<\max\limits_{1\le i\le m}\|x-\omega_i\|<\rho}} \zeta^q_\rho[\mathbf{\Omega}](x,\hat{\omega}) \tag{2.28}$$

and, for $x \in X$ and $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$,

$$\zeta^q_\rho[\mathbf{\Omega}](x,\hat{\omega}) := \limsup_{\substack{(u,\hat{v})\to(x,\hat{\omega}) \\ (u,\hat{v})\neq(x,\hat{\omega}) \\ \hat{v}=(v_1,\ldots,v_m)\in\widehat{\Omega}}} \frac{\left(\max\limits_{1\le i\le m}\|x-\omega_i\|^q - \max\limits_{1\le i\le m}\|u-v_i\|^q\right)_+}{\|(u,\hat{v})-(x,\hat{\omega})\|_\rho}. \tag{2.29}$$

(ii) *If $\hat{\zeta}^q[\mathbf{\Omega}](\bar{x}) > 0$, then $\mathbf{\Omega}$ is $[q]$-subregular at $\bar{x}$.*

Proof. (i) Let $\zeta^q[\mathbf{\Omega}](\bar{x}) < \alpha < \infty$. Choose a $\rho \in (0,1)$ and set

$$\eta := \min\left\{\frac{\rho}{2}, \frac{\rho}{\alpha}, \rho^{\frac{2}{\rho}}\right\}. \tag{2.30}$$

By (2.26), there are $x' \in B_\eta(\bar{x})$ and $\hat{\omega}' = (\omega'_1, \ldots, \omega'_m) \in \widehat{\Omega}$ such that

$$0 < f_q(x', \hat{\omega}') < \alpha d\left(x', \bigcap_{i=1}^m \Omega_i\right). \tag{2.31}$$

Denote $\varepsilon := f_q(x', \hat{\omega}')$ and $\mu := d\left(x', \bigcap_{i=1}^m \Omega_i\right)$. Then $\mu \le \|x' - \bar{x}\| \le \eta \le \frac{\rho}{2} < 1$. Observe that f_q is lower semicontinuous. Applying to f_q Lemma 10 with ε as above and

$$\lambda := \mu(1 - \mu^{\frac{\rho}{2-\rho}}), \tag{2.32}$$

we find points $x \in X$ and $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in X^m$ such that

$$\left\|(x,\hat{\omega}) - (x',\hat{\omega}')\right\|_\rho < \lambda, \; f_q(x,\hat{\omega}) \le f_q(x',\hat{\omega}'), \tag{2.33}$$

and

$$f_q(u,\hat{v}) + \frac{\varepsilon}{\lambda}\left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_\rho \geq f_q(x,\hat{\omega}), \tag{2.34}$$

for all $(u,\hat{v}) \in X \times X^m$. Thanks to (2.33), (2.32), (2.30), and (2.31), we have

$$\left\|x - x'\right\| < \lambda < \mu \leq \left\|x' - \bar{x}\right\|,$$

$$d\left(x, \bigcap_{i=1}^{m} \Omega_i\right) \geq d\left(x', \bigcap_{i=1}^{m} \Omega_i\right) - \left\|x - x'\right\| \geq \mu - \lambda = \mu^{\frac{2}{2-\rho}}, \tag{2.35}$$

$$\|x - \bar{x}\| \leq \left\|x - x'\right\| + \left\|x' - \bar{x}\right\| < 2\left\|x' - \bar{x}\right\| \leq 2\eta \leq \rho, \tag{2.36}$$

$$f_q(x,\hat{\omega}) \leq f_q(x',\hat{\omega}') < \alpha\mu \leq \alpha\eta \leq \rho. \tag{2.37}$$

It follows from (2.35), (2.36), and (2.37) that

$$\|x - \bar{x}\| < \rho,\ \hat{\omega} \in \widehat{\Omega},\ 0 < \max_{1 \leq i \leq m} \|x - \omega_i\|^q < \rho.$$

Observe that $\mu^{\frac{\rho}{2-\rho}} \leq \eta^{\frac{\rho}{2-\rho}} < \eta^{\frac{\rho}{2}} \leq \rho$, and consequently, by (2.31) and (2.32),

$$\frac{\varepsilon}{\lambda} < \frac{\alpha\mu}{\lambda} = \frac{\alpha}{1 - \mu^{\frac{\rho}{2-\rho}}} < \frac{\alpha}{1-\rho}.$$

Thanks to (2.34) and (2.27), we have

$$\max_{1 \leq i \leq m} \|x - \omega_i\|^q - \max_{1 \leq i \leq m} \|u - v_i\|^q \leq \frac{\alpha}{1-\rho}\left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_\rho$$

for all $u \in X$ and $\hat{v} = (v_1, \ldots, v_m) \in \widehat{\Omega}$. It follows that $\zeta^q_\rho[\mathbf{\Omega}](x,\hat{\omega}) \leq \dfrac{\alpha}{1-\rho}$ and consequently

$$\inf_{\substack{\|x-\bar{x}\|<\rho \\ \hat{\omega}=(\omega_1,\ldots,\omega_m)\in\widehat{\Omega} \\ 0<\max\limits_{1\leq i\leq m}\|x-\omega_i\|<\rho}} \zeta^q_\rho[\mathbf{\Omega}](x,\hat{\omega}) \leq \frac{\alpha}{1-\rho}.$$

Taking limits in the last inequality as $\rho \downarrow 0$ and $\alpha \to \zeta^q[\mathbf{\Omega}](\bar{x})$ yields the claimed inequality.

(ii) follows from (i) and Proposition 10 (ii). □

Proposition 15. *Let X be an Asplund space and Ω_1, ..., Ω_m be closed.*

(i) $\hat{\zeta}_1^{q*}[\mathbf{\Omega}](\bar{x}) \le \hat{\zeta}^q[\mathbf{\Omega}](\bar{x})$*, where* $\hat{\zeta}^q[\mathbf{\Omega}](\bar{x})$ *is given by* (2.28)*,*

$$\hat{\zeta}_1^{q*}[\mathbf{\Omega}](\bar{x}) := \lim_{\rho\downarrow 0} \inf_{\substack{\|x-\bar{x}\|<\rho \\ \hat{\omega}=(\omega_1,\ldots,\omega_m)\in\widehat{\Omega} \\ 0<\max\limits_{1\le i\le m}\|x-\omega_i\|<\rho}} \zeta_{\rho,1}^{q*}[\mathbf{\Omega}](x,\hat{\omega}) \tag{2.38}$$

and, for $x \in X$ *and* $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$*,*

$$\zeta_{\rho,1}^{q*}[\mathbf{\Omega}](x,\hat{\omega}) := \inf_{\substack{(x^*,\hat{y}^*)\in\partial f_q(x,\hat{\omega}) \\ \|\hat{y}^*\|<\rho}} \|x^*\| \tag{2.39}$$

(with the convention that the infimum over the empty set equals $+\infty$*).*

(ii) If $\hat{\zeta}_1^{q*}[\mathbf{\Omega}](\bar{x}) > 0$*, then* $\mathbf{\Omega}$ *is* $[q]$*-subregular at* $\bar{x}$*.*

Proof. (i) Let $\hat{\zeta}^q[\mathbf{\Omega}](\bar{x}) < \alpha < \infty$. Choose a $\beta \in (\hat{\zeta}^q[\mathbf{\Omega}](\bar{x}), \alpha)$ and an arbitrary $\rho > 0$. Set $\rho' = \min\{1, \alpha^{-1}\}\rho$. By (2.28) and (2.29), one can find points $x \in X$ and $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$ such that $\|x - \bar{x}\| < \rho'$, $0 < \max_{1\le i\le m} \|\omega_i - x\| < \rho'$, and

$$\max_{1\le i\le m} \|x-\omega_i\|^q - \max_{1\le i\le m} \|u-v_i\|^q \le \beta \left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_{\rho'}$$

for all $(u, \hat{v})$ with $\hat{v} = (v_1, \ldots, v_m) \in \widehat{\Omega}$ near $(x, \hat{\omega})$. In other words, $(x, \hat{\omega})$ is a local minimizer of the function

$$(u,\hat{v}) \mapsto \max_{1\le i\le m} \|u-v_i\|^q + \beta \left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_{\rho'}$$

subject to $\hat{v} = (v_1, \ldots, v_m) \in \widehat{\Omega}$. By definition (2.27), this means that $(x, \hat{\omega})$ minimizes locally the function

$$(u,\hat{v}) \mapsto f_q(u,\hat{v}) + \beta \left\|(u,\hat{v}) - (x,\hat{\omega})\right\|_{\rho'},$$

and consequently its Fréchet subdifferential at $(x, \hat{\omega})$ contains zero. Take an

$$\varepsilon \in \Big(0, \min\big\{\rho - \|x-\bar{x}\|, \frac{1}{2}\max_{1\le i\le m}\|x-\omega_i\|, \frac{1}{2}\big(\rho - \max_{1\le i\le m}\|x-\omega_i\|\big), \alpha-\beta\big\}\Big).$$

Applying Lemma 11 and Lemma 6 (ii), we can find points $x' \in X$, $\hat{\omega}' = (\omega_1', \ldots, \omega_m') \in \widehat{\Omega}$, and $(x^*, \hat{y}^*) \in \partial f_q(x', \hat{\omega}')$ such that

$$\|x' - x\| < \varepsilon, \quad \max_{1 \le i \le m} \|\omega_i' - \omega_i\| < \varepsilon, \quad \|(x^*, \hat{y}^*)\|_{\rho'} = \|x^*\| + \|\hat{y}^*\|/\rho' < \beta + \varepsilon.$$

It follows that

$$\|x' - \bar{x}\| < \rho, \;\; 0 < \max_{1 \le i \le m} \|x' - \omega_i'\| < \rho, \;\; \|x^*\| < \alpha, \;\; \text{and} \;\; \|\hat{y}^*\| < \rho'\alpha \le \rho.$$

Hence, $\zeta^{q*}_{\rho,1}[\mathbf{\Omega}](x', \hat{\omega}') < \alpha$, and consequently $\hat{\zeta}^{q*}_1[\mathbf{\Omega}](\bar{x}) < \alpha$. By letting $\alpha \to \hat{\zeta}^q[\mathbf{\Omega}](\bar{x})$, we obtain the claimed inequality.

(ii) follows from (i) and Proposition 14 (ii). □

Proposition 16. *Let X be an Asplund space and $\Omega_1, \ldots, \Omega_m$ be closed.*

(i) $\hat{\zeta}^{q*}_2[\mathbf{\Omega}](\bar{x}) \le \hat{\zeta}^{q*}_1[\mathbf{\Omega}](\bar{x})$, where $\hat{\zeta}^{q*}_1[\mathbf{\Omega}](\bar{x})$ is given by (2.38),

$$\hat{\zeta}^{q*}_2[\mathbf{\Omega}](\bar{x}) := \lim_{\rho \downarrow 0} \inf_{\substack{\|x - \bar{x}\| < \rho \\ \hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega} \\ 0 < \max_{1 \le i \le m} \|x - \omega_i\| < \rho}} \lim_{\varepsilon \downarrow 0} \inf_{\substack{\|x' - x\| < \varepsilon \\ \hat{\omega}' \in \widehat{\Omega} \\ \|\hat{\omega}' - \hat{\omega}\| < \varepsilon}} \zeta^{q*}_{\rho,\varepsilon,2}[\mathbf{\Omega}](x', \hat{\omega}') \tag{2.40}$$

and, for $x \in X$, $\hat{\omega} = (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$, and $\hat{v} := (x - \omega_1, \ldots, x - \omega_m) \neq 0$,

$$\begin{aligned} \zeta^{q*}_{\rho,\varepsilon,2}[\mathbf{\Omega}](x, \hat{\omega}) := \inf \Bigg\{ \Bigg\| \sum_{i=1}^m x_i^* \Bigg\| \;\Bigg|\; & x_i^* \in N_{\Omega_i}(\omega_i) + \rho \mathbb{B}^* \quad (i = 1, \ldots, m), \\ & x_i^* = 0 \quad \text{if} \quad \|x - \omega_i\| < \max_{1 \le j \le m} \|x - \omega_j\|, \\ & \langle x_i^*, x - \omega_i \rangle \ge \|x_i^*\| (\|x - \omega_i\| - \varepsilon), \\ & \sum_{i=1}^m \|x_i^*\| = q \, \|\hat{v}\|^{q-1} \Bigg\}. \end{aligned} \tag{2.41}$$

(ii) If $\hat{\zeta}^{q*}_2[\mathbf{\Omega}](\bar{x}) > 0$, then $\mathbf{\Omega}$ is $[q]$-subregular at $\bar{x}$.

Proof. (i) Let $\rho > 0$, $\|x - \bar{x}\| < \rho$, $\hat{\omega} := (\omega_1, \ldots, \omega_m) \in \widehat{\Omega}$ with $0 < \max_{1 \le i \le m} \|x - \omega_i\| < \rho$, $(u^*, \hat{v}^*) \in \partial f_q(x, \hat{\omega})$, where f_q is given by (2.27), and $\|\hat{v}^*\| < \rho$. Denote $\hat{v} := (x - \omega_1, \ldots, x -$

$\omega_m)$. Then $0 < \|\hat{v}\| < \rho$. Observe that function f_q is the sum of two functions on X^{m+1}:

$$(x, \hat{x}) \mapsto \varphi(x, \hat{x}) := \|(x - x_1, \ldots, x - x_m)\|^q \quad \text{and} \quad (x, \hat{x}) \mapsto \delta_{\widehat{\Omega}}(\hat{x}),$$

where $\hat{x} := (x_1, \ldots, x_m)$ and $\delta_{\widehat{\Omega}}$ is the indicator function of $\widehat{\Omega}$. The first function is Lipschitz continuous near $(x, \hat{\omega})$ (since $\hat{v} \neq 0$), while the second one is lower semicontinuous. One can apply Lemma 11. For any $\varepsilon > 0$, there exist points $x' \in X$, $\hat{x} := (x_1, \ldots, x_m) \in X^m$, $\hat{\omega}' := (\omega_1', \ldots, \omega_m') \in \widehat{\Omega}$, $(x^*, \hat{y}^*) \in \partial\varphi(x', \hat{x})$, and $\hat{\omega}^* \in N_{\widehat{\Omega}}(\hat{\omega}')$ such that

$$\begin{aligned} &\|x' - x\| < \varepsilon, \quad \|\hat{x} - \hat{\omega}\| < \frac{\varepsilon}{4}, \quad \|\hat{\omega}' - \hat{\omega}\| < \frac{\varepsilon}{4}, \\ &\|(u^*, \hat{v}^*) - (x^*, \hat{y}^*) - (0, \hat{\omega}^*)\| < \varepsilon. \end{aligned} \tag{2.42}$$

Taking a smaller ε if necessary, one can ensure that $\hat{v}' := (x' - \omega_1', \ldots, x' - \omega_m') \neq 0$, $\hat{v}'' := (x' - x_1, \ldots, x' - x_m) \neq 0$, and

$$\|\hat{v}^*\| + \varepsilon < \rho \left(\frac{\|\hat{v}'\|}{\|\hat{v}''\|} \right)^{1-q} \tag{2.43}$$

and, for any $i = 1, \ldots, m$, $\|x' - x_i\| < \max_{1 \le j \le m} \|x' - x_j\|$ if and only if $\|x' - \omega_i'\| < \max_{1 \le j \le m} \left\|x' - \omega_j'\right\|$. By Lemmas 8 and 7,

$$\hat{x}^* := -\hat{y}^* \left(\frac{\|\hat{v}''\|}{\|\hat{v}'\|} \right)^{1-q} \in q \left\|\hat{v}'\right\|^{q-1} J(\hat{v}'') \quad \text{and} \quad x^* = x_1^* + \ldots + x_m^*$$

where $\hat{x}^* = (x_1^*, \ldots, x_m^*)$. By Proposition 13,

$$\sum_{i=1}^m \|x_i^*\| = q \left\|\hat{v}'\right\|^{q-1},$$

$$x_i^* = 0 \quad \text{if} \quad \left\|x' - \omega_i'\right\| < \max_{1 \le j \le m} \left\|x' - \omega_j'\right\|,$$

$$\begin{aligned} \langle x_i^*, x' - \omega_i' \rangle &\ge \langle x_i^*, x' - x_i \rangle - \|x_i^*\| \, \|x_i - \omega_i'\| = \|x_i^*\|(\|x' - x_i\| - \|x_i - \omega_i'\|) \\ &\ge \|x_i^*\|(\|x' - \omega_i'\| - 2\|x_i - \omega_i'\|) \ge \|x_i^*\|(\|x' - \omega_i'\| - \varepsilon) \quad (i = 1, \ldots, m). \end{aligned}$$

Inequalities (2.42) and (2.43) yield the estimates:

$$\|u^*\| > \|x^*\| - \varepsilon, \quad \left\| \hat{x}^* - \hat{\omega}^* \left(\frac{\|\hat{v}''\|}{\|\hat{v}'\|} \right)^{1-q} \right\| < (\|\hat{v}^*\| + \varepsilon) \left(\frac{\|\hat{v}''\|}{\|\hat{v}'\|} \right)^{1-q} < \rho$$

and consequently

$$\|u^*\| > \left\| \sum_{i=1}^{m} x_i^* \right\| - \varepsilon, \quad \hat{x}^* \in N_{\widehat{\Omega}}(\hat{\omega}') + \rho \mathbb{B}_m^*.$$

It follows from Lemma 9 and definitions (2.39) and (2.41) that

$$\zeta_{\rho,1}^{q*}[\mathbf{\Omega}](x, \hat{\omega}) \geq \zeta_{\rho,\varepsilon,2}^{q*}[\mathbf{\Omega}](x', \hat{\omega}') - \varepsilon.$$

The claimed inequality is a consequence of the last one and definitions (2.38) and (2.40).

(ii) follows from (i) and Proposition 15 (ii). □

Proof of Theorem 5. (i) follows from Proposition 16 (ii) and definitions (2.40) and (2.41).

(ii) follows from [35, Theorem 4] thanks to Remark 14. □

Remark 20. One of the main tools in the proof of Theorem 5 is the fuzzy sum rule (Lemma 11) for Fréchet subdifferentials in Asplund spaces. The statements can be extended to general Banach spaces. For that, one has to replace Fréchet subdifferentials (and normal cones) with some other kind of subdifferentials satisfying a certain set of natural properties including the sum rule (not necessarily fuzzy) – cf. [36, p. 345].

If the sets $\Omega_1, \ldots \Omega_m$ are convex or the norm of X is Fréchet differentiable away from 0, then the fuzzy sum rule can be replaced in the proof by either the convex sum rule (Moreau–Rockafellar formula) or the simple (exact) differentiable rule (see, e.g., [31, Corollary 1.12.2]), respectively, to produce dual sufficient conditions for $[q]$-regularity of collections of sets in general Banach spaces in terms of either normals in the sense of convex analysis or Fréchet normals.

Remark 21. Since uniform $[q]$-regularity is a stronger property than $[q]$-subregularity (Remark 13), the criterion in part (ii) of Theorem 5 is also sufficient for the $[q]$-subregularity (with any $q \in (0, 1]$) of the collection of sets in part (i).

For an example illustrates application of Theorem 5 (i) for detecting subregularity of collections of sets, see [39, Example 4.13].

2.4 $[q]$-regularity of set-valued mappings

In this section, we present relationships between $[q]$-regularity properties of collections of sets and the corresponding properties of set-valued mappings. Nonlinear regularity properties of set-valued mappings have been investigated, cf., e.g., [2, 11, 19, 20, 25, 40, 44, 55].

Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$.

Definition 4. (i) F is *metrically* $[q]$*-semiregular* at $(\bar{x}, \bar{y})$ if there exist positive numbers γ and δ such that

$$\gamma d\left(\bar{x}, F^{-1}(y)\right) \leq d^q(y, \bar{y}), \ \forall y \in B_\delta(\bar{y}). \tag{2.44}$$

The exact upper bound of all numbers γ such that (2.44) is satisfied will be denoted by $\theta^q[F](\bar{x}, \bar{y})$.

(ii) F is *metrically* $[q]$*-subregular* at $(\bar{x}, \bar{y})$ if there exist positive numbers γ and δ such that

$$\gamma d\left(x, F^{-1}(\bar{y})\right) \leq d^q(\bar{y}, F(x)), \ \forall x \in B_\delta(\bar{x}). \tag{2.45}$$

The exact upper bound of all numbers γ such that (2.45) is satisfied will be denoted by $\zeta^q[F](\bar{x}, \bar{y})$.

(iii) F is *metrically* $[q]$*-regular* at $(\bar{x}, \bar{y})$ if there exist positive numbers γ and δ such that

$$\gamma d\left(x, F^{-1}(y)\right) \leq d^q\left(y, F(x)\right), \ \forall (x, y) \in B_\delta(\bar{x}, \bar{y}). \tag{2.46}$$

The exact upper bound of all numbers γ such that (2.46) is satisfied will be denoted by $\hat{\theta}^q[F](\bar{x}, \bar{y})$.

Remark 22. Property (ii) and especially property (iii) in Definition 4 with $q = 1$ are very well known and widely used in variational analysis; see, e.g., [13, 14, 15, 23, 35, 49, 51, 54, 57, 59, 60]. Property (i) (with $q = 1$) was introduced in [35]. In [2, 3], it is referred to as *metric hemiregularity*.

For a collection of sets $\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\}$ in a normed linear space X, one can consider the set-valued mapping $F : X \rightrightarrows X^m$ defined by (cf. [23, Proposition 5], [33, Theorem 3], [34, Proposition 8], [41, p. 491], [21, Proposition 33])

$$F(x) := (\Omega_1 - x) \times \ldots \times (\Omega_m - x), \ \forall x \in X.$$

It is easy to check that, for $x \in X$ and $u = (u_1, \ldots, u_m) \in X^m$, it holds

$$x \in \bigcap_{i=1}^{m} \Omega_i \iff 0 \in F(x), \quad F^{-1}(u) = \bigcap_{i=1}^{m} (\Omega_i - u_i).$$

The next proposition is a consequence of Theorem 4.

Proposition 17. *Consider $\mathbf{\Omega}$ and F as above and a point $\bar{x} \in \bigcap_{i=1}^{m} \Omega_i$.*

(i) $\mathbf{\Omega}$ is $[q]$-semiregular at $\bar{x}$ if and only if F is metrically $[q]$-semiregular at $(\bar{x}, 0)$. Moreover, $\theta^q[\mathbf{\Omega}](\bar{x}) = \theta^q[F](\bar{x}, 0)$.

(ii) $\mathbf{\Omega}$ is $[q]$-subregular at $\bar{x}$ if and only if F is metrically $[q]$-subregular at $(\bar{x}, 0)$. Moreover, $\zeta^q[\mathbf{\Omega}](\bar{x}) = \zeta^q[F](\bar{x}, 0)$.

(iii) $\mathbf{\Omega}$ is uniformly $[q]$-regular at $\bar{x}$ if and only if F is metrically $[q]$-regular at $(\bar{x}, 0)$. Moreover, $\hat{\theta}^q[\mathbf{\Omega}](\bar{x}) = \hat{\theta}^q[F](\bar{x}, 0)$.

For a further discussion of the relationships between regularity properties of $\mathbf{\Omega}$ and F see [39, Remark 5.4].

Conversely, regularity properties of set-valued mappings between normed linear spaces can be treated as realizations of the corresponding properties of certain collections of two sets.

For a given set-valued mapping $F : X \rightrightarrows Y$ between normed linear spaces and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, one can consider the collection $\mathbf{\Omega}$ of two sets $\Omega_1 = \operatorname{gph} F$ and $\Omega_2 = X \times \{\bar{y}\}$ in $X \times Y$. It is clear that $(\bar{x}, \bar{y}) \in \Omega_1 \cap \Omega_2$.

Proposition 18. *Consider F and $\mathbf{\Omega}$ as above.*

(i) F is metrically $[q]$-semiregular at $(\bar{x}, \bar{y})$ if and only if $\mathbf{\Omega}$ is $[q]$-semiregular at $(\bar{x}, \bar{y})$. Moreover,

$$\frac{\theta^q[F](\bar{x}, \bar{y})}{\theta^q[F](\bar{x}, \bar{y}) + 2^q} \le \theta^q[\mathbf{\Omega}](\bar{x}, \bar{y}) \le \theta^q[F](\bar{x}, \bar{y})/2^q. \tag{2.47}$$

(ii) F is metrically $[q]$-subregular at $(\bar{x}, \bar{y})$ if and only if $\mathbf{\Omega}$ is $[q]$-subregular at $(\bar{x}, \bar{y})$. Moreover,

$$\frac{\zeta^q[F](\bar{x}, \bar{y})}{\zeta^q[F](\bar{x}, \bar{y}) + 2^q} \le \zeta^q[\mathbf{\Omega}](\bar{x}, \bar{y}) \le \zeta^q[F](\bar{x}, \bar{y})/2^q. \tag{2.48}$$

(iii) F is metrically $[q]$-regular at $(\bar{x}, \bar{y})$ if and only if $\mathbf{\Omega}$ is uniformly $[q]$-regular at $(\bar{x}, \bar{y})$. Moreover,

$$\frac{\hat{\theta}^q[F](\bar{x}, \bar{y})}{\hat{\theta}^q[F](\bar{x}, \bar{y}) + 2^q} \le \hat{\theta}^q[\mathbf{\Omega}](\bar{x}, \bar{y}) \le \hat{\theta}^q[F](\bar{x}, \bar{y})/2^q. \tag{2.49}$$

Proof. (i) Suppose F is metrically $[q]$-semiregular at $(\bar{x}, \bar{y})$, i.e., $\theta^q[F](\bar{x}, \bar{y}) > 0$. Fix a $\gamma \in (0, \theta^q[F](\bar{x}, \bar{y}))$. Then there exists a number $\delta' > 0$ such that (2.44) is satisfied for all $y \in B_{\delta'}(\bar{y})$. Set an $\alpha := \frac{\gamma}{\gamma + 2^q}$ (so $2^q\alpha/\gamma + \alpha^{\frac{1}{q}} < 1$) and a $\delta := \min\left\{\frac{\delta'^q}{2^q\alpha}, 1\right\}$. We are going to check that

$$(\Omega_1 - (u_1, v_1)) \bigcap (\Omega_2 - (u_2, v_2)) \bigcap B_\rho(\bar{x}, \bar{y}) \neq \emptyset \tag{2.50}$$

for all $\rho \in (0, \delta)$ and $(u_1, v_1), (u_2, v_2) \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$. Indeed, take any $\rho \in (0, \delta)$ and $(u_1, v_1), (u_2, v_2) \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$. We need to find a point $(x, y) \in B_\rho(\bar{x}, \bar{y})$ satisfying

$$\begin{cases} (x, y) + (u_1, v_1) \in \operatorname{gph} F, \\ y = \bar{y} - v_2. \end{cases}$$

We set $y' := \bar{y} - v_2 + v_1$, so $y' \in B_{\delta'}(\bar{y})$ as $\|y' - \bar{y}\| = \|v_1 - v_2\| \le 2(\alpha\rho)^{\frac{1}{q}} < 2(\alpha\delta)^{\frac{1}{q}} = \delta'$. Then, by (2.44), there is an $x' \in F^{-1}(y')$ such that

$$\|\bar{x} - x'\| \le \frac{1}{\gamma}\|\bar{y} - y'\|^q.$$

Put $y := y' - v_1 = \bar{y} - v_2$ and $x := x' - u_1$. Then it holds $(x, y) + (u_1, v_1) = (x', y') \in \operatorname{gph} F$,

$\|y-\bar{y}\| = \|v_2\| \le (\alpha\rho)^{\frac{1}{q}} < \rho$, and

$$\|x-\bar{x}\| \le \|x-x'\| + \|x'-\bar{x}\| \le \|u_1\| + \frac{1}{\gamma}\|\bar{y}-y'\|^q$$
$$= \|u_1\| + \frac{1}{\gamma}\|v_1-v_2\|^q \le (2^q\alpha/\gamma + \alpha^{\frac{1}{q}})\rho < \rho.$$

Hence, (2.50) is proved.

The above reasoning also yields the first inequality in (2.47).

To prove the inverse implication, we suppose $\mathbf{\Omega}$ is $[q]$-semiregular at $(\bar{x},\bar{y})$, i.e., $\theta^q[\mathbf{\Omega}](\bar{x},\bar{y}) > 0$. Fix an $\alpha \in (0, \theta^q[\mathbf{\Omega}](\bar{x},\bar{y}))$. Then there exists $\delta' > 0$ such that (2.50) holds true for all $\rho \in (0,\delta')$ and $(u_1,v_1),(u_2,v_2) \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$. Set $\gamma := 2^q\alpha$ and $\delta < (\alpha\delta')^{\frac{1}{q}}$. We are going to check that (2.44) is satisfied. Take any $y \in B_\delta(\bar{y})$, i.e., $\|y-\bar{y}\| \le \delta < (\alpha\delta')^{\frac{1}{q}}$. Set $r \in (0,\delta')$ such that $\|y-\bar{y}\| = (\alpha r)^{\frac{1}{q}}$. Then, applying (2.50) for $\rho := \frac{r}{2^q} \in (0,\delta')$, and $(u_1,v_1) := \left(0, \frac{y-\bar{y}}{2}\right)$, $(u_2,v_2) := \left(0, \frac{\bar{y}-y}{2}\right) \in \left(\alpha\frac{r}{2^q}\right)^{\frac{1}{q}}\mathbb{B}$, we can find $(x_1,y_1) \in \operatorname{gph} F$ and $(x_2,\bar{y}) \in \Omega_2$ satisfying

$$(x_1,y_1) - (u_1,v_1) = (x_2,\bar{y}) - (u_2,v_2) \in B_{\frac{r}{2^q}}(\bar{x},\bar{y}).$$

This implies that $y_1 = y$, $x_1 \in F^{-1}(y)$, and

$$\|x_1-\bar{x}\| \le \frac{r}{2^q} = \frac{1}{2^q\alpha}\|y-\bar{y}\|^q = \frac{1}{\gamma}\|y-\bar{y}\|^q.$$

Hence, (2.44) holds true.

The last reasoning also yields the second inequality in (2.47).

(ii) Suppose F is metrically $[q]$-subregular at $(\bar{x},\bar{y})$, i.e., $\zeta^q[F](\bar{x},\bar{y}) > 0$. Fix a $\gamma \in (0, \zeta^q[F](\bar{x},\bar{y}))$. Then there exists a $\delta' > 0$ (one can take $\delta' \in (0,1)$) such that (2.45) is satisfied for all $x \in B_{\delta'}(\bar{x})$. Set an $\alpha := \frac{\gamma}{\gamma+2^q}$ (so $2^q\alpha/\gamma + \alpha^{\frac{1}{q}} < 1$) and a $\delta > 0$ satisfying $(\alpha\delta)^{\frac{1}{q}} + \delta < \delta'$. We are going to check that

$$\left(\Omega_1 + (\alpha\rho)^{\frac{1}{q}}\mathbb{B}\right) \bigcap \left(\Omega_2 + (\alpha\rho)^{\frac{1}{q}}\mathbb{B}\right) \bigcap B_\delta(\bar{x},\bar{y}) \subseteq \Omega_1 \cap \Omega_2 + \rho\mathbb{B} \tag{2.51}$$

for all $\rho \in (0, \delta)$. Indeed, take any

$$(x, y) \in \left(\Omega_1 + (\alpha\rho)^{\frac{1}{q}}\mathbb{B}\right) \bigcap \left(\Omega_2 + (\alpha\rho)^{\frac{1}{q}}\mathbb{B}\right) \bigcap B_\delta(\bar{x}, \bar{y}).$$

Then $(x, y) = (x_1, y_1) + (u_1, v_1) = (x_2, \bar{y}) + (u_2, v_2)$ for some $(x_1, y_1) \in \operatorname{gph} F$, $x_2 \in X$, and (u_1, v_1), $(u_2, v_2) \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$. Since

$$\|x_1 - \bar{x}\| \le \|u_1\| + \|x - \bar{x}\| \le (\alpha\rho)^{\frac{1}{q}} + \delta < \delta',$$

by (2.45), there exists an $x' \in F^{-1}(\bar{y})$ such that $\|x_1 - x'\| \le \frac{1}{\gamma}\|\bar{y} - y_1\|^q$. Then

$$\begin{aligned} \left\|x_1 - x' + u_1\right\| \le& \frac{1}{\gamma}\|\bar{y} - y_1\|^q + \|u_1\| = \frac{1}{\gamma}\|v_1 - v_2\|^q + \|u_1\| \\ \le& \frac{2^q\alpha\rho}{\gamma} + (\alpha\rho)^{\frac{1}{q}} \le \left(\frac{2^q\alpha}{\gamma} + \alpha^{\frac{1}{q}}\right)\rho < \rho, \\ \|v_2\| \le& (\alpha\rho)^{\frac{1}{q}} \le \alpha^{\frac{1}{q}}\rho < \rho. \end{aligned}$$

Hence, $(x, y) = (x', \bar{y}) + (x_1 - x' + u_1, v_2) \in \Omega_1 \cap \Omega_2 + \rho\mathbb{B}$.

The above reasoning also yields the first inequality in (2.48).

To prove the inverse implication, we suppose that $\boldsymbol{\Omega}$ is $[q]$-subregular at $(\bar{x}, \bar{y})$, i.e., $\zeta^q[\boldsymbol{\Omega}](\bar{x}, \bar{y}) > 0$. Fix an $\alpha \in (0, \zeta^q[\boldsymbol{\Omega}](\bar{x}, \bar{y}))$. Then there exists a $\delta' > 0$ such that (2.51) holds true for all $\rho \in (0, \delta')$. Set $\gamma := 2^q\alpha > 0$ and $\delta := \min\left\{\delta', \gamma\delta', \frac{2^q\delta'^q}{\gamma}\right\}$. We are going to check that (2.45) holds true. Take any $x \in B_\delta(\bar{x})$. Because $d(x, F^{-1}(\bar{y})) \le \|x - \bar{x}\| \le \delta$, it is sufficient to consider the case $0 < d(\bar{y}, F(x)) < (\gamma\delta)^{\frac{1}{q}}$. We take a $y \in F(x)$ such that $d(\bar{y}, F(x)) \le \|y - \bar{y}\| := r < (\gamma\delta)^{\frac{1}{q}}$. Then

$$\left(x, \frac{y + \bar{y}}{2}\right) = (x, y) + \left(0, \frac{\bar{y} - y}{2}\right) = (x, \bar{y}) + \left(0, \frac{y - \bar{y}}{2}\right), \quad \left\|\frac{\bar{y} - y}{2}\right\| = \frac{r}{2} < \delta',$$

and consequently

$$\left(x, \frac{y + \bar{y}}{2}\right) \in \left(\Omega_1 + \frac{r}{2}\mathbb{B}\right) \bigcap \left(\Omega_2 + \frac{r}{2}\mathbb{B}\right) \bigcap B_{\delta'}(\bar{x}, \bar{y}). \tag{2.52}$$

Take $\rho := \frac{r^q}{2^q\alpha} < \delta \le \delta'$. Then $\frac{r}{2} = (\alpha\rho)^{\frac{1}{q}}$, and it follows from (2.51) and (2.52) that

$$\left(x, \frac{y+\bar{y}}{2}\right) \in \Omega_1 \cap \Omega_2 + \frac{r^q}{2^q\alpha}\mathbb{B} = F^{-1}(\bar{y}) \times \{\bar{y}\} + \frac{\|y-\bar{y}\|^q}{\gamma}\mathbb{B}.$$

Hence, there is an $x' \in F^{-1}(\bar{y})$ such that

$$\|x - x'\| \le \frac{1}{\gamma}\|y - \bar{y}\|^q.$$

Taking infimum in the last inequality over $x' \in F^{-1}(\bar{y})$ and $y \in F(x)$, we arrive at (2.45).

(iii) Suppose F is metrically $[q]$-regular at $(\bar{x}, \bar{y})$, i.e., $\hat{\theta}^q[F](\bar{x}, \bar{y}) > 0$. Fix a $\gamma \in (0, \hat{\theta}^q[F](\bar{x}, \bar{y}))$. Then there exists a $\delta' > 0$ (one can take $\delta' \in (0, 1)$) such that (2.46) is satisfied for all $(x, y) \in B_{\delta'}(\bar{x}, \bar{y})$. Set an $\alpha := \frac{\gamma}{\gamma + 2^q}$ (so $2^q\alpha/\gamma + \alpha^{\frac{1}{q}} < 1$) and a $\delta := \frac{\delta'}{2\alpha^{\frac{1}{q}}+1}$. We are going to check that

$$(\Omega_1 - (x_1, y_1) - (u_1, v_1)) \bigcap (\Omega_2 - (x_2, \bar{y}) - (u_2, v_2)) \bigcap (\rho\mathbb{B}) \ne \emptyset \tag{2.53}$$

for all $\rho \in (0, \delta)$, $(x_1, y_1) \in \Omega_1 \cap B_\delta(\bar{x}, \bar{y})$, $x_2 \in B_\delta(\bar{x})$, and $(u_1, v_1), (u_2, v_2) \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$. Take any such $\rho, (x_1, y_1), x_2, (u_1, v_1)$, and (u_2, v_2). We need to find $(a, b) \in \rho\mathbb{B}$ satisfying

$$\begin{cases} (x_1, y_1) + (u_1, v_1) + (a, b) \in \operatorname{gph} F, \\ b = -v_2. \end{cases}$$

We set $y' = y_1 - v_2 + v_1$, so $y' \in B_{\delta'}(\bar{y})$ as

$$\|y' - \bar{y}\| \le \|y' - y_1\| + \|y_1 - \bar{y}\| \le \|v_1 - v_2\| + \delta \le 2(\alpha\rho)^{\frac{1}{q}} + \delta < (2\alpha^{\frac{1}{q}} + 1)\delta = \delta'.$$

Then, applying (2.46) for $(x_1, y') \in B_{\delta'}(\bar{x}, \bar{y})$, we find $x' \in F^{-1}(y')$ such that

$$\|x_1 - x'\| \le \frac{1}{\gamma}d^q(y', F(x_1)) \le \frac{1}{\gamma}\|y' - y_1\|^q = \frac{1}{\gamma}\|v_1 - v_2\|^q \le \frac{2^q\alpha\rho}{\gamma}.$$

Put $a = x' - x_1 - u_1$ and $b = -v_2$. Then $\|a\| \le \|x' - x_1\| + \|u_1\| \le (2^q\alpha/\gamma + \alpha^{\frac{1}{q}})\rho < \rho$, $\|b\| \le (\alpha\rho)^{\frac{1}{q}} < \rho$, and it holds $(x_1, y_1) + (u_1, v_1) + (a, b) = (x', y') \in \operatorname{gph} F$.

Hence, (2.53) is proved.

The above reasoning also yields the first inequality in (2.49).

To prove the inverse implication, we suppose that $\mathbf{\Omega}$ is uniformly $[q]$-regular at $(\bar{x}, \bar{y})$, i.e., $\hat{\theta}^q[\mathbf{\Omega}](\bar{x}, \bar{y}) > 0$. Fix an $\alpha \in (0, \hat{\theta}^q[\mathbf{\Omega}](\bar{x}, \bar{y}))$. Then there exists a $\delta' > 0$ (one can take $\delta' \in (0, 1)$) such that (2.53) holds true for all $\rho \in (0, \delta')$, $(x_1, y_1) \in \Omega_1 \cap B_{\delta'}(\bar{x}, \bar{y}), x_2 \in B_{\delta'}(\bar{x})$, and $(u_1, v_1), (u_2, v_2) \in (\alpha\rho)^{\frac{1}{q}}\mathbb{B}$. Set $\gamma := 2^q\alpha > 0$. Because $\theta^q[\mathbf{\Omega}](\bar{x}, \bar{y}) \geq \hat{\theta}^q[\mathbf{\Omega}](\bar{x}, \bar{y})$ (see Remark 13), assertion (i) implies that there exists a $\delta^* > 0$ such that (2.44) is satisfied for all $y \in B_{\delta^*}(\bar{y})$. Choose a positive number δ satisfying the following conditions

$$\begin{cases} \delta \leq \delta^*, \\ 2^q\delta + \frac{\delta^q}{\alpha} \leq \delta', \\ (2^q\alpha\delta + \delta^q)^{\frac{1}{q}} + \delta \leq \delta'. \end{cases} \tag{2.54}$$

Now, take any $(x, y) \in B_\delta(\bar{x}, \bar{y})$. We are going to check that (2.46) is satisfied. Because (2.44) implies

$$\gamma d(x, F^{-1}(y)) \leq \gamma\|x - \bar{x}\| + \gamma d(\bar{x}, F^{-1}(y)) \leq \gamma\delta + \|y - \bar{y}\|^q \leq \gamma\delta + \delta^q,$$

it suffices to consider the case $d(y, F(x)) < (\gamma\delta + \delta^q)^{\frac{1}{q}}$ (note that $\gamma\delta + \delta^q \leq \alpha\delta'$ by (2.54).) Choose a $y' \in F(x)$ such that

$$d(y, F(x)) \leq \|y - y'\| < (\gamma\delta + \delta^q)^{\frac{1}{q}}$$

and set $r \in (0, \delta')$ such that $\|y - y'\| = (\alpha r)^{\frac{1}{q}}$. Then

$$\|y' - \bar{y}\| \leq \|y' - y\| + \|y - \bar{y}\| < (\gamma\delta + \delta^q)^{\frac{1}{q}} + \delta \leq \delta'$$

due to (2.54). Applying (2.53) with

$$(x_1, y_1) := (x, y') \in \operatorname{gph} F \cap B_{\delta'}(\bar{x}, \bar{y}), \quad (x_2, y_2) := (\bar{x}, \bar{y}),$$

$$(u_1, v_1) := \left(0, \frac{y - y'}{2}\right), \quad (u_2, v_2) := \left(0, \frac{y' - y}{2}\right) \in \left(\alpha\frac{r}{2^q}\right)^{\frac{1}{q}}\mathbb{B},$$

we can find $(\tilde{x}, \tilde{y}) \in \operatorname{gph} F$ and $(z, \bar{y}) \in \Omega_2$ satisfying

$$(\tilde{x}, \tilde{y}) - (x_1, y_1) - (u_1, v_1) = (z, \bar{y}) - (x_2, \bar{y}) - (u_2, v_2) \in \frac{r}{2^q}\mathbb{B}.$$

This implies $\tilde{x} - x_1 \in \frac{r}{2^q}\mathbb{B}$ and $\tilde{y} = y_1 + v_1 - v_2 = y$, so $\tilde{x} \in F^{-1}(y)$. Then we obtain

$$d(x, F^{-1}(y)) \leq \|x - \tilde{x}\| \leq \frac{r}{2^q} = \frac{1}{2^q\alpha}\|y - y'\|^q = \frac{1}{\gamma}\|y - y'\|^q.$$

Taking infimum in the last inequality over $y' \in F(x)$, we arrive at (2.46).

The last reasoning also yields the second inequality in (2.49). □

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Chapter 3

About uniform regularity of collections of sets

We further investigate the uniform regularity property of collections of sets via primal and dual characterizing constants. These constants play an important role in determining convergence rates of projection algorithms for solving feasibility problems.

3.1 Introduction

Regularity properties of collections of sets play an important role in several areas of variational analysis and optimization like coderivative-subdifferential calculus, constraint qualifications, stability of solutions, and convergence of numerical algorithms.

Various regularity properties of collections of sets have proved to be useful: *(bounded) linear regularity* [2, 3, 4, 5, 6, 8, 30, 35, 40, 41], *metric inequality* [15, 16, 36], *(strong) conical hull intersection property* [2, 5, 6, 9, 10, 13, 30], *Jameson's property* (G) [5, 28]. We refer the readers to [2, 5, 23] for the relationships between these properties and the overview of the areas of their applications in analysis and optimization.

The *uniform regularity* property introduced recently in [22] and further developed in [23, 24, 25] is stronger than local linear regularity even in the convex case. It corresponds to the *metric regularity* property of set-valued mappings and is closely related to the *(extended) extremal principle*. The most recent development is the application of this property in convergence analysis of projection algorithms by Lewis et al. [29], Attouch et al. [1], Luke [31, 32], and Hesse and Luke [14].

Uniform regularity of a collection of sets in a normed linear space is characterized quantitatively in [22, 23, 24, 25] by certain nonnegative constants defined in terms of elements of the primal or dual spaces. In the setting of a finite dimensional Euclidean space, Lewis et al. [29] introduced another nonnegative constant characterizing the uniform regularity of a collection of two sets and used it when formulating convergence rates of averaged and alternating projections.

In the current note, we consider a (not necessarily nonnegative) modification of the constant from [29] in the setting of an arbitrary Hilbert space and establish its relationship with the dual space constant from [22, 23, 24, 25]. The latter constant admits a simplified equivalent representation in Hilbert spaces. As an application, we employ these constants to establish convergence results of projection algorithms.

The structure of this chapter is as follows. In Section 3.2, we recall the uniform regularity property of a finite collection of sets in a normed linear space, its main characterizations and connections with some other properties. In Section 3.3, we consider the case of a collection of two sets in a Hilbert space and establish the relationship between the dual space constants from [22, 23, 24, 25] and [29]. The final Section 3.4 is dedicated to the convergence estimates of projection algorithms.

Our basic notation is standard, cf. [33, 38]. For a normed linear space X, its topological dual is denoted X^* while $\langle \cdot, \cdot \rangle$ denotes the bilinear form defining the pairing between the two spaces. The closed unit ball and the unit sphere in a normed space are denoted $\mathbb{B}$ and $\mathbb{S}$, respectively. $B_\delta(x)$ stands for the closed ball with radius δ and center x.

3.2 Uniform regularity of a collection of sets

In this section, we recall the *uniform regularity* property of a finite collection $\boldsymbol{\Omega} := \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ $(m > 1)$ of sets in a normed linear space X near a given point $\bar{x} \in \cap_{i=1}^m \Omega_i$. The property was introduced in [22] (under a different name) and further developed in [23, 24, 25].

Definition 5. $\mathbf{\Omega}$ is *uniformly regular* at $\bar{x}$ if there exist numbers $\delta, \alpha > 0$ such that

$$\bigcap_{i=1}^{m}(\Omega_i - \omega_i - a_i) \bigcap (\rho\mathbb{B}) \neq \emptyset$$

for any $\rho \in (0;\delta], \omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $a_i \in (\alpha\rho)\mathbb{B}$, $i = 1, 2, \ldots, m$.

Uniform regularity of a collection of sets can be equivalently characterized in terms of certain nonnegative constants:

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \sup\left\{ r \geq 0 \,\middle|\, \bigcap_{i=1}^{m}(\Omega_i - a_i) \bigcap B_\rho(\bar{x}) \neq \emptyset,\ \max_{1\leq i\leq m} \|a_i\| \leq r \right\},\ \rho \in (0;\infty],$$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\rho\downarrow 0,\ \omega_i \xrightarrow{\Omega_i} \bar{x}\ (1\leq i\leq m)} \frac{\theta_\rho[\Omega_1 - \omega_1, \Omega_2 - \omega_2, \ldots, \Omega_m - \omega_m](0)}{\rho}.$$

Here $\omega_i \xrightarrow{\Omega_i} \bar{x}$ means that $\omega_i \to \bar{x}$ with $\omega_i \in \Omega_i$.

These constants characterize the mutual arrangement of sets Ω_i $(1 \leq i \leq m)$ in the primal space and are convenient for defining their extremality, stationarity and regularity properties.

The next proposition follows directly from the definitions.

Proposition 19. $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if* $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$.

When constant $\hat{\theta}[\mathbf{\Omega}](\bar{x})$ is positive, it provides a quantitative characterization of the uniform regularity property. It coincides with the supremum of all α in Definition 5.

The case $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = 0$, i.e., the absence of the uniform regularity, corresponds to *approximate stationarity* [20, 21, 22, 23, 24] of $\mathbf{\Omega}$ at $\bar{x}$, the latter property being a relaxation of the *extremality* property introduced and investigated in [27]. We refer the reader to [25, Section 3] for a modern summary of extremality, stationarity, and regularity conditions for finite collections of sets.

Another nonnegative primal space constant (being a slight modification of the corresponding one introduced in [22]) can be used for characterizing the uniform regularity:

$$\hat{\vartheta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\substack{x\to\bar{x},\ x_i\to 0\ (1\leq i\leq m)\\ x\notin \bigcap_{i=1}^{m}(\Omega_i - x_i)}} \frac{\max_{1\leq i\leq m} d(x + x_i, \Omega_i)}{d\left(x, \bigcap_{i=1}^{m}(\Omega_i - x_i)\right)}.$$

The next proposition corresponds to [22, Theorem 1].

Proposition 20. $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\vartheta}[\mathbf{\Omega}](\bar{x})$.

As a consequence, $\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ if and only if $\hat{\vartheta}[\mathbf{\Omega}](\bar{x}) > 0$.

It was shown in [22, 23, 24] that the uniform regularity of a collection of sets can be interpreted as the direct analogue of the fundamental in variational analysis *metric regularity* property of set-valued mappings.

Regularity properties can also be characterized in terms of elements of the dual space using appropriate concepts of *normal* elements. Given a subset Ω of X, a point $\bar{x}$ in Ω, and a number $\delta \geq 0$, the sets (cf. [20, 33])

$$N_\Omega(\bar{x}) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\},$$

$$\widehat{N}_\Omega(\bar{x}, \delta) := \bigcup_{x \in \Omega \cap B_\delta(\bar{x})} N_\Omega(x),$$

$$\overline{N}_\Omega(\bar{x}) := \limsup_{x \xrightarrow{\Omega} \bar{x}} N_\Omega(x) = \bigcap_{\delta > 0} \operatorname{cl}^* \widehat{N}_\Omega(\bar{x}, \delta)$$

denote the *Fréchet normal cone*, the *strict δ-normal cone*, and the *limiting normal cone* to Ω at $\bar{x}$, respectively. The denotation $u \xrightarrow{\Omega} x$ in the above formulas means that $u \to x$ with $u \in \Omega$ while cl^* denotes the sequential weak* closure in X^*.

In the Asplund space setting, the uniform regularity of a collection of sets can be characterized using the next dual space constant:

$$\hat{\eta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\delta \downarrow 0} \left\{ \left\| \sum_{i=1}^m x_i^* \right\| \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \sum_{i=1}^m \|x_i^*\| = 1 \right\}, \tag{3.1}$$

where it is assumed that the infimum over the empty set equals 1; this corresponds to all cones $\widehat{N}_{\Omega_i}(\bar{x}, \delta)$ $(1 \leq i \leq m)$ being trivial for some $\delta > 0$ ($\bar{x}$ can be an interior point of $\cap_{i=1}^m \Omega_i$.)

The next theorem corresponds to [24, Theorem 4 (v)–(vi)].

Theorem 6. *(i)* $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \leq \hat{\eta}[\mathbf{\Omega}](\bar{x})$.

(ii) Suppose X is Asplund and the sets Ω_i $(1 \leq i \leq m)$ are closed. Then $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x})$.

As a consequence, $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if* $\hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$*, i.e., there exist* $\alpha > 0$ *and* $\delta > 0$ *such that*

$$\left\| \sum_{i=1}^{m} x_i^* \right\| \geq \alpha \sum_{i=1}^{m} \|x_i^*\| \tag{3.2}$$

for all $x_i \in \Omega_i \cap B_\delta(\bar{x})$ *and* $x_i^* \in N_{\Omega_i}(x_i)$ $(1 \leq i \leq m)$*.*

The dual characterization of the uniform regularity in Theorem 6 (ii) is sometimes referred to as *(Fréchet) normal uniform regularity*, cf. [24, 25]. Constant $\hat{\eta}[\mathbf{\Omega}](\bar{x})$ coincides with the supremum of all α in the definition of this property.

Part (i) of Theorem 6 was proved in [21], while part (ii) was established in [24]. A slightly weaker estimate can be found in [21, 23].

Remark 23. In finite dimensions, constant (3.1) coincides with the corresponding one defined in terms of limiting normals:

$$\bar{\eta}[\mathbf{\Omega}](\bar{x}) := \min \left\{ \left\| \sum_{i=1}^{m} x_i^* \right\| \mid x_i^* \in \overline{N}_{\Omega_i}(\bar{x}), \sum_{i=1}^{m} \|x_i^*\| = 1 \right\}$$

(with the similar natural convention about the minimum over the empty set.) The dual uniform regularity criterion in Theorem 6 (ii) takes the following "exact" ("at the point") form:

there exists $\alpha > 0$ *such that* (3.2) *holds true for all* $x_i^* \in \overline{N}_{\Omega_i}(\bar{x})$ $(1 \leq i \leq m)$*,*

or equivalently,

$$\left.\begin{array}{l} x_i^* \in \overline{N}_{\Omega_i}(\bar{x}) \ (1 \leq i \leq m) \\ x_1^* + x_2^* + \ldots + x_n^* = 0 \end{array}\right\} \quad \Longrightarrow \quad x_1^* = x_2^* = \ldots = x_n^* = 0.$$

This is a well known qualification condition, cf. [33, Corollary 3.37].

Apart from the formulated in Theorem 6 (ii) necessary and sufficient characterization of the uniform regularity, equality $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x})$ implies also an equivalent characterization of approximate stationarity.

Corollary 3 (Extended extremal principle [20, 21]). *Suppose X is Asplund and the sets Ω_i $(1 \le i \le m)$ are closed. $\mathbf{\Omega}$ is approximately stationary at $\bar{x}$ if and only if $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 0$, i.e., for any $\varepsilon > 0$ there exist $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(1 \le i \le m)$ such that*

$$\left\| \sum_{i=1}^{m} x_i^* \right\| < \varepsilon \sum_{i=1}^{m} \|x_i^*\|.$$

This result extends the *extremal principle* [27, 34] and can be considered as a generalization of the convex *separation theorem* to collections of nonconvex sets. Some earlier formulations of Corollary 3 can be found in [18, 19, 17].

Remark 24. Corollary 3 provides also an equivalent characterization of Asplund spaces, cf. [24, Theorem 5]. Theorem 6 (ii) can be extended from Asplund to arbitrary Banach spaces if Fréchet normal cones are replaced by some other kind of normal cones satisfying certain natural properties, e.g., Clarke normal cones, cf. [25].

Remark 25. Theorem 6 can be extended to infinite collections of sets. This allows us to treat infinite and semi-infinite optimization problems, cf. [25, 26].

Verifying the uniform regularity (and several other properties) of a finite collection of sets can always be reduced to that of two sets in the product space.

Proposition 21 ([22], Proposition 4). *$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ if and only if the collection of two sets*

$$\Omega := \Omega_1 \times \Omega_2 \times \ldots \times \Omega_m \quad \text{and} \quad L := \{(x, x, \ldots, x) \mid x \in X\} \tag{3.3}$$

in X^m (with any norm compatible with that in X) is uniformly regular at the point $(\bar{x}, \bar{x}, \ldots, \bar{x})$.

Note the following simple representations of the Fréchet normal cones to the sets in (3.3).

Proposition 22. *(i) Suppose $x_i \in \Omega_i$ $(1 \le i \le m)$. Then*

$$N_\Omega(z) = \prod_{i=1}^{m} N_{\Omega_i}(x_i),$$

where $z = (x_1, x_2, \ldots, x_m)$.

(ii) Suppose $x \in X$. Then

$$N_L(z) = L^{\perp} = \left\{ z^* = (x_1^*, \ldots, x_m^*) \in (X^*)^m \mid \sum_{i=1}^m x_i^* = 0 \right\},$$

where $z = Ax := (x, x, \ldots, x)$.

Proof. The first assertion follows directly from the definition while proving the second one is a simple exercise on application of standard tools of convex analysis. □

3.3 Uniform regularity in a Hilbert space

In this section, we limit ourselves to the case when X is a Hilbert space. For the collection of sets $\boldsymbol{\Omega} = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ $(m > 1)$, denote

$$\hat{c}[\boldsymbol{\Omega}](\bar{x}) := 1 - 2(\hat{\eta}[\boldsymbol{\Omega}](\bar{x}))^2, \tag{3.4}$$

where $\hat{\eta}[\boldsymbol{\Omega}](\bar{x})$ is the dual space regularity constant defined by (3.1). By Theorem 6 (ii), the uniform regularity of $\boldsymbol{\Omega}$ at $\bar{x}$ is equivalent to the inequality $\hat{c}[\boldsymbol{\Omega}](\bar{x}) < 1$. Note that constant (3.4) can be negative: $\hat{c}[\boldsymbol{\Omega}](\bar{x}) \geq -1$.

Lemma 12. *Suppose $\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$. Then, for any $c' > \hat{c}[\boldsymbol{\Omega}](\bar{x})$, there is $\delta > 0$ such that, for any $i, j \in \{1, 2, \ldots, m\}$, $i \neq j$, and any $u \in \widehat{N}_{\Omega_i}(\bar{x}, \delta) \cap \mathbb{S}, v \in \widehat{N}_{\Omega_j}(\bar{x}, \delta) \cap \mathbb{S}$, it holds:*

$$-\langle u, v \rangle < c'. \tag{3.5}$$

Proof. By definition (3.1), for any $c' > \hat{c}[\boldsymbol{\Omega}](\bar{x}))$, there is $\delta > 0$ such that

$$2\left\| \sum_{k=1}^m x_k^* \right\|^2 > 1 - c' \text{ for all } x_k^* \in \widehat{N}_{\Omega_k}(\bar{x}, \delta) \text{ with } \sum_{k=1}^m \|x_k^*\| = 1.$$

Choose any $i, j \in \{1, 2, \ldots, m\}$, $i \neq j$, and any $u \in \widehat{N}_{\Omega_i}(\bar{x}, \delta) \cap \mathbb{S}, v \in \widehat{N}_{\Omega_j}(\bar{x}, \delta) \cap \mathbb{S}$. Set $x_i^* = u/2$, $x_j^* = v/2$, and $x_k^* = 0$ for $k \in \{1, 2, \ldots, m\} \setminus \{i, j\}$. Then $x_k^* \in \widehat{N}_{\Omega_k}(\bar{x}, \delta)$

($k \in \{1, 2, \ldots, m\}$) and $\sum_{k=1}^{m} \|x_k^*\| = 1$. It follows that

$$\|u + v\|^2 = 4 \left\| \sum_{k=1}^{m} x_k^* \right\|^2 > 2(1 - c'),$$

or equivalently

$$2 + 2\langle u, v\rangle > 2(1 - c').$$

In its turn, the last inequality is equivalent to (3.5). □

In the rest of the section, we assume that $m = 2$, i.e., $\mathbf{\Omega} = \{\Omega_1, \Omega_2\}$. Definition (3.1) of the constant characterizing the uniform regularity of a collection of sets can be simplified.

Proposition 23. *The following representation holds true:*

$$\hat{\eta}[\mathbf{\Omega}](\bar{x}) = \lim_{\delta \downarrow 0} \inf \left\{ \|x_1^* + x_2^*\| \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \|x_i^*\| = \frac{1}{2} \ (i = 1, 2) \right\}, \tag{3.6}$$

where it is assumed that the infimum over the empty set equals 1.

Proof. If, for some $\delta > 0$, one of the cones $\widehat{N}_{\Omega_1}(\bar{x}, \delta)$ or $\widehat{N}_{\Omega_2}(\bar{x}, \delta)$ is trivial, then $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 1$ and the equality is satisfied automatically. Take arbitrary nonzero $x_1^* \in \widehat{N}_{\Omega_1}(\bar{x}, \delta)$ and $x_2^* \in \widehat{N}_{\Omega_2}(\bar{x}, \delta)$ such that $\|x_1^*\| + \|x_2^*\| = 1$. Then

$$\begin{aligned}(\|x_1^*\| - \|x_2^*\|)^2 &= \|x_1^*\|^2 + \|x_2^*\|^2 - 2\|x_1^*\|\|x_2^*\|,\\ 1 &= \|x_1^*\|^2 + \|x_2^*\|^2 + 2\|x_1^*\|\|x_2^*\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_1^*\|^2 + \|x_2^*\|^2 &= \frac{1 + (\|x_1^*\| - \|x_2^*\|)^2}{2},\\ \|x_1^*\|\|x_2^*\| &= \frac{1 - (\|x_1^*\| - \|x_2^*\|)^2}{4}.\end{aligned}$$

Set

$$z_1^* := \frac{x_1^*}{2\|x_1^*\|} \quad \text{and} \quad z_2^* := \frac{x_2^*}{2\|x_2^*\|}.$$

Then $z_1^* \in \widehat{N}_{\Omega_1}(\bar{x},\delta)$, $z_2^* \in \widehat{N}_{\Omega_2}(\bar{x},\delta)$, $\|z_i^*\| = \|z_2^*\| = \frac{1}{2}$, and

$$\|z_1^* + z_2^*\|^2 = \frac{1}{2} + \frac{\langle x_1^*, x_2^*\rangle}{2\,\|x_1^*\|\,\|x_2^*\|}.$$

Next we show that

$$\|x_1^* + x_2^*\| \geq \|z_1^* + z_2^*\|.$$

Indeed,

$$\begin{aligned}
\|x_1^* + x_2^*\|^2 - \|z_1^* + z_2^*\|^2 &= \|x_1^*\|^2 + \|x_2^*\|^2 + 2\langle x_1^*, x_2^*\rangle - \frac{1}{2} - \frac{\langle x_1^*, x_2^*\rangle}{2\,\|x_1^*\|\,\|x_2^*\|} \\
&= \frac{1 + (\|x_1^*\| - \|x_2^*\|)^2}{2} - \frac{1}{2} + 2\langle x_1^*, x_2^*\rangle - \frac{\langle x_1^*, x_2^*\rangle}{2\,\|x_1^*\|\,\|x_2^*\|} \\
&= \frac{(\|x_1^*\| - \|x_2^*\|)^2}{2} + \frac{4\,\|x_1^*\|\,\|x_2^*\| - 1}{2\,\|x_1^*\|\,\|x_2^*\|}\langle x_1^*, x_2^*\rangle \\
&= \frac{(\|x_1^*\| - \|x_2^*\|)^2}{2} - \frac{(\|x_1^*\| - \|x_2^*\|)^2}{2\,\|x_1^*\|\,\|x_2^*\|}\langle x_1^*, x_2^*\rangle \\
&= \frac{(\|x_1^*\| - \|x_2^*\|)^2}{2}\left(1 - \frac{\langle x_1^*, x_2^*\rangle}{\|x_1^*\|\,\|x_2^*\|}\right) \geq 0.
\end{aligned}$$

This completes the proof. □

The following example shows that the conclusion of Proposition 23 is not true in non-Hilbert spaces.

Example 10. Consider $\mathbb{R}^2$ with the sum norm, $\|(x,y)\| = |x| + |y|$, and take $\Omega_1 = \{(x_1,x_2) \mid x_2 \leq 0\}$, $\Omega_2 = \{(x_1,x_2) \mid x_2 \geq 2x_1\}$ and $\bar{x} = (0,0) \in \Omega_1 \cap \Omega_2$. Then, for any $\delta > 0$, we have

$$\widehat{N}_{\Omega_1}(\bar{x},\delta) = \{t(0,1) \mid t \in \mathbb{R}^+\},$$

$$\widehat{N}_{\Omega_2}(\bar{x},\delta) = \{t(2,-1) \mid t \in \mathbb{R}^+\}.$$

Thus,

$$z_1^* \in \widehat{N}_{\Omega_1}(\bar{x},\delta) \text{ with } \|z_1^*\| = \frac{1}{2} \Longrightarrow z_1^* = (0,1/2),$$

$$z_2^* \in \widehat{N}_{\Omega_2}(\bar{x},\delta) \text{ with } \|z_2^*\| = \frac{1}{2} \Longrightarrow z_2^* = (1/3,-1/6),$$

and the right-hand side of (3.6) equals $\|z_1^* + z_2^*\| = \|(1/3,1/3)\| = 2/3$. At the same time, with $x_1^* = (0, 1/4) \in \widehat{N}_{\Omega_1}(\bar{x}, \delta)$ and $x_2^* = (\frac{1}{2}, -1/4) \in \widehat{N}_{\Omega_2}(\bar{x}, \delta)$ it holds $\|x_1^*\| + \|x_2^*\| = 1$ and $\|x_1^* + x_2^*\| = \frac{1}{2}$. Hence, $\hat{\eta}[\mathbf{\Omega}](\bar{x}) \le \|x_1^* + x_2^*\| < 2/3$.

The next proposition provides an equivalent representation of constant (3.4).

Proposition 24. *The following representation holds true:*

$$\hat{c}[\mathbf{\Omega}](\bar{x}) = \limsup_{\delta \downarrow 0} \left\{ -\langle x_1^*, x_2^* \rangle \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \|x_i^*\| = 1 \; (i = 1, 2) \right\}. \tag{3.7}$$

where it is assumed that the supremum over the empty set equals -1.

Proof. If, for some $\delta > 0$, one of the cones $\widehat{N}_{\Omega_1}(\bar{x}, \delta)$ or $\widehat{N}_{\Omega_2}(\bar{x}, \delta)$ is trivial, then $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 1$, the right-hand side of (3.7) equals -1 and coincides with $\hat{c}[\mathbf{\Omega}](\bar{x})$ computed in accordance with definition (3.4). Let both cones be nontrivial for any $\delta > 0$. Then, by (3.4), (3.6), and (3.7),

$$\begin{aligned}
\hat{c}[\mathbf{\Omega}](\bar{x}) &= \limsup_{\delta \downarrow 0} \left\{ 1 - 2\|x_1^* + x_2^*\|^2 \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \|x_i^*\| = \frac{1}{2} \; (i = 1, 2) \right\} \\
&= \limsup_{\delta \downarrow 0} \left\{ -\langle 2x_1^*, 2x_2^* \rangle \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \|x_i^*\| = \frac{1}{2} \; (i = 1, 2) \right\} \\
&= \hat{c}[\mathbf{\Omega}](\bar{x}).
\end{aligned}$$

□

Another dual space constant can be used alongside (3.6) and (3.7) for characterizing the uniform regularity of a collection of two sets in a Hilbert space:

$$\hat{\nu}[\mathbf{\Omega}](\bar{x}) := \limsup_{\delta \downarrow 0} \left\{ \|x_1^* - x_2^*\| \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \|x_i^*\| = \frac{1}{2} \; (i = 1, 2) \right\}, \tag{3.8}$$

where it is assumed that the supremum over the empty set equals 0; this corresponds to one of the cones $\widehat{N}_{\Omega_1}(\bar{x}, \delta)$ or $\widehat{N}_{\Omega_2}(\bar{x}, \delta)$ being trivial for some $\delta > 0$ ($\bar{x}$ can be an interior point of either Ω_1 or Ω_2.)

Remark 26. Unlike constants $\hat{\eta}[\mathbf{\Omega}](\bar{x})$ and $\hat{c}[\mathbf{\Omega}](\bar{x})$, the definition of constant $\hat{\nu}[\mathbf{\Omega}](\bar{x})$ is specific for the case of two sets.

Remark 27. Condition $\|x_i^*\| = \frac{1}{2}$, $i = 1, 2$, in definition (3.8) cannot be replaced by $\|x_1^*\| + \|x_2^*\| = 1$ (as in (3.1)): it would always be equal to 1.

Theorem 7. *The following relations hold true:*

(i) $(\hat{\eta}[\mathbf{\Omega}](\bar{x}))^2 + (\hat{\nu}[\mathbf{\Omega}](\bar{x}))^2 = 1$*;*

(ii) $1 + \hat{c}[\mathbf{\Omega}](\bar{x}) = 2(\hat{\nu}[\mathbf{\Omega}](\bar{x}))^2$.

Proof. If, for some $\delta > 0$, one of the cones $\widehat{N}_{\Omega_1}(\bar{x}, \delta)$ or $\widehat{N}_{\Omega_2}(\bar{x}, \delta)$ is trivial, then $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 1$, $\hat{\nu}[\mathbf{\Omega}](\bar{x}) = 0$, $\hat{c}[\mathbf{\Omega}](\bar{x}) = -1$, and equalities (i) and (ii) are satisfied automatically. Let both cones be nontrivial for any $\delta > 0$. Fix an arbitrary $\varepsilon > 0$.

(i) By definition (3.8), there exists $\delta > 0$ such that

$$\|x_1^* - x_2^*\| \le \hat{\nu}[\mathbf{\Omega}](\bar{x}) + \varepsilon$$

for any $x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)$ with $\|x_i^*\| = \frac{1}{2}$ $(i = 1, 2)$. At the same time, by (3.6), there are elements $x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)$ with $\|x_i^*\| = \frac{1}{2}$ $(i = 1, 2)$ such that

$$\|x_1^* + x_2^*\| \le \hat{\eta}[\mathbf{\Omega}](\bar{x}) + \varepsilon.$$

Hence,

$$(\hat{\eta}[\mathbf{\Omega}](\bar{x}) + \varepsilon)^2 + (\hat{\nu}[\mathbf{\Omega}](\bar{x}) + \varepsilon)^2 \ge \|x_1^* - x_2^*\|^2 + \|x_1^* + x_2^*\|^2 = 1.$$

Since ε is arbitrary, we have

$$\hat{\eta}[\mathbf{\Omega}](\bar{x})^2 + \hat{\nu}[\mathbf{\Omega}](\bar{x})^2 \ge 1.$$

Similarly, by (3.6) and (3.8), we find elements $x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)$ with $\|x_i^*\| = \frac{1}{2}$ $(i = 1, 2)$ such that

$$\|x_1^* - x_2^*\| \ge \hat{\nu}[\mathbf{\Omega}](\bar{x}) - \varepsilon,$$

$$\|x_1^* + x_2^*\| \ge \hat{\eta}[\mathbf{\Omega}](\bar{x}) - \varepsilon.$$

This yields

$$(\hat{\nu}[\mathbf{\Omega}](\bar{x}) - \varepsilon)^2 + (\hat{\eta}[\mathbf{\Omega}](\bar{x}) - \varepsilon)^2 \le 1,$$

and consequently,

$$\hat{\eta}[\mathbf{\Omega}](\bar{x})^2 + \hat{\nu}[\mathbf{\Omega}](\bar{x})^2 \leq 1.$$

(ii) follows immediately from (i) and definition (3.4). $\square$

Corollary 4. *$\{\Omega_1, \Omega_2\}$ is uniformly regular at $\bar{x} \in \Omega_1 \cap \Omega_2$ if and only if one of the following equivalent conditions holds true:*

(i) $\hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$;

(ii) $\hat{\nu}[\mathbf{\Omega}](\bar{x}) < 1$;

(iii) $\hat{c}[\mathbf{\Omega}](\bar{x}) < 1$.

The next example shows that the equality in Theorem 7 (ii) remains true when $\hat{c}[\mathbf{\Omega}](\bar{x}) \leq 0$.

Example 11. In $\mathbb{R}^2$ with the Euclidean norm, we fix $\Omega_1 = \{(x_1, x_2) \mid x_2 \leq 0\}$ and $\bar{x} = (0, 0)$. Then, for any $\delta > 0$, $\widehat{N}_{\Omega_1}(\bar{x}, \delta) = \{t(0, 1) \mid t \geq 0\}$. We consider the following two cases of Ω_2:

Case 1. $\Omega_2 = \{(x_1, x_2) \mid x_1 \leq 0\}$. For any $\delta > 0$, $\widehat{N}_{\Omega_2}(\bar{x}, \delta) = \{t(1, 0) \mid t \geq 0\}$. Then $\hat{c}[\mathbf{\Omega}](\bar{x}) = 0$ and $\hat{\nu}[\mathbf{\Omega}](\bar{x}) = \frac{\sqrt{2}}{2}$.

Case 2. $\Omega_2 = \{(x_1, x_2) \mid x_1 + x_2 \leq 0\}$. For any $\delta > 0$, $\widehat{N}_{\Omega_2}(\bar{x}, \delta) = \{t(1, 1) \mid t \geq 0\}$. Then $\hat{c}[\mathbf{\Omega}](\bar{x}) = -\frac{1}{\sqrt{2}}$ and $\hat{\nu}[\mathbf{\Omega}](\bar{x}) = \frac{\sqrt{2-\sqrt{2}}}{2}$.

In both cases the equality in Theorem 7 (ii) holds true.

Remark 28. In finite dimensions, constants (3.6)–(3.7) coincide with the corresponding ones defined in terms of limiting normals:

$$\bar{\eta}[\mathbf{\Omega}](\bar{x}) := \min\left\{\|x_1^* + x_2^*\| \mid x_i^* \in \overline{N}_{\Omega_i}(\bar{x}), \|x_i^*\| = \frac{1}{2}\ (i = 1, 2)\right\},$$

$$\bar{\nu}[\mathbf{\Omega}](\bar{x}) := \max\left\{\|x_1^* - x_2^*\| \mid x_i^* \in \overline{N}_{\Omega_i}(\bar{x}), \|x_i^*\| = \frac{1}{2}\ (i = 1, 2)\right\}, \tag{3.9}$$

$$\bar{c}[\mathbf{\Omega}](\bar{x}) := \max\left\{-\langle x_1^*, x_2^*\rangle \mid x_i^* \in \overline{N}_{\Omega_i}(\bar{x}), \|x_i^*\| = 1\ (i = 1, 2)\right\} \tag{3.10}$$

(with the similar natural conventions about the minimum and maximum over the empty set.) The relations amongst the above constants are consequences of those in Theorem 7:

(i) $(\bar{\eta}[\mathbf{\Omega}](\bar{x}))^2 + (\bar{\nu}[\mathbf{\Omega}](\bar{x}))^2 = 1$;

(ii) $1 + \bar{c}[\mathbf{\Omega}](\bar{x}) = 2(\bar{\nu}[\mathbf{\Omega}](\bar{x}))^2$;

(iii) $1 - \bar{c}[\mathbf{\Omega}](\bar{x}) = 2(\bar{\eta}[\mathbf{\Omega}](\bar{x}))^2$.

Remark 29. Constant (3.10) is closely related with the one introduced in [29]:

$$\bar{c} := \max\left\{-\langle x_1^*, x_2^*\rangle \mid x_i^* \in \overline{N}_{\Omega_i}(\bar{x}) \cap \mathbb{B}\ (i = 1, 2)\right\}.$$

Indeed, $\bar{c} = (\bar{c}[\mathbf{\Omega}](\bar{x}))_+$, where $(\alpha)_+ := \max\{\alpha, 0\}$.

Given a collection of m sets $\mathbf{\Omega} = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ in a finite dimensional Hilbert space X and a point $\bar{x} \in \cap_{i=1}^m \Omega_i$, one can consider the Hilbert space X^m with the norm

$$\|(x_1, x_2, \ldots, x_n)\| = \left(\sum_{i=1}^m \|x_i\|^2\right)^{\frac{1}{2}}$$

and compute constants (3.6), (3.7), and (3.8) corresponding to the collection $\mathbf{\Omega}' := \{\Omega, L\}$ and the point $\bar{z} := A\bar{x} = (\bar{x}, \bar{x}, \ldots, \bar{x}) \in \Omega \cap L$, where Ω and L are defined by (3.3).

Proposition 25. *The following representations hold true:*

$$\hat{\eta}[\mathbf{\Omega}'](\bar{z}) = \liminf_{\delta\downarrow 0}\left\{\left(\frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{m}\|x_1^* + \ldots + x_m^*\|^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)\ (1 \le i \le m), \sum_{i=1}^m \|x_i^*\|^2 = 1\right\}, \quad (3.11)$$

$$\hat{\nu}[\mathbf{\Omega}'](\bar{z}) = \limsup_{\delta\downarrow 0}\left\{\left(\frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{m}\|x_1^* + \ldots + x_m^*\|^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)\ (1 \le i \le m), \sum_{i=1}^m \|x_i^*\|^2 = 1\right\}, \quad (3.12)$$

$$\hat{c}[\mathbf{\Omega}'](\bar{z}) = \limsup_{\delta\downarrow 0}\left\{\left(1 - \frac{1}{m}\|x_1^* + \ldots + x_m^*\|^2\right)^{\frac{1}{2}} \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)\ (1 \le i \le m), \sum_{i=1}^m \|x_i^*\|^2 = 1\right\}. \quad (3.13)$$

Proof. If $z_1 = (x_1, x_2, \ldots, x_n)$, $z_2 = (u_1, u_2, \ldots, u_n) \in X^m$, then

$$\|z_1 + z_2\|^2 = \sum_{i=1}^m \|x_i\|^2 + \sum_{i=1}^m \|u_i\|^2 + 2\sum_{i=1}^m \langle x_i, u_i\rangle.$$

By the structure of $\mathbf{\Omega}'$ and (3.6), we have

$$\begin{aligned}\hat{\eta}[\mathbf{\Omega}'](\bar{z}) &= \liminf_{\delta\downarrow 0}\Bigg\{\left(\frac{1}{2} + 2\sum_{i=1}^m \langle x_i^*, u_i\rangle\right)^{\frac{1}{2}} \mid \sum_{i=1}^m \|x_i^*\|^2 = \sum_{i=1}^m \|u_i\|^2 = \frac{1}{4}, \\ &\qquad x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \sum_{i=1}^m u_i = 0 \ (1 \le i \le m)\Bigg\} \\ &= \liminf_{\delta\downarrow 0}\Bigg\{\left(\frac{1}{2} + \frac{1}{2}\sum_{i=1}^m \langle x_i^*, u_i\rangle\right)^{\frac{1}{2}} \mid \sum_{i=1}^m \|x_i^*\|^2 = \sum_{i=1}^m \|u_i\|^2 = 1, \\ &\qquad x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta), \sum_{i=1}^m u_i = 0 \ (1 \le i \le m)\Bigg\}. \end{aligned} \tag{3.14}$$

Fix any $x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)$ $(1 \le i \le m)$ with $\sum_{i=1}^m \|x_i^*\|^2 = 1$ and denote

$$x_0^* := \frac{1}{m}\sum_{i=1}^m x_i^*. \tag{3.15}$$

Consider the following minimization problem in X^m which is an important component of (3.14):

$$\begin{aligned} &\text{minimize} \quad f(u) := \sum_{i=1}^m \langle x_i^*, u_i\rangle \\ &\text{subject to} \quad \sum_{i=1}^m u_i = 0 \text{ and } \sum_{i=1}^m \|u_i\|^2 = 1. \end{aligned}$$

Since f is continuous and the constraint set is compact, the above problem has a solution $u^\circ = (u_1^\circ, u_2^\circ, \ldots, u_m^\circ)$. In accordance with the Lagrange multiplier rule, there exist multiplies λ_0, $\lambda_1 \in \mathbb{R}$ and $u^* \in X$, not all zero, such that

$$\lambda_0 x_i^* + 2\lambda_1 u_i^\circ + u^* = 0 \quad (1 \le i \le m). \tag{3.16}$$

Adding the equalities together and taking into account that $\sum_{i=1}^m u_i^\circ = 0$, we obtain

$$\lambda_0 \sum_{i=1}^m x_i^* + mu^* = 0. \tag{3.17}$$

If $\lambda_0 = 0$, then $u^* = 0$ and consequently $\lambda_1 \neq 0$ and, by (3.16), $u_i^\circ = 0$ for all $i \in \{1, 2, \ldots, m\}$, which is impossible thanks to $\sum_{i=1}^m \|u_i^\circ\|^2 = 1$. Hence, $\lambda_0 \neq 0$ and we can take $\lambda_0 = 1$. It follows from (3.16), (3.17), and (3.15) that

$$x_i^* + 2\lambda_1 u_i^\circ = x_0^* \quad (1 \le i \le m), \tag{3.18}$$

and consequently

$$\begin{aligned} 4\lambda_1^2 = \sum_{i=1}^m \|x_0^* - x_i^*\|^2 &= m\|x_0^*\|^2 + \sum_{i=1}^m \|x_i^*\|^2 - 2\left\langle \sum_{i=1}^m x_i^*, x_0^* \right\rangle \\ &= \sum_{i=1}^m \|x_i^*\|^2 - m\|x_0^*\|^2. \end{aligned} \tag{3.19}$$

At the same time,

$$\begin{aligned} 2\lambda_1 f(u^\circ) = \sum_{i=1}^m \langle x_i^*, 2\lambda_1 u_i^\circ \rangle &= \sum_{i=1}^m \langle x_i^*, x_0^* - x_i^* \rangle \\ &= \left(\left\langle \sum_{i=1}^m x_i^*, x_0^* \right\rangle - \sum_{i=1}^m \|x_i^*\|^2 \right) \\ &= \left(m\|x_0^*\|^2 - \sum_{i=1}^m \|x_i^*\|^2 \right) = -4\lambda_1^2. \end{aligned}$$

This yields either $f(u^\circ) = -2\lambda_1$ or $\lambda_1 = 0$. In the last case, by (3.18), $x_i^* = x_0^*$ for all $i \in \{1, 2, \ldots, m\}$, and consequently

$$f(u^\circ) = \sum_{i=1}^m \langle x_0^*, u_i^\circ \rangle = \left\langle x_0^*, \sum_{i=1}^m u_i^\circ \right\rangle = 0.$$

Hence, in both cases, $f(u^\circ) = -2\lambda_1$. Since u° is a point of minimum, λ_1 must be nonnegative, and consequently, by (3.19),

$$f(u^\circ) = -\left(\sum_{i=1}^{m} \|x_i^*\|^2 - m\|x_0^*\|^2\right)^{\frac{1}{2}} = -\left(1 - m\|x_0^*\|^2\right)^{\frac{1}{2}}.$$

Combining this with (3.14), we get (3.11).

(3.12) and (3.13) follow from (3.11) thanks to Theorem 7. □

Corollary 5. *The following estimates hold true:*

$$0 \le \hat{\eta}[\mathbf{\Omega}'](\bar{z}) \le \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1}{m}}\right)^{\frac{1}{2}};$$
$$\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{m}}\right)^{\frac{1}{2}} \le \hat{\nu}[\mathbf{\Omega}'](\bar{z}) \le 1;$$
$$\sqrt{1 - \frac{1}{m}} \le \hat{c}[\mathbf{\Omega}'](\bar{z}) \le 1.$$

Proof. The estimates follow from Proposition 25 due to the fact that

$$\min\{\|x_1 + x_2 + \ldots + x_m\| \mid \|x_1\|^2 + \|x_2\|^2 + \ldots + \|x_m\|^2 = 1\} \le 1.$$ □

Dual space constants (3.11), (3.12), and (3.13) can be used to characterize the uniform regularity of collections of m sets.

The next corollary follows from Proposition 21 and Corollary 4.

Corollary 6. $\mathbf{\Omega}$ *is uniformly regular at* $\bar{x} \in \cap_{i=1}^m \Omega_i$ *if and only if one of the following equivalent conditions holds true:*

(i) $\hat{\eta}[\mathbf{\Omega}'](\bar{z}) > 0$*;*

(ii) $\hat{\nu}[\mathbf{\Omega}'](\bar{z}) < 1$*;*

(iii) $\hat{c}[\mathbf{\Omega}'](\bar{z}) < 1$*.*

Observe that, when $m = 2$, constants (3.11), (3.12), and (3.13) do not coincide with the corresponding constants (3.6), (3.8), and (3.7) .

Corollary 7. *When $m = 2$, the following relations hold true:*

$$\begin{aligned}
\hat{\eta}[\mathbf{\Omega}'](\bar{z}) = \liminf_{\delta \downarrow 0} \Big\{ \Big(\frac{1 - \|x_1^* - x_2^*\|}{2} \Big)^{\frac{1}{2}} \mid \quad & x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta) \ (i = 1, 2), \\
& \|x_1^*\|^2 + \|x_2^*\|^2 = \frac{1}{2} \Big\}, \\
\hat{\nu}[\mathbf{\Omega}'](\bar{z}) = \limsup_{\delta \downarrow 0} \Big\{ \Big(\frac{1 + \|x_1^* - x_2^*\|}{2} \Big)^{\frac{1}{2}} \mid \quad & x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta) \ (i = 1, 2), \\
& \|x_1^*\|^2 + \|x_2^*\|^2 = \frac{1}{2} \Big\}, \\
\hat{c}[\mathbf{\Omega}'](\bar{z}) = \limsup_{\delta \downarrow 0} \Big\{ \|x_1^* - x_2^*\| \quad \mid \quad & x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta) \ (i = 1, 2), \\
& \|x_1^*\|^2 + \|x_2^*\|^2 = \frac{1}{2} \Big\}.
\end{aligned} \tag{3.20}$$

Proof. From Proposition 25, we have

$$\begin{aligned}
\hat{c}[\mathbf{\Omega}'](\bar{z}) = \limsup_{\delta \downarrow 0} \Big\{ \Big(1 - \frac{1}{2} \|x_1^* + x_2^*\|^2 \Big)^{1/2} \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta) \ (i = 1, 2), \\
\|x_1^*\|^2 + \|x_2^*\|^2 = 1 \Big\}.
\end{aligned}$$

In the above formula,

$$\begin{aligned}
1 - \frac{1}{2} \|x_1^* + x_2^*\|^2 &= \frac{1}{2} (2 - \|x_1^* + x_2^*\|^2) \\
&= \frac{1}{2} \big(2(\|x_1^*\|^2 + \|x_2^*\|^2) - (\|x_1^*\|^2 + \|x_2^*\|^2 + 2\langle x_1^*, x_2^* \rangle) \big) \\
&= \frac{1}{2} (\|x_1^*\|^2 + \|x_2^*\|^2 - 2\langle x_1^*, x_2^* \rangle) \\
&= \frac{1}{2} \|x_1^* - x_2^*\|^2 = \left\| \frac{x_1^*}{\sqrt{2}} - \frac{x_2^*}{\sqrt{2}} \right\|^2
\end{aligned}$$

and

$$\left\| \frac{x_1^*}{\sqrt{2}} \right\|^2 + \left\| \frac{x_2^*}{\sqrt{2}} \right\|^2 = \frac{1}{2}.$$

This proves (3.20), which also implies the other relations. □

The next relation between $\hat{c}[\mathbf{\Omega}'](\bar{z})$ and $\hat{\nu}[\mathbf{\Omega}](\bar{x})$ can be of interest.

Proposition 26. *When $m = 2$, it holds:*

$$\hat{c}[\mathbf{\Omega}'](\bar{z}) \geq \hat{\nu}[\mathbf{\Omega}](\bar{x}). \tag{3.21}$$

Furthermore, (3.21) holds as an equality whenever $\hat{\nu}[\mathbf{\Omega}](\bar{x}) > 1/\sqrt{2}$.

Proof. In view of (3.20) and (3.8), inequality (3.21) is always true.

We prove the second assertion. Suppose $\hat{\nu}[\mathbf{\Omega}](\bar{x}) > 1/\sqrt{2}$. By (3.8), for any $\delta > 0$, one can find $x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)$ with $\|x_i^*\| = \frac{1}{2}$ $(i = 1, 2)$ such that $\|x_1^* - x_2^*\| > 1/\sqrt{2}$.

Observe that, for any x_1^* and x_2^* with $\|x_1^*\|^2 + \|x_2^*\|^2 = \frac{1}{2}$, it holds

$$\|x_1^* - x_2^*\|^2 = \frac{1}{2} - 2\langle x_1^*, x_2^* \rangle.$$

Hence, maximizing $\|x_1^* - x_2^*\|$ is equivalent to minimizing $\langle x_1^*, x_2^* \rangle$, and condition $\|x_1^* - x_2^*\| > 1/\sqrt{2}$ is equivalent to $\langle x_1^*, x_2^* \rangle < 0$. Under the assumptions made,

$$\begin{aligned}
&\sup\left\{ \|x_1^* - x_2^*\| \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)\,(i = 1, 2),\ \|x_1^*\|^2 + \|x_2^*\|^2 = \frac{1}{2} \right\} \\
=&\sup\left\{ \|x_1^* - x_2^*\| \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta)\,(i = 1, 2),\ \|x_1^*\|^2 + \|x_2^*\|^2 = \frac{1}{2},\ \langle x_1^*, x_2^* \rangle < 0 \right\} \\
=&\sup\left\{ \|x_1^* - x_2^*\| \mid x_i^* \in \widehat{N}_{\Omega_i}(\bar{x}, \delta),\ \|x_i^*\| = \frac{1}{2}\,(i = 1, 2),\ \langle x_1^*, x_2^* \rangle < 0 \right\},
\end{aligned}$$

and it follows from (3.20) that $\hat{c}[\mathbf{\Omega}'](\bar{z}) = \hat{\nu}[\mathbf{\Omega}](\bar{x})$. □

3.4 Applications in projection algorithms

Inspired by [29], we are making an attempt to extend convergence results of the alternating projections for solving feasibility problems to those of the cyclic projection algorithms in Hilbert spaces. Recall that a feasibility problem consists in finding common points of a collection of sets with nonempty intersection. This model incorporates many important optimization problems.

We first recall some basic facts about projections. Given a nonempty set Ω in a normed linear space X, the distance function and projection mapping are defined, for $x \in X$, respectively, as follows:

$$d(x,\Omega) := \inf_{\omega\in\Omega} \|x-\omega\|\,,$$

$$P_\Omega(x) := \{\omega\in\Omega \mid \|x-\omega\| = d(x,\Omega)\}\,.$$

Lemma 13 ([11]). *$\omega \in P_\Omega(x) \Longrightarrow x-\omega \in N_\Omega(\omega)$.*

From now on, we are considering a finite collection of closed sets $\boldsymbol{\Omega} = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ $(m>1)$ and assuming the existence of a point $\bar{x} \in \cap_{i=1}^m \Omega_i$.

Definition 6. A sequence (x_k) is generated by

(i) the *averaged projections* for $\boldsymbol{\Omega}$ if

$$x_{k+1} \in \frac{1}{m}\sum_{i=1}^m P_{\Omega_i}(x_k), \quad k=0,1,\ldots; \tag{3.22}$$

(ii) the *cyclic projections* for $\boldsymbol{\Omega}$ if

$$x_{k+1} \in P_{\Omega_{k+1}}(x_k), \quad k=0,1,\ldots, \tag{3.23}$$

with the convention $\Omega_{i+nm} = \Omega_i$ for all $i=1,\ldots,m$ and $n\in\mathbb{N}$.

Note that the existence of the sequences in Definition 6 cannot be guaranteed in general, unless the space is finite dimensional.

From now on, we are assuming that X is a Hilbert space. The next regularity property is needed in our analysis.

Definition 7 ([29], Definition 4.3)**.** A closed set Ω is super-regular at $\bar{x}\in\Omega$ if, for any $\gamma>0$, any two points x,z sufficiently close to $\bar{x}$ with $z\in\Omega$, and any point $y\in P_\Omega(x)$, it holds $\langle z-y, x-y\rangle \le \gamma\|z-y\|\cdot\|x-y\|$.

Lemma 14 ([29], Proposition 4.4)**.** *A closed set Ω is super-regular at $\bar{x}\in\Omega$ if and only if for any $\gamma>0$, there is $\delta>0$ such that*

$$\langle u, z-x\rangle \le \gamma\|u\|\cdot\|z-x\|, \quad \forall z,x\in\Omega\cap B_\delta(\bar{x}), u\in N_\Omega(x).$$

Remark 30. Similar to the well known *prox-regularity* property (the projection mapping associated with the set being single-valued around the reference point; cf. [7, 12, 37, 39]), the super-regularity one in Definition 7 is a way of describing sets being locally "almost" convex. It is weaker than the prox-regularity while stronger than the Clarke regularity and fits well the convergence analysis of projections algorithms. For a detailed discussion and characterizations of this property we refer the reader to [29].

Theorem 8. *Suppose $\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ with*

$$\hat{c}[\mathbf{\Omega}](\bar{x}) < \frac{1}{m-1} \tag{3.24}$$

and Ω_1 is super-regular at $\bar{x}$. Then, for any $c \in ((m-1)\hat{c}[\mathbf{\Omega}](\bar{x}), 1)$, a sequence (x_k) generated by cyclic projections for $\mathbf{\Omega}$ linearly converges to some point in $\cap_{i=1}^m \Omega_i$ with rate $\sqrt[m]{c}$, provided that

$$\|x_{k+2} - x_{k+1}\| \le \|x_{k+1} - x_k\| \quad (k = 1, 2, \ldots,) \tag{3.25}$$

and x_0 is sufficiently close to $\bar{x}$.

Proof. Let $c \in ((m-1)\hat{c}[\mathbf{\Omega}](\bar{x}), 1)$. Choose $c' > \hat{c}[\mathbf{\Omega}](\bar{x})$ and $\gamma > 0$ such that $(m-1)c' + m\gamma < c$ and $\delta > 0$ such that the conclusions of Lemmas 12 and 14 (with $\Omega = \Omega_1$) are satisfied.

Let $x_0 \in X$ be such that

$$\|x_0 - \bar{x}\| < \frac{\delta(1-c)}{2(m+1)}.$$

Then

$$\alpha := \|x_1 - \bar{x}\| \le \|x_1 - x_0\| + \|x_0 - \bar{x}\| \le 2\|x_0 - \bar{x}\| < \frac{\delta(1-c)}{m+1} \tag{3.26}$$

and, by (3.25),

$$\begin{aligned} \|x_i - \bar{x}\| &\le \|x_i - x_{i-1}\| + \ldots + \|x_1 - \bar{x}\| \\ &\le (i-1)\|x_2 - x_1\| + \|x_1 - \bar{x}\| \le i\|x_1 - \bar{x}\| \\ &= i\alpha \le (m+1)\alpha \quad (i = 2, \ldots, m+1). \end{aligned} \tag{3.27}$$

We are going to prove by induction that, for all $k = 0, 1, \ldots$,

$$\|x_{km+i} - \bar{x}\| \leq (m+1)\alpha\frac{1-c^{k+1}}{1-c} \quad (i = 2, \ldots, m+1). \tag{3.28}$$

When $k = 0$, the required inequalities have been established in (3.27). Supposing that the inequalities are true for all $k = 0, \ldots, l$ where $l \geq 0$, we show that they hold true for $k = l+1$.

We first prove that

$$\|x_{(k+1)m+1} - x_{(k+1)m}\| \leq c\|x_{km+2} - x_{km+1}\| \quad (k = 0, \ldots, l). \tag{3.29}$$

Indeed, if $x_{(k+1)m+1} = x_{(k+1)m}$, the inequality is trivially satisfied. If $x_{km+2} = x_{km+1}$, then, by condition (3.25), $x_{(k+1)m+1} = x_{(k+1)m}$, and the inequality is satisfied too. Otherwise, by (3.26) and (3.28), $\|x_{km+i} - \bar{x}\| < \delta$ $(i = 2, \ldots, m+1)$ and we have by Lemmas 12 and 13, condition (3.25) and definition of projections:

$$\begin{aligned}
&\langle x_{(k+1)m} - x_{(k+1)m+1}, x_{km+i+1} - x_{km+i}\rangle \\
&\qquad < c'\|x_{(k+1)m} - x_{(k+1)m+1}\| \cdot \|x_{km+i+1} - x_{km+i}\| \\
&\qquad \leq c'\|x_{(k+1)m} - x_{(k+1)m+1}\| \cdot \|x_{km+2} - x_{km+1}\| \quad (i = 1, \ldots, m-1).
\end{aligned}$$

Adding the above inequalities, we obtain

$$\begin{aligned}
&\langle x_{(k+1)m} - x_{(k+1)m+1}, x_{(k+1)m} - x_{km+1}\rangle \\
&\qquad\qquad < (m-1)c'\|x_{(k+1)m} - x_{(k+1)m+1}\| \cdot \|x_{km+2} - x_{km+1}\|.
\end{aligned} \tag{3.30}$$

At the same time, by Lemma 14, the triangle inequality and condition (3.25),

$$\begin{aligned}
&\langle x_{(k+1)m} - x_{(k+1)m+1}, x_{km+1} - x_{(k+1)m+1}\rangle \\
&\qquad\qquad \leq \gamma\|x_{(k+1)m} - x_{(k+1)m+1}\| \cdot \|x_{(k+1)m+1} - x_{km+1}\|, \\
&\|x_{(k+1)m+1} - x_{km+1}\| \leq \sum_{i=1}^{m}\|x_{km+i+1} - x_{km+i}\| \leq m\|x_{km+2} - x_{km+1}\|,
\end{aligned}$$

and consequently,

$$\langle x_{(k+1)m} - x_{(k+1)m+1}, x_{km+1} - x_{(k+1)m+1}\rangle$$
$$\leq m\gamma \|x_{(k+1)m} - x_{(k+1)m+1}\| \cdot \|x_{km+2} - x_{km+1}\|. \quad (3.31)$$

Adding (3.30) and (3.31), we get

$$\|x_{(k+1)m} - x_{(k+1)m+1}\|^2 < c\|x_{(k+1)m} - x_{(k+1)m+1}\| \cdot \|x_{km+2} - x_{km+1}\|,$$

or equivalently

$$\|x_{(k+1)m+1} - x_{(k+1)m}\| < c\|x_{km+2} - x_{km+1}\|.$$

This proves (3.29).

Now with $k = l+1$ and taking into account (3.29), we have for $i = 2, \ldots, m+1$:

$$\begin{aligned}
\|x_{(l+1)m+i} - \bar{x}\| &\leq \|x_{(l+1)m+i} - x_{(l+1)m+i-1}\| + \ldots + \|x_{(l+1)m} - \bar{x}\| \\
&\leq i\|x_{(l+1)m+1} - x_{(l+1)m}\| + \|x_{(l+1)m} - \bar{x}\| \\
&\leq ic^{l+1}\|x_2 - x_1\| + \|x_{lm+m} - \bar{x}\| \\
&\leq (m+1)\alpha c^{l+1} + (m+1)\alpha\frac{1-c^{l+1}}{1-c} = (m+1)\alpha\frac{1-c^{l+2}}{1-c}.
\end{aligned}$$

Finally we prove that (x_n) converges to some point $\tilde{x}$ in $\cap_{i=1}^m \Omega_i$ with rate $\sqrt[m]{c}$. Take any $k, r \in \mathbb{N}$ with $k > r$ and choose $n \in \mathbb{N}$ and $i \in \{0, 1, \ldots, m-1\}$ such that $r = nm + i$. We have

$$\begin{aligned}
\|x_k - x_r\| &\leq \sum_{j=r}^{k-1} \|x_{j+1} - x_j\| \leq \sum_{j=nm}^{\infty} \|x_{j+1} - x_j\| \\
&\leq \sum_{j=n}^{\infty}\sum_{i=0}^{m-1} \|x_{mj+i+1} - x_{mj+i}\| \leq m\sum_{j=n}^{\infty} \|x_{mj+1} - x_{mj}\| \\
&\leq m\|x_2 - x_1\|\sum_{j=n}^{\infty} c^j \leq \frac{m\alpha c^n}{1-c}. \quad (3.32)
\end{aligned}$$

Hence, $\|x_k - x_r\| \to 0$ as $k, r \to \infty$, and consequently (x_n) is a Cauchy sequence and, therefore, converges to some point $\tilde{x} \in X$. It follows from (3.32) that

$$\|\tilde{x} - x_r\| \leq \frac{m\alpha c^n}{1-c} = \frac{m\alpha}{(1-c)c^{\frac{i}{m}}} c^{\frac{r}{m}} \leq \frac{m\alpha}{(1-c)c} (\sqrt[m]{c})^r.$$

Finally, we check that $\tilde{x} \in \cap_{i=1}^m \Omega_i$. Indeed, for any $i \in \{1, 2, \ldots, m\}$, $x_{nm+i} \in \Omega_i$. At the same time, $x_{nm+i} \to \tilde{x}$ as $n \to \infty$, and consequently, by the closedness of Ω_i, $\tilde{x} \in \Omega_i$. □

Remark 31. When $m = 2$, conditions (3.25) and (3.24) are satisfied automatically.

The convergence result of the alternating projection method, i.e., the cyclic projection method (3.23) when $m = 2$, established in [29, Theorem 5.16] is a consequence of Theorem 8.

Corollary 8. *Suppose that $\mathbf{\Omega}$ is uniformly regular at $\bar{x} \in \Omega_1 \cap \Omega_2$ and Ω_1 is super-regular at this point. Then, any sequence generated by the alternating projections for $\mathbf{\Omega}$ linearly converges to some point in the intersection provided that x_0 is sufficiently close to $\bar{x}$.*

Now, we derive from Corollary 8 another convergence result of the averaged projection algorithm for a collection of m sets. Given a collection of sets $\mathbf{\Omega} = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ in X, we consider the collection $\mathbf{\Omega}' := \{\Omega, L\}$ of two sets in X^m given by (3.3). For $x \in X$, denote $Ax := (x, x, \ldots, x) \in L$.

Lemma 15. *(i) For any $x \in X$,*

$$P_\Omega(Ax) = (P_{\Omega_1}(x), P_{\Omega_2}(x), \ldots, P_{\Omega_m}(x)).$$

(ii) For any $(x_1, x_2, \ldots, x_m) \in X^m$,

$$P_L(x_1, x_2, \ldots, x_m) = A\left(\frac{x_1 + x_2 + \ldots + x_m}{m}\right).$$

Proof. The first assertion is straightforward (cf. [11, Exercise 1.8]). To prove the second one, we consider the real-valued function $f : X \to \mathbb{R}$ defined by

$$f(x) := \sum_{i=1}^{m} \|x - x_i\|^2.$$

It is obvious that $Ax \in P_L(x_1, x_2, \ldots, x_m)$ if and only if x is a minimizer of f. The conclusion follows from the first order optimality condition. □

Corollary 9 ([29], Theorem 7.3)**.** *Suppose that $\mathbf{\Omega}$ is uniformly regular at $\bar{x} \in \cap_{i=1}^m \Omega_i$. Then any sequence (y_k) generated by algorithm (3.22) linearly converges to some point in $\cap_{i=1}^m \Omega_i$ provided that the initial point y_0 is sufficiently close to $\bar{x}$.*

Proof. Let (z_n) be the sequence generated by the alternating projections for the two sets Ω and L with the initial point $z_1 := Ay_1$. By Lemma 15, $z_{2k} = Ay_k$, $k = 1, 2, \ldots$, for some sequence $(y_n) \subset X$. At the same time, $\{\Omega, L\}$ is uniformly regular at $A\bar{x}$ by Proposition 21. Therefore, when y_0 is sufficiently close to $\bar{x}$, Corollary 8 implies that the sequence (z_n) linearly converges to some point $A\tilde{x} \in \Omega_1 \cap \Omega_2$. It follows that the subsequence $(z_{2k} = Ay_k)$ also linearly converges to $A\tilde{x}$. Hence, (y_k) linearly converges to $\tilde{x} \in \cap_{i=1}^m \Omega_i$. □

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Chapter 4

Regularity of collections of sets and convergence of inexact alternating projections

We study the usage of regularity properties of collections of sets in convergence analysis of alternating projection methods for solving feasibility problems. Several equivalent characterizations of these properties are provided. Two settings of inexact alternating projections are considered and the corresponding convergence estimates are established and discussed.

4.1 Introduction

In this chapter we study the usage of regularity properties of collections of sets in convergence analysis of alternating projection methods for solving *feasibility problems*, i.e., finding a point in the intersection of several sets.

Given a set A and a point x in a metric space, the (metric) *projection* of x on A is defined as follows:

$$P_A(x) := \{a \in A \mid d(x,a) = d(x,A)\},$$

where $d(x,A) := \inf_{a\in A} d(x,a)$ is the distance from x to A. If A is a closed subset of a finite dimensional space, then $P_A(x) \neq \emptyset$. If A is a closed convex subset of a Euclidean space, then

$P_A(x)$ is a singleton.

Given a collection $\{A, B\}$ of two subsets of a metric space, we can talk about *alternating projections.*

Definition 8 (Alternating projections)**.** $\{x_n\}$ is a sequence of alternating projections for $\{A, B\}$ if

$$x_{2n+1} \in P_B(x_{2n}) \quad \text{and} \quad x_{2n+2} \in P_A(x_{2n+1}) \quad (n = 0, 1, \ldots).$$

Investigations of convergence of the alternating projections to a point in the intersection of closed sets in the setting of a Hilbert space, or more often a finite dimensional Euclidean space, have long history which can be traced back to von Neumann; see the historical comments in [10, 19, 22]. In the convex case, the key convergence estimates were established by Bregman [6] and Bauschke & Borwein [3]. In the nonconvex case, in the finite dimensional setting, linear convergence of the method was shown by Lewis et al. [19, Theorem 5.16] under the assumptions of the *uniform regularity* of the collection $\{A, B\}$ and *super-regularity* of one of the sets; see the definitions and characterizations of these properties in Section 4.2.

Throughout this chapter, we assume that A and B are closed.

Theorem 9 (Linear convergence of alternating projections)**.** *Let X be a finite dimensional Euclidean space. Suppose that*

(i) $\{A, B\}$ is uniformly regular at $\bar{x} \in A \cap B$, i.e.,

$$\bar{c} := \sup \left\{ -\langle u, v \rangle \mid u \in \overline{N}_A(\bar{x}) \cap \mathbb{B},\ v \in \overline{N}_B(\bar{x}) \cap \mathbb{B} \right\} < 1; \tag{4.1}$$

(ii) A is super-regular at $\bar{x}$.

Then, for any $c \in (\bar{c}, 1)$, a sequence of alternating projections for $\{A, B\}$ with initial point sufficiently close to $\bar{x}$ converges to a point in $A \cap B$ with $R-$linear rate $\sqrt{c}$.

$\overline{N}_A(\bar{x})$ and $\overline{N}_B(\bar{x})$ in (4.1) stand for the *limiting normal cones* to the corresponding sets at $\bar{x}$; see definition (4.12) below.

Observe that $-\langle u, v \rangle$ in (4.1) can be interpreted as the cosine of the angle between the cones $\overline{N}_A(\bar{x})$ and $-\overline{N}_B(\bar{x})$.

The role of the regularity (transversality-like) property (i) of $\{A, B\}$ and convexity-like property (ii) of A in the convergence proof is analysed in Drusvyatskiy et al. [10] and Noll & Rondepierre [22]. It has well been recognized that the uniform regularity assumption is far from being necessary for the linear convergence of alternating projections. For example, as observed in [10], it fails when the affine span of $A \cup B$ is not equal to the whole space.

The drawback of the uniform regularity property as defined by (4.1) from the point of view of the alternating projections is the fact that it takes into account all (limiting) normals to each of the sets while in many situations (like the one in the above example) some normals are irrelevant to the idea of projections.

Recently, there have been several successful attempts to relax the discussed above uniform regularity property by restricting the set of involved (normal) directions to only those relevant for characterizing alternating projections. All the newly introduced regularity properties still possess some uniformity in the sense that they take into account directions originated from points in a neighbourhood of the reference point and some estimate is required to hold uniformly over all such directions.

Bauschke et al. [5, 4] suggested restricting the set of normals participating in (4.1) by replacing $\overline{N}_A(\bar{x})$ and $\overline{N}_B(\bar{x})$ with *restricted limiting normal cones* $\overline{N}_A^B(\bar{x})$ and $\overline{N}_B^A(\bar{x})$ depending on both sets and attuned to the method of alternating projections. For example, the cone $\overline{N}_A^B(\bar{x})$ consists of limits of sequences of the type $t_k(b_k - a_k)$ where $t_k > 0$, $b_k \in B$, a_k is a projection of b_k on A, and $a_k \to \bar{x}$; cf. definitions (4.17) and (4.18). Bauschke et al. also adjusted (weakened) the notion of super-regularity accordingly (by considering *joint super-restricted regularity* taking into account the other set) and, under these weaker assumptions, arrived at the same conclusion as in Theorem 9; cf. [5, Theorem 3.14] and Theorem 10 below.

The idea of Bauschke et al. has been further refined by Drusvyatskiy et al. [10, Definition 4.4] who observed that it is sufficient to consider only sequences $t_k(b_k - a_k)$ as above with $b_k \to \bar{x}$; cf. Definition 13 below. In this case, $a_k \to \bar{x}$ automatically.

In [10], the authors suggested also another way of weakening the uniform regularity condition (4.1). Instead of measuring the angles between (usual or restricted in some sense) normals (and negative normals) to the sets, they measure the angles between vectors of the type $a - b$ with $a \in A$ and $b \in B$ and each of the cones $N_B^{\text{prox}}(b)$ and $-N_A^{\text{prox}}(a)$. At least one of the angles must be sufficiently large when a and b are sufficiently close to $\bar{x}$; cf. [10, Definition 2.2]. Assuming this property and using a different technique, Drusvyatskiy et al. produced a significant advancement in convergence analysis of projection algorithms by establishing (see [10, Corollary 4.2]) $R-$linear convergence of alternating projections without the assumption of super-regularity of one of the sets (and with a slightly different convergence estimate). The idea is closely related to the more general approach, where the feasibility problem is reformulated as a problem of minimizing a coupling function, and the property introduced in [10] is sufficient for the coupling function to satisfy the Kurdyka-Łojasiewicz inequality [1, Proposition 4.1].

The two relaxed regularity properties introduced in [10] are in general independent; cf. Examples 12 and 13.

The next step has been made by Noll and Rondepierre [22, Definition 1]. They noticed that, when dealing with alternating projections, the main building block of the method consists of two successive projections:

$$a_1 \in A, \quad b \in P_B(a_1) \quad \text{and} \quad a_2 \in P_A(b) \tag{4.2}$$

and it is sufficient to consider only the (proximal) normal directions determined by $a_1 - b$ and $b - a_2$ for all a_1, b, a_2 in a neighbourhood of the reference point satisfying (4.2). In fact, in [22], a more general setting is studied which allows for nonlinear convergence estimates under more subtle nonlinear regularity assumptions.

Another important advancement in this area is considering in [19] of *inexact* alternating projections. Arguing that finding an exact projection of a point on a closed set is in general a difficult problem by itself, Lewis et al. relaxed the requirements to the sequence $\{x_n\}$ in Definition 8 by allowing the points belonging to one of the sets to be "almost" projections. Assuming that the other set is super-regular at the reference point, they established in [19, Theorem 6.1] an inexact version of Theorem 9.

In the next section, we discuss and compare the uniform regularity property of collections of sets and its relaxations mentioned above. Several equivalent characterizations of these properties are provided in a uniform way simplifying the comparison.

The terminology employed in [4, 5, 10, 19, 22] for various regularity properties is not always consistent. We have not found a better way of handling the situation, but to use the terms *BLPW-restricted regularity*, *DIL-restricted regularity*, and *NR-restricted regularity* for the properties introduced in Bauschke, Luke, Phan, and Wang [4, 5], Drusvyatskiy, Ioffe and Lewis [10], and Noll and Rondepierre [22], respectively. The refined version of *BLPW-restricted regularity* due to Drusvyatskiy et al. [10] is referred to in this chapter as *BLPW-DIL-restricted regularity*.

In Section 4.3, we study two settings of inexact alternating projections under the assumptions of DIL-restricted regularity and uniform regularity, respectively, and establish and discuss the corresponding convergence estimates.

Our basic notation is standard; cf. [9, 21, 24]. For a normed linear space X, its topological dual is denoted X^* while $\langle\cdot,\cdot\rangle$ denotes the bilinear form defining the pairing between the two spaces. If X is a Hilbert space, X^* is identified with X while $\langle\cdot,\cdot\rangle$ denotes the scalar product. If $\dim X < \infty$, then X is usually assumed equipped with the Euclidean norm. The open and closed unit balls and the unit sphere in a normed space are denoted $\mathbb{B}$, $\overline{\mathbb{B}}$ and $\mathbb{S}$, respectively. $\mathbb{B}_\delta(x)$ stands for the open ball with radius $\delta > 0$ and center x. We use the convention $\mathbb{B}_0(x) = \{x\}$.

4.2 Uniform regularity and related regularity properties

In this section, we discuss and compare the uniform regularity property of collections of sets and its several relaxations which are used in convergence analysis of projection methods.

4.2.1 Uniform regularity

The uniform regularity property has been studied in [14, 15, 16, 17, 18]. Below we consider the case of a collection $\{A, B\}$ of two nonempty closed subsets of a normed linear space.

Definition 9. Suppose X is a normed linear space. The collection $\{A, B\}$ is uniformly regular at $\bar{x} \in A \cap B$ if there exist positive numbers α and δ such that

$$(A - a - x)\bigcap(B - b - y)\bigcap(\rho\mathbb{B}) \neq \emptyset$$

for all $\rho \in (0, \delta)$, $a \in A \cap \mathbb{B}_\delta(\bar{x})$, $b \in B \cap \mathbb{B}_\delta(\bar{x})$, and $x, y \in (\alpha\rho)\mathbb{B}$.

The supremum of all α in Definition 9 is denoted $\hat{\theta}[A,B](\bar{x})$ and provides a quantitative characterization of the uniformly regularity property, the latter one being equivalent to the inequality $\hat{\theta}[A,B](\bar{x}) > 0$. It is easy to check from the definition that

$$\hat{\theta}[A,B](\bar{x}) = \liminf_{a \xrightarrow{A} \bar{x}, b \xrightarrow{B} \bar{x}, \rho \downarrow 0} \frac{\theta_\rho[A-a, B-b](0)}{\rho}, \tag{4.3}$$

where

$$\theta_\rho[A,B](\bar{x}) := \sup \left\{ r \geq 0 \mid (A-x) \bigcap (B-y) \bigcap \mathbb{B}_\rho(\bar{x}) \neq \emptyset, \ \forall x, y \in r\mathbb{B} \right\}$$

and $a \xrightarrow{A} \bar{x}$ means that $a \to \bar{x}$ with $a \in A$.

The next proposition contains several characterizations of the uniform regularity property from [14, 15, 16, 17, 18]. In its parts (ii) and (iii), $N_A(a)$ stands for the *Fréchet normal cone* to A at $a \in A$:

$$N_A(a) := \left\{ u \in X^* \mid \limsup_{x \xrightarrow{A} a} \frac{\langle u, x-a \rangle}{\|x-a\|} \leq 0 \right\}. \tag{4.4}$$

Proposition 27. *Let A and B be closed subsets of X.*

(i) Suppose X is a normed linear space.

<u>Metric characterization</u>:

$$\hat{\theta}[A,B](\bar{x}) = \liminf_{\substack{z \to \bar{x};\, x,y \to 0 \\ z \notin (A-x)\cap(B-y)}} \frac{\max\{d(z, A-x), d(z, B-y)\}}{d\left(z, (A-x) \bigcap (B-y)\right)}. \tag{4.5}$$

$\{A, B\}$ is uniformly regular at $\bar{x}$ if and only if there exist positive numbers α and δ such that

$$\alpha d\left(z, (A-x) \bigcap (B-y)\right) \leq \max\{d(z, A-x), d(z, B-y)\} \tag{4.6}$$

for all $z \in \mathbb{B}_\delta(\bar{x})$ and $x, y \in \delta\mathbb{B}$.

(ii) Suppose X is an Asplund space.

Dual characterization.

$$\hat{\theta}[A,B](\bar{x}) = \liminf_{\rho\downarrow 0} \Big\{ \|u+v\| \mid u \in N_A(a),\ v \in N_B(b),$$
$$a \in A\cap \mathbb{B}_\rho(\bar{x}),\ b \in B\cap\mathbb{B}_\rho(\bar{x}),\ \|u\|+\|v\|=1\Big\}. \quad (4.7)$$

$\{A,B\}$ is uniformly regular at $\bar{x}$ if and only if there exist positive numbers α and δ such that

$$\alpha\left(\|u\|+\|v\|\right) \le \|u+v\| \tag{4.8}$$

for all $a \in A\cap\mathbb{B}_\delta(\bar{x})$, $b\in B\cap\mathbb{B}_\delta(\bar{x})$, $u \in N_A(a)$, and $v \in N_B(b)$.

(iii) Suppose X is a Hilbert space.

Angle characterization. If either $\bar{x} \in \operatorname{bd} A \cap \operatorname{bd} B$ or $\bar{x} \in \operatorname{int}(A\cap B)$, then

$$\hat{\theta}^2[A,B](\bar{x}) = \frac{1}{2}\left(1-\hat{c}[A,B](\bar{x})\right), \tag{4.9}$$

where

$$\hat{c}[A,B](\bar{x}) := \limsup_{\rho\downarrow 0}\Big\{ -\langle u,v\rangle \mid u\in N_A(a)\cap\mathbb{S},\ v \in N_B(b)\cap\mathbb{S},$$
$$a\in A\cap\mathbb{B}_\rho(\bar{x}),\ b\in B\cap\mathbb{B}_\rho(\bar{x})\Big\}. \quad (4.10)$$

Otherwise, $\hat{\theta}[A,B](\bar{x}) = 1$ and $\hat{c}[A,B](\bar{x}) = -\infty$.

$\{A,B\}$ is uniformly regular at $\bar{x}$ if and only if $\hat{c}[A,B](\bar{x}) < 1$, i.e., there exist numbers $\alpha < 1$ and $\delta > 0$ such that

$$-\langle u,v\rangle < \alpha \tag{4.11}$$

for all $a\in A\cap\mathbb{B}_\delta(\bar{x})$, $b \in B\cap\mathbb{B}_\delta(\bar{x})$, $u\in N_A(a)\cap\mathbb{S}$, and $v\in N_B(b)\cap\mathbb{S}$.

Remark 32. 1. Regularity criteria (4.6) and (4.8) are formulated in terms of distances in the primal space and in terms of Fréchet normals, respectively. This explains why we talk about, respectively, the metric and the dual characterizations in parts (i) and (ii) of Proposition 27. The term "angle characterization" in part (iii) comes from the observation that $-\langle u,v\rangle$ in criterion (4.11) can be interpreted as the cosine of the angle between the unit vectors u and

$-v$.

2. Constant (4.3) is nonnegative while constant (4.10) can take negative values. It is easy to see from (4.5) that $\hat{\theta}[A,B](\bar{x}) \le 1$ if $\bar{x} \notin \operatorname{int}(A \cap B)$ and $\hat{\theta}[A,B](\bar{x}) = \infty$ otherwise. Similarly, $|\hat{c}[A,B](\bar{x})| \le 1$ if $\bar{x} \in \operatorname{bd} A \cap \operatorname{bd} B$ and $\hat{c}[A,B](\bar{x}) = -\infty$ otherwise.

3. Unlike [17], we assume in (4.5), (4.7), and (4.10) the standard conventions that the infimum and supremum of the empty set in $\mathbb{R}$ equal $+\infty$ and $-\infty$, respectively. As a result, an additional assumption that either $\bar{x} \in \operatorname{bd} A \cap \operatorname{bd} B$ or $\bar{x} \in \operatorname{int}(A \cap B)$ is needed in part (iii) to ensure equality (4.9).

4. Equality (4.5) was proved in [13, Theorem 1] while equality (4.7) was established in [15, Theorem 4(vi)]; see also [13, Theorem 4] for a slightly weaker result containing inequality estimates. Equality (4.9) is a direct consequence of [17, Theorem 2]. It can be also easily checked directly.

If $\dim X < \infty$, then representations (4.7) and (4.10) as well as the corresponding criteria in parts (ii) and (iii) of Proposition 27 can be simplified by using the limiting version of the Fréchet normal cones (4.4). If, additionally, X is a Euclidean space, then one can also make use of proximal normals.

Recall (cf., e.g., [21]) that, in a Euclidean space, the *limiting (Fréchet) normal cone* to A at $\bar{x}$ and the *proximal normal cone* to A at $a \in A$ are defined as follows:

$$\overline{N}_A(\bar{x}) := \operatorname*{Lim\,sup}_{a \xrightarrow{A} \bar{x}} N_A(a) = \left\{ x^* = \lim x_k^* \mid x_k^* \in N_A(a_k),\ a_k \xrightarrow{A} \bar{x} \right\}, \tag{4.12}$$

$$N_A^{\text{prox}}(a) := \operatorname{cone}\left(P_A^{-1}(a) - a\right) = \{\lambda(x-a) \mid \lambda \ge 0,\ a \in P_A(x)\}. \tag{4.13}$$

Their usage is justified by the following simple observations:

$$N_A^{\text{prox}}(a) \subset N_A(a) \quad \text{and} \quad \overline{N}_A(\bar{x}) = \operatorname*{Lim\,sup}_{a \xrightarrow{A} \bar{x}} N_A^{\text{prox}}(a). \tag{4.14}$$

Proposition 28. *Let A and B be closed subsets of X.*

(i) Suppose $\dim X < \infty$.

Dual characterizations.

$$\begin{aligned}\hat{\theta}[A,B](\bar{x}) &= \inf\left\{\|u+v\| \mid u\in \overline{N}_A(\bar{x}),\ v\in \overline{N}_B(\bar{x}),\ \|u\|+\|v\|=1\right\}\\ &= \lim_{\rho\downarrow 0}\inf\Big\{\|u+v\| \mid u\in N_A^{\rm prox}(a),\ v\in N_B^{\rm prox}(b),\\ &\qquad a\in A\cap\mathbb{B}_\rho(\bar{x}),\ b\in B\cap\mathbb{B}_\rho(\bar{x}),\ \|u\|+\|v\|=1\Big\}.\end{aligned}$$

$\{A,B\}$ is uniformly regular at $\bar{x}$ if and only if one of the following two equivalent conditions is satisfied:

(a) $\overline{N}_A(\bar{x})\cap\left(-\overline{N}_B(\bar{x})\right)=\{0\}$*;*

(b) there exist positive numbers α and δ such that inequality (4.8) holds true for all $a\in A\cap\mathbb{B}_\delta(\bar{x})$, $b\in B\cap\mathbb{B}_\delta(\bar{x})$, $u\in N_A^{\rm prox}(a)$, and $v\in N_B^{\rm prox}(b)$.

(ii) Suppose X is a Euclidean space.

Angle characterizations.

$$\begin{aligned}\hat{c}[A,B](\bar{x}) &= \sup\left\{-\langle u,v\rangle \mid u\in \overline{N}_A(\bar{x})\cap\mathbb{S},\ v\in \overline{N}_B(\bar{x})\cap\mathbb{S}\right\} \qquad (4.15)\\ &= \lim_{\rho\downarrow 0}\sup\Big\{-\langle u,v\rangle \mid u\in N_A^{\rm prox}(a)\cap\mathbb{S},\ v\in N_B^{\rm prox}(b)\cap\mathbb{S},\\ &\qquad a\in A\cap\mathbb{B}_\rho(\bar{x}),\ b\in B\cap\mathbb{B}_\rho(\bar{x})\Big\}.\end{aligned}$$

$\{A,B\}$ is uniformly regular at $\bar{x}$ if and only if one of the following two equivalent conditions is satisfied:

(a) $\left\{(u,v)\in\left(\overline{N}_A(\bar{x})\cap\mathbb{S}\right)\times\left(\overline{N}_B(\bar{x})\cap\mathbb{S}\right) \mid \langle u,v\rangle=-1\right\}=\emptyset$*;*

(b) there exist numbers $\alpha<1$ and $\delta>0$ such that inequality (4.11) holds true for all $a\in A\cap\mathbb{B}_\delta(\bar{x})$, $b\in B\cap\mathbb{B}_\delta(\bar{x})$, $u\in N_A^{\rm prox}(a)\cap\mathbb{S}$, and $v\in N_B^{\rm prox}(b)\cap\mathbb{S}$.

Remark 33. 1. Condition (a) in part (i) of the above proposition is a ubiquitous qualification condition in optimization and variational analysis; cf. *basic qualification condition* [21] and *transversality* condition [10, 20].

2. If one replaces $\mathbb{S}$ with $\overline{\mathbb{B}}$ in representation (4.15), one will get nonnegative constant (4.1). The relationship between the two constants is straightforward: $\bar{c}=\max\{\hat{c}[A,B](\bar{x}),0\}$.

4.2.2 Super-regularity

In the next several subsections, we follow [19] and [4, 5], respectively. Although some definitions and assertions are valid in arbitrary Hilbert spaces, in accordance with the setting of [19] and [4, 5], we assume in these two subsections that X is a finite dimensional Euclidean space.

Unlike the uniform regularity, the super-regularity property is defined for a single set. The next definition contains a list of equivalent characterizations of this property which come from [19, Definition 4.3, Proposition 4.4, and Corollary 4.10], respectively.

Definition 10. A closed subset $A \subset X$ is super-regular at a point $\bar{x} \in A$ if one of the following equivalent conditions is satisfied:

(i) for any $\gamma > 0$, there exists a $\delta > 0$ such that

$$\langle x - x_A, a - x_A \rangle \le \gamma \|x - x_A\| \, \|a - x_A\|$$

for all $x \in \mathbb{B}_\delta(\bar{x})$, $x_A \in P_A(x)$, and $a \in A \cap \mathbb{B}_\delta(\bar{x})$;

(ii) for any $\gamma > 0$, there exists a $\delta > 0$ such that

$$\langle u, x - a \rangle \le \gamma \|u\| \, \|x - a\| \tag{4.16}$$

for all $x, a \in A \cap \mathbb{B}_\delta(\bar{x})$ and $u \in N_A(a)$;

(iii) for any $\gamma > 0$, there exists a $\delta > 0$ such that

$$\langle v - u, y - x \rangle \ge -\gamma \|y - x\|$$

for all $x, y \in A \cap \mathbb{B}_\delta(\bar{x})$ and $u \in N_A(x)$ and $v \in N_A(y)$.

Remark 34. 1. Super-regularity is a kind of local "near convexity" property, refining or complementing a number of properties of this kind: *Clarke regularity* [7, 24], *amenability* [24], *prox-regularity* [23, 24], and *subsmoothness* [2] (cf. *first order Shapiro property* [25]). For a detailed discussion and comparing of the properties we refer the reader to [19].

2. Super-regularity of one of the sets is an important ingredient of the convergence analysis of projection methods following the scheme initiated in Lewis et al. [19]; cf. Theorems 9 and 12. In fact, a weaker "quantified" version of this property corresponding to fixing $\gamma > 0$ in Definition 10 (and Definition 11 below), i.e., a kind of *γ-super-regularity* is sufficient for this type of analysis; cf. [4, Definition 8.1] and [22, Definition 2] (The latter definition introduces a more advanced Hölder version of this property.) Of course for alternating projections to converge, γ must be small and the convergence rate depends on γ.

4.2.3 Restricted normal cones and restricted super-regularity

There have been several successful attempts to relax the discussed above regularity properties by restricting the set of involved (normal) directions to only those relevant for characterizing alternating projections.

The definitions of *restricted normal cones* to a set introduced in [4] take into account another set and generalize proximal and limiting normal cones (4.13) and (4.12) in the setting of a Euclidean space:

$$N_A^{B-\text{prox}}(a) := \text{cone}\left((P_A^{-1}(a) \cap B) - a\right), \tag{4.17}$$

$$\overline{N}_A^B(\bar{x}) := \operatorname{Lim\,sup}_{a \xrightarrow{A} \bar{x}} N_A^{B-\text{prox}}(a). \tag{4.18}$$

Sets (4.17) and (4.18) are called, respectively, the *B-proximal normal cone* to A at $a \in A$ and *B-limiting normal cone* to A at $\bar{x}$. When B is the whole space, they obviously coincide with (4.13) and (4.12) (cf. the representation of the limiting normal cone given by the equality in (4.14)). Note that cones (4.17) and (4.18) can be empty.

Similarly to (4.17), one can define also the *B-Fréchet normal cone* to A at $a \in A$:

$$N_A^B(a) := N_A(a) \cap \text{cone}\,(B - a)$$

and the corresponding limiting one. The following inclusions are straightforward:

$$N_A^{B-\text{prox}}(a) \subset N_A^B(a) \subset N_A(a).$$

Definition 11. A closed subset $A \subset X$ is B-super-regular at a point $\bar{x} \in A$ if, for any $\gamma > 0$, there exists a $\delta > 0$ such that condition (4.16) holds true for all $x, a \in A \cap \mathbb{B}_\delta(\bar{x})$ and $u \in N_A^{B-\text{prox}}(a)$.

Remark 35. As observed in [4], B-proximal normals in Definition 11 can be replaced with B-limiting ones. Similarly, in Definition 10(ii) and (iii), one can replace Fréchet normals with limiting ones.

4.2.4 BLPW-restricted regularity

The next definition introduces a modification of the property used in the angle characterization of the uniform regularity in Proposition 27(iii). This new property and its subsequent characterizations and application in convergence estimate (Theorem 10) originate in Bauschke, Luke, Phan, and Wang [4, 5]. We are going to use for the regularity property of a collection of two sets discussed below the term *BLPW-restricted regularity*.

Definition 12. A collection of closed sets $\{A, B\}$ is BLPW-restrictedly regular at $\bar{x} \in A \cap B$ if

$$\hat{c}_1[A, B](\bar{x}) := \limsup_{\rho \downarrow 0} \Big\{ -\langle u, v\rangle \mid u \in N_A^{B-\text{prox}}(a) \cap \mathbb{S},\ v \in N_B^{A-\text{prox}}(b) \cap \mathbb{S},$$
$$a \in A \cap \mathbb{B}_\rho(\bar{x}),\ b \in B \cap \mathbb{B}_\rho(\bar{x}) \Big\} < 1, \tag{4.19}$$

i.e., there exist numbers $\alpha < 1$ and $\delta > 0$ such that condition (4.11) holds for all $a \in A \cap \mathbb{B}_\delta(\bar{x})$, $b \in B \cap \mathbb{B}_\delta(\bar{x})$, $u \in N_A^{B-\text{prox}}(a) \cap \mathbb{S}$, and $v \in N_B^{A-\text{prox}}(b) \cap \mathbb{S}$.

Proposition 29. *(i) The following representation holds true:*

$$\hat{c}_1[A, B](\bar{x}) = \sup \Big\{ -\langle u, v\rangle \mid u \in \overline{N}_A^B(\bar{x}) \cap \mathbb{S},\ v \in \overline{N}_B^A(\bar{x}) \cap \mathbb{S} \Big\}. \tag{4.20}$$

(ii) If either $\overline{N}_A^B(\bar{x}) \cap \mathbb{S} \neq \emptyset$ and $\overline{N}_B^A(\bar{x}) \cap \mathbb{S} \neq \emptyset$, or $\overline{N}_A^B(\bar{x}) \cap \mathbb{S} = \overline{N}_B^A(\bar{x}) \cap \mathbb{S} = \emptyset$, then

$$\hat{c}_1[A, B](\bar{x}) = 1 - 2\hat{\theta}_1^2[A, B](\bar{x}),$$

where

$$\hat{\theta}_1[A,B](\bar{x}) = \liminf_{\rho\downarrow 0}\Big\{ \|u+v\| \mid u \in N_A^{B-\text{prox}}(a),\ v \in N_B^{A-\text{prox}}(b),$$
$$a \in A\cap \mathbb{B}_\rho(\bar{x}),\ b \in B\cap\mathbb{B}_\rho(\bar{x}),\ \|u\|+\|v\|=1\Big\}$$
$$= \inf\Big\{\|u+v\| \mid u\in \overline{N}_A^B(a),\ v\in \overline{N}_B^A(b),\ \|u\|+\|v\|=1\Big\}.$$

(iii) A collection of closed sets $\{A,B\}$ is BLPW-restrictedly regular at $\bar{x}\in A\cap B$ if and only if one of the following conditions holds true:

$$\bar{c}_1 := \sup\Big\{-\langle u,v\rangle \mid u\in\overline{N}_A^B(\bar{x})\cap\mathbb{B},\ v\in\overline{N}_B^A(\bar{x})\cap\mathbb{B}\Big\} < 1, \tag{4.21}$$
$$\hat{\theta}_1[A,B](\bar{x}) > 0,$$
$$\overline{N}_A^B(\bar{x})\cap\Big(-\overline{N}_B^A(\bar{x})\Big) \subseteq \{0\}. \tag{4.22}$$

Remark 36. 1. The difference between formula (4.20) and definition of $\bar{c}_1$ in (4.21) is that, in the latter one, closed unit balls are used instead of spheres. As a result, $\bar{c}_1$ is either nonnegative or equal $-\infty$. (The latter case is possible because restricted normal cones can be empty.) At the same time, conditions $\hat{c}_1[A,B](\bar{x}) < 1$ and $\bar{c}_1 < 1$ are equivalent and $\bar{c}_1$ can be used for characterizing BLPW-restricted regularity. The inequality $\bar{c}_1 \le \bar{c}$, where $\bar{c}$ is given by (4.1), is obvious. It can be strict; cf. [4, Example 7.1].

2. In [4], a more general setting of four sets $A, B, \widetilde{A}, \widetilde{B}$ is considered with the A- and B-proximal and limiting normals cones in Definition 12 and Proposition 29 replaced by their $\widetilde{A}$ and $\widetilde{B}$ versions. As described in [5, Subsection 3.6], this provides additional flexibility in applications when determining regularity properties. To simplify the presentation, in this chapter we set $\widetilde{A} = A$ and $\widetilde{B} = B$.

3. Condition (4.22) is referred to in [4] as (A,B)-*qualification condition* while constant (4.19) is called the *limiting CQ number*.

Theorem 10. *Let X be a finite dimensional Euclidean space. Suppose that*

(i) $\{A,B\}$ is BLPW-restrictedly regular at $\bar{x}\in A\cap B$;

(ii) A is B-super-regular at $\bar{x}$.

Then, for any $c \in (\bar{c}_1, 1)$, a sequence of alternating projections for $\{A, B\}$ with initial point sufficiently close to $\bar{x}$ converges to a point in $A \cap B$ with $R-$linear rate $\sqrt{c}$.

4.2.5 BLPW-DIL-restricted regularity

The concept of BLPW-restricted regularity was further refined in Drusvyatskiy, Ioffe and Lewis [10, Definition 4.4]. We are going to call the amended property *BLPW-DIL-restricted regularity.*

Definition 13. A collection of closed sets $\{A, B\}$ is BLPW-DIL-restrictedly regular at $\bar{x} \in A \cap B$ if

$$\hat{c}_2[A,B](\bar{x}) := \limsup_{\rho \downarrow 0} \Big\{ -\frac{\langle a - b_a, b - a_b \rangle}{\|a - b_a\| \, \|b - a_b\|} \mid b_a \in P_B(a),\ a_b \in P_A(b),$$
$$a \in (A \setminus B) \cap \mathbb{B}_\rho(\bar{x}),\ b \in (B \setminus A) \cap \mathbb{B}_\rho(\bar{x}) \Big\} < 1, \tag{4.23}$$

i.e., there exist numbers $\alpha < 1$ and $\delta > 0$ such that

$$-\langle a - b_a, b - a_b \rangle < \alpha \|a - b_a\| \, \|b - a_b\|$$

for all $a \in (A \setminus B) \cap \mathbb{B}_\delta(\bar{x})$, $b \in (B \setminus A) \cap \mathbb{B}_\delta(\bar{x})$, $b_a \in P_B(a)$, and $a_b \in P_A(b)$.

Remark 37. The property in Definition 13 is referred to in [10] as *inherent transversality.*

An analogue of Proposition 29 holds true with constant $\hat{\theta}_1[A,B](\bar{x})$ replaced by

$$\hat{\theta}_2[A,B](\bar{x}) = \frac{1}{2} \liminf_{\rho \downarrow 0} \left\{ \left\| \frac{a - b_a}{\|a - b_a\|} + \frac{b - a_b}{\|b - a_b\|} \right\| \mid b_a \in P_B(a),\ a_b \in P_A(b),\right.$$
$$\left. a \in (A \setminus B) \cap \mathbb{B}_\rho(\bar{x}),\ b \in (B \setminus A) \cap \mathbb{B}_\rho(\bar{x}) \right\}$$

and appropriate limiting objects.

It is easy to see that a BLPW-restrictedly regular collection is also BLPW-DIL-restrictedly regular, but the converse is not true in general.

4.2.6 DIL-restricted regularity

The next definition and its subsequent characterizations originate in Drusvyatskiy, Ioffe and Lewis [10]. We are going to use for the regularity property of a collection of two sets discussed below the term *DIL-restricted regularity*.

Unlike [10], if not specified otherwise, we adopt in this subsection the setting of a general Hilbert space.

Definition 14. A collection of closed sets $\{A, B\}$ is DIL-restrictedly regular at $\bar{x} \in A \cap B$ if

$$\hat{\theta}_4[A, B](\bar{x}) := \lim_{\rho \downarrow 0} \inf_{a \in (A \setminus B) \cap \mathbb{B}_\rho(\bar{x}),\, b \in (B \setminus A) \cap \mathbb{B}_\rho(\bar{x})} \max\left\{ d\left(\frac{b-a}{\|a-b\|}, N_A(a)\right), d\left(\frac{a-b}{\|a-b\|}, N_B(b)\right)\right\} > 0, \quad (4.24)$$

i.e., there exist positive numbers γ and $\delta > 0$ such that

$$\max\left\{ d\left(\frac{b-a}{\|a-b\|}, N_A(a)\right), d\left(\frac{a-b}{\|a-b\|}, N_B(b)\right)\right\} > \gamma \quad (4.25)$$

for all $a \in (A \setminus B) \cap \mathbb{B}_\delta(\bar{x})$ and $b \in (B \setminus A) \cap \mathbb{B}_\delta(\bar{x})$.

Proposition 30. *A collection of closed sets $\{A, B\}$ is DIL-restrictedly regular at $\bar{x} \in A \cap B$ if and only if*

$$\hat{c}_4[A, B](\bar{x}) := \lim_{\rho \downarrow 0} \sup \left\{ \frac{\min\{\langle b-a, u\rangle, \langle a-b, v\rangle\}}{\|a-b\|} \mid u \in N_A(a) \cap \mathbb{S}, v \in N_B(b) \cap \mathbb{S},\ a \in (A \setminus B) \cap \mathbb{B}_\rho(\bar{x}),\ b \in (B \setminus A) \cap \mathbb{B}_\rho(\bar{x}) \right\} < 1, \quad (4.26)$$

i.e., there exist numbers $\alpha < 1$ and $\delta > 0$ such that

$$\min\{\langle b-a, u\rangle, \langle a-b, v\rangle\} < \alpha \|a-b\|$$

for all $a \in (A \setminus B) \cap \mathbb{B}_\delta(\bar{x})$, $b \in (B \setminus A) \cap \mathbb{B}_\delta(\bar{x})$, $u \in N_A(a) \cap \mathbb{S}$, and $v \in N_B(b) \cap \mathbb{S}$.

Moreover, $(\hat{c}_4[A, B](\bar{x}))^2 + (\hat{\theta}_4[A, B](\bar{x}))^2 = 1$.

Remark 38. 1. If $\dim X < \infty$, then, as usual, the Fréchet normals in (4.24) and (4.26) can be replaced by the proximal ones:

$$\hat{\theta}_4[A,B](\bar{x}) = \lim_{\rho\downarrow 0} \inf_{a\in(A\setminus B)\cap\mathbb{B}_\rho(\bar{x}),\, b\in(B\setminus A)\cap\mathbb{B}_\rho(\bar{x})} \max\left\{ d\left(\frac{b-a}{\|a-b\|}, N_A^{\text{prox}}(a)\right), d\left(\frac{a-b}{\|a-b\|}, N_B^{\text{prox}}(b)\right)\right\},$$

$$\hat{c}_4[A,B](\bar{x}) = \lim_{\rho\downarrow 0}\sup\left\{\frac{\min\{\langle b-a,u\rangle, \langle a-b,v\rangle\}}{\|a-b\|} \mid u\in N_A^{\text{prox}}(a)\cap\mathbb{S},\right.$$
$$\left. v\in N_B^{\text{prox}}(b)\cap\mathbb{S},\ a\in(A\setminus B)\cap\mathbb{B}_\rho(\bar{x}),\ b\in(B\setminus A)\cap\mathbb{B}_\rho(\bar{x})\right\}.$$

2. In [10], the property in Definition 14 is referred to as *intrinsic transversality*.

The next two examples show that DIL-restricted regularity is in general independent of BLPW-DIL-restricted regularity.

Example 12 (BLPW-DIL-restricted regularity but not DIL-restricted regularity; Figure 4.1)**.** Define a function $f : [0,1] \to \mathbb{R}$ by

$$f(t) := \begin{cases} 0, & \text{if } t = 0,\\ -t + 1/2^{n+1}, & \text{if } t \in (1/2^{n+1}, 3/2^{n+2}],\\ t - 1/2^n, & \text{if } t \in (3/2^{n+2}, 1/2^n],\ n = 0,1,\dots \end{cases}$$

and consider the sets: $A = \operatorname{gph} f$ and $B = \{(t,t) : t \in [0,1]\}$ and the point $\bar{x} = (0,0) = A \cap B$ in $\mathbb{R}^2$. Suppose $\mathbb{R}^2$ is equipped with the Euclidean norm.

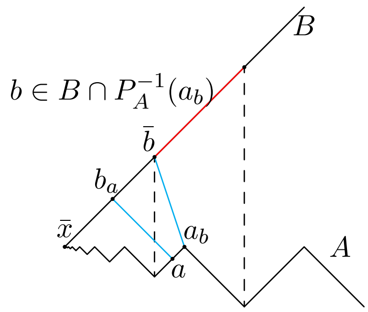

Figure 4.1: BLPW-DIL-restricted regularity but not DIL-restricted regularity

It is easy to check that f is a continuous function and consequently A is closed; $f(1/2^n) =$

0, $f(3/2^{n+2}) = -1/2^{n+2}$, $n = 0, 1, \ldots$

Take any $a \in A \setminus B$, $b \in B \setminus A$, $b_a \in P_B(a)$, and $a_b \in P_A(b)$. Thanks to the properties of the Euclidean distance, we have

$$a_b = (1/2^n, 0),$$
$$b \in B \cap P_A^{-1}(a_b) = \{(t, t) \mid t \in [3/2^{n+2}, 3/2^{n+1}]\},$$
$$a - b_a = k(1, -1)$$

for some $n \in \mathbb{N}$ and $k > 0$. Then,

$$\hat{c}_2[A, B](\bar{x}) = \max_{b \in B \cap P_A^{-1}(a_b)} \left\{ \frac{\langle (-1, 1), b - a_b \rangle}{\sqrt{2}\|b - a_b\|} \right\} = \frac{1}{\sqrt{2}} \left\langle (-1, 1), \frac{\bar{b} - a_b}{\|\bar{b} - a_b\|} \right\rangle,$$

where $\bar{b} := (3/2^{n+2}, 3/2^{n+2})$, and consequently,

$$\hat{c}_2[A, B](\bar{x}) = \frac{\langle (-1, 1), (-1, 3) \rangle}{\sqrt{2}\sqrt{10}} = \frac{2}{\sqrt{5}} > 0.$$

Hence, $\{A, B\}$ is BLPW-DIL-restrictedly regular at $\bar{x}$.

Given an $n \in \mathbb{N}$, we choose $a := (1/2^n, 0) \in A \setminus B$ and $b := (1/2^{n+1}, 1/2^{n+1}) \in B \setminus A$. Then,

$$N_A^{\text{prox}}(a) = N_A(a) = \{(t_1, t_2) : t_2 \geq |t_1|\},$$
$$N_B^{\text{prox}}(b) = N_B(b) = \mathbb{R}(1, -1),$$

and consequently,

$$a - b = 1/2^{n+1}(1, -1) \in N_B(b) \cap -N_A(a).$$

It follows that $\hat{c}_4[A, B](\bar{x}) = 1$ and $\{A, B\}$ is not DIL-restrictedly regular at $\bar{x}$. △

Example 13 (DIL-restricted regularity but not BLPW-DIL-restricted regularity; Figure 4.2). Consider two sets:

$$A = \{(t,0) : t \geq 0\} \cup \{(t,-t) : t \geq 0\},$$
$$B = \{(t,0) : t \geq 0\} \cup \{(t,t) : t \geq 0\}$$

and the point $\bar{x} = (0,0) \in A \cap B$ in $\mathbb{R}^2$. Suppose $\mathbb{R}^2$ is equipped with the Euclidean norm.

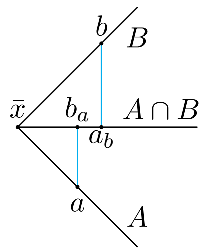

Figure 4.2: DIL-restricted regularity but not BLPW-DIL-restricted regularity

For any $a = (t_1, -t_1) \in A \setminus B$ and $b = (t_2, t_2) \in B \setminus A$, we have

$$N_A^{\text{prox}}(a) = N_A(a) = \mathbb{R}(1,1),$$
$$N_B^{\text{prox}}(b) = N_B(b) = \mathbb{R}(1,-1).$$

and consequently,

$$\hat{c}_4[A,B](\bar{x}) = \sup_{t_1>0,\, t_2>0} \frac{\min\{t_1, t_2\}}{\sqrt{t_1^2 + t_2^2}} = \frac{1}{\sqrt{2}} < 1.$$

Hence, $\{A, B\}$ is DIL-restrictedly regular at $\bar{x}$.

For any $a \in A \setminus B$, $b \in B \setminus A$, $b_a \in P_B(a)$, and $a_b \in P_A(b)$, we have

$$\frac{b - a_b}{\|b - a_b\|} = \frac{b_a - a}{\|a - b_a\|}.$$

It follows that $\hat{c}_2[A,B](\bar{x}) = 1$ and $\{A, B\}$ is not BLPW-DIL-restrictedly regular at $\bar{x}$. △

The next fact was established in [10, Proposition 4.5].

Proposition 31. *If* $\dim X < \infty$*,* $\{A, B\}$ *is BLPW-DIL-restrictedly regular at* $\bar{x}$*, and both sets* A *and* B *are super-regular at* $\bar{x}$*, then* $\{A, B\}$ *is DIL-restrictedly regular at* $\bar{x}$*.*

Remark 39. The assumption of super-regularity of both sets in Proposition 31 is essential. Indeed, in Example 12, $\{A, B\}$ is BLPW-DIL-restrictedly regular and B is super-regular (in fact, convex), while A is not and $\{A, B\}$ is not DIL-restrictedly regular.

4.2.7 NR-restricted regularity

The next step in relaxing both BLPW- and DIL-restricted regularity properties while preserving the linear convergence of alternating projections has been done in Noll and Rondepierre [22]. In what follows, the resulting property is called *NR-restricted regularity*.

Definition 15. A collection of closed sets $\{A, B\}$ is NR-restrictedly regular at $\bar{x} \in A \cap B$ if

$$\hat{c}_3[A,B](\bar{x}) := \limsup_{\rho \downarrow 0} \left\{ \frac{\langle a_1 - b, a_2 - b\rangle}{\|a_1 - b\| \, \|a_2 - b\|} \mid a_1 \in A,\, b \in P_B(a_1),\, a_2 \in P_A(b) \right. \\ \left. a_1, b, a_2 \in \mathbb{B}_\rho(\bar{x}), \right\} < 1,$$

i.e., there exist numbers $\alpha < 1$ and $\delta > 0$ such that

$$\langle a_1 - b, a_2 - b\rangle \le \alpha \|a_1 - b\| \, \|a_2 - b\|$$

for all $a_1 \in A \cap \mathbb{B}_\delta(\bar{x})$, $b \in P_B(a_1) \cap \mathbb{B}_\delta(\bar{x})$, and $a_2 \in P_A(b) \cap \mathbb{B}_\delta(\bar{x})$.

Remark 40. 1. NR-restricted regularity property is not symmetric: NR-restricted regularity of $\{A, B\}$ does not imply that $\{B, A\}$ is NR-restrictedly regular.

2. If $\{A, B\}$ is BLPW- or DIL-restrictedly regular at $\bar{x}$, then it is NR-restrictedly regular at $\bar{x}$ and the second implication can be strict [22, Propositions 1 and 2 and Example 7.6]. In fact, it is easy to check that NR-restricted regularity is implied by BLPW-DIL-restricted regularity. Example 13 shows that NR-restricted regularity can be strictly weaker.

3. Theorem 10 remains valid if the assumption of BLPW-restricted regularity is replaced by that of NR-restricted regularity and $\bar{c}_1$ is replaced by $\hat{c}_3[A, B](\bar{x})$.

4. The property in Definition 15 is referred to in [22] as *separable intersection*.

5. In [22], a more general Hölder-type property with exponent $\omega \in [0, 2)$ is considered. Definition 15 corresponds to that property with $\omega = 0$. For the convergence analysis, the authors of [22] introduce also a Hölder version of the superregularity property.

4.3 Convergence for inexact alternating projections

In this section, we study two settings of inexact alternating projections under the assumptions of DIL-restricted regularity and uniform regularity, respectively, and establish the corresponding convergence estimates.

4.3.1 Convergence for inexact alternating projections under DIL-restricted regularity

Given a point x and a set A in a Hilbert space and numbers $\tau \in (0,1]$ and $\sigma \in [0,1)$, the (τ,σ)-*projection* of x on A is defined as follows:

$$P_A^{\tau,\sigma}(x) := \{a \in A \mid \tau\|x-a\| \leq d(x,A),\ d\left(x-a, N_A(a)\right) \leq \sigma\|x-a\|\}. \tag{4.27}$$

One obviously has $P_A^{1,\sigma}(x) = P_A(x)$ for any $\sigma \in [0,1)$. Observe also that the above definition requires a to be an "almost projection" in terms of the distance $\|x-a\|$ being close to $d(x,A)$ and also $x-a$ being an "almost normal" to A.

Definition 16 (Inexact alternating projections)**.** Given $\tau \in (0,1]$ and $\sigma \in [0,1)$, $\{x_n\}$ is a sequence of (τ,σ)-alternating projections for $\{A,B\}$ if

$$x_{2n+1} \in P_B^{\tau,\sigma}(x_{2n}) \quad \text{and} \quad x_{2n+2} \in P_A^{\tau,\sigma}(x_{2n+1}) \quad (n = 0,1,\ldots).$$

The next statement is taken from [10, Theorem 5.3] where it is formulated in the setting of a finite dimensional Euclidean space. It is a version of the general metric space *Basic Lemma* from [12]. Recall that the (strong) *slope* [8] of f at a point $u \in X$ with $f(u) < +\infty$ is defined as follows:

$$|\nabla f|(u) := \limsup_{u' \to u, u' \neq u} \frac{f(u) - f(u')}{d(u',u)}.$$

Lemma 16 (Error bound)**.** *Let X be a complete metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function, $x \in X$ with $f(x) < +\infty$, $\delta > 0$, and $\alpha < f(x)$. Suppose that*

$$\mu := \inf_{u \in \mathbb{B}_\delta(x),\, \alpha < f(u) \leq f(x)} |\nabla f|(u) > \frac{f(x) - \alpha}{\delta}. \tag{4.28}$$

Then $S(f,\alpha) := \{u \in X \mid f(u) \le \alpha\} \ne \emptyset$ *and*

$$\mu d(x, S(f,\alpha)) \le f(x) - \alpha.$$

If X is an Asplund space, then a standard argument based on the *subdifferential sum rule* (cf., e.g., [11, Proposition 5(ii)] or [10, Proposition 6.9]) shows that the primal space slopes in the definition of μ in (4.28) can be replaced by the subdifferential slopes:

$$\mu = \inf_{u \in \mathbb{B}_\delta(x),\, \alpha < f(u) \le f(x)} |\partial f|(u). \tag{4.29}$$

Here

$$|\partial f|(u) := \inf_{x^* \in \partial f(u)} \|x^*\|_*,$$

where $\partial f(u)$ is the Fréchet subdifferential of f at u and $\|\cdot\|_*$ is the norm on X^* dual to the norm on X participating in the definition of the primal space slope. Note that in general $|\nabla f|(u) \le |\partial f|(u)$.

If X is a finite dimensional Euclidean space, then, instead of the Fréchet subdifferentials, one can use the *proximal* subdifferentials $\partial^{\text{prox}} f(u)$:

$$\mu = \inf_{u \in \mathbb{B}_\delta(x),\, \alpha < f(u) \le f(x)} |\partial^{\text{prox}} f|(u), \tag{4.30}$$

where

$$|\partial^{\text{prox}} f|(u) := \inf_{x^* \in \partial^{\text{prox}} f(u)} \|x^*\|.$$

The next statement is a consequence of Lemma 16. It extends slightly [10, Theorem 3.1].

Proposition 32 (Distance decrease)**.** *Let A be a closed subset of a Hilbert space X, $a \in A$, $b \notin A$, $\delta > 0$, and $\alpha < \|a-b\|$. Suppose that*

$$\mu := \inf_{\substack{u \in A \cap \mathbb{B}_\delta(a) \\ \|u-b\| \le \|a-b\|}} d\left(\frac{b-u}{\|u-b\|}, N_A(u)\right) > 0. \tag{4.31}$$

Then $d(b,A) \le \|a-b\| - \mu\delta$.

If $\dim X < \infty$*, then the Fréchet normal cones* $N_A(u)$ *in* (4.31) *can be replaced by the* *proximal ones* $N_A^{\text{prox}}(u)$.

Proof. Consider the lower semicontinuous function $f = d(\cdot, b) + \iota_A$, where ι_A is the *indicator* function of A: $\iota_A(x) = 0$ if $x \in A$ and $\iota_A(x) = +\infty$ if $x \notin A$. Then $f(a) = \|a - b\|$ and

$$\partial f(u) = \frac{u-b}{\|u-b\|} + N_A(u), \quad |\partial f|(u) = d\left(\frac{b-u}{\|u-b\|}, N_A(u)\right)$$

for any $u \in A$. It follows from the first part of Lemma 16 and representation (4.29) that $d(b, A) \le \alpha$ for any $\alpha \in (\|a-b\| - \mu\delta, \|a-b\|)$ and consequently, $d(b, A) \le \|a-b\| - \mu\delta$.

If $\dim X < \infty$, then instead of representation (4.29) one can use representation (4.30). □

The next statement is essentially [10, Lemma 3.2].

Lemma 17. *Any nonzero vectors x and y in a Hilbert space satisfy*

$$\left\| \frac{x}{\|x\|} - z \right\| \le \frac{\|x-y\|}{\|y\|},$$

where $z := \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \frac{y}{\|y\|}$ is the projection of $\frac{x}{\|x\|}$ on $\mathbb{R}y$.

Proof.

$$\begin{aligned}\left(\frac{\|x-y\|}{\|y\|}\right)^2 - \left\|\frac{x}{\|x\|} - z\right\|^2 &= \frac{\|x\|^2 - 2\langle x, y\rangle + \|y\|^2}{\|y\|^2} - 1 + \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle^2 \\ &= \frac{1}{\|y\|^2}\left(\|x\|^2 - 2\langle x, y\rangle + \frac{\langle x, y\rangle^2}{\|x\|^2}\right) \\ &= \frac{1}{\|y\|^2}\left(\|x\| - \frac{\langle x, y\rangle}{\|x\|}\right)^2 \ge 0.\end{aligned}$$

□

Theorem 11 (Convergence of inexact alternating projections)**.** *Suppose that $\{A, B\}$ is DIL-restrictedly regular at $\bar{x}$, $0 \le \sigma < \hat{\theta}_4[A, B](\bar{x})$ and $0 < \tau \le 1$. Then, for any $\gamma < \hat{\theta}_4[A, B](\bar{x})$ satisfying $0 < \gamma - \sigma \le \tau$ and*

$$c := \tau^{-1}(1 - \gamma^2 + \gamma\sigma) < 1,$$

any sequence of (τ, σ)-alternating projections for $\{A, B\}$ with initial point sufficiently close *to $\bar{x}$ converges to a point in $A \cap B$ with $R-$linear rate c.*

Proof. By Definition 14, there exists a $\rho > 0$ such that condition (4.25) holds true for all $a \in (A \setminus B) \cap \mathbb{B}_\rho(\bar{x})$ and $b \in (B \setminus A) \cap \mathbb{B}_\rho(\bar{x})$.

Let $a \in A \cap \mathbb{B}_{\rho'}(\bar{x})$ and $b \in P_B^{\tau,\sigma}(a) \cap \mathbb{B}_{\rho'}(\bar{x})$ where $\rho' := \rho/(1 + 2(\gamma - \sigma))$. We are going to show that

$$d(b, A) \leq (1 - \gamma^2 + \gamma\sigma)\|b - a\|.$$

If $b \in A$, the inequality holds true trivially. Suppose $b \notin A$ and denote $\delta := (\gamma - \sigma)\|b - a\|$. Consider any point $u \in A \cap \mathbb{B}_\delta(a)$. Since $\|u - a\| < (\gamma - \sigma)\|b - a\| \leq \tau\|b - a\| \leq d_B(a)$, we see that $u \notin B$; in particular, $a \notin B$ and $u \neq b$. Let z denote the projection of $\frac{u-b}{\|u-b\|}$ on $\mathbb{R}(a - b)$. Then $\|z\| \leq 1$ and, employing Lemma 17,

$$\begin{aligned} d\left(\frac{u - b}{\|u - b\|}, N_B(b)\right) &\leq \left\|\frac{u - b}{\|u - b\|} - z\right\| + d(z, N_B(b)) \\ &\leq \frac{\|u - a\|}{\|b - a\|} + \sigma < (\gamma - \sigma) + \sigma = \gamma. \end{aligned}$$

Since $\|u - \bar{x}\| \leq \|u - a\| + \|a - \bar{x}\| < 2(\gamma - \sigma)\rho' + \rho' = \rho$ and $\|b - \bar{x}\| < \rho' < \rho$, we get from (4.25) that $d\left(\frac{b-u}{\|u-b\|}, N_A(u)\right) > \gamma$. It follows from Proposition 32 that $d(b, A) \leq \|a - b\| - \gamma\delta = (1 - \gamma^2 + \gamma\sigma)\|a - b\|$. Hence,

$$\|a' - b\| \leq \tau^{-1} d(b, A) \leq c\|a - b\| \quad \text{for all} \quad a' \in P_A^{\tau,\sigma}(b). \tag{4.32}$$

Now we show that any sequence $\{x_n\}$ of (τ, σ)-alternating projections for $\{A, B\}$ remains in $\mathbb{B}_{\rho'}(\bar{x})$ whenever $x_0 \in \mathbb{B}_{\rho''}(\bar{x})$ where $\rho'' := \left(\frac{\tau^{-1}}{1-c} + 1\right)^{-1} \rho' < \rho'$. Indeed,

$$\|x_1 - x_0\| \leq \tau^{-1} d(x_0, B) \leq \tau^{-1}\|x_0 - \bar{x}\|.$$

Let $n \in \mathbb{N}$ and $x_k \in \mathbb{B}_{\rho'}(\bar{x})$, $k = 0, 1, \ldots n$. It follows from (4.32) that

$$\|x_{k+1} - x_k\| \leq c^k \|x_1 - x_0\| \quad (k = 0, 1, \ldots n), \tag{4.33}$$

and consequently,

$$\|x_{n+1} - x_0\| \leq \sum_{k=0}^{n} c^k \|x_1 - x_0\| \leq \frac{1}{1-c}\|x_1 - x_0\|,$$
$$\|x_{n+1} - \bar{x}\| \leq \left(\frac{\tau^{-1}}{1-c} + 1\right) \|x_0 - \bar{x}\| < \rho'.$$

Thanks to (4.33), $\{x_k\}$ is a Cauchy sequence containing two subsequences belonging to closed subsets A and B, respectively. Hence, it converges to a point in $A \cap B$ with $R-$linear rate c. □

Remark 41. 1. When inexact alternating projections are close to being exact, i.e., τ and σ are close to 1 and 0, respectively (cf. definition (4.27)), then the assumptions of Theorem 11 are easily satisfied (as long as $\hat{\theta}_4[A,B](\bar{x}) > 0$) while the convergence rate $c = \tau^{-1}(1-\gamma^2+\gamma\sigma)$ is mostly determined by the term $1-\gamma^2$. Recall that γ can be any number in $(0, \hat{\theta}_4[A,B](\bar{x}))$. Thanks to Proposition 14, $1-\gamma^2 = (\gamma')^2$ where γ' can be any number in $(\hat{c}_4[A,B](\bar{x}), 1)$.

2. When $\dim X < \infty$, the special case $\tau = 1$ and $\sigma = 0$ of Theorem 11 recaptures [10, Theorem 2.3]. The proof given above follows that of [10, Theorem 2.3].

3. It can be of interest to consider a more advanced version of inexact alternating projections than the one given in Definition 16:

$$x_{2n+1} \in P_B^{\tau_1,\sigma_1}(x_{2n}) \quad \text{and} \quad x_{2n+2} \in P_A^{\tau_2,\sigma_2}(x_{2n+1}) \quad (n = 0, 1, \ldots),$$

where $\tau_1, \tau_2 \in (0, 1]$ and $\sigma_1, \sigma_2 \in [0, 1)$. For instance, the projections on one of the sets, say, A can be required to be exact, i.e., $\tau_2 = 1$ and $\sigma_2 = 0$. Theorem 11 remains applicable to this situation with $\tau := \min\{\tau_1, \tau_2\}$ and $\sigma := \max\{\sigma_1, \sigma_2\}$. It is possible to obtain a sharper convergence estimate taking into account different "inexactness" parameters for each of the sets. For that, one needs to amend the definition of alternating projections by considering the selection of the pair $\{x_{2n+1}, x_{2n+2}\}$ as a single two-part iteration.

4.3.2 Convergence for inexact alternating projections under uniform regularity

The motivation for the discussed below version of inexact projections comes from [19, Section 6].

Given a point x and a set A in a Hilbert space and a number $\sigma \in [0,1)$, the *σ-projection* of x on A is defined as follows:

$$P_A^\sigma(x) := \{a \in A \mid d\,(x-a, N_A(a)) \le \sigma\|x-a\|\}\,. \tag{4.34}$$

Observe that

$$P_A^0(x) = \{a \in A \mid x-a \in N_A(a)\} \supset P_A(x)$$

and the inclusion can be strict even in finite dimensions. Furthermore, for any $\sigma \in [0,1)$, $P_A^\sigma(x)$ can contain points lying arbitrarily far from x.

Definition 17 (Inexact alternating projections)**.** Given a number $\sigma \in [0,1)$, $\{x_n\}$ is a sequence of σ-alternating projections for $\{A,B\}$ if

$$\begin{aligned} &x_{2n+1} \in P_B^\sigma(x_{2n}) \quad \text{and} \quad x_{2n+2} \in P_A^\sigma(x_{2n+1}),\\ &\|x_{n+2}-x_{n+1}\| \le \|x_{n+1}-x_n\| \quad (n=0,1,\ldots). \end{aligned} \tag{4.35}$$

The role of the monotonicity condition (4.35) in Definition 17 is to compensate for the lack of good projection properties of the σ-projection operator (4.34). In the case of standard alternating projections (cf. Definition 8), this condition is satisfied automatically.

Theorem 12 (Convergence of inexact alternating projections under uniform regularity)**.** *Suppose that $\{A,B\}$ is uniformly regular at $\bar{x}$, A is super-regular at $\bar{x}$ and $\sigma \in [0,1)$ satisfies*

$$c_0 := \hat{c}[A,B](\bar{x})(1-\sigma^2) + \sigma^2 + 2\sigma\sqrt{1-\sigma^2} + \sigma < 1.$$

Then, for any $c \in (c_0,1)$, any sequence $\{x_k\}$ of σ-alternating projections for $\{A,B\}$ with initial points x_0 and x_1 sufficiently close to $\bar{x}$ converges to a point in $A \cap B$ with $R-$linear *rate* $\sqrt{c}$.

Proof. Let $c \in (c_0, 1)$ and choose a $c_1 > \hat{c}[A, B](\bar{x})$ and a $\gamma > 0$ such that

$$c_1(1-\sigma^2) + \sigma^2 + (2\sigma + \gamma)\sqrt{1-\sigma^2} + \sigma < c. \tag{4.36}$$

By Proposition 27(iii) and Definition 10(ii), there exists a $\delta > 0$ such that

$$-\langle u, v\rangle \leq c_1\|u\|\,\|v\|, \tag{4.37}$$

$$\langle u, x-a\rangle \leq \gamma\|u\|\,\|x-a\| \tag{4.38}$$

for all $x, a \in A \cap \mathbb{B}_\delta(\bar{x})$, $b \in B \cap \mathbb{B}_\delta(\bar{x})$, $u \in N_A(a)$ and $v \in N_B(b)$.

Let $a_1 \in A \cap \mathbb{B}_\delta(\bar{x})$, $b \in P_B^\sigma(a_1) \cap \mathbb{B}_\delta(\bar{x})$ and $a_2 \in P_A^\sigma(b) \cap \mathbb{B}_\delta(\bar{x})$. We are going to show that

$$\|a_2 - b\| \leq c\|b - a_1\|. \tag{4.39}$$

By definition (4.34), for any $\varepsilon \in (0, 1-\sigma)$, there exist $u \in N_A(a_2)$ and $v \in N_B(b)$ such that

$$\|b - a_2 - u\| \leq (\sigma + \varepsilon)\|b - a_2\| \quad \text{and} \quad \|a_1 - b - v\| \leq (\sigma + \varepsilon)\|a_1 - b\|. \tag{4.40}$$

Additionally, one can ensure that

$$\|u\| \leq \sqrt{1-(\sigma+\varepsilon)^2}\|b - a_2\| \quad \text{and} \quad \|v\| \leq \sqrt{1-(\sigma+\varepsilon)^2}\|a_1 - b\|. \tag{4.41}$$

Indeed, take any $u \in N_A(a_2)$ satisfying the first inequality in (4.40). If $u = 0$, the first inequality in (4.41) is satisfied too. Suppose $u \neq 0$ and consider $u_1 := \langle b - a_2, u\rangle \frac{u}{\|u\|^2}$ – the projection of $b - a_2$ on $\mathbb{R}u$. Then

$$\|b - a_2\|^2 = \|u_1\|^2 + \|b - a_2 - u_1\|^2, \tag{4.42}$$

$$\|b - a_2 - u_1\| \leq \|b - a_2 - u\| \leq (\sigma + \varepsilon)\|b - a_2\| \tag{4.43}$$

and there exists a $t \in (0,1]$ such that $u_2 := tu_1$ satisfies

$$\|b - a_2 - u_2\| = (\sigma + \varepsilon)\|b - a_2\| \tag{4.44}$$

(thanks to the continuity of the function $t \mapsto \|b - a_2 - tu_1\|$). Hence, $u_2 \in N_A(a_2)$, vector $u_1 - u_2$ is a projection of $b - a_2 - u_2$ on $\mathbb{R}u$, i.e.,

$$\|b - a_2 - u_2\|^2 = \|u_1 - u_2\|^2 + \|b - a_2 - u_1\|^2, \tag{4.45}$$

and, using (4.42), (4.45), (4.44), and (4.43),

$$\begin{aligned}
\|u_2\| &= \|u_1\| - \|u_1 - u_2\| \\
&= \sqrt{\|b - a_2\|^2 - \|b - a_2 - u_1\|^2} - \sqrt{(\sigma+\varepsilon)^2\|b - a_2\|^2 - \|b - a_2 - u_1\|^2} \\
&= \frac{(1 - (\sigma+\varepsilon)^2)\|b - a_2\|^2}{\sqrt{\|b - a_2\|^2 - \|b - a_2 - u_1\|^2} + \sqrt{(\sigma+\varepsilon)^2\|b - a_2\|^2 - \|b - a_2 - u_1\|^2}} \\
&\le \frac{(1 - (\sigma+\varepsilon)^2)\|b - a_2\|^2}{\sqrt{1 - (\sigma+\varepsilon)^2}\|b - a_2\|} = \sqrt{1 - (\sigma+\varepsilon)^2}\|b - a_2\|.
\end{aligned}$$

Similarly, given any $v \in N_B(b)$ satisfying the second inequality in (4.40), one can find a $v_2 \in N_B(b)$ satisfying this inequality and, additionally, the second inequality in (4.41).

Making use of (4.37), (4.40) and (4.41), we get

$$\begin{aligned}
-\langle b - a_2, a_1 - b\rangle &= -\langle u, v\rangle - \langle u, a_1 - b - v\rangle \\
&\quad - \langle b - a_2 - u, v\rangle - \langle b - a_2 - u, a_1 - b - v\rangle \\
&\le c_1\|u\|\,\|v\| + \|u\|\,\|a_1 - b - v\| \\
&\quad + \|b - a_2 - u\|\,\|v\| + \|b - a_2 - u\|\,\|a_1 - b - v\| \\
&\le \Big(c_1(1 - (\sigma+\varepsilon)^2) + 2(\sigma+\varepsilon)(\sqrt{1 - (\sigma+\varepsilon)^2}) \\
&\quad + (\sigma+\varepsilon)^2\Big)\|b - a_2\|\,\|a_1 - b\|.
\end{aligned}$$

At the same time, making use of (4.38) and the first inequalities in (4.40) and (4.41), we have

$$\begin{aligned}\langle b-a_2,a_1-a_2\rangle &= \langle u,a_1-a_2\rangle+\langle b-a_2-u,a_1-a_2\rangle\\ &\le \gamma\|u\|\,\|a_1-a_2\|+(\sigma+\varepsilon)\|b-a_2\|\,\|a_1-a_2\|\\ &\le (\gamma\sqrt{1-(\sigma+\varepsilon)^2}+(\sigma+\varepsilon))\|b-a_2\|\,\|a_1-a_2\|,\end{aligned}$$

Adding the last two estimates and passing to limit as $\varepsilon\downarrow 0$, we obtain

$$\|b-a_2\|^2\le\Big(c_1(1-\sigma^2)+\sigma^2+(2\sigma+\gamma)\sqrt{1-\sigma^2}+\sigma\Big)\|b-a_2\|\,\|a_1-b\|.$$

Thanks to (4.36), this proves (4.39).

Now we show that a sequence $\{x_n\}$ of σ-alternating projections for $\{A,B\}$ remains in $\mathbb{B}_\delta(\bar x)$ if $x_0,x_1\in\mathbb{B}_\rho(\bar x)$ where $\rho:=\frac{1-c}{5-c}\delta<\delta$. Let $n\in\mathbb{N}$ and $x_k\in\mathbb{B}_\delta(\bar x)$, $k=0,1,\ldots,2n$. It follows from (4.39) that

$$\|x_{2k}-x_{2k-1}\|\le c^k\|x_1-x_0\|\quad(k=1,2,\ldots,n),\tag{4.46}$$

and consequently, employing also (4.35),

$$\begin{aligned}\|x_{2n+2}-x_0\| &\le \|x_{2n+2}-x_{2n+1}\|+\|x_{2n+1}-x_0\|\le\|x_{2n+2}-x_{2n+1}\|\\ &+\sum_{k=1}^{n}(\|x_{2k+1}-x_{2k}\|+\|x_{2k}-x_{2k-1}\|)+\|x_1-x_0\|\\ &\le 2\sum_{k=1}^{n}\|x_{2k}-x_{2k-1}\|+2\|x_1-x_0\|\\ &\le 2\sum_{k=0}^{n}c^k\|x_1-x_0\|\le\frac{2}{1-c}\|x_1-x_0\|.\end{aligned}$$

Thus,

$$\begin{aligned}\max\{\|x_{2n+2}-\bar x\|,\|x_{2n+1}-\bar x\|\} &\le \max\{\|x_{2n+2}-x_0\|,\|x_{2n+1}-x_0\|\}\\ &+\|x_0-\bar x\|\le\frac{2}{1-c}\|x_1-x_0\|+\|x_0-\bar x\|\\ &\le\frac{2}{1-c}\|x_1-\bar x\|+\frac{3-c}{1-c}\|x_0-\bar x\|<\frac{5-c}{1-c}\rho=\delta,\end{aligned}$$

i.e., $x_{2n+1}, x_{2n+2} \in \mathbb{B}_\delta(\bar{x})$.

Thanks to (4.46), $\{x_k\}$ is a Cauchy sequence containing two subsequences belonging to closed subsets A and B, respectively. Hence, it converges to a point in $A \cap B$ with $R-$linear rate $\sqrt{c}$. □

Remark 42. 1. When the "inexactness" parameter σ is small (cf. definition (17)), then the assumptions of Theorem 12 are easily satisfied (as long as $\hat{c}[A, B](\bar{x}) < 1$ and condition (4.35) holds) while the convergence rate is close to the one guaranteed by Theorem 9 and [19, Theorem 6.1].

2. One can also consider a more advanced version of inexact alternating projections than the one given in Definition 17:

$$x_{2n+1} \in P_B^{\sigma_1}(x_{2n}) \quad \text{and} \quad x_{2n+2} \in P_A^{\sigma_2}(x_{2n+1}), \quad (n = 0, 1, \ldots).$$

where $\sigma_1, \sigma_2 \in [0, 1)$. Theorem 12 remains applicable to this situation with $\sigma := \max\{\sigma_1, \sigma_2\}$ (cf. Remark 41.3).

3. Observe that, thanks to (4.39), for odd values of n, condition (4.35) is improved in the proof of Theorem 12:

$$\|x_{n+2} - x_{n+1}\| \leq c\|x_{n+1} - x_n\|,$$

where $c < 1$. However, the assumption is still needed to ensure that x_{n+2} is not too far from $\bar{x}$ and uniform and super-regularity conditions are applicable.

4. Constant c_1 in (4.37) is an upper estimate of the cosine of the angle φ between vectors u and $-v$ while $\sigma + \varepsilon$ in (4.40) can be interpreted as an upper estimate of the sine of the angles ψ_1 and ψ_2 between vectors $b - a_2$ and u and $a_1 - b$ and v, respectively. One can use standard trigonometric identities and inequalities (4.37) and (4.40) to obtain an upper estimate of the cosine of the angle $\varphi - \psi_1 - \psi_2$ between vectors $b - a_2$ and $b - a_1$ and possibly improve the convergence estimate in the statement of Theorem 12.

5. If both subsets A and B are super-regular, then in the proof of Theorem 12, one can

establish an analogue of (4.39) with subsets A and B interchanged:

$$\|b_2 - a\| \leq c\|a - b_1\|,$$

where $b_1 \in B \cap \mathbb{B}_\delta(\bar{x})$, $a \in P_A^\sigma(b_1) \cap \mathbb{B}_\delta(\bar{x})$ and $b_2 \in P_B^\sigma(a) \cap \mathbb{B}_\delta(\bar{x})$. This guarantees an improvement with rate c on each iteration. As a result, one obtains a better overall $R-$linear rate c.

6. The conclusion of Theorem 12 remains true if one replaces the assumptions of uniform regularity of $\{A, B\}$ (and the regularity constant $\hat{c}[A, B](\bar{x})$) and super-regularity of A with BLPW-restricted regularity (and the regularity constant $\hat{c}_1[A, B](\bar{x})$) and B-super-regularity, respectively, accompanied by appropriate adjustments in the definition of σ-projections.

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Chapter 5

An induction theorem and nonlinear regularity models

A general nonlinear regularity model for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces, is studied using special iteration procedures, going back to Banach, Schauder, Lyusternik and Graves. Namely, we revise the *induction theorem* from Khanh, *J. Math. Anal. Appl.*, 118 (1986) and employ it to obtain basic estimates for exploring regularity/openness properties. We also show that it can serve as a substitution of the Ekeland variational principle when establishing other regularity criteria. Then, we apply the induction theorem and the mentioned estimates to establish criteria for both global and local versions of regularity/openness properties for our model and demonstrate how the definitions and criteria translate into the conventional setting of a set-valued mapping $F : X \rightrightarrows Y$. An application to second-order necessary optimality conditions for a nonsmooth set-valued optimization problem with mixed constraints is provided.

5.1 Introduction

Regularity properties of set-valued mappings lie at the core of variational analysis because of their importance for establishing stability of solutions to generalized equations (initiated by Robinson [57, 58] in the 1970s), optimization and variational problems, constraint qualifications, qualification conditions in coderivative/subdifferential calculus and convergence rates of numerical algorithms; cf. books and surveys [6, 7, 10, 19, 30, 31, 33, 43, 51, 55, 60] and the references therein.

Among the variety of known regularity properties, the most recognized and widely used one is that of *metric regularity*; cf. [7, 9, 10, 19, 30, 43, 51, 53, 55, 60]. Recall that a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces is (locally) metrically regular at a point $(\bar{x}, \bar{y})$ in its graph $\operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$ with modulus $\kappa > 0$ if

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \quad \text{for all } x \text{ near } \bar{x},\ y \text{ near } \bar{y}. \tag{5.1}$$

Here $F^{-1} : Y \rightrightarrows X$ is the *inverse* mapping defined by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$. The roots of this notion can be traced back to the classical Banach-Schauder *open mapping theorem* and its subsequent generalization to nonlinear mappings known as *Lyusternik-Graves theorem*, see the survey [30] by Ioffe.

Inequality (5.1) provides a linear *error bound* estimate of metric type for the distance from x to the solution set of the generalized equation $F(u) \ni y$ corresponding to the perturbed right-hand side y in a neighbourhood of the solution $\bar{x}$ (corresponding to the right-hand side $\bar{y}$). Metric regularity is known to be equivalent to two other fundamental properties: the *openness* (or *covering*) at a linear rate and the *Aubin property* (a kind of Lipschitz-like behaviour) of the inverse mapping; cf. [5, 11, 15, 17, 19, 30, 43, 44, 51, 53, 55, 60]. Several qualitative and quantitative characterizations of the metric regularity property have been established in terms of various primal and dual space derivative-like objects: *slopes*, *graphical derivatives* (*Aubin criterion*), *subdifferentials* and *coderivatives*; cf. [6, 7, 19, 30, 43, 44, 48, 51, 52, 55, 60].

There have been many important developments of the metric regularity theory in recent years; among them clarifying the connection of the metric regularity *modulus* (the infimum of all κ such that (5.1) holds) to the *radius of metric regularity*, cf. [12, 17, 19, 25, 50, 51, 61], and the interpretation of the regularity of the subdifferential mapping via second-order growth conditions, cf. [2, 20, 21, 49, 64].

At the same time, it has been well recognized that many important variational problems do not posses conventional metric regularity. This observation has led to a significant grows of attention to more subtle regularity properties. This new development has mostly consisted in the relaxing or extension of the metric regularity property (5.1) (and the other two equivalent properties) and its characterizations along the following three directions (and their appropriate combinations).

1) Relaxing of property (5.1) by fixing one of the variables: either $y = \bar{y}$ or $x = \bar{x}$ in it. In the first case, one arrives at the very important for applications property of F known as *metric subregularity* (and respectively *calmness* of F^{-1}); cf. [1, 13, 18, 19, 27, 28, 30, 34, 46, 47, 63], while fixing the other variable (and usually also replacing $d(y, F(\bar{x}))$ with $d(y, \bar{y})$) leads to another type of relaxed regularity known as *metric semiregularity* [45] (also referred to as *metric hemiregularity* in [3]).

2) Considering nonlocal versions of (5.1), when x and y are restricted to certain subsets $U \subset X$ and $V \subset Y$, not necessarily neighbourhoods of $\bar{x}$ and $\bar{y}$, respectively, or even a subset $W \subset X \times Y$; cf. [30, 31, 32, 33]. A nonlocal regularity (covering) setting was already studied in [15].

3) Considering nonlinear versions of (5.1), when, instead of the constant modulus κ, a certain *functional modulus* $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is used in (5.1), i.e., $\kappa d(y, F(x))$ is replaced by $\mu(d(y, F(x)))$; cf. [11, 30, 33, 53]. This allows treating more subtle regularity properties arising in applications when the conventional estimates fail. The majority of researchers focus on the particular case of "power nonlinearities" when μ is of the type $\mu(t) = \lambda t^k$ with $\lambda > 0$ and $0 < k \leq 1$ [22, 23, 24, 33, 62].

Starting with Ioffe [29], most proofs of various sufficient regularity/openness criteria are based on the application of the celebrated *Ekeland variational principle* (Theorem 18); see [10, 19, 30, 51, 55, 60]. On the other hand, as observed by Ioffe in [30], the original methods used by Banach, Schauder, Lyusternik and Graves had employed special iteration procedures. This classical approach was very popular in the 1980s – early 1990s [14, 15, 16, 37, 38, 39, 56, 59]. In particular, in the series of three articles [37, 38, 39], using iteration techniques several basic statements were established which generalized many known by that time open mapping and closed graph theorems and theorems of the Lyusternik type and results on approximation and semicontinuity or their refinements. We refer to [30] for a thorough discussion and comparison of the two main techniques.

In this chapter, we demonstrate that the approach based on iteration procedures still possesses potential. In particular, we show that the *Induction theorem* [37, Theorem 1] (see Lemma 18 below), which was used as the main tool when proving the other results in [37], implies also all the main results in the subsequent articles [38, 39]. It can serve as a substitution of the Ekeland variational principle when establishing other regularity criteria. Furthermore, the latter classical result can also be established as a consequence of the Induction theorem. The sequences in the statement of this theorem as well as several other statements in Section 5.2 expose the details of iterative procedures which are usually hidden in the proofs of regularity/openness properties. This is important for the understanding of the roles played by different parameters and leaves some freedom of choice of the parameters defining iteration procedures; this can be helpful when constructing specific schemes as demonstrated in [37, 38, 39].

We consider a general nonlocal nonlinear regularity model for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces. It obviously covers the case of a parametric family of set-valued mappings; cf. [38, 39]. At the same time, the conventional setting of a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces can be imbedded into the model by defining a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$ by the equality $\mathcal{F}(x,t) := B(F(x),t) = \cup_{y\in F(x)} B(y,t)$ (with the convention $B(y,0) = \{y\}$). As observed by Ioffe [30, p. 508], this scheme is convenient for deducing regularity/openness estimates.

To define an analogue of metric regularity in this general setting, the distance $d(y, F(x))$ in the image space in the right-hand side of (5.1) is replaced by the "distance-like" quantity

$$\delta(y, F, x) := \inf\{t > 0 \mid y \in F(x,t)\}. \tag{5.2}$$

This allows one to define also a natural analogue of the covering property (but not the Aubin property!) and establish equivalence of both properties and some sufficient criteria. If $F(x,t)$ describes the set of positions of a dynamical system feasible at moment t starting at the initial point x, then constant (5.2) solves the minimal time feasibility problem.

In our study of regularity properties of set-valued mappings, we follow a three-step procedure which, in our opinion, is important for understanding the roles of particular assumptions employed in the criteria and the origins of specific regularity estimates.

1) Deducing basic regularity estimates at a fixed point $(x,t,y) \in \operatorname{gph} F$.

2) "Setting free" variable t in the basic regularity estimates obtained in step 1 and formulating the best (in terms of t) estimates. This way, the "distance-like" quantity (5.2) comes into play. The estimates are formulated at a fixed point $(x, y) \in X \times Y$.

3) "Setting free" variables x and y in the regularity estimates obtained in step 2, restricting them to a subset $W \subset X \times Y$ and formulating estimates holding for all $(x, y) \in W$. For the motivations behind such settings we refer the reader to [32, 33]. This way, we arrive at analogues of the metric regularity criteria. Under the appropriate choice of the set W, one can study various local and nonlocal settings of this property and even weaker sub- and semi-regularity versions. This line goes beyond the scope of the current chapter.

The structure of the chapter is as follows. In the next section, we give a short proof of a revised version of the Induction theorem [37, Theorem 1] and then apply it to establish several basic regularity estimates for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$ at a fixed point $(x, t, y) \in \operatorname{gph} F$. As a consequence, we obtain the two main theorems from [39] which cover the other results in [37, 38]. Next we discuss the relationship between the Induction theorem and the Ekeland variational principle. As another consequence of the aforementioned regularity estimates, we deduce several 'at a point' sufficient criteria for the regularity of F in terms of quantity (5.2). Section 5.3 is devoted to nonlinear *regularity on a set* (and the corresponding openness property) being a direct analogue of metric regularity in the conventional setting. We refrain from using the term "metric" because quantity (5.2) is not a distance in the image space. In Section 5.4, we demonstrate how the definitions and criteria from Section 5.3 translate into the conventional setting of a set-valued mapping $F : X \rightrightarrows Y$ taking the natural metric form. In Section 5.5, our general nonlinear regularity model is applied to establishing second-order necessary optimality conditions for a general nonsmooth set-valued optimization problem with mixed constraints. In line with the original idea of Lyusternik, the role of the regularity assumption is to allow handling of the constraints. This remains one of the major motivations for the development of the regularity theory. The final Section 5.6 contains some concluding remarks and a list of things to be done hopefully in not-so-distant future.

Our basic notation is standard; cf. [10, 19, 51, 55, 60]. X and Y are metric spaces. Metrics in all spaces are denoted by the same symbol $d(\cdot, \cdot)$. If x and C are a point and a subset of a metric space, then $d(x,C) := \inf_{c\in C} d(x,c)$ is the point-to-set distance from x to C, while $\overline{C}$ and $\operatorname{bd} C$ denote the closure and the boundary of C. $B(x,r)$ and $\overline{B}(x,r)$ stand for the open and closed balls of radius $r > 0$ centered at x, respectively. We use the convention that $B(x,0) = \{x\}$. If C is a subset of a linear space, then $\operatorname{cone} C := \{\lambda x \mid \lambda > 0,\ x \in C\}$ is the cone generated by C.

5.2 Regularity at a point

This section prepares the tools for the study of regularity properties of set-valued mappings in the rest of the chapter.

5.2.1 Basic estimates

The next technical lemma is a revised version of the *Induction theorem* [37, Theorem 1] and contains the core arguments used in the main results of [37, 38, 39]. For simplicity, it is formulated for mappings between metric spaces. (Most of the results in [37, 38, 39] are formulated in the more general setting of quasimetric spaces.)

Recall that a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces is called outer semicontinuous [60] at $\bar{x} \in X$ if

$$\operatorname{Lim\,sup}_{x\to\bar{x}} F(x) := \{y \in Y \mid \liminf_{x\to\bar{x}} d(y, F(x)) = 0\} \subset F(\bar{x}).$$

Lemma 18. *Let X be a complete metric space, $\Phi : \mathbb{R}_+ \rightrightarrows X$, $t > 0$ and $x \in \Phi(t)$. Suppose that Φ is outer semicontinuous at 0 and there are sequences of positive numbers (a_n) and (b_n) such that*

$$\sum_{n=0}^{\infty} b_n < \infty, \tag{5.3}$$

$$a_0 = t \quad \text{and} \quad a_n \downarrow 0 \text{ as } n \to \infty, \tag{5.4}$$

$$d(u, \Phi(a_{n+1})) < b_n \quad \text{for all} \quad u \in \Phi(a_n) \cap U_n \quad (n = 0, 1, \ldots), \tag{5.5}$$

where $U_0 := \{x\}$, $U_n := B(x, \sum_{i=0}^{n-1} b_i)$ $(n = 1, 2, \ldots)$. Then, $d(x, \Phi(0)) < \sum_{n=0}^{\infty} b_n$.

Proof. Putting $x_0 := x \in \Phi(a_0) \cap U_0$ and using (5.5) repeatedly, we obtain a sequence (x_n) satisfying $x_n \in \Phi(a_n)$ and

$$d(x_n, x_{n+1}) < b_n \quad (n = 0, 1, \ldots).$$

The above inequalities together with (5.3) imply that (x_n) is a Cauchy sequence and, as X is complete, converges to some point $z \in X$. Note that

$$d(z, x) \le \sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \sum_{n=0}^{\infty} b_n.$$

Thanks to the outer semicontinuity of Φ at 0 and (5.4), we have $z \in \Phi(0)$. Hence, $d(x, \Phi(0)) < \sum_{n=0}^{\infty} b_n$. □

Remark 43. 1. The above lemma does not talk about regularity properties of set-valued mappings. At the same time, we want the reader to observe certain similarity between the conclusion of Lemma 18 and inequality (5.1) (assuming that $\Phi(0)$ corresponds to the inverse of some set-valued mapping; this is going to be our next step). The sequences in the statement of the lemma expose iterative procedures employed in some traditional proofs of regularity properties which can be traced back to Banach and Schauder.

2. As it has been observed by many authors with regards to other regularity statements, with obvious changes, the proof of Lemma 18 remains valid if instead of the outer semicontinuity of Φ and completeness of X one assumes that $\operatorname{gph} \Phi$ is complete (in the product topology). In fact, it is sufficient to assume that $\operatorname{gph} \Phi \cap (\mathbb{R}_+ \times \overline{B}(x, \sum_{n=0}^{\infty} b_i))$ is complete.

3. In some applications, a "restricted" version of Lemma 18 can be useful. Given a subset U of X and a point $x \in \Phi(t) \cap U$, condition (5.5) can be replaced with the following "restricted" one:

$$d(u, \Phi(a_{n+1}) \cap U) < b_n \quad \text{for all} \quad u \in \Phi(a_n) \cap U_n \quad (n = 0, 1, \ldots),$$

where $U_0 := \{x\}$, $U_n := U \cap B(x, \sum_{i=0}^{n-1} b_i)$ $(n = 1, 2, \ldots)$.

4. The conclusion of Lemma 18 can be equivalently rewritten as

$$\Phi(0) \cap B\left(x, \sum_{n=0}^{\infty} b_n\right) \neq \emptyset.$$

From now on, we consider a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces, X is complete. Given a $t \in \mathbb{R}_+$, we denote $F_t := F(\cdot, t) : X \rightrightarrows Y$.

The purpose of this two-variable model is twofold. Firstly, if the second variable is interpreted as a parameter, it allows us to cover the case of a parametric family of set-valued mappings; cf. [38, 39]. Secondly, when studying regularity properties of a standard set-valued mapping $F : X \rightrightarrows Y$ between metric spaces, it can sometimes be convenient to consider its two-variable extension $(x, t) \to B(F(x), t)$; cf. Ioffe [30]. This model will be explored in Section 5.4. In this subsection we focus on the case of a parametric family of set-valued mappings and demonstrate that the main 'iterative' results of [38, 39] follow easily from Lemma 18.

The next two theorems contain the core arguments of [39, Theorems 3 and 4], respectively.

Theorem 13. *Let $t > 0$ and $(x, t, y) \in \operatorname{gph} F$. Suppose that the mapping $\tau \mapsto \Phi(\tau) := F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and there are sequences of positive numbers (b_n) and (c_n) and a function $m : (0, \infty) \to (0, \infty)$ such that condition (5.3) holds true and*

$$m(\tau) \downarrow 0 \text{ as } \tau \downarrow 0 \quad \text{and} \quad c_n \downarrow 0 \text{ as } n \to \infty, \tag{5.6}$$

$$d(x, F_{m(c_1)}^{-1}(y)) < b_0, \tag{5.7}$$

$$d(u, F_{m(c_{n+1})}^{-1}(y)) < b_n \text{ for all } u \in F_{m(c_n)}^{-1}(y) \cap B\left(x, \sum_{i=0}^{n-1} b_i\right) \quad (n = 1, 2, \ldots). \tag{5.8}$$

Then, $d(x, F_0^{-1}(y)) < \sum_{n=0}^{\infty} b_n$.

Proof. Set $a_0 := t$, $a_n := m(c_n)$ $(n = 1, 2, \ldots)$. Conditions (5.6), (5.7) and (5.8) imply (5.4) and (5.5). By Lemma 18, there exists a $z \in B(x, \sum_{n=0}^{\infty} b_n)$ such that $y \in F(z, 0)$, i.e., $z \in F_0^{-1}(y)$. □

Given a function $b : \mathbb{R}_+ \to \mathbb{R}_+$, we define, for each $t \in \mathbb{R}_+$, $b^0(t) := t$, $b^n(t) := b(b^{n-1}(t))$ $(n = 1, 2, \ldots)$.

Theorem 14. *Let $t > 0$ and $(x, t, y) \in \operatorname{gph} F$. Suppose that the mapping $\tau \mapsto \Phi(\tau) := F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and there are functions $b, m, \mu : (0, \infty) \to (0, \infty)$ such that*

$$m(\tau) \downarrow 0 \quad \Rightarrow \quad \tau \downarrow 0 \tag{5.9}$$

and, for each $\tau > 0$ with $\mu(\tau) \leq \mu(t)$,

$$\mu(\tau) \geq m(\tau) + \mu(b(\tau)), \tag{5.10}$$

$$d(u, F_{b(\tau)}^{-1}(y)) < m(\tau) \text{ for all } u \in F_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau)). \tag{5.11}$$

Then, $d(x, F_0^{-1}(y)) < \mu(t)$.

Proof. Set $a_n := b^n(t)$, $b_n := m(a_n) = m(b^n(t))$ $(n = 0, 1, \ldots)$. Adding inequalities (5.10) corresponding to $\tau = t, b(t), b^2(t), \ldots$, we obtain

$$\mu(t) \geq \sum_{n=0}^{\infty} m\left(b^n(t)\right) = \sum_{n=0}^{\infty} b_n.$$

Hence, (5.3) is satisfied and $b_n \downarrow 0$ as $n \to \infty$. Condition (5.4) is satisfied thanks to (5.9). Condition (5.11) with $\tau = a_n$ takes the following form:

$$d(u, \Phi(a_{n+1})) < b_n \text{ for all } u \in \Phi(a_n) \cap B(x, \mu(t) - \mu(a_n)). \tag{5.12}$$

For any $n > 0$, adding inequalities (5.10) corresponding to $\tau = t, b(t), \ldots, b^{n-1}(t)$, we obtain

$$\mu(t) \geq \sum_{i=0}^{n-1} b_i + \mu(a_n).$$

Hence, $\mu(a_n) \leq \mu(t)$ and condition (5.12) implies (5.5). By Lemma 18, there exists a $z \in B(x, \mu(t))$ such that $y \in F(z, 0)$. □

Remark 44. 1. The statements of Theorems 13 and 14 expose the details of iteration procedures which are usually hidden in the proofs of regularity/openness properties. For instance, the scalar function b in Theorem 14 defines the sequence of iterations corresponding to $\tau \downarrow 0$: given a value τ, the next value is $b(\tau)$ which produces a smaller than $\mu(\tau)$ value

$\mu(b(\tau))$ of the function μ with the difference $\mu(\tau) - \mu(b(\tau))$ controlling thanks to (5.10) the value $m(\tau)$ of the function m which in its turn controls thanks to (5.11) the distance between the iterations in X leading in the end to the claimed estimate. Inequalities of the type (5.10) and (5.11) (or (5.8)) are the key ingredients when istablishing regularity estimates.

Theorems 13 and 14 leave some freedom of choice of the parameters defining iteration procedures which can be helpful when constructing specific schemes as demonstrated in [37, 38, 39].

2. Instead of (5.6), it is sufficient to assume in Theorem 13 that $m(c_n) \downarrow 0$ as $n \to \infty$. In Theorem 14, this is satisfied automatically thanks to (5.9).

3. The conclusions of Theorems 13 and 14 can be equivalently rewritten as $y \in F\left(B\left(x,r\right),0\right)$ where either $r = \sum_{n=0}^{\infty} b_n$ or $r = \mu(t)$.

Theorem 14 covers a seemingly more general setting of regularity/covering on a system of balls; cf. [15, 30, 39].

Recall that a family Σ of balls in X is called a *complete system* [15, Definition 1.1] if, for any $B(x,r) \in \Sigma$, one has $B(x',r') \in \Sigma$ provided that $x' \in X$, $r' > 0$ and $d(x,x') + r' \le r$. For a subset M of X, $\Sigma(M)$ denotes a complete system of balls $B(x,r)$ in X with $B(x,r) \subset M$. Obviously the family of all balls in X forms a complete system.

Corollary 10. *Let $M \subset X$ and $\Sigma(M)$ be a complete system, $t > 0$ and $(x,t,y) \in \operatorname{gph} F$. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and there are functions $b, m, \mu : (0,\infty) \to (0,\infty)$ such that $B(x,\mu(t)) \in \Sigma(M)$, condition (5.9) is satisfied and, for each $\tau > 0$ with $\mu(\tau) \le \mu(t)$, condition (5.10) holds true and*

$$d(u, F_{b(\tau)}^{-1}(y)) < m(\tau) \text{ for all } u \in F_\tau^{-1}(y) \cap \{x' \mid B(x',\mu(\tau)) \in \Sigma(M)\}. \tag{5.13}$$

Then, $d(x, F_0^{-1}(y)) < \mu(t)$.

Proof. Since $B(x,\mu(t)) \in \Sigma(M)$, it follows that $B(x,\mu(t) - \mu(\tau)) \subset \{x' \mid B(x',\mu(\tau)) \in \Sigma(M)\}$. The conclusion follows from Theorem 14. □

The key estimates (5.11) and (5.13) in Theorem 14 and Corollary 10 are for the original space X. In some situations, one can use for that purpose also similar estimates in the image space Y.

Corollary 11. *Let $t > 0$ and $(x, t, y) \in \operatorname{gph} F$. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and there are functions $b, m, \mu : (0, \infty) \to (0, \infty)$ such that condition (5.9) is satisfied and, for each $\tau > 0$ with $\mu(\tau) \leq \mu(t)$, condition (5.10) holds true and*

$$F_0^{-1}(B(y, \tau)) \subset F_\tau^{-1}(y), \tag{5.14}$$

$$d\left(y, F_0(B(u, m(\tau)))\right) < b(\tau) \text{ for all } u \in F_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau)). \tag{5.15}$$

Then, $d(x, F_0^{-1}(y)) < \mu(t)$.

Proof. Observe that conditions (5.14) and (5.15) imply (5.11). Indeed, if $u \in F_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau))$, then, by (5.15), there exists a $z \in B(u, m(\tau))$ such that $d(y, F_0(z)) < b(\tau)$, or equivalently, $z \in F_0^{-1}(B(y, b(\tau)))$. It follows from (5.14) that $z \in F_{b(\tau)}^{-1}(y)$. Hence, $d(u, F_{b(\tau)}^{-1}(y)) < m(\tau)$. The conclusion follows from Theorem 14. □

Remark 45. 1. Instead of (5.9), it is sufficient to assume in Theorem 14 and Corollaries 10 and 11 that $b^n(t) \downarrow 0$ as $n \to \infty$. The last condition is satisfied, e.g., when $b(t) = \lambda t$ with $\lambda \in (0, 1)$.

2. If condition (5.10) holds true for all $\tau > 0$ with $\mu(\tau) \leq \mu(t)$, then $\mu(\tau) \geq \sum_{n=0}^{\infty} m\left(b^n(\tau)\right)$. On the other hand, if the last condition holds true as equality (for all $\tau > 0$ with $\mu(\tau) \leq \mu(t)$), then condition (5.10) is satisfied (as equality). Hence, condition (5.10) in Theorem 14 and Corollaries 10 and 11 can be replaced by the following definition of the smallest function μ satisfying (5.10):

$$\mu(\tau) := \sum_{n=0}^{\infty} m\left(b^n(\tau)\right), \tag{5.16}$$

thus producing the strongest conclusion.

3. It is sufficient to assume in Theorem 14 and Corollaries 10 and 11 that conditions (5.10), (5.11), (5.13), (5.14) and (5.15) are satisfied only for $\tau = t, b(t), b^2(t), \ldots$ In particular, if this sequence is monotone (as in the typical example mentioned in part 1 above or, thanks to (5.10) when μ is nondecreasing), then the conclusions of all the statements remain true when conditions (5.10), (5.11), (5.13), (5.14) and (5.15) are satisfied for all $\tau \in (0, t]$.

4. Thanks to part 3, instead of conditions (5.11), (5.13) and (5.15), one can require that, for each $n = 0, 1, \ldots$, the following conditions hold true, respectively:

$$d(u, F^{-1}_{b^{n+1}(t)}(y)) < m(b^n(t)) \text{ for all } u \in F^{-1}_{b^n(t)}(y) \cap B(x, \mu(t) - \mu(b^n(t))), \tag{5.17}$$

$$d(u, F^{-1}_{b^{n+1}(t)}(y)) < m(b^n(t)) \text{ for all } u \in F^{-1}_{b^n(t)}(y) \cap \{x' \mid B(x', \mu(b^n(t))) \in \Sigma(M)\},$$

$$d\,(y, F_0(B(u, m(b^n(t))))) < b^{n+1}(t) \text{ for all } u \in F^{-1}_{b^n(t)}(y) \cap B(x, \mu(t) - \mu(b^n(t))). \tag{5.18}$$

If μ is given by (5.16), then conditions (5.17) and (5.18) can be equivalently rewritten as follows:

$$d(u, F^{-1}_{b^{n+1}(t)}(y)) < m(b^n(t)) \text{ for all } u \in F^{-1}_{b^n(t)}(y) \cap B(x, \sum_{i=0}^{n-1} b^i(t)),$$

$$d\,(y, F_0(B(u, m(b^n(t))))) < b^{n+1}(t) \text{ for all } u \in F^{-1}_{b^n(t)}(y) \cap B(x, \sum_{i=0}^{n-1} b^i(t)).$$

5. The conclusions of Theorem 14 and Corollaries 10 and 11 can be equivalently rewritten as $y \in F(B(x, \mu(t)), 0)$.

The next two theorems are the (slightly improved) original results of [39, Theorems 3 and 4] reformulated in the setting of metric spaces and adopting the terminology and notation of the current chapter. These theorems, which follow immediately from Theorems 13 and 14, respectively, imply all the other results of [37, 38, 39] as well as many open mapping and closed graph theorems and theorems of the Lyusternik type and results on approximation and semicontinuity or their refinements; cf. the references in [37, 38, 39].

Theorem 15. *Let $t > 0$ and $(x, t) \in \operatorname{dom} F$. Suppose that, for each $y \in Y$,*

$$F_0^{-1}(y) = \operatorname{Lim\,sup}_{t \downarrow 0} F_t^{-1}(y) \tag{5.19}$$

and there are positive numbers ρ, s *and* b_n $(n = 1, 2, \ldots)$, *such that*

$$\sum_{n=1}^{\infty} b_n + s \le \rho. \tag{5.20}$$

Suppose also that, for each $y \in F(x,t)$, *there are numbers* $c_n > 0$ $(n = 1, 2, \ldots)$ *and a function* $m : (0,\infty) \to (0,\infty)$ *satisfying* (5.6) *and*

$$d(u, F^{-1}_{m(c_1)}(y)) < s \quad \textit{for all} \quad u \in F^{-1}_t(y) \cap B(x, \rho - s), \tag{5.21}$$

$$d(u, F^{-1}_{m(c_{n+1})}(y)) < b_n \;\; \textit{for all} \;\; u \in F^{-1}_{m(c_n)}(y) \cap B(x, \rho - b_n) \; (n = 1, 2, \ldots) \tag{5.22}$$

Then, $F(x,t) \subset F(B(x,\rho), 0)$.

Proof. Set $b_0 := s$ and take any $y \in F(x,t)$. It follows from (5.19) that the mapping $\tau \mapsto F^{-1}_\tau(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0. Condition (5.20) obviously implies (5.3). Observe that $\sum_{i=0}^{n-1} b_i \le \rho - \sum_{i=n}^{\infty} b_i < \rho - b_n$ $(n = 0, 1, \ldots)$ Hence, conditions (5.21) and (5.22) imply (5.7) and (5.8), respectively. By Theorem 13, $y \in F(B(x,\rho), 0)$. □

Theorem 16. *Let* $M \subset X$ *and* $\Sigma(M)$ *be a complete system. Let a function* $b : (0,\infty) \to (0,\infty)$ *be given. Suppose that, for each* $y \in Y$, *condition* (5.19) *holds true and there exists a function* $m : (0,\infty) \to (0,\infty)$ *satisfying condition* (5.9) *and, for all* $\tau \in (0,\infty)$ *and* $x \in X$ *with* $(x,t,y) \in \operatorname{gph} F$ *and* $B(x,\mu(\tau)) \in \Sigma(M)$, *conditions* (5.13) *and* (5.16) *are satisfied. Then, for any* $(x,t,y) \in \operatorname{gph} F$ *with* $t > 0$ *and* $B(x,\mu(t)) \in \Sigma(M)$, *one has* $y \in F(B(x,\mu(t)), 0)$.

Proof. Take any $(x,t,y) \in \operatorname{gph} F$ with $t > 0$ and $B(x,\mu(t)) \in \Sigma(M)$ and a function m satisfying the assumptions of the theorem. Condition (5.19) obviously implies that the mapping $\tau \mapsto F^{-1}_\tau(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0. Thanks to Remark 45.2, all the assumptions of Corollary 10 are satisfied. Hence, $y \in F(B(x,\mu(t)), 0)$. □

Remark 46. Comparing the statements of Theorem 16 and [39, Theorem 4], one can notice that the latter one looks stronger: it is formulated without assumption (5.9) and with the stronger conclusion $F(x,t) \subset F(B(x,\mu(t)), 0)$. However assumption (5.9) is implicitly used in the proof of [39, Theorem 4] and the conclusion is established for a fixed $y \in F(x,t)$ satisfying $B(x,\mu(t)) \in \Sigma(M)$. (Observe that function m in Theorem 16 and consequently function μ defined by (5.16) depend on the choice of $y \in F(x,t)$.)

Unlike the setting of the current chapter, in [39] mapping F was assumed to be defined not on $X \times \mathbb{R}_+$, but on $X \times [0, t_0]$ where t_0 is a given positive number. This difference can be easily eliminated by setting $F(x,t) := \emptyset$ when $t > t_0$ and making appropriate minor changes in the statements.

5.2.2 Lemma 18 and Ekeland variational principle

Lemma 18 which lies at the core of the proofs of the statements in the previous subsection can serve as a substitution of the Ekeland variational principle which is a traditional tool when establishing regularity criteria. This is demonstrated by the proof of such a criterion in the following theorem.

Theorem 17. *Let $t > 0$ and $(x,t,y) \in \operatorname{gph} F$. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$ is outer semicontinuous on $[0,t)$ and there is a continuous nondecreasing function $\mu : [0,t] \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$ and, for each pair $(u,\tau) \in F^{-1}(y)$ with $\tau \in (0,t]$ and $d(x,u) \le \mu(t) - \mu(\tau)$, there exists a pair $(u',\tau') \in F^{-1}(y)$ such that $u' \neq u$ and*

$$\mu(\tau') \le \mu(\tau) - d(u',u). \tag{5.23}$$

Then, $d(x, F_0^{-1}(y)) \le \mu(t)$.

Proof. Set $a_0 := t$, $\bar{x} := x$ and define a sequence $\{(x_n, a_n)\}$ by induction. For any $n = 0, 1, \ldots$, let a pair $(x_n, a_n) \in F^{-1}(y)$ with $a_n \in [0,t]$ and $d(x, x_n) \le \mu(t) - \mu(a_n)$ be given. If $a_n = 0$, set $a_{n+1} := 0$ and $x_{n+1} := x_n$. Otherwise, define

$$c_n := \inf\{\mu(\tau) \mid (u,\tau) \in F^{-1}(y),\ \mu(\tau) \le \mu(a_n) - d(u, x_n)\}. \tag{5.24}$$

By the assumptions of the theorem, $0 \le c_n < \mu(a_n)$, and one can choose a pair $(x_{n+1}, a_{n+1}) \in F^{-1}(y)$ such that $x_{n+1} \neq x_n$ and

$$\mu(a_{n+1}) \le \mu(a_n) - d(x_n, x_{n+1}), \tag{5.25}$$

$$c_n \le \mu(a_{n+1}) < \frac{\mu(a_n) + c_n}{2} < \mu(a_n). \tag{5.26}$$

It also follows from (5.25) that

$$d(x, x_{n+1}) \leq d(x, x_n) + d(x_n, x_{n+1}) \leq \mu(t) - \mu(a_{n+1}).$$

If $a_n = 0$ for some $n > 0$, then, by (5.25),

$$d(x, F_0^{-1}(y)) \leq d(x, x_n) \leq \sum_{j=0}^{n-1} d(x_j, x_{j+1}) \leq \mu(t).$$

Now assume that $a_n > 0$ for all $n = 0, 1, \ldots$. Then, $\{a_n\}$ is a decreasing sequence of positive numbers which converges to some $a \geq 0$. We are going to show that $a = 0$. Suppose that $a > 0$ and denote $\hat{a}_n := a_n - a$. Obviously, $\hat{a}_n > 0$ and $\hat{a}_n \downarrow 0$. By (5.25),

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \mu(t) - \mu(a).$$

Fix an $\varepsilon > 0$ and choose numbers $b_n > d(x_n, x_{n+1})$ such that $\sum_{n=0}^{\infty} b_n < \mu(t) - \mu(a) + \varepsilon$. Set $\Phi(\hat{a}_n) := \{x_n\}$, $\Phi(\tau) := \emptyset$ for any $\tau \in (0, \infty) \setminus \{\hat{a}_0, \hat{a}_1, \ldots\}$, and let $\Phi(0)$ be the set of all cluster points of $\{x_n\}$. Then, $x \in \Phi(\hat{a}_0)$, Φ is outer semicontinuous at 0 and $d(\Phi(\hat{a}_n), \Phi(\hat{a}_{n+1})) < b_n$. It follows from Lemma 18 that there exists a $z \in \Phi(0)$ satisfying $d(x, z) < \mu(t) - \mu(a) + \varepsilon$. By the outer semicontinuity of Φ, $y \in F(z, a)$.

Since $a > 0$, by the assumptions of the theorem, there exists a pair $(u, \tau) \in F^{-1}(y)$ such that $u \neq z$ and

$$\mu(\tau) \leq \mu(a) - d(u, z). \tag{5.27}$$

Then, $\mu(\tau) < \mu(a)$. Observe from (5.26) that

$$2\mu(a_{n+1}) - \mu(a_n) < c_n < \mu(a_n).$$

Hence, $\{c_n\}$ converges to $\mu(a)$ and consequently $\mu(\tau) < c_n$ when n is large enough. By definition (5.24), this yields

$$\mu(\tau) > \mu(a_n) - d(u, x_n). \tag{5.28}$$

At the same time,

$$d(x_n, z) \leq \sum_{j=n}^{\infty} d(x_j, x_{j+1}) \leq \mu(a_n) - \mu(a).$$

This combined with (5.27) gives

$$\mu(\tau) \leq \mu(a_n) - d(u, x_n)$$

which is in obvious contradiction with (5.28). Hence, $a = 0$, $z \in F_0^{-1}(y)$, $d(x, z) < \mu(t) + \varepsilon$, and, as ε is arbitrary, $d(x, F_0^{-1}(y)) \leq \mu(t)$. □

The proof of Theorem 17 given above relies on Lemma 18 and uses standard arguments typical for traditional proofs of the Ekeland variational principle; cf. e.g. [10]. We next show that the latter classical result can also be established as a consequence of Lemma 18.

Theorem 18 (Ekeland variational principle)**.** *Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below. Suppose $\varepsilon > 0$, $\lambda > 0$ and $x \in X$ satisfies*

$$f(x) < \inf_X f + \varepsilon.$$

Then, there exists a $z \in X$ such that

(i) $d(z, x) < \lambda$,

(ii) $f(z) \leq f(x)$,

(iii) $f(u) + (\varepsilon/\lambda) d(u, z) \geq f(z)$ *for all* $u \in X$.

Proof. Denote $\bar{x} := x$. For $n = 0, 1, \ldots$, set

$$a_n := \sup_{u \in X} \left\{ f(x_n) - f(u) - \frac{\varepsilon}{\lambda} d(u, x_n) \right\}. \tag{5.29}$$

Obviously, $0 \leq a_n < \infty$. Choose an x_{n+1} such that

$$f(x_n) - f(x_{n+1}) - \frac{\varepsilon}{\lambda} d(x_{n+1}, x_n) \geq \frac{a_n}{2}. \tag{5.30}$$

Then, for $n = 0, 1, \ldots$,

$$f(x_{n+1}) \leq f(x_n), \quad d(x_{n+1}, x_n) \leq \frac{\lambda}{\varepsilon}(f(x_n) - f(x_{n+1}))$$

and the inequalities are strict if $a_n > 0$. It follows that

$$f(x_n) \leq f(x) \quad \text{and} \quad d(x_n, x) \leq \frac{\lambda}{\varepsilon}(f(x) - f(x_n)) < \lambda.$$

If, for some n, $a_n = 0$, then $z := x_n$ satisfies the conclusions of the theorem. Suppose that $a_n > 0$ for all $n = 0, 1, \ldots$. Then, $b_n := \frac{\lambda}{\varepsilon}(f(x_n) - f(x_{n+1})) > 0$. Set $\Phi(a_n) := \{x_n\}$, $\Phi(\tau) := \emptyset$ for any $\tau \in (0, \infty) \setminus \{a_0, a_1, \ldots\}$ and $\Phi(0) := \operatorname{Lim\,sup}_{\tau \downarrow 0} \Phi(\tau)$. Hence, Φ is outer semicontinuous at 0, $x \in \Phi(a_0)$, $\sum_{n=0}^{\infty} b_n < \lambda$ and $d(\Phi(a_n), \Phi(a_{n+1})) < b_n$. Besides, it follows from (5.29) that

$$f(x_n) - f(u) - \frac{\varepsilon}{\lambda} d(u, x_n) \leq a_n \quad \text{for all} \quad u \in X. \tag{5.31}$$

Subtracting (5.30) from the last inequality and using the triangle inequality, we conclude that

$$f(x_{n+1}) - f(u) - \frac{\varepsilon}{\lambda} d(u, x_{n+1}) \leq \frac{a_n}{2} \quad \text{for all} \quad u \in X,$$

i.e., $a_{n+1} \leq a_n/2$ and consequently $a_n \downarrow 0$ as $n \to \infty$. It follows from Lemma 18 that there exists a $z \in \Phi(0)$ satisfying (i). By the definition of $\Phi(0)$ and (5.31), we conclude that conditions (ii) and (iii) are satisfied too. □

Remark 47. Lemma 18 was used in the proof of Theorem 18 where one would normally use the convergence of a Cauchy sequence. Similarly, the Ekeland variational principle can replace the Cauchy sequence argument in the proof of Lemma 18. In fact, both Lemma 18 and Theorem 18 are in a sense equivalent to the completeness of X.

5.2.3 Regularity

Lemma 18 and the other results in Subsection 5.2.1 provide a collection of basic estimates which are going to be used when establishing regularity criteria. Theorems 13, 14 and 17 and Corollaries 10 and 11 were formulated for a fixed point $(x, t, y) \in \operatorname{gph} F$. The next step is to "set variable t free" and formulate criteria for a fixed point (x, y) such that $(x, t, y) \in \operatorname{gph} F$ for some $t > 0$. Once variable t is free, it is natural to take infimum over t in the right-hand sides of the inequalities in the conclusions of the statements in Subsection 5.2.1 to obtain the best possible estimates. Under the natural assumption of monotonicity of the function μ involved in most of the statements, this is equivalent to evaluating the infimum of $t > 0$ such that $(x, t, y) \in \operatorname{gph} F$. This way the "distance-like" quantity $\delta(y, F, x)$ defined by (5.2) comes into play.

The next several assertions are immediate consequences of Theorems 13, 14 and 17 and Corollaries 10 and 11, respectively. As an illustration, we provide a short proof of the first one.

Theorem 19. *Let $(x, y) \in X \times Y$ and $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous nondecreasing function. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and, for some $\gamma > \delta(y, F, x)$ and any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} F$, there are sequences of positive numbers (b_n) and (c_n) and a function $m : (0, \infty) \to (0, \infty)$ such that conditions (5.6)–(5.8) hold true and*

$$\sum_{n=0}^{\infty} b_n \leq \mu(t). \tag{5.32}$$

Then, $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$.

Proof. It is sufficient to notice that, for any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} F$, condition (5.32) implies (5.3) and, by Theorem 13, $d(x, F_0^{-1}(y)) < \mu(t)$. Taking the infimum in the right-hand side of the above inequality over all $t > 0$ with $(x, t, y) \in \operatorname{gph} F$ and making use of the monotonicity of μ, we arrive at the claimed conclusion. □

Theorem 20. *Let $(x, y) \in X \times Y$ and $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous nondecreasing function. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and, for some $\gamma > \delta(y, F, x)$ and any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} F$, there are functions $b, m : (0, \infty) \to (0, \infty)$ such that condition (5.9) is satisfied and, for each $\tau > 0$ with $\mu(\tau) \leq \mu(t)$, conditions (5.10) and (5.11) hold true. Then, $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$.*

Corollary 12. *Let $M \subset X$ and $\Sigma(M)$ be a complete system, $(x, y) \in X \times Y$ and $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous nondecreasing function. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$*

on $\mathbb{R}_+$ is outer semicontinuous at 0 *and, for some $\gamma > \delta(y, F, x)$ and any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} F$, one has $B(x, \mu(t)) \in \Sigma(M)$, there are functions $b, m : (0, \infty) \to (0, \infty)$ such that condition (5.9) is satisfied and, for each $\tau > 0$, conditions (5.10) and (5.13) hold true. Then, $d(x, F_0^{-1}(y)) \le \mu(\delta(y, F, x))$.*

Corollary 13. *Let $(x, y) \in X \times Y$ and $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous nondecreasing function. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at* 0 *and, for some $\gamma > \delta(y, F, x)$ and any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} F$, there are functions $b, m : (0, \infty) \to (0, \infty)$ such that condition (5.9) is satisfied and, for each $\tau > 0$ with $\mu(\tau) \le \mu(t)$, conditions (5.10), (5.14) and (5.15) hold true. Then, $d(x, F_0^{-1}(y)) \le \mu(\delta(y, F, x))$.*

Remark 48. Most of the comments in Remarks 44 and 45 are applicable to Theorems 19 and 20 and Corollaries 12 and 13.

In the next theorem, we at last get rid of the technical parameters inherited from the statements in Subsection 5.2.1 and formulate a regularity statement in a more conventional way (though still as an "at a point" condition).

Theorem 21. *Let $(x, y) \in X \times Y$, $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function and $\mu(\tau) = 0$ if and only if $\tau = 0$. Suppose that the mapping $\tau \mapsto F_\tau^{-1}(y)$ is outer semicontinuous on $[0, \delta(y, F, x)]$ and, for each pair $(u, \tau) \in F^{-1}(y)$ with $\tau \in (0, \delta(y, F, x)]$ and $d(x, u) \le \mu(\delta(y, F, x)) - \mu(\delta(y, F, u))$, there exists a pair $(u', \tau') \in F^{-1}(y)$ such that $u' \neq u$ and condition (5.23) is satisfied. Then, $d(x, F_0^{-1}(y)) \le \mu(\delta(y, F, x))$.*

Proof. If $\delta(y, F, x) = \infty$, then the conclusion holds true trivially. Otherwise, the outer semicontinuity of $\tau \mapsto F_\tau^{-1}(y)$ ensures that $y \in F(x, \delta(y, F, x))$, and the conclusion follows from Theorem 17 for $t = \delta(y, F, x)$. □

Remark 49. The conclusion of Theorems 19, 20 and 21 and Corollaries 12 and 13 reminds the inequality in the definition of the metric regularity property for a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces; cf. [19]. The difference is in the right-hand side, where $\delta(y, F, x)$ stands in place of $d(y, F(x))$. The relationship between the two settings will be explored in Section 5.4.

The conclusion of Theorems 19, 20 and 21 and Corollaries 12 and 13 can be reformulated equivalently in a "covering-like" form.

Proposition 33. *Consider the following conditions:*

(i) $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$*,*

(ii) $y \in F(B(x, t), 0)$ *for any* $t > \mu(\delta(y, F, x))$*,*

(iii) $y \in F(B(x, \mu(\delta(y, F, x))), 0)$*.*

Then, (iii) $\Rightarrow$ *(ii)* $\Leftrightarrow$ *(i).*

Proof. (iii) $\Rightarrow$ (ii) is obvious.

(i) $\Rightarrow$ (ii). By (i), for any $t > \mu(\delta(y, F, x))$, there exists a $z \in F_0^{-1}(y)$ such that $d(x, z) < t$ and consequently $y \in F(z, 0) \subset F(B(x, t), 0)$.

(ii) $\Rightarrow$ (i). $y \in F(B(x, t), 0)$ and $t > 0$ if and only if $d(x, F_0^{-1}(y)) < t$. If the last inequality holds for all $t > \mu(\delta(y, F, x))$, then $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$. □

Remark 50. Proposition 33 is true without the assumption of the completeness of X.

5.3 Regularity on a set

In this section, we continue exploring regularity properties for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces. Given a subset $W \subset X \times Y$ and an upper semicontinuous nondecreasing function $\mu : [0, +\infty] \to [0, +\infty]$, we use the statements derived in Section 5.2 to characterize regularity of F *on* W *with functional modulus* μ. We "set free" the remaining two variables x and y restricting them to the set W.

Definition 18. (i) F is regular on W with functional modulus μ if

$$d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x)) \quad \text{for all} \quad (x, y) \in W.$$

(ii) F is open on W with functional modulus μ if

$$y \in F(B(x, t), 0) \quad \text{for all} \quad (x, y) \in W \text{ and } t > \mu(\delta(y, F, x)).$$

The above properties differ from the conventional metric regularity defined for set-valued mappings between metric spaces (cf. [19]) and its nonlinear extensions (cf. [33]). The relationship between the two settings will be discussed in Section 5.4.

The next proposition is a consequence of Proposition 33 thanks to Remark 50.

Proposition 34. *The two properties in Definition 18 are equivalent.*

Remark 51. It follows from Proposition 33 that the properties in Definition 18 are implied by the following stronger version of openness:

$$y \in F(B(x, \mu(\delta(y, F, x))), 0) \quad \text{for all} \quad (x, y) \in W.$$

The criteria of regularity in the next theorem are direct consequences of Theorems 19 and 20 and Corollary 13.

Theorem 22. *Suppose that, for any $(x, y) \in W$, the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and, for some $\gamma > \delta(y, F, x)$ and any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} F$, one of the following sets of conditions is satisfied:*

(i) there are sequences of positive numbers (b_n) and (c_n) and a function $m : (0, \infty) \to (0, \infty)$ such that conditions (5.6)–(5.8) and (5.32) hold true,

(ii) there are functions $b, m : (0, \infty) \to (0, \infty)$ such that condition (5.9) is satisfied and, for any $\tau > 0$ with $\mu(\tau) \leq \mu(t)$, conditions (5.10) and (5.11) hold true,

(iii) there are functions $b, m : (0, \infty) \to (0, \infty)$ such that condition (5.9) is satisfied and, for any $\tau > 0$ with $\mu(\tau) \leq \mu(t)$, conditions (5.10), (5.14) and (5.15) hold true.

Then, F is regular on W with functional modulus μ.

In the next statement, which is a consequence of the "parameter-free" Theorem 21, $p_Y : X \times Y \to Y$ denotes the canonical projection on Y: for any $(x, y) \in X \times Y$, $p_Y(x, y) = y$. Given a pair $(x, y) \in W$, denote

$$U_{x,y} := \{u \in X \mid \delta(y, F, u) > 0,\ \mu(\delta(y, F, u)) + d(u, x) \leq \mu(\delta(y, F, x))\}.$$

Theorem 23. *Let μ be continuous, $\mu(\tau) = 0$ if and only if $\tau = 0$. Suppose that F^{-1} is closed-valued on $p_Y(W)$ and, for any $(x, y) \in W$ and $u \in U_{x,y}$, there exists a point $u' \neq u$ such that*

$$\mu(\delta(y, F, u')) \leq \mu(\delta(y, F, u)) - d(u, u'). \tag{5.33}$$

Then, F is regular on W with functional modulus μ.

Proof. Fix an arbitrary $(x, y) \in W$. We need to show that $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$. If there exists a point u such that $\delta(y, F, u) = 0$ and $d(x, u) \leq \mu(\delta(y, F, x))$ (in particular, if $\delta(y, F, x) = 0$), then, by the closedness of $F^{-1}(y)$, we have $u \in F_0^{-1}(y)$, and the inequality holds trivially.

Suppose that $\delta(y, F, u) > 0$ for any $u \in X$ such that $d(x, u) \leq \mu(\delta(y, F, x))$. Take any $u \in X$ such that $d(x, u) \leq \mu(\delta(y, F, x)) - \mu(\delta(y, F, u))$ and any $\tau \in (0, \delta(y, F, x)]$ such that $(u, \tau) \in F^{-1}(y)$. Then, $\tau \geq \delta(y, F, u) > 0$ and, by the assumption, there exists a point $u' \neq u$ satisfying (5.33). Setting $\tau' = \delta(y, F, u')$, we get $(u', \tau') \in F^{-1}(y)$ and condition (5.23) is satisfied:

$$\mu(\tau') = \mu(\delta(y, F, u')) \leq \mu(\delta(y, F, u)) - d(u, u') \leq \mu(\tau) - d(u, u').$$

The mapping $\tau \mapsto F_\tau^{-1}(y)$ is outer semicontinuous on $[0, \delta(y, F, x)]$ thanks to the closedness of $F^{-1}(y)$. The required inequality follows from Theorem 21. □

One can define seemingly more general ν-versions of the properties in Definition 18, determined by a function $\nu : W \to (0, \infty]$; see [33] for the motivations behind such properties.

Definition 19. (i) F is ν-regular on W with functional modulus μ if

$$d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x)) \text{ for all } (x, y) \in W \text{ with } \mu(\delta(y, F, x)) < \nu(x, y).$$

(ii) F is ν-open on W with functional modulus μ if

$$y \in F(B(x, t), 0) \quad \text{for all} \quad (x, y) \in W \text{ and } t \in (\mu(\delta(y, F, x)), \nu(x, y)).$$

Remark 52. Each of the properties in Definition 18 is a particular case of the corresponding one in Definition 19 with any function $\nu : W \to (0, \infty]$ satisfying $\mu(\delta(y, F, x)) < \nu(x, y)$ for all $(x,y) \in W$ with $\mu(\delta(y,F,x)) < +\infty$, e.g., one can take $\nu \equiv +\infty$. At the same time, each of the properties in Definition 19 can be considered as a particular case of the corresponding one in Definition 18 with the set W replaced by $W' := \{(x,y) \in W \mid \mu(\delta(y,F,x)) < \nu(x,y)\}$.

Proposition 35. *The two properties in Definition 19 are equivalent.*

We next formulate the corresponding criteria for ν-regularity. The next two theorems are consequences of Theorem 22 and the "parameter-free" Theorem 23, respectively, thanks to Remark 52 and the simple observation that, if $\mu(\delta(y,F,x)) < \nu(x,y)$, then, making use of the upper semicontinuity of μ, it is possible to choose a $\gamma > \delta(y,F,x)$ such that $\mu(\gamma) < \nu(x,y)$.

Theorem 24. *Suppose that, for any $(x,y) \in W$, the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and, for any $t > 0$ with $(x,t,y) \in \operatorname{gph} F$ and $\mu(t) < \nu(x,y)$, one of the three sets of conditions in Theorem 22 is satisfied. Then, F is ν-regular on W with functional modulus μ.*

Theorem 25. *Let μ be continuous, $\mu(\tau) = 0$ if and only if $\tau = 0$ and $\nu : \bigcup_{(x,y)\in W}(U_{x,y} \times \{y\}) \to (0,\infty)$ be Lipschitz continuous with modulus not greater than 1 in x for any $y \in p_Y(W)$. Suppose that F^{-1} takes closed values on $p_Y(W)$ and, for any $(x,y) \in W$ and $u \in U_{x,y}$ with $\mu(\delta(y,F,u)) < \nu(u,y)$, there exists a point $u' \neq u$ such that condition (5.33) holds true. Then, F is ν-regular on W with functional modulus μ.*

Proof. Define $W' := \{(x,y) \in W \mid \mu(\delta(y,F,x)) < \nu(x,y)\}$ and take any $(x,y) \in W'$ and $u \in U_{x,y}$. Then, taking into account the Lipschitz continuity of ν, we have:

$$\mu(\delta(y,F,u)) \le \mu(\delta(y,F,x)) - d(x,u) < \nu(x,y) - d(x,u) \le \nu(u,y).$$

Hence, there exists a point $u' \neq u$ such that (5.33) holds true. By Theorem 23, F is regular on W' and, thanks to Remark 52, ν-regular on W with functional modulus μ. □

Remark 53. The properties in Definitions 18 and 19 depend on the choice of the set W and (in the case of Definitions 19) function ν. Changing these parameters may lead to the change of the regularity modulus or even kill regularity at all; cf. [33, Example 1].

The next definition introduces the more conventional local versions of the properties in Definition 18 related to a fixed point $(\bar{x},\bar{y}) \in \operatorname{gph} F_0$.

Definition 20. (i) F is regular at $(\bar{x}, \bar{y})$ with functional modulus μ if there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x)) \quad \text{for all} \quad x \in U, \ y \in V.$$

(ii) F is open at $(\bar{x}, \bar{y})$ with functional modulus μ if there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$y \in F(B(x, t), 0) \quad \text{for all} \quad x \in U, \ y \in V \text{ and } t > \mu(\delta(y, F, x)).$$

The properties in Definition 20 are obviously equivalent to the corresponding ones in Definition 18 with $W := U \times V$. The next three statements are consequences of Proposition 34 and Theorems 22 and 23, respectively.

Proposition 36. *The two properties in Definition 20 are equivalent.*

Theorem 26. *Suppose that there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that, for any $x \in U$ and $y \in V$, the mapping $\tau \mapsto F_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and, for some $\gamma > \delta(y, F, x)$ and any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} F$, one of the three sets of conditions in Theorem 22 is satisfied. Then, F is regular at $(\bar{x}, \bar{y})$ with functional modulus μ.*

Theorem 27. *Let μ be continuous, $\mu(\tau) = 0$ if and only if $\tau = 0$. Suppose that there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that F^{-1} takes closed values on V and, for any $x \in U$, $y \in V$, and $u \in U_{x,y}$, there exists a point $u' \neq u$ such that condition (5.33) is satisfied. Then, F is regular at $(\bar{x}, \bar{y})$ with functional modulus μ.*

5.4 Conventional setting

In this section, we consider the standard in variational analysis setting of a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces. Such a mapping can be imbedded into the more general setting explored in the previous sections by defining a set-valued mapping

$\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$ as follows (cf. [30, p. 508]: for any $x \in X$ and $t \geq 0$,

$$\mathcal{F}(x,t) := B(F(x),t) = \begin{cases} \{y \in Y \mid d(y,F(x)) < t\} & \text{if } t > 0, \\ F(x) & \text{if } t = 0. \end{cases} \tag{5.34}$$

(Recall the convention: $B(y,0) = \{y\}$.) We are going to consider also mappings $\overline{F} : X \rightrightarrows Y$ and $\overline{\mathcal{F}} : X \times \mathbb{R}_+ \rightrightarrows Y$, whose values are the closures of the corresponding values of F and $\mathcal{F}$, respectively: $\overline{F}(x) := \overline{F(x)}$ and

$$\overline{\mathcal{F}}(x,t) := \overline{B}(F(x),t) = \begin{cases} \{y \in Y \mid d(y,F(x)) \leq t\} & \text{if } t > 0, \\ \overline{F(x)} & \text{if } t = 0. \end{cases} \tag{5.35}$$

The next proposition summarizes several simple facts with regard to the relationship between F, $\mathcal{F}$ and $\overline{\mathcal{F}}$.

Proposition 37. *(i)* $\mathcal{F}_0(x) = F(x)$, $\overline{\mathcal{F}}_0(x) = \overline{F(x)}$ *for all* $x \in X$.

(ii) $\delta(y,\mathcal{F},x) = \delta(y,\overline{\mathcal{F}},x) = d(y,F(x))$ *for all* $x \in X$ *and* $y \in Y$.

(iii) $\mathcal{F}_0^{-1}(B(y,t)) = F^{-1}(B(y,t)) = \mathcal{F}_t^{-1}(y)$ *for all* $y \in Y$ *and* $t \geq 0$.

(iv) $\overline{\mathcal{F}}_0^{-1}(\overline{B}(y,t)) = \overline{F}^{-1}(\overline{B}(y,t)) \subset \overline{\mathcal{F}}_t^{-1}(y)$ *for all* $y \in Y$ *and* $t \geq 0$.

(v) If F^{-1} *is closed at* y, *then the mappings* $\tau \mapsto \mathcal{F}_\tau^{-1}(y)$ *and* $\tau \mapsto \overline{\mathcal{F}}_\tau^{-1}(y)$ *on* $\mathbb{R}_+$ *are outer semicontinuous at* 0.

(vi) For any $y \in Y$ *and* $\tau > 0$, $\mathcal{F}$ *and* $\overline{\mathcal{F}}$ *satisfy condition* (5.14).

(vii) If F *is upper semicontinuous on* X, *i.e., for any* $x \in X$ *and* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $F(u) \subset B(F(x),\varepsilon)$ *for all* $u \in B(x,\delta)$, *then* $\overline{\mathcal{F}}^{-1}$ *is closed-valued. In particular, for any* $y \in Y$, *the mapping* $\tau \mapsto \overline{\mathcal{F}}_\tau^{-1}(y)$ *is outer semicontinuous on* $\mathbb{R}_+$.

Proof. (i) The equalities make part of definitions (5.34) and (5.35).

(ii) By (5.2), (5.34) and (5.35),

$$\delta(y, \mathcal{F}, x) = \inf\{t > 0 \mid d(y, F(x)) < t\} = d(y, F(x)),$$
$$\delta(y, \overline{\mathcal{F}}, x) = \inf\{t > 0 \mid d(y, F(x)) \le t\} = d(y, F(x)).$$

(iii) If $t = 0$, then $\mathcal{F}_0^{-1}(y) = F^{-1}(y)$ and both equalities hold true automatically for all $y \in Y$. If $t > 0$, then

$$x \in \mathcal{F}_t^{-1}(y) \;\Leftrightarrow\; d(y, F(x)) < t \;\Leftrightarrow\; F(x) \cap B(y, t) \neq \emptyset \;\Leftrightarrow\; x \in F^{-1}(B(y, t)).$$

Hence, $\mathcal{F}_t^{-1}(y) = F^{-1}(B(y, t))$. The other equality is satisfied because $\mathcal{F}_0^{-1}(v) = F^{-1}(v)$ for all $v \in B(y, t)$.

(iv) If $t = 0$, then $\overline{\mathcal{F}}_0^{-1}(y) = \overline{\mathcal{F}}_0^{-1}(\overline{B}(y, 0)) = \overline{F}^{-1}(y)$ for all $y \in Y$. If $t > 0$, then

$$x \in \overline{F}^{-1}(\overline{B}(y, t)) \;\Leftrightarrow\; \overline{F}(x) \cap \overline{B}(y, t) \neq \emptyset \;\Rightarrow\; d(y, F(x)) \le t \;\Leftrightarrow\; x \in \overline{\mathcal{F}}_t^{-1}(y).$$

Hence, $\overline{F}^{-1}(\overline{B}(y, t)) \subset \overline{\mathcal{F}}_t^{-1}(y)$. The claimed equality is satisfied because $\overline{\mathcal{F}}_0^{-1}(v) = \overline{F}^{-1}(v)$ for all $v \in \overline{B}(y, t)$.

(v) If $x_n \to z$ and $t_n \downarrow 0$ with $d(y, F(x_n)) < t_n$ $(n = 1, 2, \ldots)$, then, for any n, there exists a $y_n \in F(x_n)$ such that $d(y, y_n) < t_n$. Hence, $y_n \to y$ as $n \to \infty$. Since F^{-1} is closed at y, we have $z \in F^{-1}(y)$ and consequently $y \in F(z) = \mathcal{F}(z, 0)$.

Similarly, if $x_n \to z$ and $t_n \downarrow 0$ with $d(y, F(x_n)) \le t_n$ $(n = 1, 2, \ldots)$, then, for any n, there exists a $y_n \in F(x_n)$ such that $d(y, y_n) < 2t_n$. Hence, $y_n \to y$ as $n \to \infty$. Since F^{-1} is closed at y, we have $z \in F^{-1}(y)$ and consequently $y \in F(z) \subset \mathcal{F}(z, 0)$.

(vi) follows from (iii) and (iv).

(vii) If $y \in Y$, $x_n \to z$ and $t_n \to \tau$ with $d(y, F(x_n)) \le t_n$ $(n = 1, 2, \ldots)$, then, since F is upper semicontinuous,

$$d(y, F(z)) \le \liminf_{n\to\infty} d(y, F(x_n)) \le \lim_{n\to\infty} t_n = \tau,$$

that is, $y \in \overline{\mathcal{F}}(z, \tau)$. □

Thanks to parts (i) and (ii) of Proposition 37, the definitions of regularity and openness properties explored in the previous sections in the current setting can be expressed in metric terms. In the next definition, which corresponds to a group of definitions from Section 5.3, $\mu : [0, +\infty] \to [0, +\infty]$ is an upper semicontinuous nondecreasing function playing the role of a *modulus* of the corresponding property.

Definition 21. (i) Given a set $W \subset X \times Y$, mapping F is metrically regular on W with functional modulus μ if

$$d(x, F^{-1}(y)) \le \mu(d(y, F(x))) \quad \text{for all} \quad (x, y) \in W. \tag{5.36}$$

(ii) Given a set $W \subset X \times Y$, mapping F is open on W with functional modulus μ if

$$y \in F(B(x, t)) \quad \text{for all} \quad (x, y) \in W \text{ and } t > \mu(d(y, F(x))).$$

(iii) Given a set $W \subset X \times Y$ and a function $\nu : W \to (0, \infty]$, mapping F is metrically ν-regular on W with functional modulus μ if

$$d(x, F^{-1}(y)) \le \mu(d(y, F(x))) \text{ for all } (x, y) \in W \tag{5.37}$$
$$\text{with } \mu(d(y, F(x))) < \nu(x, y).$$

(iv) Given a set $W \subset X \times Y$ and a function $\nu : W \to (0, \infty]$, mapping F is ν-open on W with functional modulus μ if

$$y \in F(B(x, t)) \quad \text{for all} \quad (x, y) \in W \text{ and } t \in (\mu(d(y, F(x))), \nu(x, y)).$$

(v) F is metrically regular at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with functional modulus μ if there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \le \mu(d(y, F(x))) \quad \text{for all} \quad x \in U,\ y \in V. \tag{5.38}$$

(vi) F is open at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with functional modulus μ if there exist neighbourhoods U

of $\bar{x}$ and V of $\bar{y}$ such that

$$y \in F(B(x,t)) \quad \text{for all} \quad x \in U,\ y \in V \text{ and } t > \mu(d(y, F(x))). \tag{5.39}$$

Remark 54. If μ is strictly increasing, then condition (5.39) can be rewritten equivalently in a more conventional "openness-like" form (cf. [33]):

$$B(F(x), \mu^{-1}(t)) \cap V \subset F(B(x,t)) \quad \text{for all} \quad x \in U \text{ and } t > 0.$$

In the case $W = U \times V$, similar simplifications can be made also in parts (ii) and (iv) of the above definition.

In the linear case (μ is a linear function), the metric regularity and openness/covering properties in the above definition are well known in both local and global settings (cf., e.g., [19, 30, 51, 60]) including regularity on a set [30, 31]. The nonlinear setting in the above definition follows Ioffe [33] where the properties in parts (iii) and (iv), were mostly investigated in the particular case $W = U \times V$ where $U \subset X$ and $V \subset Y$ and the function ν depends only on x.

Observe that condition (5.36) in Definition 21 is equivalent to

$$d(x, F^{-1}(y)) \le \mu(d(y, y')) \ \text{ for all } (x,y) \in W \text{ and } y' \in F(x).$$

In its turn, condition $y' \in F(x)$ is equivalent to $x \in F^{-1}(y')$. This and similar observations regarding conditions (5.37) and (5.38) allow us to rewrite these conditions, respectively, as follows:

$$\begin{aligned}
d(x, F^{-1}(y_2)) \le \mu(d(y_1, y_2)) \ &\text{ for all } y_1, y_2 \in Y,\ x \in F^{-1}(y_1) \text{ with } (x, y_2) \in W,\\
d(x, F^{-1}(y_2)) \le \mu(d(y_1, y_2)) \ &\text{ for all } y_1, y_2 \in Y,\ x \in F^{-1}(y_1)\\
&\text{ with } (x, y_2) \in W,\ \mu(d(y_1, y_2)) < \nu(x, y_2),\\
d(x, F^{-1}(y_2)) \le \mu(d(y_1, y_2)) \ &\text{ for all } y_1 \in Y,\ y_2 \in V,\ x \in F^{-1}(y_1) \cap U.
\end{aligned}$$

Thanks to these observations, one can complement the regularity and openness properties in Definition 21 with the corresponding Hölder-like (Aubin in the linear case) properties.

In the definition below, $\mu : [0, +\infty] \to [0, +\infty]$ is again an upper semicontinuous nondecreasing function.

Definition 22. (i) Given a set $W \subset X \times Y$, mapping F is Hölder on W with functional modulus μ if

$$d(y, F(x_2)) \le \mu(d(x_1, x_2)) \text{ for all } x_1, x_2 \in X,\ y \in F(x_1) \text{ with } (x_2, y) \in W.$$

(ii) Given a set $W \subset X \times Y$ and a function $\nu : W \to (0, \infty]$, mapping F is ν-Hölder on W with functional modulus μ if

$$d(y, F(x_2)) \le \mu(d(x_1, x_2)) \text{ for all } x_1, x_2 \in X,\ y \in F(x_1)$$
$$\text{with } (x_2, y) \in W,\ \mu(d(x_1, x_2)) < \nu(x_2, y).$$

(iii) F is Hölder at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with functional modulus μ if there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(y, F(x_2)) \le \mu(d(x_1, x_2)) \text{ for all } x_1, x_2 \in U,\ y \in F(x_1) \cap V. \tag{5.40}$$

Thanks to Propositions 34, 35, 36 and the discussion before Definition 22, we have the following list of equivalences.

Theorem 28. *Suppose $\mu : [0, +\infty] \to [0, +\infty]$ is an upper semicontinuous increasing function.*

(i) Given a set $W \subset X \times Y$, properties (i) and (ii) in Definition 21 are equivalent to F^{-1} being Hölder on

$$W' := \{(y, x) \in Y \times X \mid (x, y) \in W\} \tag{5.41}$$

with functional modulus μ.

(ii) Given a set $W \subset X \times Y$, properties (iii) and (iv) in Definition 21 are equivalent to F^{-1} being ν'-Hölder on (5.41) with functional modulus μ, where $\nu' : W' \to (0, \infty]$ is defined by the equality $\nu'(y, x) = \nu(x, y)$.

(iii) *Given a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$*, properties (v) and (vi) in Definition 21 are equivalent to* F^{-1} *being Hölder at* $(\bar{y}, \bar{x})$ *with functional modulus* μ*.*

Remark 55. Most of the equivalences in Theorem 28 hold true with function μ nondecreasing. The assumption that μ is strictly increasing is only needed in part (iii). In fact, it follows from the discussion before Definition 22, that properties (v) and (vi) in Definition 21 are equivalent to a stronger version of the Hölder property of F^{-1} which corresponds to replacing condition (5.40) in Definition 22 by the following one:

$$d(y, F(x_2)) \leq \mu(d(x_1, x_2)) \text{ for all } x_1 \in X,\ x_2 \in U,\ y \in F(x_1) \cap V.$$

If μ is strictly increasing, then the two versions are equivalent.

We next formulate several regularity criteria in the conventional setting of a mapping $F : X \rightrightarrows Y$ between metric spaces. All of them are consequences of the corresponding statements in Section 5.3 thanks to the relationships in Proposition 37. From now on, we assume that X is complete.

Theorem 29. *Given a set* $W \subset X \times Y$*, suppose that, for any* $(x, y) \in W$*, the inverse mapping* F^{-1} *is closed at* y *and, for some* $\gamma > d(y, F(x))$ *and any* $t \in (0, \gamma)$*, one of the following sets of conditions is satisfied:*

(i) *there are sequences of positive numbers* (b_n) *and* (c_n) *and a function* $m : (0, \infty) \to (0, \infty)$ *such that conditions (5.6) and (5.32) hold true and*

$$\begin{aligned} d\left(x, F^{-1}(B(y, m(c_1)))\right) <& b_0, \\ d\left(u, F^{-1}(B(y, m(c_{n+1})))\right) <& b_n \\ & \textit{for all } u \in F^{-1}(B(y, m(c_n))) \cap B(x, \sum_{i=0}^{n-1} b_i)\ (n = 1, 2, \ldots), \end{aligned}$$

(ii) *there are functions* $b, m : (0, \infty) \to (0, \infty)$ *such that condition (5.9) is satisfied and, for any* $\tau > 0$ *with* $\mu(\tau) \leq \mu(t)$*, condition (5.10) holds true and*

$$d\left(u, F^{-1}(B(y, b(\tau)))\right) < m(\tau) \textit{ for all } u \in F^{-1}(B(y, \tau)) \cap B(x, \mu(t) - \mu(\tau)),$$

(iii) there are functions $b, m : (0, \infty) \to (0, \infty)$ *such that condition* (5.9) *is satisfied and, for any* $\tau > 0$ *with* $\mu(\tau) \leq \mu(t)$*, condition* (5.10) *holds true and*

$$d\left(y, F(B(u, m(\tau)))\right) < b(\tau) \text{ for all } u \in F^{-1}(B(y, \tau)) \cap B(x, \mu(t) - \mu(\tau)).$$

Then, F *is metrically regular on* W *with functional modulus* μ*.*

Theorem 30. *Let* μ *be continuous,* $\mu(\tau) = 0$ *if and only if* $\tau = 0$*. Given a set* $W \subset X \times Y$*, suppose that* F *is upper semicontinuous and, for any* $(x, y) \in W$ *and* $u \in X$ *such that* $d(y, F(u)) > 0$ *and* $\mu(d(y, F(u))) + d(u, x) \leq \mu(d(y, F(x)))$*, there exists a point* $u' \neq u$ *such that*

$$\mu(d(y, F(u'))) \leq \mu(d(y, F(u))) - d(u, u').$$

Then, F *is metrically regular on* W *with functional modulus* μ*.*

Proof. By Theorem 23 and Proposition 37(i), (ii) and (vii), set-valued mapping $\overline{\mathcal{F}}$ is regular on W with functional modulus μ. Since F is upper semicontinuous, it is closed-valued and consequently making use of Proposition 37(i) again, we have for any $y \in Y$ that $\overline{\mathcal{F}}_0^{-1}(y) = \overline{F}^{-1}(y) = F^{-1}(y)$. Hence, the regularity of $\overline{\mathcal{F}}$ is equivalent to the metric regularity of F. □

5.5 Optimality conditions

In this section, we apply our general nonlinear regularity model to establish second-order necessary optimality conditions for a nonsmooth set-valued optimization problem with mixed constraints.

Let X, Y, Z and W be Banach spaces; S a nonempty subset of X; C a proper convex ordering cone in Y expressing the objective preference in the set-valued optimization problem below ("proper" means $C \neq \emptyset$ and $C \neq Y$); D a convex cone with nonempty interior in Z; $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$, and $H : X \rightrightarrows W$ set-valued mappings. We consider the problem

$$\text{Minimize}_C \; F(x) \quad \text{subject to} \quad x \in \Omega,$$

where

$$\Omega := \{x \in X \mid x \in S,\ G(x) \cap (-D) \neq \emptyset,\ 0 \in H(x)\}.$$

A triple $(\bar{x}, \bar{y}, \bar{z})$ is said to be *feasible* if $\bar{x} \in \Omega$, $\bar{y} \in F(x)$ and $\bar{z} \in G(x) \cap (-D)$. Alongside the ordering cone C we consider another proper open cone $Q \subset Y$. A point $(\bar{x}, \bar{y}) \in X \times Y$ is called a local Q-*solution* if

$$F(U \cap \Omega) \cap (\bar{y} - Q) = \emptyset \tag{5.42}$$

for some neighbourhood U of $\bar{x}$.

The above problem subsumes various vector- and set-valued optimization problems while the concept of Q-solution, under a suitable choice of Q, subsumes various kinds of solutions; cf. [42]. For instance, if $Q = \operatorname{int} C \neq \emptyset$, then Q-solution coincides with the conventional (local) *weak* solution. If Q is an open cone such that $C \setminus \{0\} \subset Q$, then Q-solution becomes *Henig proper* solution. Similarly, setting $Q = Y \setminus (-\overline{\text{cone}}(F(U \cap \Omega) - \bar{y} + C))$ where U is a neighbourhood of $\bar{x}$, we come to the concept of *Benson proper* solution.

It is worth noting the two specific features of the second-order necessary condition we present below: the regularity condition plays an important role and the right-hand side of the multiplier rule (5.47) is not the number 0 as in the classical result (and also in many its developments until now) and it may be strictly negative in particular cases. This phenomenon, known as the *envelope-like effect* revealed by Kawasaki [36], may happen because of the presence of the closure sign in the definition of the set of critical directions (5.45). For typical contributions to optimality conditions with these two features, we refer the reader to the references [4, 8, 14, 26, 35, 36, 40, 41, 54]. Theorem 31 below is a further development of many results in these references.

We first recall several useful definitions.

(i) The (positive) *dual cone* to a cone K in X:

$$K^* := \{x^* \in X^* \mid \langle x^*, x \rangle \geq 0,\ \forall x \in K\}.$$

(ii) The *contingent*, *interior tangent* and *normal* cones to a nonempty subset $M \subset X$ at

$\bar{x} \in \overline{M}$:

$$\begin{aligned} T(M,\bar{x}) :=& \{u \in X \mid \exists \gamma_n \downarrow 0,\ u_n \to u \text{ such that } \bar{x} + \gamma_n u_n \in M,\ \forall n\}, \\ IT(M,\bar{x}) :=& \{u \in X \mid \forall \gamma_n \downarrow 0,\ u_n \to u,\ \text{it holds } \bar{x} + \gamma_n u_n \in M,\ \forall \text{ large } n\}, \\ N(M,\bar{x}) :=& -[T(M,\bar{x})]^*. \end{aligned}$$

(iii) The *second-order contingent, adjacent* and *interior* sets to a nonempty subset $M \subset X$ at $\bar{x} \in \overline{M}$ in a direction $u \in X$:

$$\begin{aligned} T^2(M,\bar{x},u) :=& \{x \in X \mid \exists \gamma_n \downarrow 0,\ x_n \to x,\ \text{s.t. } \bar{x} + \gamma_n u + \frac{1}{2}\gamma_n^2 x_n \in M,\ \forall n\}, \\ A^2(M,\bar{x},u) :=& \{x \in X \mid \forall \gamma_n \downarrow 0,\ \exists x_n \to x,\ \text{s.t. } \bar{x} + \gamma_n u + \frac{1}{2}\gamma_n^2 x_n \in M, \forall n\}, \\ IT^2(M,\bar{x},u) :=& \{x \in X \mid \forall \gamma_n \downarrow 0,\ x_n \to x,\ \text{it holds } \bar{x} + \gamma_n u + \frac{1}{2}\gamma_n^2 x_n \in M, \\ & \forall \text{ large } n\}. \end{aligned}$$

(iv) The *outer limit* and *inner limit* of a set-valued mapping $E : X \rightrightarrows Y$ at $\bar{x} \in X$:

$$\begin{aligned} \operatorname{Lim\,sup}_{x \to \bar{x}} E(x) :=& \{y \in Y \mid \liminf_{x \to \bar{x}} d(y, E(x)) = 0\}, \\ \operatorname{Lim\,inf}_{x \to \bar{x}} E(x) :=& \{y \in Y \mid \lim_{x \to \bar{x}} d(y, E(x)) = 0\}. \end{aligned}$$

(v) The *contingent* and *lower* derivatives of a set-valued mapping $E : X \rightrightarrows Y$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} E$:

$$\begin{aligned} DE(\bar{x},\bar{y})(x) :=& \operatorname{Lim\,sup}_{\gamma \downarrow 0,\, x' \to x} \gamma^{-1}[E(\bar{x} + \gamma x') - \bar{y}], \\ D_l E(\bar{x},\bar{y})(x) :=& \operatorname{Lim\,inf}_{\gamma \downarrow 0,\, x' \to x} \gamma^{-1}[E(\bar{x} + \gamma x') - \bar{y}],\ x \in X. \end{aligned}$$

(vi) The *second-order contingent* and *lower* derivatives of a set-valued mapping $E : X \rightrightarrows Y$

at $(\bar{x},\bar{y}) \in \operatorname{gph} E$ in a direction $(u,v) \in X \times Y$:

$$D^2E(\bar{x},\bar{y},u,v)(x) := \operatorname{Lim\,sup}_{\gamma\downarrow 0,\, x'\to x} 2\gamma^{-2}[E(\bar{x}+\gamma u+\frac{1}{2}\gamma^2 x') - \bar{y} - \gamma v],$$
$$D_l^2E(\bar{x},\bar{y},u,v)(x) := \operatorname{Lim\,inf}_{\gamma\downarrow 0,\, x'\to x} 2\gamma^{-2}[E(\bar{x}+\gamma u+\frac{1}{2}\gamma^2 x') - \bar{y} - \gamma v],\ x \in X.$$

Note that, if M is a convex set with $\operatorname{int} M \neq \emptyset$ and $u \in T(M,\bar{x}),u)$, then

$$T(M,\bar{x}) = \overline{IT(M,\bar{x})}, \quad A^2(M,\bar{x},u) = \overline{IT^2(M,\bar{x},u)}, \tag{5.43}$$
$$A^2(M,\bar{x},u) + T(T(M,\bar{x}),u) \subset A^2(M,\bar{x},u). \tag{5.44}$$

If K is a convex cone and $\bar{x} \in \overline{K}$, then

$$N(K,\bar{x}) = \{x^* \in -K^* \mid \langle x^*, \bar{x}\rangle = 0\}.$$

Now we return to our optimization problem. Assume that Q is an open convex cone and denote $F_+(x) := F(x) + \overline{Q}$ and $G_+(x) := G(x) + D$. For a feasible triple $(\bar{x},\bar{y},\bar{z})$, we introduce the set of *critical directions*:

$$\mathcal{C}(\bar{x},\bar{y},\bar{z}) := \{(u,v,k) \in X \times Y \times Z \mid v \in D_lF_+(\bar{x},\bar{y})(u) \cap (-\operatorname{bd} Q),$$
$$k \in D_lG_+(\bar{x},\bar{z})(u) \cap (-\overline{\operatorname{cone}}(D+\bar{z})),\ 0 \in DH(\bar{x},0)(u)\}. \tag{5.45}$$

Given a triple $(u,v,k) \in \mathcal{C}(\bar{x},\bar{y},\bar{z})$ and a point $x \in X$, we denote

$$\Delta_{(u,v,k)}(x) := \left(D_l^2F_+(\bar{x},\bar{y},u,v), D_l^2G_+(\bar{x},\bar{z},u,k), D^2H(\bar{x},0,u,0)\right)(x).$$

In what follows, we will consider an extension of the mapping H: a set-valued mapping $\mathcal{H} : X \times \mathbb{R}_+ \rightrightarrows W$ with the properties $\mathcal{H}_0(\cdot) := \mathcal{H}(\cdot,0) = H(\cdot)$ and (cf. definition (5.2))

$$\delta(0,\mathcal{H},x) := \inf\{t > 0 \mid 0 \in \mathcal{H}(x,t)\} \leq \theta d(0,H(x))$$

for some $\theta > 0$ and all x in a neighbourhood of $\bar{x}$. We will need to assume a kind of regular behaviour of this extension.

Definition 23. $\mathcal{H}$ is regular at $(\bar{x}, \bar{w})$ with functional modulus μ with respect to S if there exist neighbourhoods U of $\bar{x}$ and V of $\bar{w}$ such that

$$d(x, \mathcal{H}_0^{-1}(w) \cap S) \leq \mu(\delta(w, \mathcal{H}, x)) \quad \text{for all} \quad x \in U \cap S,\ w \in V. \tag{5.46}$$

Observe that this is exactly the regularity in the sense of Definition 20(i) for the restriction of the mapping $\mathcal{H}$ on $S \times \mathbb{R}_+$. Recall that $\mu : [0, +\infty] \to [0, +\infty]$ is assumed upper semicontinuous and nondecreasing. In what follows, we will assume additionally that $\limsup_{t \downarrow 0} \mu(t)/t < \infty$.

Theorem 31. *Let $(\bar{x}, \bar{y})$ be a local Q-solution, $\bar{z} \in G(\bar{x}) \cap (-D)$, and $\mathcal{H}$ be regular at $(\bar{x}, 0)$ with respect to S. Suppose that $(u, v, k) \in \mathcal{C}(\bar{x}, \bar{y}, \bar{z})$ and $\Delta_{(u,v,k)}(IT^2(S, \bar{x}, u))$ is a convex set with nonempty interior. Then, there exist multipliers $(v^*, k^*, w^*) \in Q^* \times N(-D, \bar{z}) \times W^* \setminus \{(0, 0, 0)\}$ such that $\langle v^*, v\rangle = \langle k^*, k\rangle = 0$ and*

$$\begin{aligned} v^* \circ D_l^2 F_+(\bar{x}, \bar{y}, u, v)(x) + k^* \circ D_l^2 G_+(\bar{x}, \bar{z}, u, k)(x) + w^* \circ D^2 H(\bar{x}, 0, u, 0)(x) \\ \geq \sup_{d \in A^2(-D, \bar{z}, k)} \langle k^*, d\rangle \quad \text{for all} \quad x \in IT^2(S, \bar{x}, u). \end{aligned} \tag{5.47}$$

Moreover, $v^* \neq 0$ if the following second-order constraint qualification holds:

$$\begin{aligned} \operatorname{cone}\big((D_l^2 G_+(\bar{x}, \bar{z}, u, k) - A^2(-D, \bar{z}, k), D^2 H(\bar{x}, 0, u, 0))(IT^2(S, \bar{x}, u))\big) \\ + \operatorname{cone}(D + \bar{z}) \times \{0\} = Z \times W. \end{aligned} \tag{5.48}$$

Proof. We split the proof into several claims.

Claim 1. $(\bar{x}, \bar{y})$ *satisfies the primal necessary condition:*

$$D_l^2 F_+(\bar{x}, \bar{y}, u, v)(T^2(\Omega, \bar{x}, u)) \cap (-\operatorname{cone}(Q + v)) = \emptyset.$$

Indeed, by the definition of Q-solution, (5.42) holds true for some neighbourhood U of $\bar{x}$. Let $x \in T^2(\Omega, \bar{x}, u)$ and $y \in D_l^2 F_+(\bar{x}, \bar{y}, u, v)(x)$. Then, there are $\gamma_n \downarrow 0$, $x_n \to x$, and $y_n \to y$ such that $\bar{x} + \gamma_n u + \frac{1}{2}\gamma_n^2 x_n \in U \cap \Omega$ for all $n \in \mathbb{N}$ and $\bar{y} + \gamma_n v + \frac{1}{2}\gamma_n^2 y_n \in F(\bar{x} + \gamma_n u + \frac{1}{2}\gamma_n^2 x_n) + \overline{Q}$ for all sufficiently large n. Thanks to (5.42), we have $\gamma_n v + \frac{1}{2}\gamma_n^2 y_n \notin -Q$, and consequently,

$y \notin -\operatorname{cone}(Q+v)$.

Claim 2. *The following lower estimate for $T^2(\Omega, \bar{x}, u)$ holds true:*

$$\{x \in IT^2(S, \bar{x}, u) \mid D^2_l G_+(\bar{x}, \bar{z}, u, k)(x) \cap IT^2(-D, \bar{z}, k) \neq \emptyset,$$
$$0 \in D^2 H(\bar{x}, 0, u, 0)(x)\} \subset T^2(\Omega, \bar{x}, u).$$

Suppose $x \in IT^2(S, \bar{x}, u)$, $z \in D^2_l G_+(\bar{x}, \bar{z}, u, k)(x) \cap IT^2(-D, \bar{z}, k)$ and $0 \in D^2 H(\bar{x}, 0, u, 0)(x)$. As $0 \in D^2 H(\bar{x}, 0, u, 0)(x)$, there are $\gamma_n \downarrow 0$, $x_n \to x$, and $w_n \to 0$ such that $\frac{1}{2}\gamma_n^2 w_n \in H(\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x_n)$ for all $n \in \mathbb{N}$. As $x \in IT^2(S, \bar{x}, u)$, it holds $\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x_n \in S$ for sufficiently large n. As $\mathcal{H}$ is regular at $(\bar{x}, 0)$ with respect to S, for large n, we have:

$$d(\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x_n, \mathcal{H}_0^{-1}(0) \cap S) \leq \mu(\delta(0, \mathcal{H}, \bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x_n))$$
$$\leq \mu(\theta d(0, H(\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x_n))) \leq \mu\left(\frac{\theta}{2}\gamma_n^2 \|w_n\|\right).$$

There exists a point $\hat{x}_n \in \hat{H}_0^{-1}(0) \cap S$ such that

$$\|\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x_n - \hat{x}_n\| \leq \mu\left(\frac{\theta}{2}\gamma_n^2 \|w_n\|\right) + \gamma_n^3.$$

By setting $x'_n := (\frac{1}{2}\gamma_n^2)^{-1}(\hat{x}_n - \bar{x} - \gamma_n u)$, one has $\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x'_n \in \mathcal{H}_0^{-1}(0) \cap S$ and

$$\|x_n - x'_n\| \leq \frac{\mu\left(\frac{\theta}{2}\gamma_n^2 \|w_n\|\right)}{\frac{1}{2}\gamma_n^2} + 2\gamma_n \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $x'_n \to x$ as $n \to \infty$. As $z \in D^2_l G_+(\bar{x}, \bar{z}, u, k)(x)$, there exists $z_n \to z$ such that $\bar{z}+\gamma_n k+\frac{1}{2}\gamma_n^2 z_n \in G(\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x'_n) + D$ for large n. Moreover, as $z \in IT^2(-D, \bar{z}, k)$, it holds $\bar{z}+\gamma_n k+\frac{1}{2}\gamma_n^2 z_n \in -D$ for large n. Hence, $(G(\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x'_n)+D) \cap (-D) \neq \emptyset$ and, as D is a convex cone, $G(\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x'_n) \cap (-D) \neq \emptyset$ for large n. Thus, $\bar{x}+\gamma_n u+\frac{1}{2}\gamma_n^2 x'_n \in \Omega$ for large n, i.e., $x \in T^2(\Omega, \bar{x}, u)$.

Claim 3. $\Delta_{(u,v,k)}(IT^2(S, \bar{x}, u)) \bigcap \left((-\operatorname{cone}(Q+v)) \times IT^2(-D, \bar{z}, k) \times \{0\}\right) = \emptyset$.

Suppose to the contrary the existence of $x \in IT^2(S, \bar{x}, u)$, $y \in -\operatorname{cone}(Q+v)$ and $z \in IT^2(-D, \bar{z}, k)$ such that $(y, z, 0) \in \Delta_{(u,v,k)}(x)$. Then, by Claim 2, $x \in T^2(\Omega, \bar{x}, u)$. We arrive at a contradiction with Claim 1.

<u>Claim 4</u>. *There exist multipliers* $(v^*, k^*, w^*) \in Q^* \times N(-D, \bar{z}) \times W^* \setminus \{(0,0,0)\}$ *such that* $\langle v^*, v\rangle = \langle k^*, k\rangle = 0$ *and* (5.47) *holds true.*

If $IT^2(-D, \bar{z}, k) = \emptyset$, then $A^2(-D, \bar{z}, k) = \emptyset$ and (5.47) holds true trivially. Let $IT^2(-D, \bar{z}, k) \neq \emptyset$. The standard separation theorem applied to the two convex sets in Claim 3 yields the existence of multipliers $(v^*, k^*, w^*) \in Y^* \times Z^* \times W^* \setminus \{(0,0,0)\}$ such that

$$\langle v^*, y\rangle + \langle k^*, z\rangle + \langle w^*, w\rangle \geq \langle v^*, q\rangle + \langle k^*, d\rangle \tag{5.49}$$

for all $x \in IT^2(S, \bar{x}, u)$, $(y, z, w) \in \Delta_{(u,v,k)}(x)$, $q \in -\operatorname{cone}(Q + v)$, and all $d \in IT^2(-D, \bar{z}, k)$. For any fixed admissible x, y, z, w and d and any $q \in \operatorname{cone}(Q + v)$ and $t > 0$, one has $-tq \in -\operatorname{cone}(Q + v)$, and consequently,

$$\langle v^*, q\rangle \geq \lim_{t\to\infty} \frac{\langle v^*, y\rangle + \langle k^*, z\rangle + \langle w^*, w\rangle - \langle k^*, d\rangle}{t} = 0.$$

Hence,

$$\langle v^*, q\rangle \geq 0 \quad \text{for all} \quad q \in \operatorname{cone}(Q + v), \tag{5.50}$$

and consequently, taking into account the second property in (5.43), inequality (5.49) implies (5.47).

Since Q is a cone, by the same argument, it follows from (5.50) that $v^* \in Q^*$. As $v \in -\operatorname{bd} Q$, we also have $\langle v^*, v\rangle = 0$. Using (5.50) and property (5.44) of the adjacent set, we obtain from (5.49) that

$$\langle v^*, y\rangle + \langle k^*, z\rangle + \langle w^*, w\rangle \geq \langle k^*, d\rangle + \langle k^*, d'\rangle$$

for all $x \in IT^2(S, \bar{x}, u)$, $(y, z, w) \in \Delta_{(u,v,k)}(x)$, $d \in A^2(-D, \bar{z}, k)$, and all $d' \in T(T(-D, \bar{z}), k)$. Using the fact that $T(T(-D, \bar{z}), k)$ is a cone, we conclude as before that $k^* \in -(T(T(-D, \bar{z}), k))^*$, and consequently, $k^* \in N(-D, \bar{z})$. As $k \in T(-D, \bar{z})$, we also have $\langle k^*, k\rangle = 0$.

Claim 5. *Under the constraint qualification* (5.48), v^* *in* (5.47) *is nonzero.*

Suppose that $v^* = 0$. Then, $(k^*, w^*) \neq (0, 0)$ and (5.47) gives

$$\langle k^*, z\rangle + \langle w^*, w\rangle \geq \langle k^*, d\rangle \tag{5.51}$$

for all $x \in IT^2(S, \bar{x}, u)$, $z \in D_l^2 G_+(\bar{x}, \bar{z}, u, k)(x)$, $w \in D^2 H(\bar{x}, 0, u, 0)(x)$ and $d \in A^2(-D, \bar{z}, k)$. Take arbitrarily $(z', w') \in Z \times W$. By virtue of (5.48), there are $x \in IT^2(S, \bar{x}, u)$, $z \in D_l^2 G_+(\bar{x}, \bar{z}, u, k)(x)$, $w \in D^2 H(\bar{x}, 0, u, 0)(x)$, $d \in A^2(-D, \bar{z}, k)$, $d' \in D$ and $\gamma_1, \gamma_2 > 0$ such that $(z', w') = \gamma_1(z - d, w) + (\gamma_2(d' + \bar{z}), 0)$. Since $k^* \in N(-D, \bar{z})$, one has $\langle k^*, d'\rangle \geq 0$ and $\langle k^*, \bar{z}\rangle = 0$. Hence, using (5.51),

$$\begin{aligned}\langle k^*, z'\rangle + \langle w^*, w'\rangle &= \gamma_1\langle k^*, z - d\rangle + \gamma_2\langle k^*, d' + \bar{z}\rangle + \gamma_1\langle w^*, w\rangle \\ &= \gamma_1(\langle k^*, z\rangle + \langle w^*, w\rangle - \langle k^*, d\rangle) + \gamma_2\langle k^*, d' + \bar{z}\rangle \\ &\geq \gamma_2\langle k^*, d' + \bar{z}\rangle \geq 0.\end{aligned}$$

As $(z', w') \in Z \times W$ is arbitrary, we have $(k^*, w^*) = (0, 0)$, a contradiction. □

Remark 56. 1. The requirements on the extension mapping $\mathcal{H}$ formulated before Definition 23 are satisfied, e.g., if

$$e(\mathcal{H}(x, t), H(x)) := \sup_{h \in \mathcal{H}(x,t)} d(h, H(x)) \leq \alpha t^k$$

for some $\alpha > 0$, $k \geq 1$ and all (x, t) in a neighbourhood of $(\bar{x}, 0)$.

2. The lower estimate for $T^2(\Omega, \bar{x}, u)$ in Claim 2 and its proof presented above are valid for any feasible triple $(\bar{x}, \bar{y}, \bar{z})$ and any $u \in X$ with $0 \in DH(\bar{x}, 0)(u)$ and $k \in D_l G_+(\bar{x}, \bar{z})(u)$. This estimate can be of importance beyond Theorem 31.

3. In the proof of Theorem 31 (see Claim 2), one can employ weaker regularity properties of the extension mapping $\mathcal{H}$ than the one given in Definition 23. Firstly, it is sufficient to require the inequality in (5.46) to hold only at the fixed point $w = 0$. This important property known as *metric subregularity* can be treated in the abstract setting of the current chapter and is going to make the topic of subsequent research. Moreover, only points of the form $\bar{x} + \gamma_n u + \frac{1}{2}\gamma_n^2 x_n$ are involved in the proof. Hence, a development of our regularity model

corresponding to *directional metric subregularity* is on the agenda. Such an extension is going to properly improve [42, Theorem 3.1].

4. Following [42], one can improve Theorem 31 by relaxing the restrictive assumption of nonemptyness of the interior of the set $\Delta_{(u,v,k)}(IT^2(S,\bar{x},u))$.

5. It is possible to develop multiplier rules similar to the one in Theorem 31 in terms of other types of generalized derivatives, for instance asymptotic derivatives, instead of the contingent-type ones. Such rules may be useful when the contingent-type derivatives do not exist in a particular problem under consideration.

5.6 Concluding remarks

This chapter considers a general regularity model for a set-valued mapping $F : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces. We demonstrate that the classical approach going back to Banach, Schauder, Lyusternik and Graves and based on iteration procedures still possesses potential. In particular, we show that the *Induction theorem* [37, Theorem 1], which was used as the main tool when proving the other results in [37], implies also all the main results in the subsequent articles [38, 39] and can serve as a substitution of the Ekeland variational principle when establishing other regularity criteria. Furthermore, the latter classical result can also be established as a consequence of the Induction theorem.

This research prompts a list of questions and problems which should be taken care of.

1) "On a set" nonlinear regularity, considered in Section 5.3 and interpreted there as a direct analogue of metric regularity in the conventional setting, is in fact a general model which covers also relaxed versions of regularity like sub- and semi-regularity.

2) The particular case of "power nonlinearities", i.e., the case when functional modulus μ is of the type $\mu(t) = \lambda t^k$ with $0 < k \leq 1$, should be treated explicitly.

3) Theorem 17 illustrates the usage of the Induction theorem as a substitution for the Ekeland variational principle when establishing regularity criteria like Theorem 30. In the last theorem which is an (indirect) consequence of Theorem 17, the mapping is assumed upper semicontinuous. This assumption can be relaxed with the help of a slightly more advanced version of Theorem 17.

4) The regularity model studied in this chapter is illustrated in Section 5.5 by an applica-

tion to second-order necessary optimality conditions for a nonsmooth set-valued optimization problem with mixed constraints. Other classes of optimization problems can be handled along the same lines using also other types of generalized derivatives. The relaxed versions of regularity mentioned in item 1 above are going to be useful in this context.

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Chapter 6

Metric subregularity - a view from the induction theorem

Iteration procedures, which go back to Banach, Schauder, Lyusternik and Graves, are used for studying metric subregularity properties of set-valued mappings in the general nonlinear setting.

6.1 Introduction

As shown in [3, 9, 10, 12, 14, 15, 16, 17] the following *induction theorem* (and its other versions, e.g., [14, Theorem 1]) containing a typical Cauchy sequence argument can serve as a substitution of the Ekeland variational principle when establishing regularity criteria for set-valued mappings. In fact, the two results are in a sense equivalent to the completeness of X.

Lemma 19. *[17, Lemma 2.1] Let X be a complete metric space, $\Phi : \mathbb{R}_+ \rightrightarrows X$, $t > 0$, and $x \in \Phi(t)$. Suppose that Φ is outer semicontinuous [27] at 0:*

$$\limsup_{\tau \to 0} \Phi(\tau) := \left\{ z \in X \mid \liminf_{\tau \to 0} d(z, \Phi(\tau)) = 0 \right\} \subset \Phi(0)$$

and there are sequences of positive numbers (a_n) *and* (b_n) *such that*

$$\sum_{n=0}^{\infty} b_n < \infty,$$

$$a_0 = t \quad \text{and} \quad a_n \downarrow 0 \text{ as } n \to \infty,$$

$$d(u, \Phi(a_{n+1})) < b_n \quad \text{for all} \quad u \in \Phi(a_n) \cap U_n \quad (n = 0, 1, \ldots),$$

where $U_0 := \{x\}$, $U_n := B\left(x, \sum_{i=0}^{n-1} b_i\right)$ $(n = 1, 2, \ldots)$. *Then,* $d(x, \Phi(0)) < \sum_{n=0}^{\infty} b_n$.

In [17], the above lemma was used as a key tool for establishing global and local regularity criteria for a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces. These regularity criteria were naturally translated into those for metric regularity/openness in the conventional setting of a set-valued mapping $F : X \rightrightarrows Y$.

In this chapter, we will demonstrate that the general regularity theory for a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$ developed in [17] can without changes be translated into the conventional setting to obtain criteria for metric subregularity property of a set-valued mapping $F : X \rightrightarrows Y$. This relaxed version of the metric regularity property is also an important property. Its outstanding role in optimization and variational analysis in relation to calmness properties, error bounds, weak sharp minima, slopes, and subdifferentials has been verified through a vast number of publications, e.g., [1, 2, 3, 4, 5, 6, 11, 12, 19, 20, 21, 24, 28]. For the interest of enriching the regularity theory for a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$, the sub-version of the regularity property in this setting will also be discussed, and as expected, it is a direct counterpart of the corresponding relaxed version of the metric regularity property in the conventional setting.

Following the lines of [3, 9, 10, 12, 14, 15, 16, 17, 25], most of the results in this chapter will be formulated for the most general model which involves certain gauge functions. Then one can straightforwardly derive those for the linear and Hölder-type models by considering the gauge function of the corresponding type. Due to the very importance of the linear and Hölder-type regularity models in applications, especially in convergence analysis of computational methods, e.g., [7, 8, 22, 23, 29, 30], we will also explicitly formulate criteria for metric subregularity property of linear and Hölder-type models.

Our basic notation is standard; cf. [5, 24, 27]. X and Y are metric spaces. Metrics in all spaces are denoted by the same symbol $d(\cdot,\cdot)$. If x and C are a point and a subset of a metric space, then $d(x,C) := \inf_{c\in C} d(x,c)$ is the point-to-set distance from x to C, while $\overline{C}$ denotes the closure of C. $B(x,r)$ and $\overline{B}(x,r)$ stand for the open and closed balls of radius $r > 0$ centered at x, respectively. We use the convention that $B(x,0) = \{x\}$.

6.2 Subregularity for $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$

In this section, we will continue to develop the regularity theory proposed in [17] for a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces. A relaxed version of the regularity property for a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$ will be discussed and their criteria will be provided.

Since $X \times \mathbb{R}_+$ is a metric space with the product metric, the set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$ is a special set-valued mapping between metric spaces. On the other hand, every set-valued mapping $F : X \rightrightarrows Y$ between metric spaces can naturally be extended to, for example, the mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$ by

$$\mathcal{F}(x,t) := \begin{cases} F(x) & \text{if} \quad t = 0 \\ \emptyset & \text{if} \quad t > 0. \end{cases}$$

From now on in this section, we consider a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$, where X and Y are metric spaces, X is complete. Given a $t \in \mathbb{R}_+$, we denote $\mathcal{F}_t := \mathcal{F}(\cdot,t) : X \rightrightarrows Y$. We define, for $(x,y) \in X \times Y$,

$$\delta(y,\mathcal{F},x) := \inf\{t > 0 \mid y \in \mathcal{F}(x,t)\}$$

with the usual convention that $\inf \emptyset = +\infty$.

Throughout the chapter, if not specifically stated, $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semicontinuous nondecreasing function.

6.2.1 Basic estimates

This subsection consists of preliminary results deduced from Lemma 19 which are the basis for establishing the main results in this chapter.

Theorem 32. *[17, Theorems 2.15, 2.16, 2.18] Given a point $(x, y) \in X \times Y$, suppose that the mapping $\tau \mapsto \mathcal{F}_\tau^{-1}(y)$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and, for some $\gamma > \delta(y, \mathcal{F}, x)$ and any $t \in (0, \gamma)$ with $(x, t, y) \in \operatorname{gph} \mathcal{F}$, one of the following sets of conditions is satisfied:*

(i) there are sequences of positive numbers (b_n) and (c_n) and a function $m : (0, \infty) \to (0, \infty)$ such that the following conditions hold true:

$$m(\tau) \downarrow 0 \text{ as } \tau \downarrow 0 \quad \text{and} \quad c_n \downarrow 0 \text{ as } n \to \infty, \tag{6.1}$$

$$d\left(x, \mathcal{F}_{m(c_1)}^{-1}(y)\right) < b_0, \tag{6.2}$$

$$d\left(u, \mathcal{F}_{m(c_{n+1})}^{-1}(y)\right) < b_n \text{ for all } u \in \mathcal{F}_{m(c_n)}^{-1}(y) \cap B\left(x, \sum_{i=0}^{n-1} b_i\right) \ (n = 1, 2, \ldots),$$

$$\sum_{n=0}^{\infty} b_n \le \mu(t). \tag{6.3}$$

(ii) there are functions $b, m : (0, \infty) \to (0, \infty)$ such that

$$m(\tau) \downarrow 0 \quad \Rightarrow \quad \tau \downarrow 0 \tag{6.4}$$

and, for each $\tau > 0$ with $\mu(\tau) \le \mu(t)$,

$$\mu(\tau) \ge m(\tau) + \mu(b(\tau)), \tag{6.5}$$

$$d\left(u, \mathcal{F}_{b(\tau)}^{-1}(y)\right) < m(\tau) \text{ for all } u \in \mathcal{F}_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau)).$$

(iii) there are functions $b, m : (0, \infty) \to (0, \infty)$ such that condition (6.4) is satisfied and, for each $\tau > 0$ with $\mu(\tau) \le \mu(t)$, conditions (6.5),

$$\mathcal{F}_0^{-1}(B(y, \tau)) \subset \mathcal{F}_\tau^{-1}(y),$$

$$d\left(y, \mathcal{F}_0(B(u, m(\tau)))\right) < b(\tau) \text{ for all } u \in \mathcal{F}_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau))$$

hold true.

Then, $d\left(x, \mathcal{F}_0^{-1}(y)\right) \le \mu(\delta(y, \mathcal{F}, x))$.

Theorem 33. *[17, Theorem 2.19] Given a point $(x, y) \in X \times Y$ and a continuous non-decreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mu(\tau) = 0$ if and only if $\tau = 0$, suppose that the mapping $\tau \mapsto \mathcal{F}_\tau^{-1}(y)$ is outer semicontinuous on $[0, \delta(y, \mathcal{F}, x)]$ and, for each pair $(u, \tau) \in \mathcal{F}^{-1}(y)$ with $\tau \in (0, \delta(y, \mathcal{F}, x)]$ and $d(x, u) \leq \mu(\delta(y, \mathcal{F}, x)) - \mu(\delta(y, \mathcal{F}, u))$, there exists a pair $(u', \tau') \in \mathcal{F}^{-1}(y)$ such that $u' \neq u$ and condition*

$$\mu(\tau') \leq \mu(\tau) - d(u', u) \tag{6.6}$$

is satisfied. Then, $d\left(x, \mathcal{F}_0^{-1}(y)\right) \leq \mu(\delta(y, \mathcal{F}, x))$.

The conclusion of Theorems 32 and 33 can be reformulated equivalently in a "covering-like" form thanks to the next Proposition.

Proposition 38. *[17, Proposition 2.22] Consider the following conditions:*

(i) $d\left(x, \mathcal{F}_0^{-1}(y)\right) \leq \mu(\delta(y, \mathcal{F}, x))$;

(ii) $y \in \mathcal{F}(B(x, t), 0)$ *for any* $t > \mu(\delta(y, \mathcal{F}, x))$;

(iii) $y \in \mathcal{F}_0(B(x, \mu(\delta(y, \mathcal{F}, x))))$.

Then, (iii) ⇒ (i) ⇔ (ii).

6.2.2 Definitions and equivalences

Definition 24. (i) $\mathcal{F}$ is subregular on a subset $U \subset X$ at a point $\bar{y} \in Y$ with functional modulus μ if one of the following equivalent conditions holds true:

$$d(x, \mathcal{F}_0^{-1}(\bar{y})) \leq \mu(\delta(\bar{y}, \mathcal{F}, x)) \quad \text{for all} \quad x \in U,$$

$$\bar{y} \in \mathcal{F}(B(x, t), 0) \quad \text{for all} \quad x \in U \text{ and } t > \mu(\delta(\bar{y}, \mathcal{F}, x)).$$

(ii) Given a subset $U \subset X$ and a function $\nu : U \to (0, \infty]$, $\mathcal{F}$ is ν-subregular on U at a point $\bar{y} \in Y$ with functional modulus μ if one of the following equivalent conditions holds true:

$$
\begin{aligned}
d(x, \mathcal{F}_0^{-1}(\bar{y})) \leq \mu(\delta(\bar{y}, \mathcal{F}, x)) \quad & \text{for all} \quad x \in U \\
& \text{with } \mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x), \\
\bar{y} \in \mathcal{F}(B(x,t), 0) \quad & \text{for all} \quad x \in U \\
& \text{and } t \in (\mu(\delta(\bar{y}, \mathcal{F}, x)), \nu(x)).
\end{aligned}
$$

(iii) $\mathcal{F}$ is subregular at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}_0$ with functional modulus μ if there exists a neighborhood U of $\bar{x}$ such that $\mathcal{F}$ is subregular on U at $\bar{y}$ with functional modulus μ.

(iv) Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}_0$ and a function $\nu : X \to \mathbb{R}$ strictly positive around $\bar{x}$, $\mathcal{F}$ is ν-subregular at $(\bar{x}, \bar{y})$ with functional modulus μ if there exists a neighborhood U of $\bar{x}$ such that $\mathcal{F}$ is ν-subregular on U at $\bar{y}$ with functional modulus μ.

Remark 57. Local properties (iii) and (iv) in Definition 24 are not a realization of the properties in [17, Definition 3.12] because the set $\{\bar{y}\}$ is not a neighborhood of $\bar{y}$ in a metric space.

The next proposition summarizes the relationships amongst the properties in Definition 24.

Proposition 39. *For the properties in Definition 24, the following statements are true:*

(i) property (i) implies property (ii) for any subset $U' \subset X$ and any function $\nu : U' \to (0, \infty]$ satisfying $\nu(x) \leq \mu(\delta(\bar{y}, \mathcal{F}, x))$ for all $x \in U' \setminus U$, in particular, property (i) implies property (ii) for the same subset U;

(ii) property (ii) implies property (i) for $U' := \{x \in U \mid \mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x)\}$, in particular, if $\mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x)$ for all $x \in U$, then the two properties are equivalent;

(iii) property (i) implies property (iii) provided that U is a neighborhood of $\bar{x}$;

(iv) property (i) implies property (iv) provided that $U \cup \{x \in X : \nu(x) \leq \mu(\delta(\bar{y}, \mathcal{F}, x))\}$ is a neighborhood of $\bar{x}$;

(v) property (ii) implies property (iii) provided that $U' := \{x \in U \mid \mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x)\}$ *is a neighborhood of* $\bar{x}$*;*

(vi) property (ii) implies property (iv) provided that $U \cup \{x \in X : \nu(x) \le \mu(\delta(\bar{y}, \mathcal{F}, x))\}$ *is a neighborhood of* $\bar{x}$*;*

(vii) property (iii) implies property (iv) and if there is a neighborhood U *of* $\bar{x}$ *such that* $\mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x)$ *for all* $x \in U$*, then the two properties are equivalent;*

(viii) properties (i), (ii), (iii) and (iv) are implied by the following slightly stronger ones, respectively:

$$\begin{aligned}
\bar{y} \in \mathcal{F}_0\left(B(x, \mu(\delta(\bar{y}, \mathcal{F}, x)))\right) \quad & \text{for all} \quad x \in U, \\
\bar{y} \in \mathcal{F}_0\left(B(x, \mu(\delta(\bar{y}, \mathcal{F}, x)))\right) \quad & \text{for all} \quad x \in U \\
& \text{with } \mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x), \\
\exists \varepsilon > 0 : \bar{y} \in \mathcal{F}_0\left(B(x, \mu(\delta(\bar{y}, \mathcal{F}, x)))\right) \quad & \text{for all} \quad x \in B(\bar{x}, \varepsilon), \\
\exists \varepsilon > 0 : \bar{y} \in \mathcal{F}_0\left(B(x, \mu(\delta(\bar{y}, \mathcal{F}, x)))\right) \quad & \text{for all} \quad x \in B(\bar{x}, \varepsilon) \\
& \text{with } \mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x).
\end{aligned}$$

6.2.3 Criteria for subregularity of $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$

The following criteria for the properties in Definition 24 are derived from the corresponding statements in Subsection 6.2.1.

Theorem 34. *Given a subset* $U \subset X$ *and a point* $\bar{y} \in Y$*, suppose that the mapping* $\tau \mapsto \mathcal{F}_\tau^{-1}(\bar{y})$ *on* $\mathbb{R}_+$ *is outer semicontinuous at* 0 *and, for any* $x \in U$*, some* $\gamma > \delta(\bar{y}, \mathcal{F}, x)$ *and any* $t \in (0, \gamma)$ *with* $(x, t, \bar{y}) \in \operatorname{gph} \mathcal{F}$*, one of the following sets of conditions is satisfied:*

(i) there are sequences of positive numbers (b_n) *and* (c_n) *and a function* $m : (0, \infty) \to (0, \infty)$ *such that conditions (6.1) and (6.3) hold true and*

$$d\left(x, \mathcal{F}^{-1}_{m(c_1)}(\bar{y})\right) < b_0,$$
$$d\left(u, \mathcal{F}^{-1}_{m(c_{n+1})}(\bar{y})\right) < b_n \text{ for all } u \in \mathcal{F}^{-1}_{m(c_n)}(\bar{y}) \cap B\left(x, \sum_{i=0}^{n-1} b_i\right) \quad (n = 1, 2, \ldots);$$

(ii) *there are functions* $b, m : (0,\infty) \to (0,\infty)$ *such that condition* (6.4) *is satisfied and, for any* $\tau > 0$ *with* $\mu(\tau) \le \mu(t)$*, condition* (6.5) *holds true and*

$$d\left(u, \mathcal{F}^{-1}_{b(\tau)}(\bar{y})\right) < m(\tau) \text{ for all } u \in \mathcal{F}^{-1}_{\tau}(\bar{y}) \cap B(x, \mu(t) - \mu(\tau));$$

(iii) *there are functions* $b, m : (0,\infty) \to (0,\infty)$ *such that condition* (6.4) *is satisfied and, for any* $\tau > 0$ *with* $\mu(\tau) \le \mu(t)$*, condition* (6.5) *holds true and*

$$\mathcal{F}^{-1}_{0}(B(\bar{y}, \tau)) \subset \mathcal{F}^{-1}_{\tau}(\bar{y}),$$
$$d\left(\bar{y}, \mathcal{F}_0(B(u, m(\tau)))\right) < b(\tau) \text{ for all } u \in \mathcal{F}^{-1}_{\tau}(\bar{y}) \cap B(x, \mu(t) - \mu(\tau)).$$

Then, $\mathcal{F}$ *is subregular on* U *at* $\bar{y}$ *with functional modulus* μ*.*

Proof. Take an arbitrary point $x \in U$. Sets of conditions (i), (ii) and (iii) ensure the corresponding ones in Theorem 32 to be satisfied for the point $(x, \bar{y}) \in X \times Y$ and so that $d(x, \mathcal{F}^{-1}_0(\bar{y})) \le \mu(\delta(\bar{y}, \mathcal{F}, x))$. Hence, $\mathcal{F}$ is subregular on U at $\bar{y}$ with functional modulus μ. □

Theorem 35. *Given a subset* $U \subset X$*, a point* $\bar{y} \in Y$ *and a function* $\nu : U \to (0,\infty]$*, suppose that the mapping* $\tau \mapsto \mathcal{F}^{-1}_{\tau}(\bar{y})$ *on* $\mathbb{R}_+$ *is outer semicontinuous at* 0 *and, for any* $x \in U$ *and* $t > 0$ *with* $(x, t, \bar{y}) \in \operatorname{gph} \mathcal{F}$ *and* $\mu(t) < \nu(x)$*, one of the sets of conditions in Theorem 34 is satisfied. Then,* $\mathcal{F}$ *is* ν*-subregular on* U *at* $\bar{y}$ *with functional modulus* μ*.*

Proof. Take an arbitrary point $x \in U$ with $\mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x)$. Choose a $\gamma > \delta(\bar{y}, \mathcal{F}, x)$ such that $\mu(\gamma) < \nu(x)$, then for all $t \in (0, \gamma)$, we have $\mu(t) < \nu(x)$. Sets of conditions (i), (ii) and (iii) ensure the corresponding ones in Theorems 32 to be satisfied for the point $(x, \bar{y}) \in X \times Y$ and so that $d(x, \mathcal{F}^{-1}_0(\bar{y})) \le \mu(\delta(\bar{y}, \mathcal{F}, x))$. Hence, $\mathcal{F}$ is ν-subregular on U at $\bar{y}$ with functional modulus μ. □

The next two statements are consequences of Theorems 34 and 35, respectively, for $U = B(\bar{x}, \varepsilon)$, a neighborhood of $\bar{x}$.

Theorem 36. *Given a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}_0$*, suppose that the mapping* $\tau \mapsto \mathcal{F}^{-1}_{\tau}(\bar{y})$ *on* $\mathbb{R}_+$ *is outer semicontinuous at* 0 *and there is an* $\varepsilon > 0$ *such that, for any* $x \in B(\bar{x}, \varepsilon)$*, some*

$\gamma > \delta(\bar{y}, \mathcal{F}, x)$ and any $t \in (0, \gamma)$ with $(x, t, \bar{y}) \in \operatorname{gph} \mathcal{F}$, one of the sets of conditions in Theorem 34 is satisfied. Then, $\mathcal{F}$ is subregular at $(\bar{x}, \bar{y})$ with functional modulus μ.

Theorem 37. *Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}_0$ and a function $\nu : X \to \mathbb{R}$ strictly positive around $\bar{x}$, suppose that the mapping $\tau \mapsto \mathcal{F}_\tau^{-1}(\bar{y})$ on $\mathbb{R}_+$ is outer semicontinuous at 0 and there is an $\varepsilon > 0$ such that, for any $x \in B(\bar{x}, \varepsilon)$ and $t > 0$ with $(x, t, \bar{y}) \in \operatorname{gph} \mathcal{F}$ and $\mu(t) < \nu(x)$, one of the sets of conditions in Theorem 34 is satisfied. Then, $\mathcal{F}$ is ν-subregular at $(\bar{x}, \bar{y})$ with functional modulus μ.*

We next formulate criteria of error bound-types for the subregularity properties in Definitions 24. Given a point $\bar{y} \in Y$, let us denote, for any point $x \in X$, the set

$$V_x := \{u \in X \mid \delta(\bar{y}, \mathcal{F}, u) > 0,\ \mu(\delta(\bar{y}, \mathcal{F}, u)) + d(u, x) \le \mu(\delta(\bar{y}, \mathcal{F}, x))\}.$$

Note that $V_x \subset B(x, \mu(\delta(\bar{y}, \mathcal{F}, x)))$.

Theorem 38. *Given a subset $U \subset X$, a point $\bar{y} \in Y$ and a continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$, suppose that $\mathcal{F}^{-1}(\bar{y})$ is closed and, for any $x \in U$ and $u \in V_x$, there exists a point $u' \neq u$ such that*

$$\mu(\delta(\bar{y}, \mathcal{F}, u')) \le \mu(\delta(\bar{y}, \mathcal{F}, u)) - d(u, u'). \tag{6.7}$$

Then, $\mathcal{F}$ is subregular on U at $\bar{y}$ with functional modulus μ.

Proof. Take an arbitrary $x \in U$. We need to show that $d(x, \mathcal{F}_0^{-1}(\bar{y})) \le \mu(\delta(\bar{y}, \mathcal{F}, x))$.

If there exists a point u such that $\delta(\bar{y}, \mathcal{F}, u) = 0$ and $d(x, u) \le \mu(\delta(\bar{y}, \mathcal{F}, x))$ (in particular, if $\delta(\bar{y}, \mathcal{F}, x) = 0$), then, by the closedness of $\mathcal{F}^{-1}(\bar{y})$, $u \in \mathcal{F}_0^{-1}(\bar{y})$, and the inequality holds trivially.

Suppose that $\delta(\bar{y}, \mathcal{F}, u) > 0$ for any $u \in X$ such that $d(x, u) \le \mu(\delta(\bar{y}, \mathcal{F}, x))$. Take any $u \in X$ such that $d(x, u) \le \mu(\delta(\bar{y}, \mathcal{F}, x)) - \mu(\delta(\bar{y}, \mathcal{F}, u))$ and any $\tau \in (0, \delta(\bar{y}, \mathcal{F}, x)]$ such that $(u, \tau) \in \mathcal{F}^{-1}(\bar{y})$. Then, $\tau \ge \delta(\bar{y}, \mathcal{F}, u) > 0$ and, by the assumption, there exists a point $u' \neq u$ satisfying (6.7). Setting $\tau' = \delta(\bar{y}, \mathcal{F}, u')$, we get $(u', \tau') \in \mathcal{F}^{-1}(\bar{y})$ and condition (6.6) is satisfied:

$$\mu(\tau') = \mu(\delta(\bar{y}, \mathcal{F}, u')) \leq \mu(\delta(\bar{y}, \mathcal{F}, u)) - d(u, u') \leq \mu(\tau) - d(u, u').$$

The mapping $\tau \mapsto \mathcal{F}_\tau^{-1}(\bar{y})$ is outer semicontinuous on $[0, \delta(\bar{y}, \mathcal{F}, x)]$ thanks to the closedness of $\mathcal{F}^{-1}(\bar{y})$. The required inequality follows from Theorem 33. □

Theorem 39. *Given a subset $U \subset X$, a point $\bar{y} \in Y$, a continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$ and a function $\nu : \bigcup_{x \in U} V_x \to (0, \infty]$ being Lipschitz continuous with modulus not greater than 1, suppose that $\mathcal{F}^{-1}(\bar{y})$ is closed and, for any $x \in U$ and $u \in V_x$ with $\mu(\delta(\bar{y}, \mathcal{F}, u)) < \nu(u)$, there exists a point $u' \neq u$ such that condition (6.7) holds true. Then, $\mathcal{F}$ is ν-subregular on U at $\bar{y}$ with functional modulus μ.*

Proof. Define $U' := \{x \in U \mid \mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x)\}$ and take any $x \in U'$ and $u \in U_x$. Then, taking into account the Lipschitz continuity of ν, we have:

$$\mu(\delta(\bar{y}, \mathcal{F}, u)) \leq \mu(\delta(\bar{y}, \mathcal{F}, x)) - d(x, u) < \nu(x) - d(x, u) \leq \nu(u).$$

Hence, there exists a point $u' \neq u$ such that (6.7) holds true. By Theorem 38, $\mathcal{F}$ is subregular on U' at $\bar{y}$ with functional modulus μ and, thanks to Proposition 39 (i), ν-subregular on U at $\bar{y}$ with functional modulus μ. □

Theorem 40. *Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}_0$ and a continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$, suppose that $\mathcal{F}^{-1}(\bar{y})$ is closed and that there is an $\varepsilon > 0$ such that, for any $u \in B(\bar{x}, \varepsilon)$, there exists a point $u' \neq u$ such that condition (6.7) is satisfied. Then, $\mathcal{F}$ is subregular at $(\bar{x}, \bar{y})$ with functional modulus μ.*

Proof. Take an arbitrary $x \in B(\bar{x}, \varepsilon/2)$. We need to show that $d(x, \mathcal{F}_0^{-1}(\bar{y})) \leq \mu(\delta(\bar{y}, \mathcal{F}, x))$.

If $d(x, \bar{x}) \leq \mu(\delta(\bar{y}, \mathcal{F}, x))$, then the inequality holds trivially because $d(x, \mathcal{F}_0^{-1}(\bar{y})) \leq d(x, \bar{x})$ thanks to $\bar{x} \in \mathcal{F}_0^{-1}(\bar{y})$. So we can suppose that $\mu(\delta(\bar{y}, \mathcal{F}, x)) < d(x, \bar{x}) < \varepsilon/2$. In this case, for any $u \in V_x$, we have

$$d(u, \bar{x}) \leq d(u, x) + d(x, \bar{x}) < \mu(\delta(\bar{y}, \mathcal{F}, x)) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

That is $u \in B(\bar{x}, \varepsilon)$. The conclusion follows from Theorem 38 for the subset $U = B(\bar{x}, \varepsilon/2)$. □

Theorem 41. *Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}_0$, a function $\nu : X \to \mathbb{R}$ strictly positive around $\bar{x}$ and a continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$, suppose that $\mathcal{F}^{-1}(\bar{y})$ is closed and that there are an $\varepsilon > 0$ and a function $\nu : B(\bar{x}, \varepsilon) \to (0, \infty)$ being Lipschitz continuous with modulus not greater than 1 such that, for any $u \in B(\bar{x}, \varepsilon)$ with $\mu(\delta(\bar{y}, \mathcal{F}, u)) < \nu(u)$, there exists a point $u' \neq u$ such that condition (6.7) is satisfied. Then, $\mathcal{F}$ is ν-subregular at $(\bar{x}, \bar{y})$ with functional modulus μ.*

Proof. Take an arbitrary $x \in B(\bar{x}, \varepsilon/2)$ with $\mu(\delta(\bar{y}, \mathcal{F}, x)) < \nu(x)$. We need to show that $d(x, \mathcal{F}_0^{-1}(\bar{y})) \le \mu(\delta(\bar{y}, \mathcal{F}, x))$.

If $d(x, \bar{x}) \le \mu(\delta(\bar{y}, \mathcal{F}, x))$, then the inequality holds trivially because

$$d(x, \mathcal{F}_0^{-1}(\bar{y})) \le d(x, \bar{x})$$

thanks to $\bar{x} \in \mathcal{F}_0^{-1}(\bar{y})$. So we can suppose that $\mu(\delta(\bar{y}, \mathcal{F}, x)) < d(x, \bar{x}) < \varepsilon/2$. In this case, for any $u \in V_x$, we have

$$d(u, \bar{x}) \le d(u, x) + d(x, \bar{x}) < \mu(\delta(\bar{y}, \mathcal{F}, x)) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

$$\mu(\delta(\bar{y}, \mathcal{F}, u)) \le \mu(\delta(\bar{y}, \mathcal{F}, x)) - d(x, u) < \nu(x) - d(x, u) \le \nu(u).$$

That is $u \in B(\bar{x}, \varepsilon)$ and $\mu(\delta(\bar{y}, \mathcal{F}, u)) < \nu(u)$. The conclusion follows from Theorem 39 for the subset $U = B(\bar{x}, \varepsilon/2)$. □

6.3 Metric subregularity for $F : X \rightrightarrows Y$

In this section, we consider the conventional setting of a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces. Such a mapping can be imbedded into the more general setting explored in the previous sections by defining a set-valued mapping $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$ as follows (cf. [9, p. 508]: for any $x \in X$ and $t \geq 0$,

$$\mathcal{F}(x,t) := B(F(x),t) = \begin{cases} \{y \in Y \mid d(y, F(x)) < t\} & \text{if } t > 0, \\ F(x) & \text{if } t = 0. \end{cases} \tag{6.8}$$

(Recall the convention: $B(y,0) = \{y\}$.) We are going to consider also mappings $\overline{F} : X \rightrightarrows Y$ and $\overline{\mathcal{F}} : X \times \mathbb{R}_+ \rightrightarrows Y$, whose values are the closures of the corresponding values of F and $\mathcal{F}$, respectively: $\overline{F}(x) := \overline{F(x)}$ and

$$\overline{\mathcal{F}}(x,t) := \overline{B}(F(x),t) = \begin{cases} \{y \in Y \mid d(y, F(x)) \leq t\} & \text{if } t > 0, \\ \overline{F(x)} & \text{if } t = 0. \end{cases} \tag{6.9}$$

The next proposition summarizes several simple facts with regard to the relationship amongst F, $\mathcal{F}$ and $\overline{\mathcal{F}}$.

Proposition 40. *[17, Proposition 4.1]*

(i) $\mathcal{F}_0(x) = F(x)$, $\overline{\mathcal{F}}_0(x) = \overline{F(x)}$ *for all* $x \in X$.

(ii) $\delta(y, \mathcal{F}, x) = \delta(y, \overline{\mathcal{F}}, x) = d(y, F(x))$ *for all* $x \in X$ *and* $y \in Y$.

(iii) $\mathcal{F}_0^{-1}(B(y,t)) = F^{-1}(B(y,t)) = \mathcal{F}_t^{-1}(y)$ *for all* $y \in Y$ *and* $t \geq 0$.

(iv) $\overline{\mathcal{F}}^{-1}(\overline{B}(y,t)) = \overline{F}^{-1}(\overline{B}(y,t)) \subset \overline{\mathcal{F}}_t^{-1}(y)$ *for all* $y \in Y$ *and* $t \geq 0$.

(v) If F^{-1} *is closed at* y, *then the mappings* $\tau \mapsto \mathcal{F}_\tau^{-1}(y)$ *and* $\tau \mapsto \overline{\mathcal{F}}_\tau^{-1}(y)$ *on* $\mathbb{R}_+$ *are outer semicontinuous at* 0.

(vi) For any $y \in Y$ *and* $\tau > 0$, $\mathcal{F}$ *and* $\overline{\mathcal{F}}$ *satisfy condition*

$$\mathcal{F}_0^{-1}(B(y,\tau)) \subset \mathcal{F}_\tau^{-1}(y).$$

(vii) If F *is upper semicontinuous, i.e., for any* $x \in X$ *and* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $F(u) \subset B(F(x), \varepsilon)$ *for all* $u \in B(x, \delta)$, *then* $\overline{\mathcal{F}}^{-1}$ *is closed-valued. In particular, for any* $y \in Y$, *the mapping* $\tau \mapsto \overline{\mathcal{F}}_\tau^{-1}(y)$ *is outer semicontinuous on* $\mathbb{R}_+$.

6.3.1 Definitions and equivalences

Thanks to parts (i) and (ii) of Proposition 40, the subregularity properties of a set-valued mapping $F : X \rightrightarrows Y$ can be stated, corresponding to Definition 24, as follows.

Definition 25. (i) F is metrically subregular on a subset $U \subset X$ at a point $\bar{y} \in Y$ with functional modulus μ if one of the following equivalent conditions holds true:

$$d(x, F^{-1}(\bar{y})) \le \mu(d(\bar{y}, F(x))) \quad \text{for all} \quad x \in U, \tag{6.10}$$

$$\bar{y} \in F(B(x,t)) \quad \text{for all} \quad x \in U$$
$$\text{and } t > \mu(d(\bar{y}, F(x))).$$

(ii) Given a subset $U \subset X$ and a function $\nu : U \to (0, \infty]$, F is metrically ν-subregular on a subset $U \subset X$ at a point $\bar{y} \in Y$ with functional modulus μ if one of the following equivalent conditions holds true:

$$d(x, F^{-1}(\bar{y})) \le \mu(d(\bar{y}, F(x))) \quad \text{for all} \quad x \in U$$
$$\text{with } \mu(d(\bar{y}, F(x))) < \nu(x), \tag{6.11}$$
$$\bar{y} \in F(B(x,t)) \quad \text{for all} \quad x \in U$$
$$\text{and } t \in (\mu(d(\bar{y}, F(x))), \nu(x)).$$

(iii) F is metrically subregular at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with functional modulus μ if there exists a neighborhood U of $\bar{x}$ such that F is metrically subregular on U at $\bar{y}$ with functional modulus μ.

(iv) Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a function $\nu : X \to \mathbb{R}$ strictly positive around $\bar{x}$, F is metrically ν-subregular at $(\bar{x}, \bar{y})$ with functional modulus μ if there exists a neighborhood U of $\bar{x}$ such that F is metrically ν-subregular on U at $\bar{y}$ with functional modulus μ.

Proposition 41. *Let $F : X \rightrightarrows Y$, $\mathcal{F}$ and $\overline{\mathcal{F}}$ be as in (6.8) and (6.9). Then $\mathcal{F}$ (equivalently, $\overline{\mathcal{F}}$) satisfies one of the properties in Definition 24 if and only if F satisfies the corresponding properties in Definition 25.*

The next proposition followed from Propositions 39 and 41 summarizes the relationships amongst the properties in Definition 25.

Proposition 42. *For the properties in Definition 25, the following statements are true:*

(i) property (i) implies property (ii) for any subset $U' \subset X$ *and any function* $\nu : U' \to (0, \infty]$ *satisfying* $\nu(x) \le \mu(d(\bar{y}, F(x)))$ *for all* $x \in U' \setminus U$*, in particular, property (i) implies property (ii) for the same subset* U*;*

(ii) property (ii) implies property (i) for $U' := \{x \in U \mid \mu(d(\bar{y}, F(x))) < \nu(x)\}$*, in particular, if* $\mu(d(\bar{y}, F(x))) < \nu(x)$ *for all* $x \in U$*, then the two properties are equivalent;*

(iii) property (i) implies property (iii) provided that U *is a neighborhood of* $\bar{x}$*;*

(iv) property (i) implies property (iv) provided that $U \cup \{x \in X : \nu(x) \le \mu(d(\bar{y}, F(x)))\}$ *is a neighborhood of* $\bar{x}$*;*

(v) property (ii) implies property (iii) provided that $U' := \{x \in U \mid \mu(d(\bar{y}, F(x))) < \nu(x)\}$ *is a neighborhood of* $\bar{x}$*;*

(vi) property (ii) implies property (iv) provided that $U \cup \{x \in X : \nu(x) \le \mu(d(\bar{y}, F(x)))\}$ *is a neighborhood of* $\bar{x}$*;*

(vii) property (iii) implies property (iv) and if there is a neighborhood U *of* $\bar{x}$ *such that* $\mu(d(\bar{y}, F(x))) < \nu(x)$ *for all* $x \in U$*, then the two properties are equivalent;*

(viii) properties (i), (ii), (iii) and (iv) are implied by the following slightly stronger ones, respectively:

$$\begin{aligned}
\bar{y} \in F\left(B(x, \mu(d(\bar{y}, F(x))))\right) \quad & \text{for all} \quad x \in U, \\
\bar{y} \in F\left(B(x, \mu(d(\bar{y}, F(x))))\right) \quad & \text{for all} \quad x \in U \\
& \text{with } \mu(d(\bar{y}, F(x))) < \nu(x), \\
\exists \varepsilon > 0 : \bar{y} \in F\left(B(x, \mu(d(\bar{y}, F(x))))\right) \quad & \text{for all} \quad x \in B(\bar{x}, \varepsilon), \\
\exists \varepsilon > 0 : \bar{y} \in F\left(B(x, \mu(d(\bar{y}, F(x))))\right) \quad & \text{for all} \quad x \in B(\bar{x}, \varepsilon) \\
& \text{with } \mu(d(\bar{y}, F(x))) < \nu(x).
\end{aligned}$$

The metric subregularity (subopenness, or pseudo-openness [2]) properties in Definition 25 have proved to be important in both theory and applications, mostly in the linear (sometimes Hölder) case in the local setting (cf. [2, 5, 7, 9, 13, 18, 22, 23, 26, 29, 30]).

Observe that condition (6.10) in Definition 25 is equivalent to

$$d(x, F^{-1}(\bar{y})) \le \mu(d(\bar{y}, y)) \text{ for all } x \in U,\ y \in F(x).$$

In its turn, condition $y \in F(x)$ is equivalent to $x \in F^{-1}(y)$. This and a similar observation with regard to condition (6.11) in Definition 25 allow us to rewrite these conditions, respectively, as follows:

$$d(x, F^{-1}(\bar{y})) \le \mu(d(\bar{y}, y)) \text{ for all } y \in Y,\ x \in F^{-1}(y) \cap U,$$
$$d(x, F^{-1}(\bar{y})) \le \mu(d(\bar{y}, y)) \text{ for all } y \in Y,\ x \in F^{-1}(y) \cap U \text{ with } \mu(d(\bar{y}, y)) < \nu(x).$$

Thanks to these observations, one can complement the properties in Definition 25 with the corresponding Hölder-like (Aubin-like in the linear case) properties.

Definition 26. (i) F is sub-Hölder on a subset $V \subset Y$ at a point $\bar{x} \in X$ with functional modulus μ if

$$d(y, F(\bar{x})) \le \mu(d(\bar{x}, x)) \text{ for all } x \in X,\ y \in F(x) \cap V.$$

(ii) Given a subset $V \subset Y$ and a function $\nu : V \to (0, \infty]$, F is ν-sub-Hölder on V at a point $\bar{x} \in X$ with functional modulus μ if

$$d(y, F(\bar{x})) \le \mu(d(\bar{x}, x)) \text{ for all } x \in X,\ y \in F(x) \cap V \text{ with } \mu(d(\bar{x}, x)) < \nu(y).$$

(iii) F is sub-Hölder at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with functional modulus μ if there exists a neighborhood V of $\bar{y}$ such that F is sub-Hölder on a subset V at $\bar{x}$ with functional modulus μ.

(iv) Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a function $\nu : Y \to \mathbb{R}$ strictly positive around $\bar{y}$, F is ν-sub-Hölder at $(\bar{x}, \bar{y})$ with functional modulus μ if there exists a neighborhood V of $\bar{y}$ such that F is ν-sub-Hölder on V at $\bar{x}$ with functional modulus μ.

Thanks to Propositions 39 and 40 and the discussion before Definition 26, we have the following list of equivalences.

Proposition 43. *(i) F is metrically subregular on $U \subset X$ at $\bar{y} \in Y$ with functional modulus μ if and only if F^{-1} is sub-Hölder on U at $\bar{y}$ with functional modulus μ.*

(ii) F is metrically ν-subregular on $U \subset X$ at $\bar{y} \in Y$ with functional modulus μ if and only if F^{-1} is ν-sub-Hölder on U at $\bar{y}$ with functional modulus μ.

(iii) F is metrically subregular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with functional modulus μ if and only if F^{-1} is sub-Hölder at $(\bar{y}, \bar{x})$ with functional modulus μ.

(iv) F is metrically ν-subregular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with functional modulus μ if and only if F^{-1} is ν-sub-Hölder at $(\bar{y}, \bar{x})$ with functional modulus μ.

6.3.2 Criteria for metric subregularity of $F : X \rightrightarrows Y$

We are going to formulate criteria for metric subregularity properties.

Theorem 42. *Given a subset $U \subset X$ and a point $\bar{y} \in Y$, suppose that F^{-1} is closed at $\bar{y}$ and, for any $x \in U \setminus F^{-1}(\bar{y})$, for some $\gamma > d(\bar{y}, F(x))$ and any $t \in [d(\bar{y}, F(x)), \gamma)$, one of the following sets of conditions is satisfied:*

(i) there are sequences of positive numbers (b_n) and (c_n) and a function $m : (0, \infty) \to (0, \infty)$ such that conditions (6.1) and (6.3) hold true and

$$d\left(x, F^{-1}(B(\bar{y}, m(c_1)))\right) < b_0, \tag{6.12}$$

$$d\left(u, F^{-1}(B(\bar{y}, m(c_{n+1})))\right) < b_n$$

$$\text{for all } u \in F^{-1}(B(\bar{y}, m(c_n))) \cap B\left(x, \sum_{i=0}^{n-1} b_i\right) \ (n = 1, 2, \ldots); \tag{6.13}$$

(ii) there are functions $b, m : (0,\infty) \to (0,\infty)$ such that condition (6.4) is satisfied and, for any $\tau > 0$ with $\mu(\tau) \le \mu(t)$, condition (6.5) holds true and

$$d\left(u, F^{-1}(B(\bar{y}, b(\tau)))\right) < m(\tau) \qquad \text{for all } u \in F^{-1}(B(\bar{y},\tau)) \cap B(x, \mu(t) - \mu(\tau)).$$

Then, F is metrically subregular on U at $\bar{y}$ with functional modulus μ.

Proof. Consider $\overline{\mathcal{F}}$ by (6.9). Then $(x,t,\bar{y}) \in \operatorname{gph}\overline{\mathcal{F}}$ if and only if $d(\bar{y}, F(x)) \le t$. The conclusion follows from Theorem 34 thanks to Propositions 40 and 41. $\square$

Similarly, the following three statements are derived immediately from Theorems 35, 36 and 37.

Theorem 43. *Given a subset $U \subset X$, a point $\bar{y} \in Y$ and a function $\nu : U \to (0,\infty]$, suppose that F^{-1} is closed at $\bar{y}$ and, for any $x \in U \setminus F^{-1}(\bar{y})$ and $t \ge d(\bar{y}, F(x))$ with $\mu(t) < \nu(x)$, one of the sets of conditions in Theorem 42 is satisfied. Then, F is metrically ν-subregular on U at $\bar{y}$ with functional modulus μ.*

Theorem 44. *Given a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$, suppose that F^{-1} is closed at $\bar{y}$ and there is an $\varepsilon > 0$ such that, for any $x \in B(\bar{x},\varepsilon)\backslash F^{-1}(\bar{y})$, some $\gamma > d(\bar{y},F(x))$ and any $t \in [d(\bar{y},F(x)),\gamma)$, one of the sets of conditions in Theorem 42 is satisfied. Then, F is metrically subregular at $(\bar{x},\bar{y})$ with functional modulus μ.*

Theorem 45. *Given a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$ and a function $\nu : X \to \mathbb{R}$ strictly positive around $\bar{x}$, suppose that F^{-1} is closed at $\bar{y}$ and there is an $\varepsilon > 0$ such that, for any $x \in B(\bar{x},\varepsilon)\backslash F^{-1}(\bar{y})$ and $t \ge d(\bar{y},F(x))$ with $\mu(t) < \nu(x)$, one of the sets of conditions in Theorem 42 is satisfied. Then, F is metrically ν-subregular at $(\bar{x},\bar{y})$ with functional modulus μ.*

Criteria of error bound-types can also be obtained in the following four statements.

Theorem 46. *Given a subset $U \subset X$, a point $\bar{y} \in Y$ and a continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$, suppose that F is upper semicontinuous and, for any $x \in U \setminus F^{-1}(\bar{y})$ and $u \in X$ with $d(\bar{y},F(u)) > 0$ and $\mu(d(\bar{y},F(u))) + d(u,x) \le$*

$\mu(d(\bar{y}, F(x)))$, there exists a point $u' \neq u$ such that

$$\mu(d(\bar{y}, F(u'))) \leq \mu(d(\bar{y}, F(u))) - d(u, u'). \tag{6.14}$$

Then, F is metrically subregular on U at $\bar{y}$ with functional modulus μ.

Proof. Consider $\overline{\mathcal{F}}$ by (6.9). The conclusion follows from Theorem 38 thanks to Propositions 40 and 41. □

Theorem 47. *Given a subset $U \subset X$, a point $\bar{y} \in Y$ and a continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$, define*

$$U_\mu = \bigcup_{x \in U} \{u \in X \mid d(u, x) \leq \mu(d(\bar{y}, F(x)))\}.$$

Let $\nu : U_\mu \to (0, \infty]$ be Lipschitz continuous with modulus not greater than 1. Suppose that F is upper semicontinuous and, for any $x \in U \backslash F^{-1}(\bar{y})$ and $u \in X$ with $0 < \mu(d(\bar{y}, F(u))) < \nu(u)$ and $\mu(d(\bar{y}, F(u))) + d(u, x) \leq \mu(d(\bar{y}, F(x)))$, there exists a point $u' \neq u$ such that condition (6.14) is satisfied. Then, F is metrically ν-subregular on U at $\bar{y}$ with functional modulus μ.

Proof. Define $U' := \{x \in U \mid \mu(d(\bar{y}, F(x))) < \nu(x)\}$ and take any $x \in U' \setminus F^{-1}(\bar{y})$ and $u \in X$ such that $d(\bar{y}, F(u)) > 0$ and $\mu(d(\bar{y}, F(u))) + d(u, x) \leq \mu(d(\bar{y}, F(x)))$. Then, taking into account the Lipschitz continuity of ν, we have

$$\mu(d(\bar{y}, F(u))) \leq \mu(d(\bar{y}, F(x))) - d(x, u) < \nu(x) - d(x, u) \leq \nu(u).$$

Hence, there exists a point $u' \neq u$ such that (6.14) holds true. By Theorem 46, F is metrically subregular on U' at $\bar{y}$ with functional modulus μ and, thanks to Proposition 42 (vi), F is metrically ν-subregular on U at $\bar{y}$ with functional modulus μ. □

Theorem 48. *Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a continuous nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(\tau) = 0$ if and only if $\tau = 0$, suppose that F is upper semicontinuous and there is an $\varepsilon > 0$ such that, for any $u \in B(\bar{x}, \varepsilon) \setminus F^{-1}(\bar{y})$, there exists a point $u' \neq u$ such that condition (6.14) is satisfied. Then, F is metrically subregular at $(\bar{x}, \bar{y})$ with functional modulus μ.*

Proof. Consider $\overline{\mathcal{F}}$ by (6.9). The conclusion follows from Theorem 40 thanks to Propositions 40 and 41. □

Theorem 49. *Given a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$*, a continuous nondecreasing function* $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $\mu(\tau) = 0$ *if and only if* $\tau = 0$ *and a function* $\nu : X \to \mathbb{R}$ *strictly positive and Lipschitz continuous with modulus not greater than 1 around* $\bar{x}$*, suppose that* F *is upper semicontinuous and there is an* $\varepsilon > 0$ *such that, for any* $u \in B(\bar{x}, \varepsilon) \setminus F^{-1}(\bar{y})$ *with* $\mu(d(\bar{y}, F(u))) < \nu(u)$*, there exists a point* $u' \neq u$ *such that condition (6.14) is satisfied. Then,* F *is metrically* ν*-subregular at* $(\bar{x}, \bar{y})$ *with functional modulus* μ*.*

Proof. Take an arbitrary $x \in B(\bar{x}, \varepsilon/2) \setminus F^{-1}(\bar{y})$ with $\mu(d(\bar{y}, F(x))) < \nu(x)$. If $\mu(d(\bar{y}, F(x))) \geq \varepsilon/2$, then

$$d(x, F^{-1}(\bar{y})) \leq d(x, \bar{x}) < \varepsilon/2 < \mu(d(\bar{y}, F(x))).$$

Otherwise, for any $u \in X$ such that $d(\bar{y}, F(u)) > 0$ and $\mu(d(\bar{y}, F(u))) + d(u, x) \leq \mu(d(\bar{y}, F(x)))$, we have

$$d(u, \bar{x}) \leq d(u, x) + d(x, \bar{x}) < \mu(d(\bar{y}, F(x))) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

$$\mu(d(\bar{y}, F(u))) \leq \mu(d(\bar{y}, F(x))) - d(x, u) < \nu(x) - d(x, u) \leq \nu(u).$$

That is $u \in B(\bar{x}, \varepsilon)$ and $\mu(d(\bar{y}, F(u))) < \nu(u)$. The conclusion follows from Theorem 47 for the subset $U = B(\bar{x}, \varepsilon/2)$. □

6.3.3 Definitions and equivalences for metric subregularity of order k

Metric subregularity properties of linear and Hölder-type (of order k) models obtained by considering the gauge function μ of the corresponding forms are very important in applications. In this section, we establish criteria for these properties. All of them are obtained as simplifications of the corresponding ones in Section 6.3 for the gauge function $\mu(\cdot) = r(\cdot)^k$. In the special case when $k = 1$, we obtain criteria for the metric subregularity properties of linear model.

Throughout this section, let $r > 0$ and $k \in (0, 1]$ be constants.

Definition 27. (i) F is metrically subregular on a subset $U \subset X$ at a point $\bar{y} \in Y$ of order k with modulus r if one of the following equivalent conditions holds true:

$$\begin{aligned} d(x, F^{-1}(\bar{y})) \le r d^k(\bar{y}, F(x)) \quad & \text{for all} \quad x \in U, \\ \bar{y} \in F(B(x,t)) \quad & \text{for all} \quad x \in U \\ & \text{and } t > r d^k(\bar{y}, F(x)). \end{aligned}$$

(ii) F is sub-Hölder on a subset $V \subset Y$ at a point $\bar{x} \in X$ of order k with modulus r if

$$d(y, F(\bar{x})) \le r d^k(\bar{x}, x) \quad \text{for all } x \in X,\ y \in F(x) \cap V.$$

(iii) Given a subset $U \subset X$ and a function $\nu : U \to (0, \infty]$, F is metrically ν-subregular on U at a point $\bar{y} \in Y$ of order k with modulus r if one of the following equivalent conditions holds true:

$$\begin{aligned} d(x, F^{-1}(\bar{y})) \le r d^k(\bar{y}, F(x)) \quad & \text{for all} \quad x \in U \\ & \text{with } r d^k(\bar{y}, F(x)) < \nu(x), \\ \bar{y} \in F(B(x,t)) \quad & \text{for all} \quad x \in U \\ & \text{and } t \in (r d^k(\bar{y}, F(x)), \nu(x)). \end{aligned}$$

(iv) Given a subset $V \subset Y$ and a function $\nu : V \to (0, \infty]$, F is ν-sub-Hölder on V at a point $\bar{x} \in X$ of order k with modulus r if

$$\begin{aligned} d(y, F(\bar{x})) \le r d^k(\bar{x}, x) \quad & \text{for all } x \in X,\ y \in F(x) \cap V \\ & \text{with } r d^k(\bar{x}, x) < \nu(y). \end{aligned}$$

(v) F is metrically subregular at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ of order k with modulus r if there exists a neighborhood U of $\bar{x}$ such that F is metrically subregular on U at $\bar{y}$ of order k with modulus r.

(vi) F is sub-Hölder at $(\bar{x}, \bar{y})$ of order k with modulus r if there exists a neighborhood V of

$\bar{y}$ such that F is sub-Hölder on V at $\bar{x}$ of order k with modulus r.

(vii) Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a function $\nu : X \to \mathbb{R}$ strictly positive around $\bar{x}$, F is metrically ν-subregular at $(\bar{x}, \bar{y})$ of order k with modulus r if there exists a neighborhood U of $\bar{x}$ such that F is metrically ν-subregular on U at $\bar{y}$ of order k with modulus r.

(viii) Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a function $\nu : Y \to \mathbb{R}$ strictly positive around $\bar{y}$, F is ν-sub-Hölder at $(\bar{x}, \bar{y})$ of order k with modulus r if there exists a neighborhood V of $\bar{y}$ such that F is ν-sub-Hölder on V at $\bar{x}$ of order k with modulus r.

The next proposition following from Proposition 42 and Theorem 43 summarizes the relationships amongst the properties in Definition 27.

Proposition 44. *For the properties in Definition 27, the following statements are true:*

(i) property (i) is equivalent to F^{-1} being sub-Hölder on U at $\bar{y}$ of order k with modulus r;

(ii) property (iii) is equivalent to F^{-1} being ν-sub-Hölder on U at $\bar{y}$ of order k with modulus r;

(iii) property (v) is equivalent to F^{-1} being sub-Hölder at $(\bar{y}, \bar{x})$ of order k with modulus r;

(iv) property (vii) is equivalent to F^{-1} being ν-sub-Hölder at $(\bar{y}, \bar{x})$ of order k with modulus r;

(v) property (i) implies property (iii) for any subset U' and any function $\nu : U' \to (0, \infty]$ satisfying $\nu(x) \le rd^k(\bar{y}, F(x))$ for all $x \in U' \setminus U$, in particular, property (i) implies property (iii) for the same subset U;

(vi) property (iii) implies property (i) for $U' := \{x \in U \mid rd^k(\bar{y}, F(x)) < \nu(x)\}$, in particular, if $rd^k(\bar{y}, F(x)) < \nu(x)$ for all $x \in U$, then the two properties are equivalent;

(vii) property (i) implies property (v) provided that U is a neighborhood of $\bar{x}$;

(viii) property (i) implies property (vii) provided that $U \cup \{x \in X : \nu(x) \le rd^k(\bar{y}, F(x))\}$ is a neighborhood of $\bar{x}$;

(ix) property (iii) implies property (v) provided that $U' := \{x \in U \mid rd^k(\bar{y}, F(x)) < \nu(x)\}$ is a neighborhood of $\bar{x}$;

(x) *property (iii) implies property (vii) provided that* $U \cup \{x \in X : \nu(x) \le rd^k(\bar{y}, F(x))\}$ *is a neighborhood of* $\bar{x}$*;*

(xi) *property (v) implies property (vii) and if there is a neighborhood* U *of* $\bar{x}$ *such that* $rd^k(\bar{y}, F(x)) < \nu(x)$ *for all* $x \in U$*, then the two properties are equivalent;*

(xii) *properties (i), (iii), (v) and (vii) are implied by the following slightly stronger ones, respectively:*

$$\begin{aligned}
\bar{y} \in F\left(B(x, rd^k(\bar{y}, F(x)))\right) \quad & \textit{for all} \quad x \in U, \\
\bar{y} \in F\left(B(x, rd^k(\bar{y}, F(x)))\right) \quad & \textit{for all} \quad x \in U \\
& \textit{with } rd^k(\bar{y}, F(x)) < \nu(x), \\
\exists \varepsilon > 0 : \bar{y} \in F\left(B(x, rd^k(\bar{y}, F(x)))\right) \quad & \textit{for all} \quad x \in B(\bar{x}, \varepsilon), \\
\exists \varepsilon > 0 : \bar{y} \in F\left(B(x, rd^k(\bar{y}, F(x)))\right) \quad & \textit{for all} \quad x \in B(\bar{x}, \varepsilon) \\
& \textit{with } rd^k(\bar{y}, F(x)) < \nu(x).
\end{aligned}$$

6.3.4 Criteria for metric subregularity of order k

We are going to formulate criteria for subregularity properties of order k defined in Definition 27.

All of them are consequences of the corresponding statements in Subsections 6.3.2 for the gauge function $\mu(\cdot) = r(\cdot)^k$.

Theorem 50. *Given a subset* $U \subset X$ *and a point* $\bar{y} \in Y$*, suppose that* F^{-1} *is closed at* $\bar{y}$ *and, for any* $x \in U \setminus F^{-1}(\bar{y})$*, for some* $\gamma > d(\bar{y}, F(x))$ *and any* $t \in [d(\bar{y}, F(x)), \gamma)$*, one of the following sets of conditions is satisfied:*

(i) *there are sequences of positive numbers* (b_n) *and* (c_n) *and a function* $m : (0, \infty) \to (0, \infty)$ *such that conditions* (6.1)*,* (6.12) *and* (6.13) *hold true and*

$$\sum_{n=0}^{\infty} b_n \le rt^k;$$

(ii) *there are functions* $b, m : (0, \infty) \to (0, \infty)$ *such that condition* (6.4) *is satisfied and, for* *any* $\tau \in (0,t]$, *condition* (6.5) *holds true and*

$$d\left(u, F^{-1}(B(\bar{y}, b(\tau)))\right) < m(\tau)$$
$$\text{for all } u \in F^{-1}(B(\bar{y},\tau)) \cap B(x, rt^k - r\tau^k);$$

Then, F *is metrically subregular on* U *at* $\bar{y}$ *of order* k *with modulus* r*.*

Theorem 51. *Given a subset* $U \subset X$ *and a function* $\nu : U \to (0,\infty]$*, suppose that* F^{-1} *is closed at* $\bar{y}$ *and, for any* $x \in U \setminus F^{-1}(\bar{y})$ *and* $t \in [d(\bar{y}, F(x)), r^{-1/k}\nu(x)^{1/k})$*, one of the sets of conditions in Theorem* 50 *is satisfied. Then,* F *is metrically* ν*-subregular on* U *at* $\bar{y}$ *of order* k *with modulus* r*.*

Theorem 52. *Given a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$*, suppose that* F^{-1} *is closed at* $\bar{y}$ *and there is an* $\varepsilon > 0$ *such that, for any* $x \in B(\bar{x},\varepsilon) \setminus F^{-1}(\bar{y})$*, some* $\gamma > d(\bar{y}, F(x))$ *and any* $t \in [d(\bar{y}, F(x)), \gamma)$*, one of the sets of conditions in Theorem* 50 *is satisfied. Then,* F *is metrically subregular at* $(\bar{x}, \bar{y})$ *of order* k *with modulus* r*.*

Theorem 53. *Given a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *and a function* $\nu : X \to \mathbb{R}$ *strictly positive around* $\bar{x}$*, suppose that* F^{-1} *is closed at* $\bar{y}$ *and there is an* $\varepsilon > 0$ *such that, for any* $x \in B(\bar{x},\varepsilon) \setminus F^{-1}(\bar{y})$ *and* $t \in [d(\bar{y}, F(x)), r^{-1/k}\nu(x)^{1/k})$*, one of the sets of conditions in Theorem* 50 *is satisfied. Then,* F *is metrically* ν*-subregular at* $(\bar{x}, \bar{y})$ *of order* k *with modulus* r*.*

Corollary 14. *Given a subset* $U \subset X$ *and a point* $\bar{y} \in Y$*, suppose that* F^{-1} *is closed at* $\bar{y}$ *and, for any* $x \in U \setminus F^{-1}(\bar{y})$*, for some* $\gamma > d(\bar{y}, F(x))$ *and any* $t \in [d(\bar{y}, F(x)), \gamma)$*, there is a constant* $\lambda \in (0,1)$ *such that, for any* $\tau \in (0,t]$*,*

$$d\left(u, F^{-1}(B(\bar{y}, \lambda\tau))\right) < r(1-\lambda^k)\tau^k$$
$$\text{for all } u \in F^{-1}(B(\bar{y},\tau)) \cap B(x, r(t^k - \tau^k)). \tag{6.15}$$

Then, F *is metrically subregular on* U *at* $\bar{y}$ *of order* k *with modulus* r*.*

Proof. The conclusion follows from Theorem 50 since conditions (6.4) and (6.5) automatically hold true for the two functions $b(\cdot) = \lambda(\cdot)$ and $m(\cdot) = r(1-\lambda^k)(\cdot)^k$. □

Corollary 15. *Given a subset $U \subset X$ and a function $\nu : U \to (0, \infty]$, suppose that F^{-1} is closed at $\bar{y}$ and, for any $x \in U \setminus F^{-1}(\bar{y})$ and $t \in [d(\bar{y}, F(x)), r^{-1/k}\nu(x)^{1/k})$, there is a constant $\lambda \in (0,1)$ such that, for any $\tau \in (0,t]$, condition (6.15) is satisfied. Then, F is metrically ν-subregular on U at $\bar{y}$ of order k with modulus r.*

Corollary 16. *Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, suppose that F^{-1} is closed at $\bar{y}$ and there is an $\varepsilon > 0$ such that, for any $x \in B(\bar{x}, \varepsilon) \setminus F^{-1}(\bar{y})$, there is a constant $\lambda \in (0,1)$ such that, for any $t \in (0, \varepsilon)$,*

$$d\left(u, F^{-1}(B(\bar{y}, \lambda t))\right) < r(1-\lambda^k)t^k \quad \text{for all } u \in F^{-1}(B(\bar{y}, t)) \cap B(\bar{x}, \varepsilon). \tag{6.16}$$

Then, F is metrically subregular at $(\bar{x}, \bar{y})$ of order k with modulus r.

Proof. Choose a number $\delta > 0$ such that $2\delta + r^{-1/k}\delta^{1/k} \le \varepsilon$. Take any $x \in B(\bar{x}, \delta) \setminus F^{-1}(\bar{y})$. If $\delta \le rd^k(\bar{y}, F(x))$, then $d(x, F^{-1}(\bar{y})) \le rd^k(\bar{y}, F(x))$ since $d(x, F^{-1}(\bar{y})) \le d(x, \bar{x}) \le \delta$. Otherwise, we define $\gamma := r^{-1/k}\delta^{1/k} > d(\bar{y}, F(x))$. The conclusion then follows from Corollary 14 thanks to the observation that for any $t < \gamma$ and $\tau \le t$, it holds $B(x, r(t^k - \tau^k)) \subset B(x, rt^k) \subset B(x, r\gamma^k) = B(x, \delta) \subset B(\bar{x}, \varepsilon)$. ∎

Corollary 17. *Given a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a function $\nu : X \to \mathbb{R}$ strictly positive around $\bar{x}$, suppose that F^{-1} is closed at $\bar{y}$ and there is an $\varepsilon > 0$ such that, for any $x \in B(\bar{x}, \varepsilon) \setminus F^{-1}(\bar{y})$ with $rd^k(\bar{y}, F(x)) < \nu(x)$, there is a constant $\lambda \in (0,1)$ such that, for any $t \in (0, \varepsilon)$, condition (6.16) is satisfied. Then, F is metrically ν-subregular at $(\bar{x}, \bar{y})$ of order k with modulus r.*

Theorem 54. *Given a subset $U \subset X$, suppose that F is upper semicontinuous and, for any $x \in U \setminus F^{-1}(\bar{y})$ and $u \in X$ with $d(\bar{y}, F(u)) > 0$ and $rd^k(\bar{y}, F(u)) + d(u, x) \le rd^k(\bar{y}, F(x))$, there exists a point $u' \neq u$ such that*

$$\mu(d(\bar{y}, F(u'))) \le rd^k(\bar{y}, F(u)) - d(u, u'). \tag{6.17}$$

Then, F is metrically subregular on U at $\bar{y}$ of order k with modulus r.

Theorem 55. *Given a subset $U \subset X$ and a function $\nu : U \to (0, \infty]$, suppose that F is upper semicontinuous and, for any $x \in U \setminus F^{-1}(\bar{y})$ and $u \in X$ with $0 < rd^k(\bar{y}, F(u)) < \nu(u)$ and*

$rd^k(\bar{y}, F(u)) + d(u, x) \leq rd^k(\bar{y}, F(x))$, *there exists a point* $u' \neq u$ *such that condition* (6.17) *is satisfied. Then,* F *is metrically* ν*-subregular on* U *at* $\bar{y}$ *of order* k *with modulus* r*.*

Theorem 56. *Given a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$*, suppose that* F *is upper semicontinuous and there is an* $\varepsilon > 0$ *such that, for any* $u \in B(\bar{x}, \varepsilon) \setminus F^{-1}(\bar{y})$*, there exists a point* $u' \neq u$ *such that condition* (6.17) *is satisfied. Then,* F *is metrically subregular at* $(\bar{x}, \bar{y})$ *of order* k *with modulus* r*.*

Theorem 57. *Given a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *and a function* $\nu : X \to (0, \infty]$ *Lipschitz continuous with modulus not greater than 1 around* $\bar{x}$*, suppose that* F *is upper semicontinuous and there is an* $\varepsilon > 0$ *such that, for any* $u \in B(\bar{x}, \varepsilon) \setminus F^{-1}(\bar{y})$ *with* $rd^k(\bar{y}, F(u)) < \nu(u)$*, there exists a point* $u' \neq u$ *such that condition* (6.17) *is satisfied. Then,* F *is metrically* ν*-subregular at* $(\bar{x}, \bar{y})$ *of order* k *with modulus* r*.*

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Chapter 7

On Hölder calmness of solution mappings of parametric equilibrium problems

We consider parametric equilibrium problems in metric spaces. Sufficient conditions for the Hölder calmness of solutions are established. We also study the Hölder well-posedness for equilibrium problems in metric spaces.

7.1 Introduction

Optimization is one of the most fertile areas of mathematics. Its conclusions and recommendations play a very important role in both theoretical and applied mathematics. Equilibrium problems were first considered in [15] and since then have been studied by many researchers all over the world. The equilibrium problem model incorporates many other important problems in optimization and other areas such as: variational inequalities, fixed point problems, complementarity, etc. There have been many studies of existence of solutions to equilibrium problems (see [11, 14, 18, 19, 20, 30]) and their stability, e.g., semi-continuity in the sense of Berge and Hausdorff (see [3, 5, 6, 10, 22, 24]) or Hölder (Lipschitzian) continuity (see [1, 4, 7, 9, 13, 27, 28, 29].)

This chapter extends [2] and studies $(l.\alpha)$-*Hölder calmness* of solutions to parametric equilibrium problems. When $\alpha = 1$, this is a kind of calmness property which is in general stronger than the property of the same name usually used in variational analysis. Calmness property of multi-valued mappings has been examined by many authors (see [16, 17, 21, 23, 26, 32]) in which subdifferentials and coderivatives play the main role. As applications we investigate conditions for Hölder calmness of solutions to optimization problems and well-posedness in the Hölder sense. The last subject is intimately related to the stability property and plays a very important role in studying optimization and variational problems.

The structure of the chapter is as follows. Section 2 presents the equilibrium problem model and materials used in the rest of this chapter. We establish in Section 3 a sufficient condition for the Hölder calmness of the solution mapping to parametric equilibrium problems. The Hölder well-posedness of equilibrium problems is studied in Section 4.

Throughout the chapter, if not explicitly stated otherwise, X, Λ, M are metric spaces and $\mathbb{R}$ is the set of all real numbers while $\mathbb{R}_+$ is the set of all positive numbers. We use $d(\cdot,\cdot)$ for all metrics.

7.2 Preliminaries

Given a subset $K \subseteq X$ and a function $f : X \times X \to \mathbb{R}$, a standard *equilibrium problem* is defined as follows:

(EP) find $\bar{x} \in K$ such that $f(\bar{x}, y) \geq 0$ for all $y \in K$.

The set of solutions to this problem is denoted by S.

In this chapter, we consider several extensions of (EP).

The constraint set K and objective function f can be perturbed by parameters $\lambda \in \Lambda$ and $\mu \in M$, respectively. Given a multi-valued mapping $K : \Lambda \rightrightarrows X$, a function $f : X \times X \times M \to \mathbb{R}$, and a pair $(\lambda, \mu) \in \Lambda \times M$, one can consider a parameterized equilibrium problem:

$(EP)_{\lambda,\mu}$ find $\bar{x} \in K(\lambda)$ such that $f(\bar{x}, y, \mu) \geq 0$ for all $y \in K(\lambda)$.

The set of solutions to problem $(EP)_{\lambda,\mu}$ is denoted by $S(\lambda, \mu)$.

The approximate version of this problem can be of interest: for each $(\lambda, \mu) \in \Lambda \times M$ and $\varepsilon > 0$,

$(\widetilde{EP})_{\varepsilon,\lambda,\mu}$ find $\bar{x} \in K(\lambda)$ such that $f(\bar{x}, y, \mu) + \varepsilon \geq 0$ for all $y \in K(\lambda)$.

We denote by $\widetilde{S}(\varepsilon, \lambda, \mu)$ the solution set of $(\widetilde{EP})_{\varepsilon,\lambda,\mu}$.

Definition 28. For a function $f : X \to \mathbb{R}$ and positive numbers l, α,

(i) f is $(l.\alpha)$-Hölder continuous on a subset $U \subset X$ if

$$|f(x_1) - f(x_2)| \leq ld^{\alpha}(x_1, x_2) \text{ for all } x_1, x_2 \in U;$$

(ii) f is $(l.\alpha)$-Hölder calm at $\bar{x}$ on a neighborhood U of $\bar{x}$ if

$$|f(x) - f(\bar{x})| \leq ld^{\alpha}(x, \bar{x}) \text{ for all } x \in U.$$

We say that f satisfies a certain property on a subset $A \subseteq X$ if it is satisfied at every point of A.

From this definition, it is obvious that Hölder continuity is stronger than Hölder calmness.

To define extensions of these properties for multi-valued mappings we recall the definitions of point-to-set and set-to-set distances.

For subsets A, B of X and a point $a \in X$,

$$d(a, B) := \inf_{b \in B} d(a, b);$$

$$H^*(A, B) := \sup_{a \in A} d(a, B);$$

$$H(A, B) := \max\{H^*(A, B), H^*(B, A)\};$$

$$\rho(A, B) := \sup_{a \in A, b \in B} d(a, b).$$

Note that H and ρ can take infinite values (if A or B is unbounded). It is also obvious that $H(A, B) \leq \rho(A, B)$ for any subsets A and B, and the inequality can be strict.

Definition 29. For a multi-valued mapping $K : \Lambda \rightrightarrows X$ and positive numbers l, α,

(i) K is $(l.\alpha)$-Hölder continuous on a subset $U \subset X$ if

$$H(K(\lambda_1), K(\lambda_2)) \leq ld^{\alpha}(\lambda_1, \lambda_2) \text{ for all } \lambda_1, \lambda_2 \in U;$$

(ii) K is $(l.\alpha)$-Hölder calm at $\bar{\lambda}$ on a neighborhood U of $\bar{\lambda}$ if

$$H(K(\lambda), K(\bar{\lambda})) \leq l d^{\alpha}(\lambda, \bar{\lambda}) \text{ for all } \lambda \in U. \tag{7.1}$$

We will also consider the versions of the properties in Definition 29 with H replaced by ρ. In this case, we will talk about the corresponding properties *with respect to* ρ.

Remark 58. The calmness in the above definition (when $\alpha = 1$) is a stronger property than the one usually considered in variational analysis. The latter corresponds to replacing H in (7.1) by H^* (see, e.g., [31]). Respectively, (l, α)-calmness is stronger than the so-called calmness $[\alpha]$ in [25].

We next define *uniform Hölder calmness* as the natural counterpart of the relative Hölder continuity in [6].

Definition 30. For positive numbers m, β, θ, a function $f : X \times X \times M \to \mathbb{R}$ is $(m.\beta)$-Hölder calm at $\bar{\mu}$ on a neighborhood V of $\bar{\mu}$, θ-uniformly over a subset $S \subseteq X$ if

$$|f(x, y, \bar{\mu}) - f(x, y, \mu)| \leq m d^{\beta}(\bar{\mu}, \mu) d^{\theta}(x, y), \quad \forall \mu \in V, \forall x, y \in S, x \neq y.$$

If $\theta = 0$, we say that f is $(m.\beta)$-Hölder calm at $\bar{\mu}$ on V, uniformly over S.

We next discuss several monotonicity properties some of which are going to play a crucial role in examining the Hölder calmness of the solution mapping of the equilibrium problems $(EP)_{\lambda,\mu}$.

Given a function $f : X \times X \to \mathbb{R}$, positive numbers h, β, and a subset $S \subseteq X$, consider the following properties.

(M_1) For all $x, y \in S, x \neq y$,

$$f(x, y) + f(y, x) + h d^{\beta}(x, y) \leq 0. \tag{7.2}$$

(M_2) For all $x, y \in S$,

$$h d^{\beta}(x, y) \leq d(f(x, y), \mathbb{R}_+) + d(f(y, x), \mathbb{R}_+). \tag{7.3}$$

(M_3) For all $x, y \in S, x \neq y$,

$$[f(x, y) \geq 0 \Longrightarrow f(y, x) + hd^\beta(x, y) \leq 0].$$

(M_4) For all $x, y \in S, x \neq y$,

$$[f(x, y) < 0 \Longrightarrow f(y, x) \geq 0].$$

If any of the above properties is fulfilled, we say that f satisfies the corresponding condition on S with constants h and β (if applicable).

Remark 59. Properties (M_1), (M_3) and (M_4) were considered in [4, 6, 8] where they were called *Hölder strong monotonicity*, *Hölder strong pseudo-monotonicity* and *quasi-monotonicity*, respectively. Property (M_2) is a particular case of the corresponding monotonicity property introduced in [6] for multi-valued mappings. This property has been employed to investigate the Hölder continuity of solution mappings in many articles (see [2, 8, 28].)

The next proposition gives the relationships between these monotonicity properties.

Proposition 45. *(i)* $(M_1) \Rightarrow (M_2) \Rightarrow (M_3)$*;*

(ii) $[(M_3)\ \&\ (M_4)] \Rightarrow (M_2)$.

Proof. The following simple observation is used in the proof:

$$d(a, \mathbb{R}_+) = \max\{-a, 0\} \geq -a.$$

$(M_1) \Rightarrow (M_2)$. If (7.2) holds for some $x \neq y$, then

$$hd^\beta(x, y) \leq -f(x, y) - f(y, x) \leq d(f(x, y), \mathbb{R}_+) + d(f(y, x), \mathbb{R}_+),$$

i.e., (7.3) holds. When $x = y$, (7.3) holds automatically.

$(M_2) \Rightarrow (M_3)$. If (7.3) holds for some $x \neq y$ and $f(x, y) \geq 0$, then $d(f(x, y), \mathbb{R}_+) = 0$ and (7.3) takes the form

$$hd^\beta(x, y) \leq d(f(y, x), \mathbb{R}_+).$$

It follows from the last inequality that $d(f(y,x),\mathbb{R}_+) > 0$ and consequently $d(f(y,x),\mathbb{R}_+) = -f(y,x)$. Hence, (M_3) holds true.

$[(M_3)\ \&\ (M_4)] \Rightarrow (M_2)$. Let (M_3) and (M_4) hold true. We only need to prove (7.3) when $x \neq y$. If $f(x,y) \geq 0$, then $d(f(x,y),\mathbb{R}_+) = 0$ and (M_3) implies

$$0 < hd^{\beta}(x,y) \leq -f(y,x) = d(f(y,x),\mathbb{R}_+).$$

Hence, (7.3) is true. If $f(x,y) < 0$, then (M_4) implies $f(y,x) \geq 0$, and we can apply (M_3) again to show that

$$0 < hd^{\beta}(x,y) \leq -f(x,y) = d(f(x,y),\mathbb{R}_+).$$

Taking into account that $d(f(y,x),\mathbb{R}_+) = 0$, we conclude that (7.3) is true in this case too. □

We now give examples showing that implications in Proposition 45 can be strict.

Example 14. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,y) = x - y$ satisfies (M_2) with $h = \beta = 1$. Indeed,

$$d(f(x,y),\mathbb{R}_+) + d(f(y,x),\mathbb{R}_+) = d(x-y,\mathbb{R}_+) + d(y-x,\mathbb{R}_+) = |x-y| = d(x,y).$$

At the same time, $f(x,y) + f(y,x) = 0$ and (7.2) is violated for any $x \neq y$. f does not satisfy (M_1). It is also obvious that f satisfies both (M_3) and (M_4).

Example 15. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,y) = -\dfrac{1}{4}(|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}})$ satisfies (M_3) with $h = \sqrt{2}$ and $\beta = \frac{1}{2}$ as $f(x,y) \geq 0$ if and only if $x = y = 0$, it does not satisfy (M_2). Indeed, for any $y = -x \neq 0$, we have

$$d(f(x,y),\mathbb{R}_+) + d(f(y,x),\mathbb{R}_+) = \frac{1}{2}(|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}}) = |x|^{\frac{1}{2}} < 2|x|^{\frac{1}{2}} = \sqrt{2}d^{\frac{1}{2}}(x,y).$$

We can see that the combination of (M_3) and (M_4) implies (M_2), but they are not equivalent by considering the function $f(x,y) = -(|x| + |y|)$. This function satisfies (M_2) with $h = \beta = 1$, but breaks (M_4).

7.3 The Hölder calmness of the solution mapping

The next theorem gives a sufficient condition for the Hölder calmness of the solution mapping of the problem $(EP)_{\lambda,\mu}$. It improves Theorem 2.1 in [2]. We always assume that solution sets $S(\lambda,\mu)$ are nonempty for all (λ,μ) in a neighborhood of the considered point $(\bar{\lambda},\bar{\mu})$.

Theorem 58. *Consider equilibrium problem $(EP)_{\lambda,\mu}$ and suppose the following conditions hold.*

(i) There exist neighborhoods $U(\bar{\lambda})$ of $\bar{\lambda}$ and $V(\bar{\mu})$ of $\bar{\mu}$ and positive numbers n_1, δ_1 and θ such that f is $(n_1.\delta_1)$-Hölder calm at $\bar{\mu}$ on $V(\bar{\mu})$, θ-uniformly over $K(U(\bar{\lambda}))$.

(ii) There exist positive numbers n_2 and δ_2 such that, for all $x \in K(U(\bar{\lambda}))$ and $\mu \in V(\bar{\mu})$, the function $f(x,\cdot,\mu)$ is $(n_2.\delta_2)$-Hölder continuous on $K(U(\bar{\lambda}))$.

(iii) $f(\cdot,\cdot,\bar{\mu})$ satisfies condition (M_2) on $K(U(\bar{\lambda}))$ with constants $h>0$ and $\beta>\theta$.

(iv) K is $(l.\alpha)$-Hölder calm at $\bar{\lambda}$ on $U(\bar{\lambda})$ with some positive l and α.

Then solutions to $(EP)_{\lambda,\mu}$ satisfy the condition of Hölder calmness with respect to ρ: there exist constants $k_1,k_2>0$ such that

$$\rho\left(S(\bar{\lambda},\bar{\mu}),S(\lambda,\mu)\right) \le k_1 d^{\alpha\delta_2/\beta}(\bar{\lambda},\lambda) + k_2 d^{\delta_1/(\beta-\theta)}(\bar{\mu},\mu),$$

for all (λ,μ) in a neighborhood of $(\bar{\lambda},\bar{\mu})$.

Proof. Take $\lambda \in U(\bar{\lambda})$ and $\mu \in V(\bar{\mu})$.

Step 1 We prove that for each $x(\lambda,\bar{\mu}) \in S(\lambda,\bar{\mu})$ and $x(\lambda,\mu) \in S(\lambda,\mu)$,

$$d_1 := d\left(x(\lambda,\bar{\mu}),x(\lambda,\mu)\right) \le \left(\frac{n_1}{h}\right)^{1/(\beta-\theta)} d^{\delta_1/(\beta-\theta)}(\bar{\mu},\mu). \tag{7.4}$$

Suppose $x(\lambda,\bar{\mu}) \neq x(\lambda,\mu)$ (if the equality holds, then (7.4) holds trivially). Because both

$x(\lambda, \bar{\mu})$ and $x(\lambda, \mu)$ belong to $K(\lambda)$ and are solutions of $(EP)_{\lambda,\mu}$, one has

$$f\left(x(\lambda, \bar{\mu}), x(\lambda, \mu), \bar{\mu}\right) \geq 0; \tag{7.5}$$

$$f\left(x(\lambda, \mu), x(\lambda, \bar{\mu}), \mu\right) \geq 0. \tag{7.6}$$

At the same time, (iii) implies

$$d\left(f\left(x(\lambda, \bar{\mu}), x(\lambda, \mu), \bar{\mu}\right), \mathbb{R}_+\right) + d\left(f\left(x(\lambda, \mu), x(\lambda, \bar{\mu}), \bar{\mu}\right), \mathbb{R}_+\right) \geq h d_1^{\beta}.$$

Combining this inequality with (7.5) and (7.6), we get

$$d\left(f\left(x(\lambda, \mu), x(\lambda, \bar{\mu}), \bar{\mu}\right), f\left(x(\lambda, \mu), x(\lambda, \bar{\mu}), \mu\right)\right) \geq h d_1^{\beta}.$$

Because f is $(n_1.\delta_1)$-Hölder calm at $\bar{\mu}$, θ-uniformly over $K(U(\bar{\lambda}))$ by (i), the above relationship implies

$$n_1 d_1^{\theta} d^{\delta_1}(\bar{\mu}, \mu) \geq h d_1^{\beta}.$$

This is equivalent to $d_1^{\beta-\theta} \leq \dfrac{n_1}{h} d^{\delta_1}(\bar{\mu}, \mu)$ from which we get (7.4) proved.

Step 2 We prove that for each $x(\bar{\lambda}, \bar{\mu}) \in S(\bar{\lambda}, \bar{\mu})$ and $x(\lambda, \bar{\mu}) \in S(\lambda, \bar{\mu})$,

$$d_2 := d\left(x(\bar{\lambda}, \bar{\mu}), x(\lambda, \bar{\mu})\right) \leq \left(\frac{2n_2 l^{\delta_2}}{h}\right)^{1/\beta} d^{\alpha\delta_2/\beta}(\bar{\lambda}, \lambda). \tag{7.7}$$

Suppose $x(\bar{\lambda}, \bar{\mu}) \neq x(\lambda, \bar{\mu})$. (iv) implies that there exist $\bar{x} \in K(\bar{\lambda})$ and $x \in K(\lambda)$ such that

$$d\left(x(\bar{\lambda}, \bar{\mu}), x\right) \leq l d^{\alpha}(\bar{\lambda}, \lambda); \tag{7.8}$$

$$d\left(x(\lambda, \bar{\mu}), \bar{x}\right) \leq l d^{\alpha}(\bar{\lambda}, \lambda). \tag{7.9}$$

We get from the definition of $(EP)_{\lambda,\mu}$,

$$f\left(x(\bar{\lambda}, \bar{\mu}), \bar{x}, \bar{\mu}\right) \geq 0; \tag{7.10}$$

$$f\left(x(\lambda, \bar{\mu}), x, \bar{\mu}\right) \geq 0. \tag{7.11}$$

At the same time, (iii) implies

$$d\left(f\left(x(\bar{\lambda},\bar{\mu}),x(\lambda,\bar{\mu}),\bar{\mu}\right),\mathbb{R}_+\right)+d\left(f\left(x(\lambda,\bar{\mu}),x(\bar{\lambda},\bar{\mu}),\bar{\mu}\right),\mathbb{R}_+\right)\geq hd_2^{\beta}.$$

Combining this inequality with (7.10) and (7.11), we get

$$d\left(f\left(x(\bar{\lambda},\bar{\mu}),x(\lambda,\bar{\mu}),\bar{\mu}\right),f\left(x(\bar{\lambda},\bar{\mu}),\bar{x},\bar{\mu}\right)\right)$$
$$+d\left(f\left(x(\lambda,\bar{\mu}),x(\bar{\lambda},\bar{\mu}),\bar{\mu}\right),f\left(x(\lambda,\bar{\mu}),x,\bar{\mu}\right)\right)\geq hd_2^{\beta}.$$

Because f is $(n_2\delta_2)$-Hölder continuous with respect to the second component in $K(U(\bar{\lambda}))$ by (ii), the last inequality implies that

$$n_2 d^{\delta_2}\left(x(\lambda,\bar{\mu})),\bar{x}\right)+n_2 d^{\delta_2}\left(x(\bar{\lambda},\bar{\mu}),x\right)\geq hd_2^{\beta}.$$

We combine this with (7.8) and (7.9) and get

$$n_2 l^{\delta_2} d^{\alpha\delta_2}(\bar{\lambda},\lambda)+n_2 l^{\delta_2} d^{\alpha\delta_2}(\bar{\lambda},\lambda)\geq hd_2^{\beta},$$

or equivalently $d_2^{\beta}\leq\dfrac{2n_2 l^{\delta_2}}{h}d^{\alpha\delta_2}(\bar{\lambda},\lambda)$. We have (7.7) proved.

Step 3 For all $x(\bar{\lambda},\bar{\mu})\in S(\bar{\lambda},\bar{\mu})$ and $x(\lambda,\mu)\in S(\lambda,\mu)$, we always have

$$d\left(x(\bar{\lambda},\bar{\mu}),x(\lambda,\mu)\right)\leq d_1+d_2.$$

From (7.4) and (7.7), by taking $k_1=\left(\dfrac{2n_2 l^{\delta_2}}{h}\right)^{1/\beta}$ and $k_2=\left(\dfrac{n_1}{h}\right)^{1/(\beta-\theta)}$, we get

$$\rho\left(S(\bar{\lambda},\bar{\mu}),S(\lambda,\mu)\right)\leq k_1 d^{\alpha\delta_2/\beta}(\bar{\lambda},\lambda)+k_2 d^{\delta_1/(\beta-\theta)}(\bar{\mu},\mu).$$

Therefore, Theorem 58 has been proved. □

By using the technique similar to the one in the proof of Theorem 2.1 in [6], we can show that, under assumption (iii), the solution to $(EP)_{\bar{\lambda},\bar{\mu}}$ is unique. However, when $(\lambda,\mu)\neq(\bar{\lambda},\bar{\mu})$, the solutions to $(EP)_{\lambda,\mu}$ do not have to be unique as demonstrated by the following example.

Example 16. Let $X=\mathbb{R}$, $\Lambda\equiv M=[0,1]$, $K(\lambda)=[0,1]$, $f(x,y,\lambda)=y-x+\lambda$ for all $\lambda\in\Lambda$, and $\bar{\lambda} = 0$.

Then $|f(x, y, \lambda) - f(x, y, \bar{\lambda})| = |\lambda|$.

Hence, f is (1.1)-Hölder calm at $\bar{\lambda}$ uniformly over $[0, 1]$. We have $|f(x, y, \lambda) - f(x, z, \lambda)| = |y - z|$ for all $y, z \in [0, 1]$.

So $f(x, \cdot, \lambda)$ is (1.1)-Hölder continuous on $[0, 1]$. Therefore, assumptions (i) and (ii) hold. It is clear that condition (iv) also holds. Assumption (iii) is fulfilled as shown in Example 14. Hence, Theorem 58 derives the Holder calmness of $S(\cdot)$ at $\bar{\lambda}$.

It is not difficult to check that $S(0) = \{0\}$ and $S(\lambda) = [0, \lambda]$ for all $\lambda \in (0, 1]$.

Normally, to receive a property of solution mappings, the problem's hypotheses are also required at the level corresponding to that property. We can see from the preceding theorem that all the hypotheses are related to Hölder continuity and Hölder calmness, except (iii), which is about monotonicity.

The next example indicates the essential role of assumption (iii) in Theorem 58.

Example 17. Take $X = \mathbb{R}$, $M \equiv \Lambda = [0, 1]$, $K(\lambda) = [-1, 1]$ for all $\lambda \in [0, 1]$. For each $\lambda \in [0, 1]$, consider the function f defined by $f(x, y, \lambda) = \lambda(x + y)$. Take $\bar{\lambda} = 0$.

We have $|f(x, y, \lambda) - f(x, y, \bar{\lambda})| = |x+y| \cdot |\lambda - \bar{\lambda}| \leq 2|\lambda - \bar{\lambda}|$ for all $x, y \in [-1, 1]$. So f is (2.1)-Hölder calm at $\bar{\lambda}$ on $[0, 1]$ uniformly over $[-1, 1]$. At the same time, $|f(x, y, \lambda) - f(x, z, \lambda)| = |\lambda| \cdot |y - z| \leq |y - z|$ for all $y, z \in [-1, 1]$. This means that $f(x, \cdot, \lambda)$ is (1.1)-Hölder continuous on $[-1, 1]$. Hence, conditions (i) and (ii) are fulfilled.

Condition (iv) is also true straightforwardly. However, we have

$$S(0) = [-1, 1], \ S(\lambda) = \{1\}, \forall \lambda \in (0, 1].$$

So $\rho(S(\lambda), S(0)) = 2$ for any $\lambda \in (0, 1]$.

Therefore, the solution mapping S is not Hölder calm at $\bar{\mu} = 0$. The reason here is that f breaks condition (M_2). Indeed,

$$d(f(1, 0, 0), \mathbb{R}_+) + d(f(0, 1, 0), \mathbb{R}_+) = 0 < h|1 - 0|^{\beta} = h, \forall h, \beta > 0.$$

Condition (M_2) in Theorem 58 is indispensable.

Remark 60. It follows from Proposition 45 that the conclusion of Theorem 58 remains true if condition (iii) is replaced by either condition (M_1) or conditions (M_3) and (M_4).

The next proposition aims to illustrate application of Theorem 58. For each $(\lambda, \mu) \in \Lambda \times M$, we consider the minimization problem:

(MP) minimize $f(x, \mu)$ subject to $x \in K(\lambda)$,

where $f : X \times M \to \mathbb{R}$ and $K : \Lambda \rightrightarrows X$.

We denote $S(\lambda, \mu) = \left\{\bar{x} \in K(\lambda) : f(\bar{x}, \mu) = \min_{x \in K(\lambda)} f(x, \mu)\right\}$ and assume that $S(\lambda, \mu) \neq \emptyset$ for all (λ, μ) near the considered point $(\bar{\lambda}, \bar{\mu})$.

Proposition 46. *Consider (MP) and suppose the following conditions hold.*

(i) There exist neighborhoods $V(\bar{\mu})$ of $\bar{\mu}$ and $U(\bar{\lambda})$ of $\bar{\lambda}$ and numbers $n_1 > 0$ and $\delta_1 > 0$ such that f is $(n_1.\delta_1)$-Hölder calm at $\bar{\mu}$ on $V(\bar{\mu})$ uniformly over $K(U(\bar{\lambda}))$, i.e.,

$$|f(x, \mu) - f(x, \bar{\mu})| \leq n_1 d^{\delta_1}(\mu, \bar{\mu})$$

for all $x \in K(U(\bar{\lambda}))$ and $\mu \in V(\bar{\mu})$.

(ii) There exist numbers $n_2 > 0$ and $\delta_2 > 0$ such that f is $(n_2.\delta_2)$-Hölder continuous in x on $K(U(\bar{\lambda}))$ uniformly over $\mu \in V(\bar{\mu})$, i.e.,

$$|f(x, \mu) - f(y, \mu)| \leq n_2 d^{\delta_2}(x, y) \tag{7.12}$$

for all $\mu \in V(\bar{\mu})$ and $x, y \in K(U(\bar{\lambda}))$, and (7.12) holds as an equality when $\mu = \bar{\mu}$.

(iii) K is $(l.\alpha)$-Hölder calm at $\bar{\lambda}$ on $U(\bar{\lambda})$ with some $l > 0$ and $\alpha > 0$.

Then the mapping S is Hölder calm with respect to ρ, i.e., there exist constants $k_1, k_2 > 0$ such that

$$\rho\left(S(\bar{\lambda}, \bar{\mu}), S(\lambda, \mu)\right) \leq k_1 d^{\alpha}(\bar{\lambda}, \lambda) + k_2 d(\bar{\mu}, \mu) \tag{7.13}$$

for all (λ, μ) in a neighborhood of $(\bar{\lambda}, \bar{\mu})$.

Proof. We define the function $g : X \times X \times M \to \mathbb{R}$ as follows

$$g(x, y, \mu) = f(y, \mu) - f(x, \mu).$$

We observe that $\bar{x} \in S(\lambda, \mu)$ if and only if $\bar{x} \in K(\lambda)$ and $g(\bar{x}, y, \mu) \geq 0, \forall y \in K(\lambda)$. So to prove the proposition, it suffices to check that g satisfies the conditions of Theorem 58.

We first check condition (i). For every $\mu \in V(\bar{\mu})$ and $x, y \in K(U(\bar{\lambda}))$ we have

$$\begin{aligned}|g(x, y, \mu) - g(x, y, \bar{\mu})| &= |f(y, \mu) - f(x, \mu) - f(y, \bar{\mu}) + f(x, \bar{\mu})| \\ &\leq |f(x, \mu) - f(x, \bar{\mu})| + |f(y, \mu) - f(y, \bar{\mu})| \leq 2n_1 d^{\delta_1}(\mu, \bar{\mu}).\end{aligned}$$

This means that g is $(2n_1.\delta_1)$-Hölder calm at $\bar{\mu}$ on $V(\bar{\mu})$ uniformly over $K(U(\bar{\lambda}))$.

We have at the same time

$$|g(x, y, \mu) - g(x, z, \mu)| = |f(y, \mu) - f(z, \mu)| \leq n_2 d^{\delta_2}(y, z),$$

i.e., g is $(n_2.\delta_2)$-Hölder continuous with respect to the second component. So conditions (i) and (ii) in Theorem 58 are fulfilled.

We now check condition (iii) in Theorem 58. For all $x, y \in K(U(\bar{\lambda}))$, we have

$$\begin{aligned}&d\left(g(x, y, \bar{\mu}), \mathbb{R}_+\right) + d\left(g(y, x, \bar{\mu}), \mathbb{R}_+\right) \\ &= d\left(f(y, \bar{\mu}) - f(x, \bar{\mu}), \mathbb{R}_+\right) + d\left(f(x, \bar{\mu}) - f(y, \bar{\mu}), \mathbb{R}_+\right) \\ &= |f(x, \bar{\mu}) - f(y, \bar{\mu})| = n_2 d^{\delta_2}(x, y).\end{aligned}$$

So g satisfies condition (M_2), and (iii) in Theorem 58 is fulfilled. Therefore, it follows from Theorem 58 that (7.13) holds true with some $k_1, k_2 > 0$. □

7.4 The Hölder well-posedness of equilibrium problems

We will denote by $(\mathcal{EP})$ the family of problems $\{(EP)_{\lambda,\mu} : (\lambda, \mu) \in \Lambda \times M\}$ and extend the concept of Lipschitzian well-posedness for optimization problems introduced in [12] to equilibrium problems.

Definition 31. $(\mathcal{EP})$ is Hölder well-posed at $(\bar{\lambda}, \bar{\mu})$ if $\widetilde{S}(0, \bar{\lambda}, \bar{\mu})$ is a singleton and $\widetilde{S}$ is Hölder calm at $(0, \bar{\lambda}, \bar{\mu})$ on a neighborhood of $(0, \bar{\lambda}, \bar{\mu})$.

The next theorem gives a sufficient condition for the Hölder well-posedness of $(\mathcal{EP})$. It improves and modifies Theorem 3.1 in [2].

Theorem 59. *Assume $S(\bar{\lambda}, \bar{\mu}) \neq \emptyset$ and the following conditions hold.*

(i) There exist neighborhoods $U(\bar{\lambda})$ of $\bar{\lambda}$ and $V(\bar{\mu})$ of $\bar{\mu}$ and positive numbers n_1, δ_1 and θ such that f is $(n_1.\delta_1)$-Hölder calm at $\bar{\mu}$ on $V(\bar{\mu})$, θ-uniformly over $K(U(\bar{\lambda}))$.

(ii) There exist positive numbers n_2 and δ_2 such that, for all $x \in K(U(\bar{\lambda}))$ and $\mu \in V(\bar{\mu})$, the function $f(x, \cdot, \mu)$ is $(n_2.\delta_2)$-Hölder continuous on $K(U(\bar{\lambda}))$.

(iii) $f(\cdot, \cdot, \bar{\mu})$ satisfies condition (M_2) on $K(U(\bar{\lambda}))$ with constants $h > 0$ and $\beta > \theta$.

(iv) K is $(l.\alpha)$-Hölder calm at $\bar{\lambda}$ on $U(\bar{\lambda})$ with some positive l and α.

Then $(\mathcal{EP})$ is Hölder well-posed at $(\bar{\lambda}, \bar{\mu})$.

Proof. Take $N = [0, +\infty) \times M$. For $\eta = (\varepsilon, \mu), \eta' = (\varepsilon', \mu') \in N$, consider a function d_N defined by

$$d_N(\eta, \eta') = \max\{|\varepsilon - \varepsilon'|, d(\mu, \mu')\}.$$

Then, (N, d_N) is a metric space. We define a function $g : X \times X \times N \to \mathbb{R}$ as follows

$$g(x, y, \eta) = f(x, y, \mu) + \varepsilon.$$

To prove the theorem, it suffices to check that g satisfies the conditions of Theorem 58.

Take any neighborhood W of 0 in $[0, 1]$. Then for all $\eta = (\varepsilon, \mu) \in W \times V(\bar{\mu})$, $\bar{\eta} = (0, \bar{\mu})$, and $x, y \in K(U(\bar{\lambda}))$, one has

$$\begin{aligned} |g(x, y, \eta) - g(x, y, \bar{\eta})| &= |f(x, y, \mu) - f(x, y, \bar{\mu}) + \varepsilon| \\ &\leq \varepsilon + |f(x, y, \mu) - f(x, y, \bar{\mu})| \leq \varepsilon + n_1 d^{\delta_1} d(\mu, \bar{\mu}) \\ &\leq \varepsilon^{\delta_1} + n_1 d^{\delta_1}(\mu, \bar{\mu}) \leq 2 \max\{1, n_1\} d_N^{\delta_1}(\eta, \bar{\eta}) \end{aligned}$$

since $\varepsilon \in V \subseteq [0,1]$ and the Hölder order $\delta_1 \leq 1$. So g is $(2\max\{1,n_1\}.\delta_1)$-Hölder calm at $\bar{\eta}$ on $W \times V(\bar{\mu})$ uniformly over $K(U(\bar{\lambda}))$.

We have, at the same time,

$$|g(x,y,\eta) - g(x,z,\eta)| = |f(x,y,\mu) - f(x,z,\mu)| \leq n_2 d^{\delta_2}(y,z),$$

or g is $(n_2.\delta_2)$-Hölder continuous with respect to the second component on $K(U(\bar{\lambda}))$. Conditions (i) and (ii) of Theorem 58 are fulfilled.

We now check condition (iii) of Theorem 58. For all $x, y \in K(U(\bar{\lambda}))$, we get

$$\begin{aligned} &d\left(g(x,y,\bar{\eta}),\mathbb{R}_+\right) + d\left(g(y,x,\bar{\eta}),\mathbb{R}_+\right) \\ &\qquad = d\left(f(x,y,\bar{\mu}),\mathbb{R}_+\right) + d\left(f(y,x,\bar{\mu}),\mathbb{R}_+\right) \geq h d^{\beta}(x,y). \end{aligned}$$

This means that g satisfies condition (iii) of Theorem 58 and we have all its hypotheses satisfied. Therefore, the mapping of solutions to $(\mathcal{EP})$ is both Hölder calm and single-valued at $(0,\bar{\eta})$ which combined with Definition 31 gives the conclusion of the theorem. □

7.5 Conclusion

Assuming Hölder calmness and Hölder continuity in Hausdorff distance, we have established the Hölder calm property of the solution mapping with respect to ρ. This obviously implies the Hölder calm property in Hausdorff distance. We have established a sufficient condition for the Hölder well-posedness of equilibrium problems. These may be extended to many other classes of problems.

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