Best approximation by downward sets with applications

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Abstract

We develop a theory of downward subsets of a Banach lattice X with a strong unit. We study best approximation in X by elements of downward sets, and give necessary and sufficient conditions for any element of best approximation by a closed subset of X. We also characterize strictly downward subsets of X, and prove that a downward subset of X is strictly downward if and only if each its boundary point is Chebyshev. The results obtained are used for examination of some proximinal and Chebyshev pairs (U, x) where U is a closed subset of X.

Key words: Best approximation, Downward sets, Proximinal sets, Chebyshev sets, Banach lattices

1 Introduction

The theory of best approximation by elements of convex and reverse convex sets (that is, complements of convex sets) is well-developed and has found

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applications in many areas of mathematics. However, convexity is sometimes a very restrictive assumption, so there is a clear need to study the best approximation by not necessarily convex sets.

In this paper we develop a theory of best approximation by elements of closed downward sets in a Banach lattice X with a strong unit. We show that a closed downward set is proximinal, that is a best approximation by this set exists for each $x \in X$ and derive necessary and sufficient conditions for the uniqueness of best approximation. A downward set is not necessarily convex. We show that this set is *abstract convex* with respect to a certain set of elementary functions (see e.g. [2,6] for the definition of abstract convexity. This fact allows us to examine separation properties of downward sets and gives necessary and sufficient conditions for best approximation. (A finite dimensional version of separation results can be found in [1].)

We show that the results obtained for downward sets can be used in more general situation. In particular we give necessary and sufficient conditions for best approximation of some points by arbitrary closed sets. We also examine best approximation by the so-called normal sets.

The structure of the paper is as follows. In Section 2, we recall main definitions. In Section 3, we investigate best approximation in X by elements of downward sets.In particular, we show that the least element of the set of best approximations exists. In section 4, we present the characterizations of downward sets in terms of separation from outside points. Strictly downward sets and strictly downward at a point sets are studied in Section 5. In this section we also introduce a notion of a Chebyshev point and show that a point of a downward set W is Chebyshev if and only if W is strictly downward at this point. Connections between downward subsets of X and normal subsets of the cone X^+ based on the notion of the downward hull of a normal set, are investigated in section 6. In section 7, we introduce notions of a proximinal pair and a Chebyshev pair and study these pairs by means of the downward hull of a corresponding set.

2 Preliminaries

Let X be a normed vector space. For a nonempty subset W of X and $x \in X$, define

$$d(x, W) = \inf_{w \in W} \|x - w\|.$$

Recall (see e.g. [7]) that a point $w_0 \in W$ is called a *best approximation* for $x \in X$ if

$$\|x - w_0\| = d(x, W).$$

If each $x \in X$ has at least one best approximation $w_0 \in W$, then W is called a *proximinal subset* of X. If each $x \in X$ has a unique best approximation $w_0 \in W$, then W is called a *Chebyshev subset* of X.

Let $W \subset X$. For $x \in X$, denote by $P_W(x)$ the set of all best approximations of x in W :

$$P_W(x) = \{w \in W : \|x - w\| = d(x, W)\}. \tag{2.1}$$

It is well-known that $P_W(x)$ is a closed and bounded subset of X. If $x \notin W$ then $P_W(x)$ is located in the boundary of W.

In this paper we shall study best approximation in Banach lattices with the strong unit. Let X be a vector lattice. Recall (see e.g [8]) that an element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda\mathbf{1}$. Using a strong unit $\mathbf{1}$ we can define a norm on X by

$$\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda\mathbf{1}\} \quad \forall\, x \in X. \tag{2.2}$$

Then

$$B(x, r) := \{y \in X : \|x - y\| \leq r\} = \{y \in X : x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}. \tag{2.3}$$

We have also

$$|x| \leq \|x\|\mathbf{1} \quad \text{for all} \quad x \in X. \tag{2.4}$$

It is well known that X equipped with the norm (2.2) is a Banach lattice and there exists a compact topological space Q such that X is isomorphic (as a vector lattice) and isometric to the space of all real-valued continuous functions $C(Q)$ with the norm $\|x\| = \max_{q \in Q} |x(q)|$. Well-known examples of Banach lattices with the strong units are the lattice of all bounded functions defined on a set X and the lattice $L_\infty(S, \Sigma, \mu)$ of all essentially bounded functions defined on a space S with a σ-algebra of measurable sets Σ and a measure μ.

We shall study in this paper downward sets. Recall that a subset W of an ordered set X is said to be *downward*, if $(w \in W, x \leq w) \implies x \in W$.

For any subset W of a normed space X, we shall denote by $\operatorname{int} W$, $\operatorname{cl} W$, and $\operatorname{bd} W$ the interior, the closure and the boundary of W, respectively. If X is a lattice and there exists the least element of W, we shall denote it by $\min W$.

3 Downward sets and their approximation properties

Let X be a Banach lattice with a strong unit $\mathbf{1}$.

Proposition 3.1. *Let W be a downward subset of X and $x \in X$. Then the following are true:*

(1) *If* $x \in W$, *then* $x - \varepsilon \mathbf{1} \in intW$ *for all* $\varepsilon > 0$.
(2) *We have* $intW = \{x \in X : x + \varepsilon \mathbf{1} \in W \text{ for some } \varepsilon > 0\}$.

Proof: (1). Let $\varepsilon > 0$ be given and $x \in W$. Let

$$V = \{y \in X : \|y - (x - \varepsilon \mathbf{1})\| < \varepsilon\}$$

be an open neighborhood of $(x - \varepsilon \mathbf{1})$. Then, by (2.3) $V = \{y \in X : x - 2\varepsilon \mathbf{1} < y < x\}$. Since W is a downward set and $x \in W$, it follows that $V \subset W$, and so $x - \varepsilon \mathbf{1} \in \operatorname{int} W$.

(2). Let $x \in \operatorname{int} W$. Then there exists $\varepsilon_0 > 0$ such that the closed ball $B(x, \varepsilon_0) \subset W$. In view of (2.3), $x + \varepsilon_0 \mathbf{1} \in W$.

Conversely, suppose that there exists $\varepsilon > 0$ such that $x + \varepsilon \mathbf{1} \in W$. Then, by the above, $x = (x + \varepsilon \mathbf{1}) - \varepsilon \mathbf{1} \in \operatorname{int} W$, which completes the proof. ∎

Corollary 3.1. *Let* W *be a closed downward subset of* X *and* $w \in W$. *Then,* $w \in bdW$ *if and only if* $\lambda \mathbf{1} + w \notin W$ *for all* $\lambda > 0$.

Lemma 3.1. *Let* W *be a closed downward subset of* X. *Then* W *is proximinal in* X.

Proof: Let $x_0 \in X \setminus W$ be arbitrary and $r := d(x_0, W) = \inf_{w \in W} \|x_0 - w\| > 0$. This implies that for each $\varepsilon > 0$ there exists $w_\varepsilon \in W$ such that $\|x_0 - w_\varepsilon\| < r + \varepsilon$. Then, by (2.3) we have

$$-(r + \varepsilon)\mathbf{1} \le w_\varepsilon - x_0 \le (r + \varepsilon)\mathbf{1}.$$

Let $w_0 = x_0 - r\mathbf{1}$. Then, we have

$$\|x_0 - w_0\| = \|r\mathbf{1}\| = r = d(x_0, W)$$

and so $w_0 - \varepsilon \mathbf{1} = x_0 - r\mathbf{1} - \varepsilon \mathbf{1} \le w_\varepsilon$. Since W is downward and $w_\varepsilon \in W$, it follows that $w_0 - \varepsilon \mathbf{1} \in W$ for all $\varepsilon > 0$. Since W is closed, we have $w_0 \in W$, and so $w_0 \in P_W(x_0)$. Thus the result follows. ∎

Remark 3.1. *We proved that for each* $x_0 \in X \setminus W$ *the set* $P_W(x_0)$ *contains the element* $w_0 = x_0 - r\mathbf{1}$ *with* $r = d(x_0, W)$. *If* $x_0 \in W$ *then* $w_0 = x_0$ *and* $P_W(x_0) = \{w_0\}$.

Proposition 3.2. *Let* W *be a closed downward subset of* X *and* $x_0 \in X$. *Then there exists the least element* $w_0 := \min P_W(x_0)$ *of the set* $P_W(x_0)$, *namely,* $w_0 = x_0 - r\mathbf{1}$, *where* $r := d(x_0, W)$.

Proof: If $x_0 \in W$, then the result holds. Assume that $x_0 \notin W$ and $w_0 = x_0 - r\mathbf{1}$. Then, by Remark 3.1, we have $w_0 \in P_W(x_0)$. Since Applying (2.3) and the equality $\|x_0 - w_0\| = r$ we get

$$x \ge x_0 - r\mathbf{1} = w_0 \quad \forall\, x \in B(x_0, r).$$

This implies that w_0 is the least element of the closed ball $B(x_0, r)$.

Now, let $w \in P_W(x_0)$ be arbitrary. Then, $\|x_0 - w\| = r$, and so $w \in B(x_0, r)$. Therefore, $w \geq w_0$. Hence, w_0 is the least element of the set $P_W(x_0)$. ■

Corollary 3.2. *Let W be a closed downward subset of X, $x_0 \in X$ and $w_0 = \min P_W(x_0)$. Then, $w_0 \leq x_0$.*

Corollary 3.3. *Let W be a closed downward subset of X and $x \in X$ be arbitrary. Then*

$$d(x, W) = \min\{\lambda \geq 0 : x - \lambda \mathbf{1} \in W\}.$$

Proof: Let $A = \{\lambda \geq 0 : x - \lambda \mathbf{1} \in W\}$. If $x \in W$, then $x - 0\mathbf{1} = x \in W$, and so $\min A = 0 = d(x, W)$. Suppose that $x \notin W$ then $r := d(x, W) > 0$. Let $\lambda > 0$ be arbitrary such that $x - \lambda \mathbf{1} \in W$. Thus, we have

$$\lambda = \|\lambda \mathbf{1}\| = \|x - (x - \lambda \mathbf{1})\| \geq d(x, W) = r.$$

Since, by Proposition 3.2, $x - r\mathbf{1} \in W$, it follows that $r \in A$. Hence, $\min A = r$, which completes the proof. ■

4 Characterization of best approximations by downward sets

Let $\varphi : X \times X \longrightarrow \mathbb{R}$ be a function defined by

$$\varphi(x, y) := \sup\{\lambda \in \mathbb{R} : \lambda \mathbf{1} \leq x + y\} \quad \forall\, x,\ y \in X. \tag{4.1}$$

Since $\mathbf{1}$ is a strong unit it follows that the set $\{\lambda \in \mathbb{R} : \lambda \leq x + y\}$ is nonempty and bounded from above (by the number $\|x + y\|$). Clearly this set is closed. It follows from the aforesaid and the definition of φ that the function φ enjoys the following properties:

$$-\infty < \varphi(x, y) \leq \|x + y\| \quad \text{for each} \quad x, y \in X \tag{4.2}$$

$$\varphi(x, y)\mathbf{1} \leq x + y \quad \text{for all} \quad x,\ y \in X \tag{4.3}$$

$$\varphi(x, y) = \varphi(y, x) \quad \text{for all} \quad x,\ y \in X; \tag{4.4}$$

$$\varphi(x, -x) = \sup\{\lambda \in \mathbb{R} : \lambda \mathbf{1} \leq x - x = 0\} = 0 \quad \text{for all} \quad x \in X. \tag{4.5}$$

For each $y \in X$, define the function $\varphi_y : X \longrightarrow \mathbb{R}$ by

$$\varphi_y(x) := \varphi(x, y) \quad \forall\, x \in X. \tag{4.6}$$

The function $f : X \to \mathbb{R}$ is called topical if this function is increasing ($x \geq y \implies f(x) \geq f(y)$) and plus-homogeneous ($f(x + \alpha \mathbf{1}) = f(x) + \alpha$ for all

$x \in X$ and $\alpha \in \mathbb{R}$). The definition of topical function in finite dimensional case can be found in [4].

Lemma 4.1. *The function φ_y defined by (4.6) is topical.*

Proof: (1). Let $x,\ z \in X$ with $x \leq z$. Then $\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + y\} \subset \{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq z + y\}$. Hence,

$$\varphi_y(x) = \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + y\} \leq \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq z + y\} = \varphi_y(z).$$

(2). Let $x \in X$ and $\alpha \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}\varphi_y(x + \alpha\mathbf{1}) &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + \alpha\mathbf{1} + y\}\\ &= \sup\{\lambda \in \mathbb{R} : (\lambda - \alpha)\mathbf{1} \leq x + y\}\\ &= \sup\{\beta + \alpha \in \mathbb{R} : \beta\mathbf{1} \leq x + y\}\\ &= \sup\{\beta \in \mathbb{R} : \beta\mathbf{1} \leq x + y\} + \alpha = \varphi_y(x) + \alpha. \blacksquare\end{aligned}$$

Proposition 4.1. *The function φ_y is Lipschitz continuous.*

Proof: Let $x,\ z \in X$ be arbitrary. Since $|x - z| \leq \|x - z\|\mathbf{1}$ it follows that

$$z - \|x - z\|\mathbf{1} \leq x \leq z + \|x - z\|\mathbf{1}.$$

In view of Lemma 4.1 we have

$$\varphi_y(z) - \|x - z\| \leq \varphi_y(x) \leq \varphi_y(z) + \|x - z\|,$$

and hence

$$|\varphi_y(x) - \varphi_y(z)| \leq \|x - z\|. \tag{4.7}$$

Therefore, φ_y is Lipschitz continuous. $\blacksquare$

Corollary 4.1. *The function φ defined by (4.1) is continuous. It follows from (4.7).*

Lemma 4.2. *Let W be a closed downward subset of X, $y_0 \in bdW$ and let φ be the function defined by (4.1). Then, $\varphi(w, -y_0) \leq 0$ for all $w \in W$.*

Proof: Assume that there exists $w_0 \in W$ such that $\varphi(w_0, -y_0) > 0$. Then $\sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq w_0 - y_0\} > 0$ so there exists $\lambda_0 > 0$ such that $\lambda_0\mathbf{1} \leq w_0 - y_0$. This means that $\lambda_0\mathbf{1} + y_0 \leq w_0$. Since W is a downward set and $w_0 \in W$, it follows that $\lambda_0\mathbf{1} + y_0 \in W$. Therefore, by Proposition 3.1 (2), we have $y_0 \in \operatorname{int} W$. This is a contradiction, which completes the proof. $\blacksquare$

We now give characterizations of downward sets in terms of separation from outside points. The proof of the following Theorem is similar to that in [4] (see Proposition 2.1) for a finite dimensional case. For an easy reference we present a version of this proof.

Theorem 4.1. *Let W be a subset of X and φ be the coupling function of* (4.1). *Then the following are equivalent:*

(1) *W is a downward set.*
(2) *For each $x \in X \setminus W$, we have*

$$\varphi(w, -x) < 0 \quad \forall\, w \in W.$$

(3) *For each $x \in X \setminus W$, there exists $l \in X$ such that*

$$\varphi(w, l) < 0 \leq \varphi(x, l) \quad \forall\, w \in W.$$

Proof: (1) $\Longrightarrow$ (2). Suppose that (1) holds and that there exist $x \in X \setminus W$ and $w \in W$ such that $\varphi(w, -x) \geq 0$. Then, by (4.3) we have $0 \leq \varphi(w, -x)\mathbf{1} \leq w - x$, and so $x \leq w$. Since W is downward and $w \in W$, it follows that $x \in W$. This is a contradiction.

(2) $\Longrightarrow$ (3). Assume that (2) holds and $x \in X \setminus W$ is arbitrary. Then, by hypothesis, we have

$$\varphi(w, -x) < 0 \quad \forall\, w \in W.$$

Now, let $l = -x \in X$. Using (4.5) we have for each $w \in W$:

$$\varphi(w, l) = \varphi(w, -x) < 0 = \varphi(x, -x) = \varphi(x, l).$$

(3) $\Longrightarrow$ (1). Suppose that (3) holds and W is not a downward set. Then there exist $w_0 \in W$ and $x_0 \in X \setminus W$ with $x_0 \leq w_0$. It follows, by hypothesis, that there exists $l \in X$ such that

$$\varphi(w, l) < 0 \leq \varphi(x_0, l) \quad \forall\, w \in W. \tag{4.8}$$

Since $\varphi(., l)$ is increasing, we have

$$0 \leq \varphi(x_0, l) \leq \varphi(w_0, l).$$

This contradicts (4.8). ∎

Theorem 4.2. *Let φ be the function defined by (4.1). Then for a subset W of X the following are equivalent:*

(1) *W is a closed downward subset of X.*

(2) *W is downward, and for each $x \in X$ the set*

$$H = \{\lambda \in \mathbb{R} : x + \lambda \mathbf{1} \in W\}$$

is closed in $\mathbb{R}$.

(3) *For each* $x \in X \setminus W$, *there exists* $l \in X$ *such that*

$$\varphi(w,l) < 0 < \varphi(x,l), \qquad (w \in W).$$

(4) *For each* $x \in X \setminus W$, *there exists* $l \in X$ *such that*

$$\sup_{w\in W} \varphi(w,l) < \varphi(x,l).$$

Proof: (1) $\Longrightarrow$ (2). Assume that (1) holds and let $x \in X$, $\lambda_k \in \mathbb{R}$, $x+\lambda_k\mathbf{1} \in W$ $(k = 1,\ 2,\ \cdots)$ and $\lambda_k \longrightarrow \lambda \in \mathbb{R}$. Then, we have

$$\|(x+\lambda_k\mathbf{1}) - (x+\lambda\mathbf{1})\| = \|(\lambda_k-\lambda)\mathbf{1}\| = |\lambda_k-\lambda| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow +\infty.$$

Since $x+\lambda_k\mathbf{1} \in W$ $(k = 1,\ 2,\ \cdots)$ and W is closed, it follows that $x+\lambda\mathbf{1} \in W$, and so $\lambda \in H$. Hence, H is a closed subset of $\mathbb{R}$.

(2) $\Longrightarrow$ (3). Suppose that (2) holds and $x \in X\setminus W$ is arbitrary. We claim that there exists $\lambda_0 > 0$ such that $-\lambda_0 \notin H$. Indeed, if $-\lambda \in H$ for all $\lambda > 0$. then due to the closedness of H, we have $0 \in H$. This implies $x = x + 0\cdot\mathbf{1} \in W$. This is a contradiction. Now, let $l = \lambda_0\mathbf{1} - x \in X$. We show that $\varphi(w,l) < 0$ for all $w \in W$. Assume that there exists $w_0 \in W$ such that $\varphi(w_0,l) \geq 0$. Then by (4.3) we have $0 \leq \varphi(w_0,l)\mathbf{1} \leq w_0 + l$, and so $w_0 \geq -l = x - \lambda_0\mathbf{1}$. Since W is downward and $w_0 \in W$, it follows that $x - \lambda_0\mathbf{1} \in W$, and consequently $-\lambda_0 \in H$. This is a contradiction. Hence, $\varphi(w,l) < 0$ for all $w \in W$.

On the other hand, we have

$$\begin{aligned}\varphi(x,l) &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + l\}\\ &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + \lambda_0\mathbf{1} - x = \lambda_0\mathbf{1}\}\\ &= \sup\{\lambda \in \mathbb{R} : (\lambda-\lambda_0)\mathbf{1} \leq 0\} = \sup\{\alpha + \lambda_0 \in \mathbb{R} : \alpha\mathbf{1} \leq 0\}\\ &= \sup\{\alpha \in \mathbb{R} : \alpha\mathbf{1} \leq 0\} + \lambda_0 = \lambda_0 > 0.\end{aligned}$$

(3) $\Longrightarrow$ (4) is obvious.

(4) $\Longrightarrow$ (1). Suppose that (4) holds and that W is not downward. Then there exist $w_0 \in W$ and $x_0 \in X\setminus W$ with $x_0 \leq w_0$. By hypothesis, there exists $l \in X$ such that

$$\sup_{w\in W} \varphi(w,l) < \varphi(x_0,l).$$

Since $\varphi(.,l)$ is increasing, it follows that

$$\varphi(x_0,l) \leq \varphi(w_0,l) \leq \sup_{w\in W} \varphi(w,l) < \varphi(x_0,l).$$

This is a contradiction. Hence, W is a downward set.

Finally, assume that W is not closed. Then there exists a sequence $\{w_n\}_{n\geq 1} \subset W$ and $x_0 \in X \setminus W$ such that $\|w_n - x_0\| \longrightarrow 0$ as $n \longrightarrow +\infty$. Since $x_0 \in X \setminus W$, by hypothesis, there exists $l \in X$ such that

$$\sup_{w \in W} \varphi(w, l) < \varphi(x_0, l).$$

Thus, we have

$$\varphi(w_n, l) \leq \sup_{w \in W} \varphi(w, l) \quad \forall\, n \geq 1.$$

By continuity of $\varphi_l = \varphi(., l)$ it follows that $\varphi(x_0, l) \leq \sup_{w \in W} \varphi(w, l)$. This is a contradiction, which completes the proof. ■

Lemma 4.3. *Let W be a closed downward subset of X, $w_0 \in bdW$ and $l = -w_0$. Let φ be defined by (4.1). Then*

$$\varphi(w, l) \leq 0 = \varphi(w_0, l) \quad \forall\, w \in W.$$

Proof: Since $w_0 \in \mathrm{bd}\, W$, it follows, by Lemma 4.2, that

$$\varphi(w, l) = \varphi(w, -w_0) \leq 0 \quad \forall\, w \in W.$$

Also, we have

$$\varphi(w_0, l) = \sup\{\lambda \in \mathbb{R} : \lambda \mathbf{1} \leq w_0 + l\} = \sup\{\lambda \in \mathbb{R} : \lambda \mathbf{1} \leq w_0 - w_0 = 0\} = 0. \blacksquare$$

Consider an arbitrary set Y and a set L of functions defined on a set Y. We need the following definition (see, e.g. [2,6]). A subset Ω of Y is called *abstract convex* with respect to L if for each point $y \in Y \setminus \Omega$ there exists $l \in L$ such that $\sup_{\omega \in \Omega} l(\omega) < l(y)$.

Consider now the space X and the set $L = \{\varphi_y : y \in X\}$ where φ_y is defined by (4.6). Theorem 4.2 demonstrates that a set $W \subset X$ is closed and downward if and only if this set is abstract convex with respect to the set L.

5 Strictly downward sets and their approximation properties

We start with the following definition :

Definition 5.1. *A downward subset W of X is called strictly downward if for each boundary point w_0 of W, the inequality $w > w_0$ implies $w \notin W$.*

This definition was introduced in [5] for finite dimensional spaces. We now present an example of a strictly downward set. Recall that a function f :

$X \longrightarrow \mathbb{R}$ is called *increasing* if $x,\ y \in X$ with $x \le y$, implies $f(x) \le f(y)$. A function $f : X \longrightarrow \mathbb{R}$ is called *strictly increasing* at a point $y \in X$ if $x < y$ implies $f(x) < f(y)$. A function, which is strictly increasing at each point $y \in X$ is called strictly increasing on X. It is easy to check that $f : X \longrightarrow \mathbb{R}$ is increasing if and only if its level sets $\mathcal{S}_c(f) = \{x \in X : f(x) \le c\}$ $(c \in \mathbb{R})$ are downward. (The empty set is downward by definition.) Let $f : X \longrightarrow \mathbb{R}$ be a continuous strictly increasing function. If $c \in \mathbb{R}$ is a number such that the level set $\mathcal{S}_c(f)$ is nonempty then the boundary $\mathrm{bd}\,\mathcal{S}_c(f)$ of this set coincides with the set $\{x \in X : f(x) = c\}$.

Lemma 5.1. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous strictly increasing function. Then all nonempty level sets $\mathcal{S}_c(f)$ $(c \in \mathbb{R})$ of f are strictly downward.*

Proof: Since f is continuous strictly increasing, it follows that

$$\mathrm{bd}\,\mathcal{S}_c(f) = \{x \in X : f(x) = c\} \ \ (c \in \mathbb{R}).$$

Let $x \in \mathrm{bd}\,\mathcal{S}_c(f)$ be arbitrary and $y \in X$ with $y > x$. Since f is strictly increasing and $f(x) = c$, then $f(y) > f(x) = c$, and so $y \notin \mathcal{S}_c(f)$. Hence, $\mathcal{S}_c(f)$ $(c \in \mathbb{R})$ is strictly downward. ■

Definition 5.2. *Let W be a downward set. We say that W is strictly downward at a point $w' \in bdW$ if for all $w_0 \in bdW$ with $w_0 \le w'$, the inequality $w > w_0$ implies $w \notin W$.*

Proposition 5.1. *Let W be a closed downward set. Then W is strictly downward at $w' \in bdW$ if and only if*

(i) $w > w' \implies w \notin W$;

(ii) $(w_0 \le w', w_0 \in bdW) \implies w_0 = w'$.

Proof: Let W be strictly downward at $w' \in \mathrm{bd}\,W \subset W$. Then (i) holds, so we need to check that relations $w_0 \le w'$, $w_0 \in \mathrm{bd}\,W$ imply $w_0 = w'$. Assume that $w_0 \ne w'$ then $w' > w_0$ and due to Definition 5.2 we have $w' \notin W$, which is a contradiction.

Assume now that (i) and (ii) hold. Let $w_0 \in \mathrm{bd}\,W$ and $w_0 \le w'$. Then due to (ii) we have $w_0 = w'$ and due to (i) we have $w > w_0$ implies $w \notin W$. ■

Proposition 5.2. *Let W be a closed downward set. Then W is strictly downward if and only if W is strictly downward at each its boundary point.*

Proof: Due to Proposition 5.1 we need only to show that for each strictly downward set W and each boundary point w' of W relations $w_0 \le w', w_0 \in \mathrm{bd}\,W$ imply $w_0 = w'$. This is true since the inequality $w_0 < w'$ leads for strictly downward sets to $w' \notin W$. ■

Proposition 5.3. *Let φ be the function defined by (4.1). Let W be a closed downward set that is strictly downward at a point $w' \in bdW$. Then there exists* *unique $l \in X$ such that*

$$\varphi(w,l) \le 0 = \varphi(w',l), \quad \forall\, w \in W.$$

Proof: Let $l = -w'$. Then, by Lemma 4.3, we have

$$\varphi(w,l) \le 0 = \varphi(w',l) \quad \forall\, w \in W.$$

Now, suppose there exists $l' \in X$ such that

$$\varphi(w,l') \le 0 = \varphi(w',l') \quad \forall\, w \in W. \tag{5.1}$$

It follows, by (4.3) that $0 = \varphi(w',l')\mathbf{1} \le w' + l'$, and so $-l' \le w'$. Since W is downward and $w' \in W$, we have $-l' \in W$. Let $\varepsilon > 0$ be given and $l_\varepsilon = -l' + \varepsilon\mathbf{1} \in X$. Then,

$$\begin{aligned}
\varphi(l_\varepsilon, l') &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \le l_\varepsilon + l'\} \\
&= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \le -l' + \varepsilon\mathbf{1} + l' = \varepsilon\mathbf{1}\} \\
&= \sup\{\lambda \in \mathbb{R} : (\lambda - \varepsilon)\mathbf{1} \le 0\} = \sup\{\alpha + \varepsilon \in \mathbb{R} : \alpha\mathbf{1} \le 0\} \\
&= \sup\{\alpha \in \mathbb{R} : \alpha\mathbf{1} \le 0\} + \varepsilon = \varepsilon > 0.
\end{aligned}$$

Due to (5.1) we have that $l_\varepsilon := -l' + \varepsilon\mathbf{1} \notin W$ for all $\varepsilon > 0$. Then, by Corollary 3.1, $-l' \in \operatorname{bd} W$. Since W is strictly downward at each its boundary point, $w' \ge -l'$ and $-l' \in \operatorname{bd} W$, it follows from Proposition 5.1 that $w' = -l'$. Hence, $l' = l = -w'$. Thus the result follows. ■

Theorem 5.1. *Let φ be the function defined by (4.1). Then for a closed downward subset W of X the following assertions are equivalent:*

(1) W is strictly downward.
(2) for each $w_0 \in bd W$ there exists unique $l \in X$ such that

$$\varphi(w,l) \le 0 = \varphi(w_0,l) \quad \forall\, w \in W.$$

Proof: The implication (1) $\Longrightarrow$ (2) follows from Proposition 5.1 and Proposition 5.3. We now prove the implication (2) $\Longrightarrow$ (1). Assume that for each $w_0 \in \operatorname{bd} W$ there exists unique $l \in X$ such that

$$\varphi(w,l) \le 0 = \varphi(w_0,l) \quad \forall\, w \in W.$$

Let $w_0 \in \operatorname{bd} W$ and $y \in X$ with $y > w_0$. Assume that $y \in W$. Since $y > w_0$, it follows that $\lambda\mathbf{1} < y - w_0$ for all $\lambda < 0$. This implies that

$$y + \lambda\mathbf{1} > w_0 \quad \forall\, \lambda > 0. \tag{5.2}$$

We claim that $y + \lambda\mathbf{1} \notin W$ for all $\lambda > 0$. Suppose that there exists $\lambda_0 > 0$ such that $y + \lambda_0\mathbf{1} \in W$. Let

$$V = \{x \in X : \|x - w_0\| \leq \frac{1}{2}\lambda_0\}.$$

It is clear that V is a neighborhood of w_0. By (2.3) we have

$$V = \{x \in X : w_0 - \frac{1}{2}\lambda_0\mathbf{1} \leq x \leq w_0 + \frac{1}{2}\lambda_0\mathbf{1}\}.$$

Applying (5.2) we conclude that

$$x \leq w_0 + \frac{1}{2}\lambda_0\mathbf{1} < y + \frac{1}{2}\lambda_0\mathbf{1} + \frac{1}{2}\lambda_0\mathbf{1} = y + \lambda_0\mathbf{1},$$

for each $x \in V$. Since W is downward and $y + \lambda_0\mathbf{1} \in W$, it follows that $V \subset W$, and so $w_0 \in \operatorname{int} W$. This is a contradiction, and hence the claim is true.

Then, by Corollary 3.1, $y \in \operatorname{bd} W$. Let $l = -y$. It follows from Lemma 4.3 that

$$\varphi(w, l) \leq 0 = \varphi(y, l) \quad \forall\, w \in W. \tag{5.3}$$

On the other hand, applying Lemma 4.3 to the point w_0 we have for $l' = -w_0$:

$$\varphi(w, l') \leq 0 = \varphi(w_0, l') \quad \forall\, w \in W. \tag{5.4}$$

Since $w_0 < y$ and $\varphi(., l')$ is increasing, it follows that $0 = \varphi(w_0, l') \leq \varphi(y, l') \leq 0$. This, together with (5.4) imply that

$$\varphi(w, l') \leq 0 = \varphi(y, l') \quad \forall\, w \in W. \tag{5.5}$$

. Since $w_0 \neq y$ it follows that $l' \neq l$, hence (5.3) and (5.5) contradict the uniqueness of l. We have demonstrated that the assumption $y \in W$ leads to contradictions. Thus $y \notin W$. This means that W is strictly downward. ■

Corollary 5.1. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous strictly increasing function and φ be the function defined by (4.1). Then for each $x \in X$ there exists unique $l = -x$ such that*

$$\varphi(w, l) \leq 0 = \varphi(x, l) \quad \forall\, w \in \mathcal{S}_c(f).$$

Proof: This is an immediate consequence of Lemma 5.1 and Theorem 5.1. ■

Definition 5.3. *Let W be a downward set. A point $w' \in bdW$ is said to be a Chebyshev point if for each $w_0 \in bdW$ with $w_0 \leq w'$ and for each $x_0 \notin W$ such that $w_0 \in P_W(x_0)$ it follows that $P_W(x_0) = \{w_0\}$, that is best approximation of x_0 is unique.*

Theorem 5.2. *Let W be a closed downward subset of X and $w' \in bdW$. Then the following assertions are equivalent:*

(1) *w' is a Chebyshev point of W .*
(2) *W is a strictly downward set at w'.*

Proof: (1) $\Longrightarrow$ (2). Suppose that (1) holds and that W is not strictly downward at w'. Then we can find $w_0 \in \mathrm{bd}\, W$ such that $w_0 \leq w'$ and there exists $w \in W$ with $w > w_0$. Let $r \geq \|w - w_0\| > 0$. It follows from (2.4) that

$$w - w_0 \leq |w - w_0| \leq \|w - w_0\|\mathbf{1} \leq r\mathbf{1},$$

so $w \leq w_0 + r\mathbf{1}$. Let $x_0 = w_0 + r\mathbf{1} \in X$. Then $\|x_0 - w_0\| = \|r\mathbf{1}\| = r$.

We claim that $d(x_0, W) = r$. Suppose this does not hold. Then there exists $y \in W$ such that $\|x_0 - y\| < r$. (We have $x_0 \neq y$, otherwise $x_0 = y \in W$.) Then there exists $r_0 \in (0, r)$ such that $\|x_0 - y\| \leq r_0$. Hence, by (2.3) we have $x_0 \leq y + r_0\mathbf{1}$. Since $x_0 = w_0 + r\mathbf{1}$, it follows that

$$w_0 + \lambda_0\mathbf{1} \leq y \qquad \text{with} \quad \lambda_0 = (r - r_0) > 0.$$

Since W is downward and $y \in W$, it follows that $w_0 + \lambda_0\mathbf{1} \in W$, and so, by Proposition 3.1 (2), $w_0 \in \mathrm{int}\, W$. This is a contradiction. Therefore, $d(x_0, W) = r = \|x_0 - w_0\|$, that is, $w_0 \in P_W(x_0)$.

On the other hand, we have $w \leq w_0 + r\mathbf{1} = x_0$. Since $w_0 < w$ it follows that $0 \leq x_0 - w < x_0 - w_0 = r\mathbf{1}$. Hence,

$$\|x_0 - w\| \leq \|r\mathbf{1}\| = r = d(x_0, W) \leq \|x_0 - w\|.$$

Then, $\|x_0 - w\| = d(x_0, W)$, and so $w \in P_W(x_0)$ with $w \neq w_0$. Thus there exists a point $w_0 \in \mathrm{bd}\, W$ with $w_0 \leq w'$ such that $P_W(x_0)$ contains w_0 and also at least one point different from w_0. This is impossible since w' is a Chebyshev point.

(2) $\Longrightarrow$ (1). Assume that W is strictly downward at point $w' \in \mathrm{bd}\, W$. Then for each point $w_0 \leq w', w_0 \in \mathrm{bd}\, W$ we have $w_0 = w'$. So we need only to check that $P_W(x_0) = \{w'\}$ for each $x_0 \notin W$ such that $w' \in P_W(x_0)$. Let x_0 be such an element. Applying Proposition 3.2, we conclude that the least element w_0 of the set $P_W(x_0)$ exists and $w_0 = x_0 - r\mathbf{1}$ with $r = d(x_0, W)$. We have $w_0 \in \mathrm{bd}\, W, w_0 \leq w'$, hence $w_0 = w'$. Since W is strictly downward at $w_0 = w'$, $w \geq w'$ for all $w \in P_W(x_0) \subset W$, we have in view of Proposition 5.1 that $w' = w$ for all $w \in P_W(x_0)$. Hence, $P_W(x_0) = \{w'\}$, and so w' is a Chebyshev point of W. ∎

Corollary 5.2. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous strictly increasing function. Then, $\mathcal{S}_c(f)$ is a Chebyshev subset of X.*

Proof: This is an immediate consequence of Lemma 5.1 and Theorem 5.2. ■

6 Connection between downward sets, normal sets, and their approximation properties

In the rest of the paper we will apply the results obtained in previous sections for examination of best approximation by some closed subsets of the space X. We will use for this purpose the downward hall U_* of a set $U \subset X$. By definition U_* coincides with the intersection of all downward sets containing U. Since the intersection of an arbitrary family of downward sets is downward it follows that U_* is downward. Clearly U_* is the least (by inclusion) downward set, which contains U.

Proposition 6.1. *([6], Proposition 2.3) Let $U \subset X$. Then $U_* = U - X^+ := \{u - v : u \in U, v \geq 0\}$.*

In this section we use the results obtained for the examination of best approximation by normal sets. Recall (see, for example, [2,3]) that a subset G of the positive cone $X^+ := \{x \in X : x \geq 0\}$ is called *normal,* if $(g \in G,\ x \in X^+,\ x \leq g) \implies x \in G$. For any subset A of X we shall use the notation $A^+ = \{a^+ : a \in A\}$, where $a^+ = \sup(a, 0)$. We also use notation $a^- = -\inf(a, 0)$.

Proposition 6.2. *Let G be a normal subset of X^+ and $G_* \subset X$ be the downward hull of the set G. Then the following are true:*
(1) $G_* = \{x \in X : x^+ \in G\}$.
(2) $G = G_* \cap X^+$.
(3) *G is closed if and only if G_* is closed.*
(4) $(G_*)^+ = G$.

This proposition was proved in [1,2] for finite dimensional space X. The proof from [1,2] is valid also in the case under consideration so we omit this proof.

Proposition 6.3. *Let $y_0 \in X$ and $x_0 \in X^+$. Then $\|x_0 - y_0\| \geq \|x_0 - {y_0}^+\|$.*

Proof: We have $x_0 = {x_0}^+$ and $y_0 = {y_0}^+ - {y_0}^-$. Then

$$x_0 - y_0 = (x_0 + {y_0}^-) - {y_0}^+.$$

Therefore,

$$|x_0 - y_0| = (x_0 - y_0)^+ + (x_0 - y_0)^- = (x_0 + {y_0}^-) + {y_0}^+$$

$$= (x_0 + {y_0}^+) + {y_0}^- \geq (x_0 + {y_0}^+) = |x_0 - {y_0}^+|.$$

Hence, $\|x_0 - y_0\| \geq \|x_0 - {y_0}^+\|$. ■

Corollary 6.1. *Let $G \subset X^+$ be a normal set and $G_* \subset X$ be the downward hull of G, and $x_0 \in X^+$. Then, $d(x_0, G_*) = d(x_0, G)$.*

Proof: Since $G \subset G_*$, we have $d(x_0, G_*) \leq d(x_0, G)$. On the other hand, let $g \in G_*$ be arbitrary. Then, $g^+ \in G$ (by Proposition 6.2). Therefore, by Proposition 6.3, it follows that

$$\|x_0 - g\| \geq \|x_0 - g^+\| \geq d(x_0, G) \quad \forall\, g \in G_*.$$

Hence, $d(x_0, G_*) \geq d(x_0, G)$. Consequently, $d(x_0, G) = d(x_0, G_*)$. ■

Proposition 6.4. *Let G be a closed normal subset of X^+ and $x_0 \in X^+$. Then there exists the least element $g_0 := \min P_G(x_0)$ of the set $P_G(x_0)$, namely, $g_0 = {w_0}^+$, where w_0 is the least element of the set $P_{G_*}(x_0)$ and $G_* \subset X$ is the downward hull of G.*

Proof: Since G is closed in X^+, it follows, by Proposition 6.2, that G_* is closed in X. Then, by Proposition 3.2, $w_0 = x_0 - r\mathbf{1}$ the least element of the set $P_{G_*}(x_0)$ exists, where $r := d(x_0, G_*)$. Also, by Corollary 6.1, we have $r = d(x_0, G_*) = d(x_0, G)$.

Now, let $g_0 = {w_0}^+ = (x_0 - r\mathbf{1})^+$. Since $w_0 \in G_*$, by Proposition 6.2, we have $g_0 = {w_0}^+ \in G$, and hence

$$\|x_0 - g_0\| = \|x_0 - {w_0}^+\| \geq d(x_0, G) = r.$$

On the other hand, by Proposition 6.3, we have

$$r = d(x_0, G_*) = \|x_0 - w_0\| \geq \|x_0 - {w_0}^+\| = \|x_0 - g_0\|.$$

Hence, $\|x_0 - g_0\| = r = d(x_0, G)$, and so $g_0 \in P_G(x_0)$. Let $g \in P_G(x_0)$ be arbitrary. Then, $\|x_0 - g\| = d(x_0, G) = r = d(x_0, G_*)$. Since $g \in G \subset G_*$, it follows that $g \in P_{G_*}(x_0)$, and so $g \geq w_0$. Since also $g \geq 0$ we have $g \geq \sup(w_0, 0) = {w_0}^+ = g_0$. Hence, $g_0 = {w_0}^+$ is the least element of the set $P_G(x_0)$. ■

Theorem 6.1. *Let G be a closed normal subset of X^+, $G_* \subset X$ be the downward hull of G and $x_0 \in X$. Then, $d(x_0, G_*) = d({x_0}^+, G)$.*

Proof: Let $w \in G_*$ be arbitrary and $w = w^+ - w^-$. Then, by Proposition 6.2, $w^+ \in G$. Since

$$|x_0 - w| = |{x_0}^+ - w^+| + |{x_0}^- - w^-| \geq |{x_0}^+ - w^+|$$

it follows that

$$\|x_0 - w\| \geq \|{x_0}^+ - w^+\| \geq d({x_0}^+, G) \quad \forall\, w \in G_*.$$

Then, $d(x_0, G_*) \geq d({x_0}^+, G)$. On the other hand, since G is a closed normal subset of X^+ and ${x_0}^+ \in X^+$, it follows, by Proposition 6.4, that $g_0 = ({x_0}^+ - r\mathbf{1})^+$ the least element of the set $P_G({x_0}^+)$ exists, where $r := d({x_0}^+, G_*)$.

Now, define

$$g_{x_0} = g_0 - {x_0}^-.$$

Thus, we have (because ${x_0}^- \geq 0$) $g_{x_0} \leq g_0$. Since G_* is downward and $g_0 \in G \subset G_*$, it follows that $g_{x_0} \in G_*$. Also, we have $x_0 - g_{x_0} = {x_0}^+ - g_0$. Hence, $d(x_0, G_*) \leq \|x_0 - g_{x_0}\| = \|{x_0}^+ - g_0\| = d({x_0}^+, G)$. Consequently, we have $d(x_0, G_*) = d({x_0}^+, G)$. ■

Using Proposition 6.4 and Theorem 6.1 we can extend results obtained for downward sets in previous sections for best approximation by normal sets.

7 Proximinal and Chebyshev pairs

Definition 7.1. *Let $U \subset X$ and $x \in X$. We say that a pair (U, x) is a proximinal, if there exists a best approximation of x by U. A proximinal pair is called Chebyshev if there is a unique best approximation of x by U.*

In this section we shall study some proximinal and Chebyshev pairs. We shall use the function p define on X by

$$p(x) = \inf\{\lambda \in \mathbb{R} : x \leq \lambda \mathbf{1}\} \quad \forall\, x \in X.$$

Since the set $\{\lambda \in \mathbb{R} : x \leq \lambda\mathbf{1}\}$ is closed and bounded, it follows that $p(x) = \min\{\lambda \in \mathbb{R} : x \leq \lambda\mathbf{1}\}$, hence $x \leq p(x)\mathbf{1}$. We have

$$\|x\| = \max(p(x), p(-x)) \ \ (x \in X).$$

Indeed, assume for the sake of definiteness that $p(x) \geq p(-x)$. Then, $x \leq p(x)\mathbf{1}$, $-x \leq p(-x)\mathbf{1} \leq p(x)\mathbf{1}$ and $p(x)$ is the least number λ such that both $x \leq \lambda\mathbf{1}$, $-x \leq \lambda\mathbf{1}$, hence $\|x\| = p(x) = \max(p(x), p(-x))$.

It is well-known and easy to check that:

1) p is sublinear: $p(x + y) \leq p(x) + p(y)$ for all $x,\ y \in X$.
2) p is increasing.
3) $p(x + \mu\mathbf{1}) = p(x) + \mu$ for all $x \in X$ and all $\mu \in \mathbb{R}$. In particular, $p(\mathbf{1}) = p(0 + 1) = 1$.

Now, consider the set

$$Z = \{z \in X : p(z) \geq p(-z)\}.$$

We have

$$-Z = \{z \in X : -z \in Z\} = \{z \in X : p(-z) \geq p(z)\}.$$

Clearly, $Z \cup (-Z) = X$ and $Z \cap (-Z) = \{z \in X : p(z) = p(-z)\}$. It is easy to check that $(\mathrm{int}\, Z) \cap (\mathrm{int}\,(-Z)) = \emptyset$. The set Z is upward. Indeed, assume that $z \in Z$ and $x \in X$ with $z \le x$. Since p is increasing, it follows that

$$p(x) \ge p(z) \ge p(-z) \ge p(-x),$$

and hence $x \in Z$. Also, we have $\|z\| = p(z)$ for all $z \in Z$.

Recall that the Banach lattice X with a strong unit is isomorphic and isometric to the space $C(Q)$ of all real-valued continuous functions defined on a compact topological space Q. If $X = C(Q)$ then

$$p(x) = \max_{q \in Q} x(q), \quad p(-x) = -\min_{q \in Q} x(q),$$

and

$$\|x\| = \max(\max_{q \in Q} x(q), -\min_{q \in Q} x(q)) = \max_{q \in Q} |x(q)| \quad (x \in X).$$

We also have

$$Z = \{x \in C(Q) : \max_{q \in Q} x(q) \ge -\min_{q \in Q} x(q)\}.$$

Lemma 7.1. *Let φ be the function defined by (4.1). Then*

$$\varphi(x, y) = -p(-x - y) \quad \forall\, x,\ y \in X.$$

Proof: We have for all $x,\ y \in X$:

$$\begin{aligned}\varphi(x, y) &= \sup\{\lambda \in \mathbb{R} : \lambda \mathbf{1} \le x + y\} = \sup\{\lambda \in \mathbb{R} : -x - y \le -\lambda \mathbf{1}\} \\ &= \sup\{-\alpha \in \mathbb{R} : -x - y \le \alpha \mathbf{1}\} = -\inf\{\alpha \in \mathbb{R} : -x - y \le \alpha \mathbf{1}\} \\ &= -p(-x - y). \ \blacksquare\end{aligned}$$

Let U be an arbitrary closed subset of X and let U_* be the downward hull of U.

Proposition 7.1. *Let $x_0 \in X$ be an element such that $x_0 - U \subset Z$. Then, $d(x_0, U) = d(x_0, U_*)$.*

Proof: Let $r = d(x_0, U_*)$. Since $U \subset U_*$ it follows that $r \le d(x_0, U)$, so we need only to check the reverse inequality. Let $u_* \in U_*$. Then, by Proposition 6.1, there exists $u \in U$ and $v \ge 0$ such that $u_* = u - v$. Hence, $x_0 - u_* = x_0 - u + v = x_1 - u$ with $x_1 \ge x_0$. By hypothesis, we have $x_0 - u \in Z$. Since $x_1 \ge x_0$ and Z is upward, it follows that $x_1 - u \in Z$. Since $\|z\| = p(z)$ for all $z \in Z$ and p is increasing, we have

$$\|x_0 - u_*\| = \|x_1 - u\| = p(x_1 - u) \ge p(x_0 - u) = \|x_0 - u\|.$$

Thus for each $u_* \in U_*$ there exists $u \in U$ such that $\|x_0 - u_*\| \ge \|x_0 - u\|$. This means that $r := d(x_0, U_*) \ge d(x_0, U)$. We proved that $d(x_0, U) = r$. $\blacksquare$

Proposition 7.2. *Let $x_0 \in X$ be an element such that $x_0 - U \subset Z$ and let U_* be a closed set. Then (U, x_0) is a proximinal pair.*

Proof: Since U_* is a closed downward set in X, it follows, by Proposition 3.2, that the least element w_0 of the set $P_{U_*}(x_0)$ exists and $w_0 = x_0 - r\mathbf{1}$, where $r = d(x_0, U_*)$. In view of Proposition 7.1 we have $r = d(x, U)$. Since $w_0 \in U_*$, by Proposition 6.1, there exist $u \in U$ and $v \geq 0$ such that $w_0 = x_0 - r\mathbf{1} = u - v$. We have $x_0 - u = r\mathbf{1} - v$. Applying properties of p we conclude that

$$p(x_0 - u) = p(r\mathbf{1} - v) \leq p(r\mathbf{1}) = r.$$

Since, by hypothesis, $x_0 - u \in Z$, it follows that $\|x_0 - u\| = p(x_0 - u) \leq r$. On the other hand, $\|x_0 - u\| \geq d(x_0, U) = r$. Hence, $\|x_0 - u\| = r$, and so $u \in P_U(x_0)$, which completes the proof. ■

Example 7.1. (i). *Let U_* be closed and let U be bounded from above, then $-U$ is bounded from below. Since Z is upward, it follows that for all large enough elements x_0, we have $x_0 - U \subset Z$, hence the best approximation by U exists.*
(ii). *Let U_* be closed and let $U \subset (-Z)$ and $x_0 \geq 0$. Then, $-U \subset Z$. Since Z is upward, it follows that $x_0 - U \subset Z$, hence the best approximation by U exists.*

We now indicate some classes of sets U for which the downward hull U_* is closed.

1) Let U be a compact set. Then $U_* = U - X_+$ is closed.
2) Let U be a closed normal subset of X_+. Then U_* is closed. It follows from Proposition 6.2.
3) Assume that there exists a set V such that $V \subset U \subset V_*$ and V_* is closed. Then $U_* = V_*$, hence U_* is closed. In particular U_* is closed if there exists either compact or closed normal set V such that $V \subset U \subset V_*$.

Using results obtained for downward sets, we can prove the following fact.
Theorem 7.1. *Let (U, x_0) be a proximinal pair and let $x_0 - U \subset Z$. Let $u_0 \in U$ and $r_0 := \|x_0 - u_0\|$. Assume that φ is the function defined by (4.1). Then the following assertions are equivalent:*

(1) $u_0 \in P_U(x_0)$.
(2) *There exists $l \in X$ such that*

$$\varphi(u, l) \leq 0 \leq \varphi(y, l) \quad \forall\, u \in U,\ y \in B(x_0, r_0). \tag{7.1}$$

Moreover, if (7.1) holds with $l = -u_0$, then, $u_0 = \min P_U(x_0)$.

Proof: (1) $\Longrightarrow$ (2). Suppose that $u_0 \in P_U(x_0)$. Then, $r_0 = \|x_0 - u_0\| = d(x_0, U)$. Consider the closure $\mathrm{cl}\, U_*$ of the downward hull U_*. It is easy to check that $\mathrm{cl}\, U_*$ is downward. Using Proposition 3.1 we can easily prove that $d(x_0, U_*) = d(x_0, \mathrm{cl}\, U_*)$. Applying Proposition 7.1, we conclude that $d(x_0, \mathrm{cl}\, U_*) = d(x_0, U_*) = d(x_0, U) = r_0$. Since $u_0 \in P_U(x_0)$ it follows that $u_0 \in P_{\mathrm{cl}\, U_*}(x_0)$. In view of Proposition 3.2, the least element $w_0 = x_0 - r_0\mathbf{1}$ of the set $P_{\mathrm{cl}\, U_*}(x_0)$ exists. Let $l = -w_0 \in X$ and let $y \in B(x_0, r_0)$ be arbitrary. Thus, by (2.3) we have $-r_0\mathbf{1} \leq y - x_0$. This implies that $-r_0 \in \{\alpha \in \mathbb{R} : \alpha\mathbf{1} \leq y - x_0\}$. Therefore,

$$\begin{aligned}\varphi(y, l) &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y + l\} = \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y - w_0\} \\ &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y - (x_0 - r_0\mathbf{1})\} \\ &= \sup\{\lambda \in \mathbb{R} : (\lambda - r_0)\mathbf{1} \leq y - x_0\} \\ &= \sup\{\alpha + r_0 \in \mathbb{R} : \alpha\mathbf{1} \leq y - x_0\} \\ &= \sup\{\alpha \in \mathbb{R} : \alpha\mathbf{1} \leq y - x_0\} + r_0 \\ &\geq -r_0 + r_0 = 0.\end{aligned}$$

On the other hand, since $w_0 \in P_{\mathrm{cl}\, U_*}(x_0)$, it follows that $w_0 \in \mathrm{bd}\, \mathrm{cl}\, U_*$. Then, by Lemma 4.2, we have $\varphi(u_*, -w_0) \leq 0$ for all $u_* \in \mathrm{cl}\, U_*$, and so $\varphi(u, l) \leq 0$ for all $u \in U$ because $U \subset \mathrm{cl}\, U_*$.

(2) $\Longrightarrow$ (1). Assume that there exists $l \in X$ such that

$$\varphi(u, l) \leq 0 \leq \varphi(y, l) \quad \forall\, u \in U,\ y \in B(x_0, r_0).$$

Due to (2.3) we have

$$B(x_0, r_0) = \{y \in X : x_0 - r_0\mathbf{1} \leq y \leq x_0 + r_0\mathbf{1}\}.$$

This implies that $x_0 - r_0\mathbf{1} \in B(x_0, r_0)$. Then, by hypothesis, $\varphi(x_0 - r_0\mathbf{1}, l) \geq 0$, and hence, since $\varphi(., l)$ is topical, we have $\varphi(x_0, l) \geq r_0$. Due to (4.3) we get

$$r_0\mathbf{1} \leq \varphi(x_0, l)\mathbf{1} \leq x_0 + l. \tag{7.2}$$

Now, let $u \in U$ be arbitrary. Then $x_0 - u \in Z$, and hence $\|x_0 - u\| = p(x_0 - u)$. But, by Lemma 7.1, we have

$$\varphi(x, y) = -p(-x - y) \quad \forall\, x,\ y \in X.$$

Since $\varphi(u, .)$ is topical, using hypothesis and (7.2) we have

$$-\|x_0 - u\| = \varphi(u, -x_0) \leq \varphi(u, l - r_0\mathbf{1}) = \varphi(u, l) - r_0 \leq 0 - r_0 = -r_0.$$

Thus, $r_0 \leq \|x_0 - u\|$ for all $u \in U$. This implies that $\|x_0 - u_0\| = d(x_0, U)$, and hence $u_0 \in P_U(x_0)$.

Finally, suppose that (7.1) holds with $l = -u_0$. Then, by the implication (2) $\Longrightarrow$ (1), we have $u_0 \in P_U(x_0)$. Now, let $u \in P_U(x_0)$ be arbitrary. Thus,

$\|x_0 - u\| = d(x_0, U) = \|x_0 - u_0\| = r_0$, that is, $u \in B(x_0, r_0)$. It follows, by hypothesis, that $\varphi(u, -u_0) \geq 0$, and therefore (by (4.3)), $0 \leq \varphi(u, -u_0)\mathbf{1} \leq u - u_0$. This implies that $u_0 \leq u$ for all $u \in P_U(x_0)$. Hence, $u_0 = \min P_U(x_0)$, which completes the proof. ■

Proposition 7.3. *Let $U \subset X$ be a closed set and let $x_0 \in X$ be an element such that $x_0 - U \subset Z$. Assume that U_* is a closed set. Consider the following assertions:*
(1) *(U_*, x_0) is a Chebyshev pair.*
(2) *(U, x_0) is a Chebyshev pair.*
Then (1) $\Longrightarrow$ (2). *If each boundary point of U_* is Chebyshev, then* (2) $\Longrightarrow$ (1).

Proof: (1) $\Longrightarrow$ (2). Suppose (1) holds and if possible that (U, x_0) is not a Chebyshev pair. Then there exist u_1 and $u_2 \in P_U(x_0)$ with $u_1 \neq u_2$. Since, by hypothesis, $x_0 - U \subset Z$, it follows from Proposition 7.1 that $d(x_0, U) = d(x_0, U_*)$. Hence $u_1,\ u_2 \in P_{U_*}(x_0)$ because $U \subset U_*$. This is a contradiction.

(2) $\Longrightarrow$ (1). Suppose (2) holds and that each boundary point of U_* is Chebyshev. Assume on the contrary that the pair (U_*, x_0) is not Chebyshev. Then there exist u_* and $v_* \in P_{U_*}(x_0)$ with $u_* \neq v_*$. Due to Proposition 6.1 there exist u and $v \in U$ such that $u_* \leq u$ and $v_* \leq v$. Therefore, since $d(x_0, U) = d(x_0, U_*)$, we have

$$d(x_0, U) = \|x_0 - u_*\| \geq p(x_0 - u_*) \geq p(x_0 - u) = \|x_0 - u\| \geq d(x_0, U),$$

and

$$d(x_0, U) = \|x_0 - v_*\| \geq p(x_0 - v_*) \geq p(x_0 - v) = \|x_0 - v\| \geq d(x_0, U).$$

Hence $u,\ v \in P_U(x_0) \subset P_{U_*}(x_0)$.

Since U_* is a closed downward set and each its boundary point is Chebyshev, it follows from Theorem 5.2 and Proposition 5.1 that $u \neq v$, and so the pair (U, x_0) is not Chebyshev. This is a contradiction, which completes the proof.
■

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