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# AN INDUCTION THEOREM AND NONLINEAR REGULARITY MODELS\*

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**Abstract.** A general nonlinear regularity model for a set-valued mapping  $F : X \times \mathbb{R}_+ \rightrightarrows Y$ , where  $X$  and  $Y$  are metric spaces, is considered using special iteration procedures, going back to Banach, Schauder, Lusternik and Graves. Namely, we revise the *induction theorem* from Khanh, *J. Math. Anal. Appl.*, 118 (1986) and employ it to obtain basic estimates for studying regularity/openness properties. We also show that it can serve as a substitution of the Ekeland variational principle when establishing other regularity criteria. Then, we apply the induction theorem and the mentioned estimates to establish criteria for both global and local versions of regularity/openness properties for our model and demonstrate how the definitions and criteria translate into the conventional setting of a set-valued mapping  $F : X \rightrightarrows Y$ .

**Key words.** metric regularity, induction theorem, Ekeland variational principle

**AMS subject classifications.** 47H04, 49J53, 90C31

**1. Introduction.** Regularity properties of set-valued mappings lie at the core of variational analysis because of their importance for establishing stability of solutions to generalized equations (initiated by Robinson [40, 41] in the 1970s), optimization and variational problems, constraint qualifications, qualification conditions in coderivative/subdifferential calculus and convergence rates of numerical algorithms; cf. books and surveys [4, 5, 7, 15, 22, 23, 25, 30, 35, 38, 43] and the references therein.

Among the variety of known regularity properties, the most recognized and widely used one is that of *metric regularity*; cf. [5, 6, 7, 15, 22, 30, 35, 37, 38, 43].

Recall that a set-valued mapping  $F : X \rightrightarrows Y$  between metric spaces is (locally) metrically regular at a point  $(\bar{x}, \bar{y})$  in its graph  $\text{gph}F := \{(x, y) \in X \times Y \mid y \in F(x)\}$  with modulus  $\kappa > 0$  if

$$(1.1) \quad d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \quad \text{for all } x \text{ near } \bar{x}, y \text{ near } \bar{y}.$$

(Here  $F^{-1} : Y \rightrightarrows X$  is the *inverse* mapping defined by  $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ .) The roots of this notion can be traced back to the classical Banach-Schauder *open mapping theorem* and its subsequent generalization to nonlinear mappings known as *Ljusternik-Graves theorem*, see the survey [22] by Ioffe.

Inequality (1.1) provides a linear *error bound* estimate of metric type for the distance from  $x$  to the solution set of the generalized equation  $F(u) \ni y$  corresponding to the perturbed right-hand side  $y$  in a neighbourhood of the solution  $\bar{x}$  (corresponding to the right-hand side  $\bar{y}$ ). Metric regularity is known to be equivalent to two other fundamental properties: the *openness* (or *covering*) at a linear rate and the *Aubin property* (a kind of Lipschitz-like behaviour) of the inverse mapping; cf. [3, 8, 11, 13, 15, 22, 30, 31, 35, 37, 38, 43].

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Several characterizations of the metric regularity property have been established in terms of various primal and dual space derivative-like objects: *slopes*, *graphical derivatives* (*Aubin criterion*), *subdifferentials* and *coderivatives*; cf. [4, 5, 15, 22, 30, 31, 34, 35, 36, 38, 43].

The development of the regularity theory in recent years has mostly consisted in the relaxing or extension of the metric regularity property (1.1) (and the other two equivalent properties) and its characterizations along the following three directions (and their appropriate combinations).

1) Relaxing of property (1.1) by fixing one of the variables: either  $y = \bar{y}$  or  $x = \bar{x}$  in it. In the first case, one arrives at the very important for applications property of  $F$  known as *metric subregularity* (and respectively *calmness* of  $F^{-1}$ ); cf. [1, 9, 14, 15, 19, 20, 22, 26, 33, 44], while fixing the other variable (and usually also replacing  $d(y, F(\bar{x}))$  with  $d(y, \bar{y})$ ) leads to another type of relaxed regularity known as *metric semiregularity* [32] (also referred to as *metric hemiregularity* in [2]).

2) Considering nonlocal versions of (1.1), when  $x$  and  $y$  are restricted to certain subsets  $U \subset X$  and  $V \subset Y$ , not necessarily neighbourhoods of  $\bar{x}$  and  $\bar{y}$ , respectively, or even a subset  $W \subset X \times Y$ ; cf. [22, 23, 24, 25]. A nonlocal regularity (covering) setting was already studied in [11].

3) Considering nonlinear versions of (1.1), when, instead of the constant modulus  $\kappa$ , a certain *functional modulus*  $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is used in (1.1), i.e.,  $\kappa d(y, F(x))$  is replaced by  $\mu(d(y, F(x)))$ ; cf. [8, 22, 25, 37]. This allows treating more subtle regularity properties arising in applications when the conventional estimates fail. The majority of researchers focus on the particular case of “power nonlinearities” when  $\mu$  is of the type  $\mu(t) = \lambda t^k$  with  $k \geq 1$  [16, 17, 18, 25].

Starting with Ioffe [21], the majority of proofs of various sufficient regularity/openness criteria are based on the application of the celebrated *Ekeland variational principle* (Theorem 2.13); see [7, 15, 22, 35, 38, 43]. On the other hand, as observed by Ioffe in [22], the original methods used by Banach, Schauder, Lyusternik and Graves had employed special iteration procedures. This classical approach was very popular in the 1980s – early 1990s [10, 11, 12, 27, 28, 29, 39, 42]. In particular, in the series of three articles [27, 28, 29], several basic statements were established which generalized many known by that time open mapping and closed graph theorems and theorems of the Lusternik type and results on approximation and semicontinuity or their refinements. We refer to [22] for a thorough discussion and comparison of the two main techniques.

In this article, we demonstrate that the approach based on iteration procedures still possesses potential. In particular, we show that the *Induction theorem* [27, Theorem 1] (see Lemma 2.1 in the current article), which was used as the main tool when proving the other results in [27], implies also all the main results in the subsequent articles [28, 29]. It can serve as a substitution of the Ekeland variational principle when establishing other regularity criteria. Furthermore, the latter classical result can also be established as a consequence of the Induction theorem.

We consider a general regularity model for a set-valued mapping  $F : X \times \mathbb{R}_+ \rightrightarrows Y$ , where  $X$  and  $Y$  are metric spaces. The conventional setting of a set-valued mapping  $F : X \rightrightarrows Y$  between metric spaces can be imbedded into it by defining a set-valued mapping  $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$  by the equality  $\mathcal{F}(x, t) := B(F(x), t)$ . To define an analogue of metric regularity in this general setting, the distance  $d(y, F(x))$  in the image space in the right-hand side of (1.1) is replaced by the “distance-like” quantity

$$(1.2) \quad \delta(y, F, x) := \inf\{t > 0 \mid y \in F(x, t)\}.$$

This allows one to define also a natural analogue of the covering property (But not the Aubin property!) and establish equivalence of both properties and some sufficient criteria.

Motivated by Ioffe [25], we investigate general nonlocal nonlinear regularity models.

The structure of the article is as follows. In the next section, we give a short proof of a revised version of the Induction theorem [27, Theorem 1] and then apply it to establish several basic regularity estimates for a set-valued mapping  $F : X \times \mathbb{R}_+ \rightrightarrows Y$  at a fixed point  $(x, t, y) \in \text{gph}F$ . As a consequence, we obtain the two main theorems from [29]. Next we discuss the relationship between the Induction theorem and the Ekeland variational principle. As another consequence of the aforementioned regularity estimates, we deduce several ‘at a point’ sufficient criteria for the regularity of  $F$  in terms of quantity (1.2).

Section 3 is devoted to nonlinear *regularity on a set* (and the corresponding openness property) being a direct analogue of metric regularity in the conventional setting. We refrain from using the term “metric” because quantity (1.2) is not a distance in the image space.

In Section 4, we demonstrate how the definitions and criteria from Section 3 translate into the conventional setting of a set-valued mapping  $F : X \rightrightarrows Y$  taking the natural metric form.

The final Section 5 contain some concluding remarks and a list of things to be done hopefully in not-so-distant future.

Our basic notation is standard; cf. [15, 35, 43].  $X$  and  $Y$  are metric spaces. Metrics in all spaces are denoted by the same symbol  $d(\cdot, \cdot)$ . If  $x$  and  $C$  are a point and a subset of a metric space, then  $d(x, C) := \inf_{c \in C} d(x, c)$  is the point-to-set distance from  $x$  to  $C$ , while  $\overline{C}$  denotes the closure of  $C$ .  $B(x, r)$  and  $\overline{B}(x, r)$  stand for the open and closed balls of radius  $r > 0$  centered at  $x$ , respectively. We use the convention that  $B(x, 0) = \{x\}$ .

## 2. Regularity at a point.

**2.1. Basic estimates.** The next lemma is a revised version of the *Induction theorem* [27, Theorem 1] and contains the core arguments used in the main results of [27, 28, 29]. For simplicity, it is formulated for mappings between metric spaces. (Most of the results in [27, 28, 29] were established in the more general setting of quasimetric spaces.)

Recall that a set-valued mapping  $F : X \rightrightarrows Y$  between metric spaces is called outer semicontinuous [43] at  $\bar{x} \in X$  if

$$\text{Lim sup}_{x \rightarrow \bar{x}} F(x) := \{y \in Y \mid \liminf_{x \rightarrow \bar{x}} d(y, F(x)) = 0\} \subset F(\bar{x}).$$

LEMMA 2.1. *Let  $X$  be a complete metric space,  $\Phi : \mathbb{R}_+ \rightrightarrows X$ ,  $t > 0$  and  $x \in \Phi(t)$ . Suppose that  $\Phi$  is outer semicontinuous at 0 and there are sequences of positive numbers  $(a_n)$  and  $(b_n)$  such that*

$$(2.1) \quad \sum_{n=0}^{\infty} b_n < \infty,$$

$$(2.2) \quad a_0 = t \quad \text{and} \quad a_n \downarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

$$(2.3) \quad d(u, \Phi(a_{n+1})) < b_n \quad \text{for all} \quad u \in \Phi(a_n) \cap U_n \quad (n = 0, 1, \dots),$$

where  $U_0 := \{x\}$ ,  $U_n := B(x, \sum_{i=0}^{n-1} b_i)$  ( $n = 1, 2, \dots$ ). Then,  $d(x, \Phi(0)) < \sum_{n=0}^{\infty} b_n$ .

*Proof.* Putting  $x_0 := x \in \Phi(a_0) \cap U_0$  and using (2.3) repeatedly, we obtain a sequence  $(x_n)$  satisfying  $x_n \in \Phi(a_n)$  and

$$d(x_n, x_{n+1}) < b_n \quad (n = 0, 1, \dots).$$

The above inequalities together with (2.1) imply that  $(x_n)$  is a Cauchy sequence and, as  $X$  is complete, converges to some point  $z \in X$ . Note that

$$d(z, x) \leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \sum_{n=0}^{\infty} b_n.$$

Thanks to the outer semicontinuity of  $\Phi$  at 0 and (2.2), we have  $z \in \Phi(0)$ . Hence,  $d(x, \Phi(0)) < \sum_{n=0}^{\infty} b_n$ .  $\square$

*Remark 2.2.* 1. With obvious changes, the proof of Lemma 2.1 remains valid if instead of the outer semicontinuity of  $\Phi$  and completeness of  $X$  one assumes that  $\text{gph}\Phi$  is complete (in the product topology). In fact, it is sufficient to assume that  $\text{gph}\Phi \cap (\mathbb{R}_+ \times \overline{B}(x, \sum_{n=0}^{\infty} b_i))$  is complete.

2. In some applications, a “restricted” version of Lemma 2.1 can be useful. Given a subset  $U$  of  $X$  and a point  $x \in \Phi(t) \cap U$ , condition (2.3) can be replaced with the following “restricted” one:

$$d(u, \Phi(a_{n+1}) \cap U) < b_n \quad \text{for all } u \in \Phi(a_n) \cap U_n \quad (n = 0, 1, \dots),$$

where  $U_0 := \{x\}$ ,  $U_n := U \cap B(x, \sum_{i=0}^{n-1} b_i)$  ( $n = 1, 2, \dots$ ).

3. The conclusion of Lemma 2.1 can be equivalently rewritten as

$$\Phi(0) \cap B\left(x, \sum_{n=0}^{\infty} b_n\right) \neq \emptyset.$$

From now on, we consider a set-valued mapping  $F : X \times \mathbb{R}_+ \rightrightarrows Y$ , where  $X$  and  $Y$  are metric spaces,  $X$  is complete. Given a  $t \in \mathbb{R}_+$ , we denote  $F_t := F(\cdot, t) : X \rightrightarrows Y$ .

The next two theorems contain the core arguments of [29, Theorems 3 and 4], respectively.

**THEOREM 2.3.** *Let  $t > 0$  and  $(x, t, y) \in \text{gph}F$ . Suppose that the mapping  $\tau \mapsto \Phi(\tau) := F_{\tau}^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and there are sequences of positive numbers  $(b_n)$  and  $(c_n)$  and a function  $m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.1) holds true and*

$$(2.4) \quad m(\tau) \downarrow 0 \text{ as } \tau \downarrow 0 \quad \text{and} \quad c_n \downarrow 0 \text{ as } n \rightarrow \infty,$$

$$(2.5) \quad d(x, F_{m(c_1)}^{-1}(y)) < b_0,$$

$$(2.6) \quad d(u, F_{m(c_{n+1})}^{-1}(y)) < b_n \text{ for all } u \in F_{m(c_n)}^{-1}(y) \cap B(x, \sum_{i=0}^{n-1} b_i) \quad (n = 1, 2, \dots).$$

Then,  $d(x, F_0^{-1}(y)) < \sum_{n=0}^{\infty} b_n$ .

*Proof.* Set  $a_0 := t$ ,  $a_n := m(c_n)$  ( $n = 1, 2, \dots$ ). Conditions (2.4), (2.5) and (2.6) imply (2.2) and (2.3). By Lemma 2.1, there exists a  $z \in B(x, \sum_{n=0}^{\infty} b_n)$  such that  $y \in F(z, 0)$ , i.e.,  $z \in F_0^{-1}(y)$ .  $\square$

*Remark 2.4.* 1. Instead of (2.4), it is sufficient to assume in Theorem 2.3 that  $m(c_n) \downarrow 0$  as  $n \rightarrow \infty$ .

2. The conclusion of Theorem 2.3 can be equivalently rewritten as

$$y \in F \left( B \left( x, \sum_{n=0}^{\infty} b_n \right), 0 \right).$$

Given a function  $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , we define, for each  $t \in \mathbb{R}_+$ ,  $b^0(t) := t$ ,  $b^n(t) := b(b^{n-1}(t))$  ( $n = 1, 2, \dots$ ).

**THEOREM 2.5.** *Let  $t > 0$  and  $(x, t, y) \in \text{gph}F$ . Suppose that the mapping  $\tau \mapsto \Phi(\tau) := F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and there are functions  $b, m, \mu : (0, \infty) \rightarrow (0, \infty)$  such that*

$$(2.7) \quad m(\tau) \downarrow 0 \quad \Rightarrow \quad \tau \downarrow 0$$

and, for each  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ ,

$$(2.8) \quad \mu(\tau) \geq m(\tau) + \mu(b(\tau)),$$

$$(2.9) \quad d(u, F_{b(\tau)}^{-1}(y)) < m(\tau) \text{ for all } u \in F_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau)).$$

Then,  $d(x, F_0^{-1}(y)) < \mu(t)$ .

*Proof.* Set  $a_n := b^n(t)$ ,  $b_n := m(a_n) = m(b^n(t))$  ( $n = 0, 1, \dots$ ). Adding inequalities (2.8) corresponding to  $\tau = t, b(t), b^2(t), \dots$ , we obtain

$$\mu(t) \geq \sum_{n=0}^{\infty} m(b^n(t)) = \sum_{n=0}^{\infty} b_n.$$

Hence, (2.1) is satisfied and  $b_n \downarrow 0$  as  $n \rightarrow \infty$ . Condition (2.2) is satisfied thanks to (2.7). Condition (2.9) with  $\tau = a_n$  takes the following form:

$$(2.10) \quad d(u, \Phi(a_{n+1})) < b_n \text{ for all } u \in \Phi(a_n) \cap B(x, \mu(t) - \mu(a_n)).$$

For any  $n > 0$ , adding inequalities (2.8) corresponding to  $\tau = t, b(t), \dots, b^{n-1}(t)$ , we obtain

$$\mu(t) \geq \sum_{i=0}^{n-1} b_i + \mu(a_n).$$

Hence,  $\mu(a_n) \leq \mu(t)$  and condition (2.10) implies (2.3). By Lemma 2.1, there exists a  $z \in B(x, \mu(t))$  such that  $y \in F(z, 0)$ .  $\square$

Recall that a family  $\Sigma$  of balls in  $X$  is called a *complete system* [11, Definition 1.1] if, for any  $B(x, r) \in \Sigma$ , one has  $B(x', r') \in \Sigma$  provided that  $x' \in X$ ,  $r' > 0$  and  $d(x, x') + r' \leq r$ . For a subset  $M$  of  $X$ ,  $\Sigma(M)$  denotes a complete system of balls  $B(x, r)$  in  $X$  with  $B(x, r) \subset M$ . Obviously the family of all balls in  $X$  forms a complete system.

**COROLLARY 2.6.** *Let  $M \subset X$  and  $\Sigma(M)$  be a complete system,  $t > 0$  and  $(x, t, y) \in \text{gph}F$ . Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and there are functions  $b, m, \mu : (0, \infty) \rightarrow (0, \infty)$  such that  $B(x, \mu(t)) \in \Sigma(M)$ , condition (2.7) is satisfied and, for each  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , condition (2.8) holds true and*

$$(2.11) \quad d(u, F_{b(\tau)}^{-1}(y)) < m(\tau) \text{ for all } u \in F_\tau^{-1}(y) \cap \{x' \mid B(x', \mu(\tau)) \in \Sigma(M)\}.$$

Then,  $d(x, F_0^{-1}(y)) < \mu(t)$ .

*Proof.* Since  $B(x, \mu(t)) \in \Sigma(M)$ , it follows that  $B(x, \mu(t) - \mu(\tau)) \subset \{x' \mid B(x', \mu(\tau)) \in \Sigma(M)\}$ . The conclusion follows from Theorem 2.5.  $\square$

Key estimates (2.9) and (2.11) in Theorem 2.5 and Corollary 2.6 are for the original space  $X$ . In some situations, one can use for that purpose also similar estimates in the image space  $Y$ .

**COROLLARY 2.7.** *Let  $t > 0$  and  $(x, t, y) \in \text{gph}F$ . Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and there are functions  $b, m, \mu : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for each  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , condition (2.8) holds true and*

$$(2.12) \quad F_0^{-1}(B(y, \tau)) \subset F_\tau^{-1}(y),$$

$$(2.13) \quad d(y, F_0(B(u, m(\tau)))) < b(\tau) \text{ for all } u \in F_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau)).$$

Then,  $d(x, F_0^{-1}(y)) < \mu(t)$ .

*Proof.* Observe that conditions (2.12) and (2.13) imply (2.9). Indeed, if  $u \in F_\tau^{-1}(y) \cap B(x, \mu(t) - \mu(\tau))$ , then, by (2.13), there exists a  $z \in B(u, m(\tau))$  such that  $d(y, F_0(z)) < b(\tau)$ , or equivalently,  $z \in F_0^{-1}(B(y, b(\tau)))$ . It follows from (2.12) that  $z \in F_{b(\tau)}^{-1}(y)$ . Hence,  $d(u, F_{b(\tau)}^{-1}(y)) < m(\tau)$ . The conclusion follows from Theorem 2.5.  $\square$

**Remark 2.8.** 1. *Instead of (2.7), it is sufficient to assume in Theorem 2.5 and Corollaries 2.6 and 2.7 that  $b^n(t) \downarrow 0$  as  $n \rightarrow \infty$ .*

*The last condition is satisfied, e.g., when  $b(t) = \lambda t$  with  $\lambda \in (0, 1)$ .*

2. *If condition (2.8) holds true for all  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , then  $\mu(\tau) \geq \sum_{n=0}^{\infty} m(b^n(\tau))$ . On the other hand, if the last condition holds true as equality (for all  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ ), then condition (2.8) is satisfied (as equality). Hence, condition (2.8) in Theorem 2.5 and Corollaries 2.6 and 2.7 can be replaced by the following definition of the smallest function  $\mu$  satisfying (2.8):*

$$(2.14) \quad \mu(\tau) := \sum_{n=0}^{\infty} m(b^n(\tau)),$$

*thus producing the strongest conclusion.*

3. *It is sufficient to assume in Theorem 2.5 and Corollaries 2.6 and 2.7 that conditions (2.8), (2.9), (2.11), (2.12) and (2.13) are satisfied only for  $\tau = t, b(t), b^2(t), \dots$ . In particular, if this sequence is monotone (as in the typical example mentioned in part 1 above or, thanks to (2.8) when  $\mu$  is nondecreasing), then the conclusions of all the statements remain true when conditions (2.8), (2.9), (2.11), (2.12) and (2.13) are satisfied for all  $\tau \in (0, t]$ .*

4. *Thanks to part 3, instead of conditions (2.9), (2.11) and (2.13), one can require that, for each  $n = 0, 1, \dots$ , the following conditions hold true, respectively:*

$$(2.15) \quad \begin{aligned} d(u, F_{b^{n+1}(t)}^{-1}(y)) &< m(b^n(t)) \text{ for all } u \in F_{b^n(t)}^{-1}(y) \\ &\cap B(x, \mu(t) - \mu(b^n(t))), \\ d(u, F_{b^{n+1}(t)}^{-1}(y)) &< m(b^n(t)) \text{ for all } u \in F_{b^n(t)}^{-1}(y) \\ &\cap \{x' \mid B(x', \mu(b^n(t))) \in \Sigma(M)\}, \\ d(y, F_0(B(u, m(b^n(t)))))) &< b^{n+1}(t) \text{ for all } u \in F_{b^n(t)}^{-1}(y) \\ (2.16) \quad &\cap B(x, \mu(t) - \mu(b^n(t))). \end{aligned}$$

If  $\mu$  is given by (2.14), then conditions (2.15) and (2.16) can be equivalently rewritten as follows:

$$d(u, F_{b^{n+1}(t)}^{-1}(y)) < m(b^n(t)) \text{ for all } u \in F_{b^n(t)}^{-1}(y) \cap B(x, \sum_{i=0}^{n-1} b^i(t)),$$

$$d(y, F_0(B(u, m(b^n(t)))))) < b^{n+1}(t) \text{ for all } u \in F_{b^n(t)}^{-1}(y) \cap B(x, \sum_{i=0}^{n-1} b^i(t)).$$

5. The conclusions of Theorem 2.5 and Corollaries 2.6 and 2.7 can be equivalently rewritten as  $y \in F(B(x, \mu(t)), 0)$ .

The next two theorems are the (slightly improved) original results of [29, Theorems 3 and 4] reformulated in the setting of metric spaces and adopting the terminology and notation of the current article. These theorems, which follow immediately from Theorems 2.3 and 2.5, respectively, imply all the other results of [27, 28, 29] as well as many open mapping and closed graph theorems and theorems of the Lusternik type and results on approximation and semicontinuity or their refinements; cf. the references in [27, 28, 29].

**THEOREM 2.9.** *Let  $t > 0$  and  $(x, t) \in \text{dom } F$ . Suppose that, for each  $y \in Y$ ,*

$$(2.17) \quad F_0^{-1}(y) = \text{Lim sup}_{t \downarrow 0} F_t^{-1}(y)$$

and there are positive numbers  $\rho$ ,  $s$  and  $b_n$  ( $n = 1, 2, \dots$ ), such that

$$(2.18) \quad \sum_{n=1}^{\infty} b_n + s \leq \rho.$$

Suppose also that, for each  $y \in F(x, t)$ , there are numbers  $c_n > 0$  ( $n = 1, 2, \dots$ ) and a function  $m : (0, \infty) \rightarrow (0, \infty)$  satisfying (2.4) and

$$(2.19) \quad d(u, F_{m(c_1)}^{-1}(y)) < s \text{ for all } u \in F_t^{-1}(y) \cap B(x, \rho - s),$$

$$(2.20) \quad d(u, F_{m(c_{n+1})}^{-1}(y)) < b_n \text{ for all } u \in F_{m(c_n)}^{-1}(y) \cap B(x, \rho - b_n) \text{ } (n = 1, 2, \dots)$$

Then,  $F(x, t) \subset F(B(x, \rho), 0)$ .

*Proof.* Set  $b_0 := s$  and take any  $y \in F(x, t)$ . It follows from (2.17) that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0. Condition (2.18) obviously implies (2.1). Observe that  $\sum_{i=0}^{n-1} b_i \leq \rho - \sum_{i=n}^{\infty} b_i < \rho - b_n$  ( $n = 0, 1, \dots$ ). Hence, conditions (2.19) and (2.20) imply (2.5) and (2.6), respectively. By Theorem 2.3,  $y \in F(B(x, \rho), 0)$ .  $\square$

**THEOREM 2.10.** *Let  $M \subset X$  and  $\Sigma(M)$  be a complete system. Let a function  $b : (0, \infty) \rightarrow (0, \infty)$  be given. Suppose that, for each  $y \in Y$ , condition (2.17) holds true and there exists a function  $m : (0, \infty) \rightarrow (0, \infty)$  satisfying condition (2.7) and, for all  $\tau \in (0, \infty)$  and  $x \in X$  with  $(x, t, y) \in \text{gph} F$  and  $B(x, \mu(\tau)) \in \Sigma(M)$ , conditions (2.11) and (2.14) are satisfied. Then, for any  $(x, t, y) \in \text{gph} F$  with  $t > 0$  and  $B(x, \mu(t)) \in \Sigma(M)$ , one has  $y \in F(B(x, \mu(t)), 0)$ .*

*Proof.* Take any  $(x, t, y) \in \text{gph} F$  with  $t > 0$  and  $B(x, \mu(t)) \in \Sigma(M)$  and a function  $m$  satisfying the assumptions of the theorem. Condition (2.17) obviously implies that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0.



Thanks to Remark 2.8.2, all the assumptions of Corollary 2.6 are satisfied. Hence,  $y \in F(B(x, \mu(t)), 0)$ .  $\square$

*Remark 2.11.* Comparing the statements of Theorem 2.10 and [29, Theorem 4], one can notice that the latter one looks stronger: it is formulated without assumption (2.7) and with the stronger conclusion  $F(x, t) \subset F(B(x, \mu(t)), 0)$ . However assumption (2.7) is implicitly used in the proof of [29, Theorem 4] and the conclusion is established for a fixed  $y \in F(x, t)$  satisfying  $B(x, \mu(t)) \in \Sigma(M)$ . (Observe that function  $m$  in Theorem 2.10 and consequently function  $\mu$  defined by (2.14) depend on the choice of  $y \in F(x, t)$ .)

Unlike the setting of the current article, in [29] mapping  $F$  was assumed to be defined not on  $X \times \mathbb{R}_+$ , but on  $X \times [0, t_0]$  where  $t_0$  is a given positive number. This difference can be easily eliminated by setting  $F(x, t) := \emptyset$  when  $t > t_0$  and making appropriate minor changes in the statements.

**2.2. Lemma 2.1 and Ekeland variational principle.** Lemma 2.1 which lies at the core of the proofs of Theorems 2.3 and 2.5 can serve as a substitution of the Ekeland variational principle which is a traditional tool when establishing regularity criteria. This is demonstrated by the proof of the following theorem.

**THEOREM 2.12.** *Let  $t > 0$  and  $(x, t, y) \in \text{gph}F$ . Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  is outer semicontinuous on  $[0, t)$  and there is a continuous nondecreasing function  $\mu : [0, t] \rightarrow \mathbb{R}_+$  satisfying  $\mu(\tau) = 0$  if and only if  $\tau = 0$  and, for each pair  $(u, \tau) \in F^{-1}(y)$  with  $\tau \in (0, t]$  and  $d(x, u) \leq \mu(t) - \mu(\tau)$ , there exists a pair  $(u', \tau') \in F^{-1}(y)$  such that  $u' \neq u$  and*

$$(2.21) \quad \mu(\tau') \leq \mu(\tau) - d(u', u).$$

Then,  $d(x, F_0^{-1}(y)) \leq \mu(t)$ .

*Proof.* Set  $a_0 := t$ ,  $x_0 := x$  and define a sequence  $\{(x_n, a_n)\}$  by induction. For any  $n = 0, 1, \dots$ , let a pair  $(x_n, a_n) \in F^{-1}(y)$  with  $a_n \in [0, t]$  and  $d(x, x_n) \leq \mu(t) - \mu(a_n)$  be given. If  $a_n = 0$ , set  $a_{n+1} := 0$  and  $x_{n+1} := x_n$ . Otherwise, define

$$(2.22) \quad c_n := \inf\{\mu(\tau) \mid (u, \tau) \in F^{-1}(y), \mu(\tau) \leq \mu(a_n) - d(u, x_n)\}.$$

By the assumptions of the theorem,  $0 \leq c_n < \mu(a_n)$ , and one can choose a pair  $(x_{n+1}, a_{n+1}) \in F^{-1}(y)$  such that  $x_{n+1} \neq x_n$  and

$$(2.23) \quad \mu(a_{n+1}) \leq \mu(a_n) - d(x_n, x_{n+1}),$$

$$(2.24) \quad c_n \leq \mu(a_{n+1}) < \frac{\mu(a_n) + c_n}{2} < \mu(a_n).$$

It also follows from (2.23) that

$$d(x, x_{n+1}) \leq d(x, x_n) + d(x_n, x_{n+1}) \leq \mu(t) - \mu(a_{n+1}).$$

If  $a_n = 0$  for some  $n > 0$ , then, by (2.23),

$$d(x, F_0^{-1}(y)) \leq d(x, x_n) \leq \sum_{j=0}^{n-1} d(x_j, x_{j+1}) \leq \mu(t).$$

Now assume that  $a_n > 0$  for all  $n = 0, 1, \dots$ . Then,  $\{a_n\}$  is a decreasing sequence of positive numbers which converges to some  $a \geq 0$ . We are going to show that  $a = 0$ .

Suppose that  $a > 0$  and denote  $\hat{a}_n := a_n - a$ . Obviously,  $\hat{a}_n > 0$  and  $\hat{a}_n \downarrow 0$ . By (2.23),

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \mu(t) - \mu(a).$$

Fix an  $\varepsilon > 0$  and choose numbers  $b_n > d(x_n, x_{n+1})$  such that  $\sum_{n=0}^{\infty} b_n < \mu(t) - \mu(a) + \varepsilon$ . Set  $\Phi(\hat{a}_n) := \{x_n\}$ ,  $\Phi(\tau) := \emptyset$  for any  $\tau \in (0, \infty) \setminus \{\hat{a}_0, \hat{a}_1, \dots\}$ , and let  $\Phi(0)$  be the set of all cluster points of  $\{x_n\}$ . Then,  $x \in \Phi(\hat{a}_0)$ ,  $\Phi$  is outer semicontinuous at 0 and  $d(\Phi(\hat{a}_n), \Phi(\hat{a}_{n+1})) < b_n$ . It follows from Lemma 2.1 that there exists a  $z \in \Phi(0)$  satisfying  $d(x, z) < \mu(t) - \mu(a) + \varepsilon$ . By the outer semicontinuity of  $\Phi$ ,  $y \in F(z, a)$ .

Since  $a > 0$ , by the assumptions of the theorem, there exists a pair  $(u, \tau) \in F^{-1}(y)$  such that  $u \neq z$  and

$$(2.25) \quad \mu(\tau) \leq \mu(a) - d(u, z).$$

Then,  $\mu(\tau) < \mu(a)$ . Observe from (2.24) that

$$2\mu(a_{n+1}) - \mu(a_n) < c_n < \mu(a_n).$$

Hence,  $\{c_n\}$  converges to  $\mu(a)$  and consequently  $\mu(\tau) < c_n$  when  $n$  is large enough. By definition (2.22), this yields

$$(2.26) \quad \mu(\tau) > \mu(a_n) - d(u, x_n).$$

At the same time,

$$d(x_n, z) \leq \sum_{j=n}^{\infty} d(x_j, x_{j+1}) \leq \mu(a_n) - \mu(a).$$

This combined with (2.25) gives

$$\mu(\tau) \leq \mu(a_n) - d(u, x_n)$$

which is in obvious contradiction with (2.26). Hence,  $a = 0$ ,  $z \in F_0^{-1}(y)$ ,  $d(x, z) < \mu(t) + \varepsilon$ , and, as  $\varepsilon$  is arbitrary,  $d(x, F_0^{-1}(y)) \leq \mu(t)$ .  $\square$

The proof of Theorem 2.12 given above uses standard arguments typical for traditional proofs of the Ekeland variational principle; cf. e.g. [7]. We next show that the latter classical result can also be established as a consequence of Lemma 2.1.

**THEOREM 2.13** (Ekeland variational principle). *Let  $X$  be a complete metric space and  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be lower semicontinuous and bounded from below. Suppose  $\varepsilon > 0$ ,  $\lambda > 0$  and  $x \in X$  satisfies*

$$f(x) < \inf_X f + \varepsilon.$$

*Then, there exists a  $z \in X$  such that*

- (i)  $d(z, x) < \lambda$ ,
- (ii)  $f(z) \leq f(x)$ ,
- (iii)  $f(u) + (\varepsilon/\lambda)d(u, z) \geq f(z)$  for all  $u \in X$ .

*Proof.* Denote  $x_0 := x$ . For  $n = 0, 1, \dots$ , set

$$(2.27) \quad a_n := \sup_{u \in X} \left\{ f(x_n) - f(u) - \frac{\varepsilon}{\lambda} d(u, x_n) \right\}.$$

Obviously,  $0 \leq a_n < \infty$ . Choose an  $x_{n+1}$  such that

$$(2.28) \quad f(x_n) - f(x_{n+1}) - \frac{\varepsilon}{\lambda} d(x_{n+1}, x_n) \geq \frac{a_n}{2}.$$

Then, for  $n = 0, 1, \dots$ ,

$$f(x_{n+1}) \leq f(x_n), \quad d(x_{n+1}, x_n) \leq \frac{\lambda}{\varepsilon} (f(x_n) - f(x_{n+1}))$$

and the inequalities are strict if  $a_n > 0$ . It follows that

$$f(x_n) \leq f(x) \quad \text{and} \quad d(x_n, x) \leq \frac{\lambda}{\varepsilon} (f(x) - f(x_n)) < \lambda.$$

If, for some  $n$ ,  $a_n = 0$ , then  $z := x_n$  satisfies the conclusions of the theorem. Suppose that  $a_n > 0$  for all  $n = 0, 1, \dots$ . Then,  $b_n := \frac{\lambda}{\varepsilon} (f(x_n) - f(x_{n+1})) > 0$ . Set  $\Phi(a_n) := \{x_n\}$ ,  $\Phi(\tau) := \emptyset$  for any  $\tau \in (0, \infty) \setminus \{a_0, a_1, \dots\}$  and  $\Phi(0) := \text{Lim sup}_{\tau \downarrow 0} \Phi(\tau)$ . Hence,  $\Phi$  is outer semicontinuous at 0,  $x \in \Phi(a_0)$ ,  $\sum_{n=0}^{\infty} b_n < \lambda$  and  $d(\Phi(a_n), \Phi(a_{n+1})) < b_n$ . Besides, it follows from (2.27) that

$$(2.29) \quad f(x_n) - f(u) - \frac{\varepsilon}{\lambda} d(u, x_n) \leq a_n \quad \text{for all } u \in X.$$

Subtracting (2.28) from the last inequality and using the triangle inequality, we conclude that

$$f(x_{n+1}) - f(u) - \frac{\varepsilon}{\lambda} d(u, x_{n+1}) \leq \frac{a_n}{2} \quad \text{for all } u \in X,$$

i.e.,  $a_{n+1} \leq a_n/2$  and consequently  $a_n \downarrow 0$  as  $n \rightarrow \infty$ . It follows from Lemma 2.1 that there exists a  $z \in \Phi(0)$  satisfying (i). By the definition of  $\Phi(0)$  and (2.29), we conclude that conditions (ii) and (iii) are satisfied too.  $\square$

*Remark 2.14.* Lemma 2.1 was used in the proof of Theorem 2.13 where one would normally use the convergence of a Cauchy sequence. Similarly, the Ekeland variational principle can replace the Cauchy sequence argument in the proof of Lemma 2.1. In fact, both Lemma 2.1 and Theorem 2.13 are in a sense equivalent to the completeness of  $X$ .

**2.3. Regularity.** Theorems 2.3, 2.5 and 2.12 and Corollaries 2.6 and 2.7 were formulated for a fixed point  $(x, t, y) \in \text{gph}F$ . The next step is to “set variable  $t$  free” and formulate criteria in terms of (fixed)  $x$  and  $y$  only.

The next assertion establishes a regularity estimate in terms of the “distance-like” quantity  $\delta(y, F, x)$  defined by (1.2). It is an immediate consequence of Theorem 2.3.

**THEOREM 2.15.** *Let  $(x, y) \in X \times Y$  and  $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an upper semicontinuous nondecreasing function. Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and, for some  $\gamma > \delta(y, F, x)$  and any  $t \in (0, \gamma)$  with  $(x, t, y) \in \text{gph}F$ , there are sequences of positive numbers  $(b_n)$  and  $(c_n)$  and a function  $m : (0, \infty) \rightarrow (0, \infty)$  such that conditions (2.4)–(2.6) hold true and*

$$(2.30) \quad \sum_{n=0}^{\infty} b_n \leq \mu(t).$$

Then,  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ .

*Proof.* It is sufficient to notice that, for any  $t \in (0, \gamma)$  with  $(x, t, y) \in \text{gph}F$ , condition (2.30) implies (2.1) and, by Theorem 2.3,  $d(x, F_0^{-1}(y)) < \mu(t)$ . Taking the infimum in the right-hand side of the above inequality over all  $t > 0$  with  $(x, t, y) \in \text{gph}F$  and making use of the monotonicity of  $\mu$ , we arrive at the claimed conclusion.  $\square$

The next several assertions are consequences of Theorems 2.5 and 2.12 and Corollaries 2.6 and 2.7, respectively, thanks to the same arguments as those used in the proof of Theorem 2.15.

**THEOREM 2.16.** *Let  $(x, y) \in X \times Y$  and  $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an upper semicontinuous nondecreasing function. Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and, for some  $\gamma > \delta(y, F, x)$  and any  $t \in (0, \gamma)$  with  $(x, t, y) \in \text{gph}F$ , there are functions  $b, m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for each  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , conditions (2.8) and (2.9) hold true. Then,  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ .*

**COROLLARY 2.17.** *Let  $M \subset X$  and  $\Sigma(M)$  be a complete system,  $(x, y) \in X \times Y$  and  $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an upper semicontinuous nondecreasing function. Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and, for some  $\gamma > \delta(y, F, x)$  and any  $t \in (0, \gamma)$  with  $(x, t, y) \in \text{gph}F$ , one has  $B(x, \mu(t)) \in \Sigma(M)$ , there are functions  $b, m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for each  $\tau > 0$ , conditions (2.8) and (2.11) hold true. Then,  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ .*

**COROLLARY 2.18.** *Let  $(x, y) \in X \times Y$  and  $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an upper semicontinuous nondecreasing function. Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and, for some  $\gamma > \delta(y, F, x)$  and any  $t \in (0, \gamma)$  with  $(x, t, y) \in \text{gph}F$ , there are functions  $b, m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for each  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , conditions (2.8), (2.12) and (2.13) hold true. Then,  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ .*

*Remark 2.19.* Most of the comments in Remarks 2.4 and 2.8 are applicable to Theorems 2.15 and 2.16 and Corollaries 2.17 and 2.18.

**THEOREM 2.20.** *Let  $(x, y) \in X \times Y$ ,  $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous nondecreasing function and  $\mu(\tau) = 0$  if and only if  $\tau = 0$ . Suppose that the mapping  $\tau \mapsto F_\tau^{-1}(y)$  is outer semicontinuous on  $[0, \delta(y, F, x)]$  and, for each pair  $(u, \tau) \in F^{-1}(y)$  with  $\tau \in (0, \delta(y, F, x)]$  and  $d(x, u) \leq \mu(\delta(y, F, x)) - \mu(\delta(y, F, u))$ , there exists a pair  $(u', \tau') \in F^{-1}(y)$  such that  $u' \neq u$  and condition (2.21) is satisfied. Then,  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ .*

*Proof.* If  $\delta(y, F, x) = \infty$ , then the conclusion holds true trivially. Otherwise, the outer semicontinuity of  $\tau \mapsto F_\tau^{-1}(y)$  ensures that  $y \in F(x, \delta(y, F, x))$ , and the conclusion follows from Theorem 2.12 for  $t = \delta(y, F, x)$ .  $\square$

*Remark 2.21.* The conclusion of Theorems 2.15, 2.16 and 2.20 and Corollaries 2.17 and 2.18 reminds the inequality in the definition of the metric regularity property for a set-valued mapping  $F : X \rightrightarrows Y$  between metric spaces; cf. [15]. The difference is in the right-hand side, where  $\delta(y, F, x)$  stands in place of  $d(y, F(x))$ . The relationship between the two settings will be explored in Section 4.

The conclusion of Theorems 2.15, 2.16 and 2.20 and Corollaries 2.17 and 2.18 can be reformulated equivalently in a ‘‘covering-like’’ form.

**PROPOSITION 2.22.** *Consider the following conditions:*

- (i)  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ ,
- (ii)  $y \in F(B(x, t), 0)$  for any  $t > \mu(\delta(y, F, x))$ ,
- (iii)  $y \in F(B(x, \mu(\delta(y, F, x))), 0)$ .

Then, (iii)  $\Rightarrow$  (ii)  $\Leftrightarrow$  (i).

*Proof.* (iii)  $\Rightarrow$  (ii) is obvious.

(i)  $\Rightarrow$  (ii). By (i), for any  $t > \mu(\delta(y, F, x))$ , there exists a  $z \in F_0^{-1}(y)$  such that  $d(x, z) < t$  and consequently  $y \in F(z, 0) \subset F(B(x, t), 0)$ .

(ii)  $\Rightarrow$  (i).  $y \in F(B(x, t), 0)$  and  $t > 0$  if and only if  $d(x, F_0^{-1}(y)) < t$ . If the last inequality holds for all  $t > \mu(\delta(y, F, x))$ , then  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ .  $\square$

*Remark 2.23.* Proposition 2.22 is true without the assumption of the completeness of  $X$ .

**3. Regularity on a set.** In this section, we continue exploring regularity properties for a set-valued mapping  $F : X \times \mathbb{R}_+ \rightrightarrows Y$ , where  $X$  and  $Y$  are metric spaces. Given a subset  $W \subset X \times Y$  and an upper semicontinuous nondecreasing function  $\mu : [0, +\infty] \rightarrow [0, +\infty]$ , we use the statements derived in Section 2 to characterize regularity of  $F$  on  $W$  with functional modulus  $\mu$ .

DEFINITION 3.1.

(i)  $F$  is regular on  $W$  with functional modulus  $\mu$  if

$$d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x)) \quad \text{for all } (x, y) \in W.$$

(ii)  $F$  is open on  $W$  with functional modulus  $\mu$  if

$$y \in F(B(x, t), 0) \quad \text{for all } (x, y) \in W \text{ and } t > \mu(\delta(y, F, x)).$$

The above properties differ from the conventional metric regularity defined for set-valued mappings between metric spaces (cf. [15]) and its nonlinear extensions (cf. [25]). The relationship between the two settings will be discussed in Section 4.

The next proposition is a consequence of Proposition 2.22 thanks to Remark 2.23.

PROPOSITION 3.2. *The two properties in Definition 3.1 are equivalent.*

*Remark 3.3.* It follows from Proposition 2.22 that the properties in Definition 3.1 are implied by the following stronger version of openness:

$$y \in F(B(x, \mu(\delta(y, F, x))), 0) \quad \text{for all } (x, y) \in W.$$

The criteria of regularity in the next theorem are direct consequences of Theorems 2.15 and 2.16 and Corollary 2.18.

THEOREM 3.4. *Suppose that, for any  $(x, y) \in W$ , the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and, for some  $\gamma > \delta(y, F, x)$  and any  $t \in (0, \gamma)$  with  $(x, t, y) \in \text{gph}F$ , one of the following sets of conditions is satisfied:*

- (i) *there are sequences of positive numbers  $(b_n)$  and  $(c_n)$  and a function  $m : (0, \infty) \rightarrow (0, \infty)$  such that conditions (2.4)–(2.6) and (2.30) hold true,*
- (ii) *there are functions  $b, m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for any  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , conditions (2.8) and (2.9) hold true,*
- (iii) *there are functions  $b, m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for any  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , conditions (2.8), (2.12) and (2.13) hold true.*

Then,  $F$  is regular on  $W$  with functional modulus  $\mu$ .

In the next statement,  $p_Y : X \times Y \rightarrow Y$  denotes the canonical projection on  $Y$ : for any  $(x, y) \in X \times Y$ ,  $p_Y(x, y) = y$ . Given a pair  $(x, y) \in W$ , denote

$$U_{x,y} := \{u \in X \mid \delta(y, F, u) > 0, \mu(\delta(y, F, u)) + d(u, x) \leq \mu(\delta(y, F, x))\}.$$

**THEOREM 3.5.** *Let  $\mu$  be continuous,  $\mu(\tau) = 0$  if and only if  $\tau = 0$ . Suppose that  $F^{-1}$  is closed-valued on  $p_Y(W)$  and, for any  $(x, y) \in W$  and  $u \in U_{x,y}$ , there exists a point  $u' \neq u$  such that*

$$(3.1) \quad \mu(\delta(y, F, u')) \leq \mu(\delta(y, F, u)) - d(u, u').$$

*Then,  $F$  is regular on  $W$  with functional modulus  $\mu$ .*

*Proof.* Fix an arbitrary  $(x, y) \in W$ . We need to show that  $d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x))$ .

If there exists a point  $u$  such that  $\delta(y, F, u) = 0$  and  $d(x, u) \leq \mu(\delta(y, F, x))$  (in particular, if  $\delta(y, F, x) = 0$ ), then, by the closedness of  $F^{-1}(y)$ ,  $u \in F_0^{-1}(y)$ , and the inequality holds trivially.

Suppose that  $\delta(y, F, u) > 0$  for any  $u \in X$  such that  $d(x, u) \leq \mu(\delta(y, F, x))$ . Take any  $u \in X$  such that  $d(x, u) \leq \mu(\delta(y, F, x)) - \mu(\delta(y, F, u))$  and any  $\tau \in (0, \delta(y, F, x)]$  such that  $(u, \tau) \in F^{-1}(y)$ . Then,  $\tau \geq \delta(y, F, u) > 0$  and, by the assumption, there exists a point  $u' \neq u$  satisfying (3.1). Setting  $\tau' = \delta(y, F, u')$ , we get  $(u', \tau') \in F^{-1}(y)$  and condition (2.21) is satisfied:

$$\mu(\tau') = \mu(\delta(y, F, u')) \leq \mu(\delta(y, F, u)) - d(u, u') \leq \mu(\tau) - d(u, u').$$

The mapping  $\tau \mapsto F_\tau^{-1}(y)$  is outer semicontinuous on  $[0, \delta(y, F, x)]$  thanks to the closedness of  $F^{-1}(y)$ . The required inequality follows from Theorem 2.20.  $\square$

Following [25], one can define seemingly more general  $\nu$ -versions of the properties in Definition 3.1, determined by a function  $\nu : W \rightarrow (0, \infty]$ .

**DEFINITION 3.6.**

(i)  *$F$  is  $\nu$ -regular on  $W$  with functional modulus  $\mu$  if*

$$d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x)) \text{ for all } (x, y) \in W \text{ with } \mu(\delta(y, F, x)) < \nu(x, y).$$

(ii)  *$F$  is  $\nu$ -open on  $W$  with functional modulus  $\mu$  if*

$$y \in F(B(x, t), 0) \text{ for all } (x, y) \in W \text{ and } t \in (\mu(\delta(y, F, x)), \nu(x, y)).$$

*Remark 3.7.* Each of the properties in Definition 3.1 is a particular case of the corresponding one in Definition 3.6 with any function  $\nu : W \rightarrow (0, \infty]$  satisfying  $\mu(\delta(y, F, x)) < \nu(x, y)$  for all  $(x, y) \in W$  with  $\mu(\delta(y, F, x)) < +\infty$ , e.g., one can take  $\nu \equiv +\infty$ . At the same time, each of the properties in Definition 3.6 can be considered as a particular case of the corresponding one in Definition 3.1 with the set  $W$  replaced by  $W' := \{(x, y) \in W \mid \mu(\delta(y, F, x)) < \nu(x, y)\}$ .

**PROPOSITION 3.8.** *The two properties in Definition 3.6 are equivalent.*

We next formulate the corresponding criteria for  $\nu$ -regularity. The next theorem is an immediate consequence of Theorem 3.4 thanks to Remark 3.7 and the simple observation that, if  $\mu(\delta(y, F, x)) < \nu(x, y)$ , then, thanks to the upper semicontinuity of  $\mu$ , it is possible to choose a  $\gamma > \delta(y, F, x)$  such that  $\mu(\gamma) < \nu(x, y)$ .

**THEOREM 3.9.** *Suppose that, for any  $(x, y) \in W$ , the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and, for any  $t > 0$  with  $(x, t, y) \in \text{gph}F$  and  $\mu(t) < \nu(x, y)$ , one of the three sets of conditions in Theorem 3.4 is satisfied. Then,  $F$  is  $\nu$ -regular on  $W$  with functional modulus  $\mu$ .*

**THEOREM 3.10.** *Let  $\mu$  be continuous,  $\mu(\tau) = 0$  if and only if  $\tau = 0$  and  $\nu : \bigcup_{(x,y) \in W} (U_{x,y} \times \{y\}) \rightarrow (0, \infty)$  be Lipschitz continuous with modulus not greater than 1 in  $x$  for any  $y \in p_Y(W)$ . Suppose that  $F^{-1}$  takes closed values on  $p_Y(W)$  and, for*

any  $(x, y) \in W$  and  $u \in U_{x,y}$  with  $\mu(\delta(y, F, u)) < \nu(u, y)$ , there exists a point  $u' \neq u$  such that condition (3.1) holds true. Then,  $F$  is  $\nu$ -regular on  $W$  with functional modulus  $\mu$ .

*Proof.* Define  $W' := \{(x, y) \in W \mid \mu(\delta(y, F, x)) < \nu(x, y)\}$  and take any  $(x, y) \in W'$  and  $u \in U_{x,y}$ . Then, taking into account the Lipschitz continuity of  $\nu$ , we have:

$$\mu(\delta(y, F, u)) \leq \mu(\delta(y, F, x)) - d(x, u) < \nu(x, y) - d(x, u) \leq \nu(u, y).$$

Hence, there exists a point  $u' \neq u$  such that (3.1) holds true. By Theorem 3.5,  $F$  is regular on  $W'$  and, thanks to Remark 3.7,  $\nu$ -regular on  $W$  with functional modulus  $\mu$ .  $\square$

*Remark 3.11.* Properties in Definitions 3.1 and 3.6 depend on the choice of the set  $W$  and (in the case of Definitions 3.6) function  $\nu$ . Changing these parameters may lead to the change of the regularity modulus or even kill regularity at all; cf. [25, Example 1].

The next definition introduces the local versions of the properties in Definition 3.1 related to a fixed point  $(\bar{x}, \bar{y}) \in \text{gph}F_0$ .

DEFINITION 3.12.

- (i)  $F$  is regular at  $(\bar{x}, \bar{y})$  with functional modulus  $\mu$  if there exist neighbourhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that

$$d(x, F_0^{-1}(y)) \leq \mu(\delta(y, F, x)) \quad \text{for all } x \in U, y \in V.$$

- (ii)  $F$  is open at  $(\bar{x}, \bar{y})$  with functional modulus  $\mu$  if there exist neighbourhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that

$$y \in F(B(x, t), 0) \quad \text{for all } x \in U, y \in V \text{ and } t > \mu(\delta(y, F, x)).$$

The properties in Definition 3.12 are obviously equivalent to the corresponding ones in Definition 3.1 with  $W := U \times V$ . The next three statements are consequences of Proposition 3.2 and Theorems 3.4 and 3.5, respectively.

PROPOSITION 3.13. *The two properties in Definition 3.12 are equivalent.*

THEOREM 3.14. *Suppose that there exist neighbourhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that, for any  $x \in U$  and  $y \in V$ , the mapping  $\tau \mapsto F_\tau^{-1}(y)$  on  $\mathbb{R}_+$  is outer semicontinuous at 0 and, for some  $\gamma > \delta(y, F, x)$  and any  $t \in (0, \gamma)$  with  $(x, t, y) \in \text{gph}F$ , one of the three sets of conditions in Theorem 3.4 is satisfied. Then,  $F$  is regular at  $(\bar{x}, \bar{y})$  with functional modulus  $\mu$ .*

THEOREM 3.15. *Let  $\mu$  be continuous,  $\mu(\tau) = 0$  if and only if  $\tau = 0$ . Suppose that there exist neighbourhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that  $F^{-1}$  takes closed values on  $V$  and, for any  $x \in U$ ,  $y \in V$ , and  $u \in U_{x,y}$ , there exists a point  $u' \neq u$  such that condition (3.1) is satisfied. Then,  $F$  is regular at  $(\bar{x}, \bar{y})$  with functional modulus  $\mu$ .*

**4. Conventional setting.** In this section, we consider the standard in variational analysis setting of a set-valued mapping  $F : X \rightrightarrows Y$  between metric spaces. Such a mapping can be imbedded into the more general setting explored in the previous sections by defining a set-valued mapping  $\mathcal{F} : X \times \mathbb{R}_+ \rightrightarrows Y$  as follows (cf. [22, p. 508]: for any  $x \in X$  and  $t \geq 0$ ,

$$(4.1) \quad \mathcal{F}(x, t) := B(F(x), t) = \begin{cases} \{y \in Y \mid d(y, F(x)) < t\} & \text{if } t > 0, \\ F(x) & \text{if } t = 0. \end{cases}$$

(Recall the convention:  $B(y, 0) = \{y\}$ .) We are going to consider also mappings  $\overline{F} : X \rightrightarrows Y$  and  $\overline{\mathcal{F}} : X \times \mathbb{R}_+ \rightrightarrows Y$ , whose values are the closures of the corresponding values of  $F$  and  $\mathcal{F}$ , respectively:  $\overline{F}(x) := \overline{F(x)}$  and

$$(4.2) \quad \overline{\mathcal{F}}(x, t) := \overline{B}(F(x), t) = \begin{cases} \{y \in Y \mid d(y, F(x)) \leq t\} & \text{if } t > 0, \\ \overline{F(x)} & \text{if } t = 0. \end{cases}$$

The next proposition summarizes several simple facts with regard to the relationship between  $F$ ,  $\mathcal{F}$  and  $\overline{\mathcal{F}}$ .

PROPOSITION 4.1.

- (i)  $\mathcal{F}_0(x) = F(x)$ ,  $\overline{\mathcal{F}}_0(x) = \overline{F(x)}$  for all  $x \in X$ .
- (ii)  $\delta(y, \mathcal{F}, x) = \delta(y, \overline{\mathcal{F}}, x) = d(y, F(x))$  for all  $x \in X$  and  $y \in Y$ .
- (iii)  $\mathcal{F}_0^{-1}(B(y, t)) = F^{-1}(B(y, t)) = \mathcal{F}_t^{-1}(y)$  for all  $y \in Y$  and  $t \geq 0$ .
- (iv)  $\overline{\mathcal{F}}^{-1}(\overline{B}(y, t)) = \overline{F}^{-1}(\overline{B}(y, t)) \subset \overline{\mathcal{F}}_t^{-1}(y)$  for all  $y \in Y$  and  $t \geq 0$ .
- (v) If  $F^{-1}$  is closed at  $y$ , then the mappings  $\tau \mapsto \mathcal{F}_\tau^{-1}(y)$  and  $\tau \mapsto \overline{\mathcal{F}}_\tau^{-1}(y)$  on  $\mathbb{R}_+$  are outer semicontinuous at 0.
- (vi) For any  $y \in Y$  and  $\tau > 0$ ,  $\mathcal{F}$  and  $\overline{\mathcal{F}}$  satisfy condition (2.12).
- (vii) If  $F$  is upper semicontinuous, i.e., for any  $x \in X$  and  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $F(u) \subset B(F(x), \varepsilon)$  for all  $u \in B(x, \delta)$ , then  $\overline{\mathcal{F}}^{-1}$  is closed-valued. In particular, for any  $y \in Y$ , the mapping  $\tau \mapsto \overline{\mathcal{F}}_\tau^{-1}(y)$  is outer semicontinuous on  $\mathbb{R}_+$ .

*Proof.* (i) The equalities make part of definitions (4.1) and (4.2).

(ii) By (1.2), (4.1) and (4.2),

$$\begin{aligned} \delta(y, \mathcal{F}, x) &= \inf\{t > 0 \mid d(y, F(x)) < t\} = d(y, F(x)), \\ \delta(y, \overline{\mathcal{F}}, x) &= \inf\{t > 0 \mid d(y, F(x)) \leq t\} = d(y, F(x)). \end{aligned}$$

(iii) If  $t = 0$ , then  $\mathcal{F}_0^{-1}(y) = F^{-1}(y)$  and both equalities hold true automatically for all  $y \in Y$ . If  $t > 0$ , then

$$x \in \mathcal{F}_t^{-1}(y) \Leftrightarrow d(y, F(x)) < t \Leftrightarrow F(x) \cap B(y, t) \neq \emptyset \Leftrightarrow x \in F^{-1}(B(y, t)).$$

Hence,  $\mathcal{F}_t^{-1}(y) = F^{-1}(B(y, t))$ . The other equality is satisfied because  $\mathcal{F}_0^{-1}(v) = F^{-1}(v)$  for all  $v \in B(y, t)$ .

(iv) If  $t = 0$ , then  $\overline{\mathcal{F}}_0^{-1}(y) = \overline{F}^{-1}(\overline{B}(y, 0)) = \overline{F}^{-1}(y)$  for all  $y \in Y$ . If  $t > 0$ , then

$$x \in \overline{\mathcal{F}}_t^{-1}(y) \Leftrightarrow \overline{F}(x) \cap \overline{B}(y, t) \neq \emptyset \Rightarrow d(y, F(x)) \leq t \Leftrightarrow x \in \overline{\mathcal{F}}_t^{-1}(y).$$

Hence,  $\overline{\mathcal{F}}_t^{-1}(y) \subset \overline{\mathcal{F}}_t^{-1}(y)$ . The claimed equality is satisfied because  $\overline{\mathcal{F}}_0^{-1}(v) = \overline{F}^{-1}(v)$  for all  $v \in \overline{B}(y, t)$ .

(v) If  $x_n \rightarrow z$  and  $t_n \downarrow 0$  with  $d(y, F(x_n)) < t_n$  ( $n = 1, 2, \dots$ ), then, for any  $n$ , there exists a  $y_n \in F(x_n)$  such that  $d(y, y_n) < t_n$ . Hence,  $y_n \rightarrow y$  as  $n \rightarrow \infty$ . Since  $F^{-1}$  is closed at  $y$ , we have  $z \in F^{-1}(y)$  and consequently  $y \in F(z) = \mathcal{F}(z, 0)$ .

Similarly, if  $x_n \rightarrow z$  and  $t_n \downarrow 0$  with  $d(y, F(x_n)) \leq t_n$  ( $n = 1, 2, \dots$ ), then, for any  $n$ , there exists a  $y_n \in F(x_n)$  such that  $d(y, y_n) < 2t_n$ . Hence,  $y_n \rightarrow y$  as  $n \rightarrow \infty$ . Since  $F^{-1}$  is closed at  $y$ , we have  $z \in F^{-1}(y)$  and consequently  $y \in F(z) \subset \mathcal{F}(z, 0)$ .

(vi) follows from (iii) and (iv).

(vii) If  $y \in Y$ ,  $x_n \rightarrow z$  and  $t_n \rightarrow \tau$  with  $d(y, F(x_n)) \leq t_n$  ( $n = 1, 2, \dots$ ), then, since  $F$  is upper semicontinuous,

$$d(y, F(z)) \leq \liminf_{n \rightarrow \infty} d(y, F(x_n)) \leq \lim_{n \rightarrow \infty} t_n = \tau,$$



that is,  $y \in \overline{\mathcal{F}}(z, \tau)$ .  $\square$

Thanks to parts (i) and (ii) of Proposition 4.1, the definitions of regularity and openness properties explored in the previous sections in the current setting can be expressed in metric terms. In the next definition, which corresponds to a group of definitions from Section 3,  $\mu : [0, +\infty] \rightarrow [0, +\infty]$  is an upper semicontinuous nondecreasing function playing the role of a *modulus* of the corresponding property.

DEFINITION 4.2.

- (i) *Given a set  $W \subset X \times Y$ , mapping  $F$  is metrically regular on  $W$  with functional modulus  $\mu$  if*

$$(4.3) \quad d(x, F^{-1}(y)) \leq \mu(d(y, F(x))) \quad \text{for all } (x, y) \in W.$$

- (ii) *Given a set  $W \subset X \times Y$ , mapping  $F$  is open on  $W$  with functional modulus  $\mu$  if*

$$y \in F(B(x, t)) \quad \text{for all } (x, y) \in W \text{ and } t > \mu(d(y, F(x))).$$

- (iii) *Given a set  $W \subset X \times Y$  and a function  $\nu : W \rightarrow (0, \infty]$ , mapping  $F$  is metrically  $\nu$ -regular on  $W$  with functional modulus  $\mu$  if*

$$(4.4) \quad \begin{aligned} d(x, F^{-1}(y)) &\leq \mu(d(y, F(x))) \quad \text{for all } (x, y) \in W \\ &\text{with } \mu(d(y, F(x))) < \nu(x, y). \end{aligned}$$

- (iv) *Given a set  $W \subset X \times Y$  and a function  $\nu : W \rightarrow (0, \infty]$ , mapping  $F$  is  $\nu$ -open on  $W$  with functional modulus  $\mu$  if*

$$y \in F(B(x, t)) \quad \text{for all } (x, y) \in W \text{ and } t \in (\mu(d(y, F(x))), \nu(x, y)).$$

- (v)  *$F$  is metrically regular at a point  $(\bar{x}, \bar{y}) \in \text{gph}F$  with functional modulus  $\mu$  if there exist neighbourhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that*

$$(4.5) \quad d(x, F^{-1}(y)) \leq \mu(d(y, F(x))) \quad \text{for all } x \in U, y \in V.$$

- (vi)  *$F$  is open at  $(\bar{x}, \bar{y}) \in \text{gph}F$  with functional modulus  $\mu$  if there exist neighbourhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that*

$$(4.6) \quad y \in F(B(x, t)) \quad \text{for all } x \in U, y \in V \text{ and } t > \mu(d(y, F(x))).$$

*Remark 4.3. If  $\mu$  is strictly increasing, then condition (4.6) can be rewritten equivalently in a more conventional ‘‘openness-like’’ form (cf. [25]):*

$$B(F(x), \mu^{-1}(t)) \cap V \subset F(B(x, t)) \quad \text{for all } x \in U \text{ and } t > 0.$$

*In the case  $W = U \times V$ , similar simplifications can be made also in parts (ii) and (iv) of the above definition.*

In the linear case, the metric regularity and openness/covering properties in the above definition are very well known in both local and global settings (cf., e.g., [15, 22, 35, 43] including regularity on a set [22, 23]). The nonlinear setting in the above definition follows Ioffe [25] where the properties in parts (iii) and (iv), were mostly investigated in the particular case  $W = U \times V$  where  $U \subset X$  and  $V \subset Y$  and the function  $\nu$  depends only on  $x$ .

Observe that condition (4.3) in Definition 4.2 is equivalent to

$$d(x, F^{-1}(y)) \leq \mu(d(y, y')) \quad \text{for all } (x, y) \in W \text{ and } y' \in F(x).$$

In its turn, condition  $y' \in F(x)$  is equivalent to  $x \in F^{-1}(y')$ . This and similar observations regarding conditions (4.4) and (4.5) allow us to rewrite these conditions, respectively, as follows:

$$\begin{aligned} d(x, F^{-1}(y_2)) &\leq \mu(d(y_1, y_2)) \text{ for all } y_1, y_2 \in Y, x \in F^{-1}(y_1) \text{ with } (x, y_2) \in W, \\ d(x, F^{-1}(y_2)) &\leq \mu(d(y_1, y_2)) \text{ for all } y_1, y_2 \in Y, x \in F^{-1}(y_1) \\ &\quad \text{with } (x, y_2) \in W, \mu(d(y_1, y_2)) < \nu(x, y_2), \\ d(x, F^{-1}(y_2)) &\leq \mu(d(y_1, y_2)) \text{ for all } y_1 \in Y, y_2 \in V, x \in F^{-1}(y_1) \cap U. \end{aligned}$$

Thanks to these observations, one can complement the regularity and openness properties in Definition 4.2 with the corresponding Hölder-like (Aubin in the linear case) properties.

In the definition below,  $\mu : [0, +\infty] \rightarrow [0, +\infty]$  is again an upper semicontinuous nondecreasing function.

DEFINITION 4.4.

- (i) *Given a set  $W \subset X \times Y$ , mapping  $F$  is Hölder on  $W$  with functional modulus  $\mu$  if*

$$d(y, F(x_2)) \leq \mu(d(x_1, x_2)) \text{ for all } x_1, x_2 \in X, y \in F(x_1) \text{ with } (x_2, y) \in W.$$

- (ii) *Given a set  $W \subset X \times Y$  and a function  $\nu : W \rightarrow (0, \infty]$ , mapping  $F$  is  $\nu$ -Hölder on  $W$  with functional modulus  $\mu$  if*

$$\begin{aligned} d(y, F(x_2)) &\leq \mu(d(x_1, x_2)) \text{ for all } x_1, x_2 \in X, y \in F(x_1) \\ &\quad \text{with } (x_2, y) \in W, \mu(d(x_1, x_2)) < \nu(x_2, y). \end{aligned}$$

- (iii)  *$F$  is Hölder at a point  $(\bar{x}, \bar{y}) \in \text{gph}F$  with functional modulus  $\mu$  if there exist neighbourhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that*

$$(4.7) \quad d(y, F(x_2)) \leq \mu(d(x_1, x_2)) \text{ for all } x_1, x_2 \in U, y \in F(x_1) \cap V.$$

Thanks to Propositions 3.2, 3.8, 3.13 and the discussion before Definition 4.4, we have the following list of equivalences.

THEOREM 4.5. *Suppose  $\mu : [0, +\infty] \rightarrow [0, +\infty]$  is an upper semicontinuous increasing function.*

- (i) *Given a set  $W \subset X \times Y$ , properties (i) and (ii) in Definition 4.2 are equivalent to  $F^{-1}$  being Hölder on*

$$(4.8) \quad W' := \{(y, x) \in Y \times X \mid (x, y) \in W\}$$

*with functional modulus  $\mu$ .*

- (ii) *Given a set  $W \subset X \times Y$ , properties (iii) and (iv) in Definition 4.2 are equivalent to  $F^{-1}$  being  $\nu'$ -Hölder on (4.8) with functional modulus  $\mu$ , where  $\nu' : W' \rightarrow (0, \infty]$  is defined by equality  $\nu'(y, x) = \nu(x, y)$ .*

- (iii) *Given a point  $(\bar{x}, \bar{y}) \in \text{gph}F$ , properties (v) and (vi) in Definition 4.2 are equivalent to  $F^{-1}$  being Hölder at  $(\bar{y}, \bar{x})$  with functional modulus  $\mu$ .*

Remark 4.6. *Most of the equivalences in Theorem 4.5 hold true with function  $\mu$  nondecreasing. The assumption that  $\mu$  is strictly increasing is only needed in part (iii). In fact, it follows from the discussion before Definition 4.4, that properties (v) and (vi) in Definition 4.2 are equivalent to a stronger version of the Hölder property*

of  $F^{-1}$  which correspond to replacing condition (4.7) in Definition 4.4 by the following one:

$$d(y, F(x_2)) \leq \mu(d(x_1, x_2)) \text{ for all } x_1 \in X, x_2 \in U, y \in F(x_1) \cap V.$$

If  $\mu$  is strictly increasing, then the two versions are equivalent.

We next formulate several regularity criteria in the conventional setting of a mapping  $F : X \rightrightarrows Y$  between metric spaces. All of them are consequences of the corresponding statements in Section 3 thanks to the relationships in Proposition 4.1. From now on, we assume that  $X$  is complete.

**THEOREM 4.7.** *Given a set  $W \subset X \times Y$ , suppose that, for any  $(x, y) \in W$ ,  $F^{-1}$  is closed at  $y$  and, for some  $\gamma > d(y, F(x))$  and any  $t \in (0, \gamma)$ , one of the following sets of conditions is satisfied:*

- (i) *there are sequences of positive numbers  $(b_n)$  and  $(c_n)$  and a function  $m : (0, \infty) \rightarrow (0, \infty)$  such that conditions (2.4) and (2.30) hold true and*

$$d(x, F^{-1}(B(y, m(c_1)))) < b_0,$$

$$d(u, F^{-1}(B(y, m(c_{n+1})))) < b_n$$

$$\text{for all } u \in F^{-1}(B(y, m(c_n))) \cap B(x, \sum_{i=0}^{n-1} b_i) \text{ (} n = 1, 2, \dots \text{),}$$

- (ii) *there are functions  $b, m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for any  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , condition (2.8) holds true and*

$$d(u, F^{-1}(B(y, b(\tau)))) < m(\tau) \text{ for all } u \in F^{-1}(B(y, \tau)) \cap B(x, \mu(t) - \mu(\tau)),$$

- (iii) *there are functions  $b, m : (0, \infty) \rightarrow (0, \infty)$  such that condition (2.7) is satisfied and, for any  $\tau > 0$  with  $\mu(\tau) \leq \mu(t)$ , condition (2.8) holds true and*

$$d(y, F(B(u, m(\tau)))) < b(\tau) \text{ for all } u \in F^{-1}(B(y, \tau)) \cap B(x, \mu(t) - \mu(\tau)).$$

Then,  $F$  is metrically regular on  $W$  with functional modulus  $\mu$ .

**THEOREM 4.8.** *Let  $\mu$  be continuous,  $\mu(\tau) = 0$  if and only if  $\tau = 0$ . Given a set  $W \subset X \times Y$ , suppose that  $F$  is upper semicontinuous and, for any  $(x, y) \in W$  and  $u \in X$  such that  $d(y, F(u)) > 0$  and  $\mu(d(y, F(u))) + d(u, x) \leq \mu(d(y, F(x)))$ , there exists a point  $u' \neq u$  such that*

$$\mu(d(y, F(u'))) \leq \mu(d(y, F(u))) - d(u, u').$$

Then,  $F$  is metrically regular on  $W$  with functional modulus  $\mu$ .

*Proof.* By Theorem 3.5 and Proposition 4.1(i), (ii) and (vii), set-valued mapping  $\overline{F}$  is regular on  $W$  with functional modulus  $\mu$ . Since  $F$  is upper semicontinuous, it is closed-valued and consequently making use of Proposition 4.1(i) again, we have for any  $y \in Y$  that  $\overline{F}_0^{-1}(y) = \overline{F}^{-1}(y) = F^{-1}(y)$ . Hence, the regularity of  $\overline{F}$  is equivalent to the metric regularity of  $F$ .  $\square$

**5. Concluding remarks.** This article considers a general regularity model for a set-valued mapping  $F : X \times \mathbb{R}_+ \rightrightarrows Y$ , where  $X$  and  $Y$  are metric spaces. We demonstrate that the classical approach going back to Banach, Schauder, Lyusternik and Graves and based on iteration procedures still possesses potential. In particular, we show that the *Induction theorem* [27, Theorem 1], which was used as the main

tool when proving the other results in [27], implies also all the main results in the subsequent articles [28, 29] and can serve as a substitution of the Ekeland variational principle when establishing other regularity criteria. Furthermore, the latter classical result can also be established as a consequence of the Induction theorem.

This research prompts a list of questions and problems which should be taken care of.

1) “On a set” nonlinear regularity, considered in Section 3 and interpreted there as a direct analogue of metric regularity in the conventional setting, is in fact a general model which covers also relaxed versions of regularity like sub- and semi-regularity.

2) The particular case of “power nonlinearities”, i.e., the case when functional modulus  $\mu$  is of the type  $\mu(t) = \lambda t^k$  with  $k \geq 1$ , should be treated explicitly.

3) Theorem 2.12 illustrates the usage of the Induction theorem as a substitution for the Ekeland variational principle when establishing regularity criteria like Theorem 4.8. In the last theorem which is an (indirect) consequence of Theorem 2.12, the mapping is assumed upper semicontinuous. This assumption can be relaxed with the help of a slightly more advanced version of Theorem 2.12.

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