COPYRIGHT NOTICE

FedUni ResearchOnline
https://researchonline.federation.edu.au

On the Aubin property of a class of parameterized variational systems

H. Gfrerer · **J.V. Outrata**

Abstract The paper deals with a new sharp criterion ensuring the Aubin property of solution maps to a class of parameterized variational systems. This class includes parameter-dependent variational inequalities with non-polyhedral constraint sets and also parameterized generalized equations with conic constraints. The new criterion requires computation of directional limiting coderivatives of the normal-cone mapping for the so-called critical directions. The respective formulas have the form of a second-order chain rule and extend the available calculus of directional limiting objects. The suggested procedure is illustrated by means of examples.

1 Introduction

In [8], the authors have developed a new sufficient condition ensuring the Aubin property of solution maps to general implicitly defined multifunctions. This property itself has been introduced in [1] and became gradually one of the most important stability notions for multifunctions. It is widely used in *post-optimal analysis*, as a useful *qualification condition* in generalized differentiation and it is closely connected with several important classical results like, e.g., the theorems of Lyusternik and Graves [5, pp. 275-276].

H. Gfrerer
Institute of Computational Mathematics, Johannes Kepler University Linz, A-4040 Linz, Austria, E-mail: helmut.gfrerer@jku.at

J.V. Outrata
Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 18208 Prague, Czech Republic, and Centre for Informatics and Applied Optimization, Federation University of Australia, POB 663, Ballarat, Vic 3350, Australia, E-mail: outrata@utia.cas.cz

This paper is focused on the Aubin property of *solution maps* to parameter-dependent variational systems and extends the currently available results collected, e.g., in [5]. An efficient application of the new criterion in case of standard variational systems requires our ability to compute graphical derivatives and directional limiting coderivatives of normal-cone mappings to the considered constraint sets. Unfortunately, the calculus of directional limiting objects is not yet sufficiently developed and also in computation of graphical derivatives of normal-cone mappings one often meets various too restrictive assumptions. In this paper we will compute graphical derivatives and directional limiting coderivatives of normal cone mappings associated with the sets Γ of the form

$$\Gamma = g^{-1}(D) \tag{1}$$

under reasonable assumptions imposed on the mapping g and the set D.

To this aim we will significantly improve the results from [12] and [13] concerning the graphical derivative and from [13, Theorem 4.1] concerning the regular coderivative of the normal-cone mapping associated with (1). The resulting new second-order chain rules are valid under substantially relaxed reducibility and nondegeneracy assumptions compared with the preceding results of this type and are thus important for their own sake, not only in the context of this paper. Concretely, the new formula for the graphical derivative could be used, e.g., in testing the so-called isolated calmness of solution maps to variational systems ([9], [12], [13]).

The main result (Theorem 5) represents a variant of [8, Theorem 4.4] tailored to a broad class of parameterized variational systems. It improves the sharpness of the currently available criteria for the Aubin property in the frequently arising case when the considered parametrization is not *ample*, cf. [4, Definition 1.1].

The plan of the paper is as follows. In Section 2 we summarize the needed notions from variational analysis, state the main problem and recall [8, Theorem 4.4] which will be used as the main tool in our development. Section 3 is devoted to the new results concerning the mentioned graphical derivatives and directional limiting coderivatives of the normal-cone mapping related to Γ. In Section 4 we will formulate the resulting new criteria for the Aubin property of the considered solution maps and illustrate their application by means of an example. It shows the ability of the presented approach to deal with Γ given by nonlinear programming (NLP) constraints. Section 5 contains some amendments which may be useful for genuine conic constraints. In particular, we consider the case when D amounts to the Carthesian product of Lorentz cones.

Our notation is standard. For a set A, lin$A$ denotes the *linearity space* of A, i.e., the largest linear space contained in A, sp$A$ is the linear hull of A and $P_A(\cdot)$ stands for the mapping of metric projection onto A. For a multifunction F, gph$F$ denotes its graph and rge$F$ denotes its range, i.e., rge$F := \{y | y \in F(x) \text{ for } x \in \text{dom} F\}$. For a cone K, K° is the (negative) polar cone, $\mathbb{B}, \mathbb{S}$ are the unit ball and the unit sphere, respectively, and for a vector a, $[a]$ stands for the linear subspace generated by a. Finally, $\xrightarrow{A}$ means the convergence within a set A.

2 Problem formulation and preliminaries

In the first part of this section we introduce some notions from variational analysis which will be extensively used throughout the whole paper. Consider first a general closed-graph multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^z$ and its inverse $F^{-1} : \mathbb{R}^z \rightrightarrows \mathbb{R}^n$ and assume that $(\bar{u}, \bar{v}) \in \mathrm{gph}\, F$.

Definition 1 We say that F has the *Aubin property* around $(\bar{u}, \bar{v})$, provided there are neighborhoods U of $\bar{u}$, V of $\bar{v}$ and a modulus $\kappa > 0$ such that

$$F(u_1) \cap V \subset F(u_2) + \kappa \|u_1 - u_2\| \mathbb{B} \text{ for all } u_1, u_2 \in U.$$

F is said to be *calm* at $(\bar{u}, \bar{v})$, provided there is a neighborhood V of $\bar{v}$ and a modulus $\kappa > 0$ such that

$$F(u) \cap V \subset F(\bar{u}) + \kappa \|u - \bar{u}\| \mathbb{B} \text{ for all } u \in \mathbb{R}^n.$$

It is clear that the calmness is substantially weaker (less restrictive) than the Aubin property. Furthermore, it is known that F is calm at $(\bar{u}, \bar{v})$ if and only if F^{-1} is *metrically subregular at* $(\bar{u}, \bar{v})$, i.e., there is a neighborhood V of $\bar{v}$ and a modulus $\kappa > 0$ such that

$$d(v, F(\bar{u})) \leq \kappa d(\bar{u}, F^{-1}(v)) \text{ for all } v \in V,$$

cf. [5, Exercise 3H.4].

To conduct a thorough analysis of the above stability notions one typically makes use of some basic notions of generalized differentiation, whose definitions are presented below.

Definition 2 Let A be a closed set in $\mathbb{R}^n$ and $\bar{x} \in A$.

(i)
$$T_A(\bar{x}) := \mathrm{Lim}\sup_{t \searrow 0} \frac{A - \bar{x}}{t}$$

is the *tangent (contingent, Bouligand) cone* to A at $\bar{x}$ and

$$\hat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ$$

is the *regular (Fréchet) normal cone* to A at $\bar{x}$.

(ii)
$$N_A(\bar{x}) := \mathrm{Lim}\sup_{\substack{A \\ x \to \bar{x}}} \hat{N}_A(x)$$

is the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$ and, given a direction $d \in \mathbb{R}^n$,

$$N_A(\bar{x}; d) := \mathrm{Lim}\sup_{\substack{t \searrow 0 \\ d' \to d}} \hat{N}_A(\bar{x} + t d')$$

is the *directional limiting normal cone* to A at $\bar{x}$ *in direction* d .

The symbol "Limsup" stands for the outer (upper) set limit in the sense of Painlevé-Kuratowski, cf. [16, Chapter 4B]. If A is convex, then both the regular and the limiting normal cones coincide with the classical normal cone in the sense of convex analysis. Therefore we will use in this case the notation N_A.

By the definition, the limiting normal cone coincides with the directional limiting normal cone in direction 0, i.e., $N_A(\bar{x}) = N_A(\bar{x};0)$, and $N_A(\bar{x};d) = \emptyset$ whenever $d \notin T_A(\bar{x})$.

The above listed cones enable us to describe the local behavior of multifunctions via various generalized derivatives. Consider again the multifunction F and the point $(\bar{u},\bar{v}) \in \operatorname{gph} F$.

Definition 3 (i) The multifunction $DF(\bar{u},\bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^z$, defined by

$$DF(\bar{u},\bar{v})(d) := \{h \in \mathbb{R}^z | (d,h) \in T_{\operatorname{gph} F}(\bar{u},\bar{v})\}, d \in \mathbb{R}^n$$

is called the *graphical derivative* of F at $(\bar{u},\bar{v})$;
(ii) The multifunction $\hat{D}^*F(\bar{u},\bar{v}) : \mathbb{R}^z \rightrightarrows \mathbb{R}^n$, defined by

$$\hat{D}^*F(\bar{u},\bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*,-v^*) \in \hat{N}_{\operatorname{gph} F}(\bar{u},\bar{v})\}, v^* \in \mathbb{R}^z$$

is called the *regular (Fréchet) coderivative* of F at $(\bar{u},\bar{v})$.
(iii) The multifunction $D^*F(\bar{u},\bar{v}) : \mathbb{R}^z \rightrightarrows \mathbb{R}^n$, defined by

$$D^*F(\bar{u},\bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*,-v^*) \in N_{\operatorname{gph} F}(\bar{u},\bar{v})\}, v^* \in \mathbb{R}^z$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{u},\bar{v})$.
(iv) Finally, given a pair of directions $(d,h) \in \mathbb{R}^n \times \mathbb{R}^z$, the multifunction $D^*F((\bar{u},\bar{v});(d,h)) : \mathbb{R}^n \rightrightarrows \mathbb{R}^z$, defined by

$$D^*F((\bar{u},\bar{v});(d,h))(v^*) := \{u^* \in \mathbb{R}^n | (u^*,-v^*) \in N_{\operatorname{gph} F}((\bar{u},\bar{v});(d,h))\}, v^* \in \mathbb{R}^z \tag{2}$$

is called the *directional limiting coderivative* of F at $(\bar{u},\bar{v})$ in direction (d,h).

For the properties of the cones $T_A(\bar{x})$, $\hat{N}_A(\bar{x})$ and $N_A(\bar{x})$ from Definition 2 and generalized derivatives (i), (ii) and (iii) from Definition 3 we refer the interested reader to the monographs [16] and [10]. The directional limiting normal cone and coderivative were introduced by the first author in [7] and various properties of these objects can be found in [8] and the references therein. Note that $D^*F((\bar{u},\bar{v})) = D^*F((\bar{u},\bar{v});(0,0))$ and that $\operatorname{dom} D^*F((\bar{u},\bar{v});(d,h)) = \emptyset$ whenever $h \notin DF(\bar{u},\bar{v})(d)$.

Let now $M : \mathbb{R}^l \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a given multifunction with a closed graph and $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ be the associated *implicit multifunction* given by

$$S(p) := \{x \in \mathbb{R}^m | 0 \in M(p,x)\}. \tag{3}$$

In what follows, p will be called the *parameter* and x will be the *decision variable*. Given a *reference pair* $(\bar{p},\bar{x}) \in \operatorname{gph} S$, one has the following criterion for the Aubin property of S around $(\bar{p},\bar{x})$.

Theorem 1 *([8, Theorem 4.4, Corollary 4.5]). Assume that*

(i)

$$\{u \,|\, 0 \in DM(\bar{p},\bar{x},0)(q,u)\} \neq \emptyset \text{ for all } q \in \mathbb{R}^l; \tag{4}$$

(ii) M is metrically subregular at $(\bar{p},\bar{x},0)$;
(iii) For every nonzero $(q,u) \in \mathbb{R}^l \times \mathbb{R}^n$ verifying $0 \in DM(\bar{p},\bar{x},0)(q,u)$ one has the implication

$$(q^*,0) \in D^*M((\bar{p},\bar{x},0);(q,u,0))(v^*) \Rightarrow q^* = 0. \tag{5}$$

Then S has the Aubin property around $(\bar{p},\bar{x})$ and for any $q \in \mathbb{R}^l$

$$DS(\bar{p},\bar{x})(q) = \{u \,|\, 0 \in DM(\bar{p},\bar{x},0)(q,u)\}. \tag{6}$$

The above assertions remain true provided assumptions (ii), (iii) are replaced by

(iv) For every nonzero $(q,u) \in \mathbb{R}^l \times \mathbb{R}^n$ verifying $0 \in DM(\bar{p},\bar{x},0)(q,u)$ one has the implication

$$(q^*,0) \in D^*M((\bar{p},\bar{x},0);(q,u,0))(v^*) \Rightarrow \begin{cases} q^* = 0 \\ v^* = 0. \end{cases} \tag{7}$$

In this paper we will consider the case of *variational systems* where

$$M(p,x) := H(p,x) + \hat{N}_\Gamma(x), \ \Gamma = g^{-1}(D). \tag{8}$$

In (8), $H : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, $g : \mathbb{R}^n \to \mathbb{R}^s$ is twice continuously differentiable and $D \subset \mathbb{R}^s$ is a closed set.

We recall from [8] that Theorem 1 provides us in case of M given by (8) with sharper (more restrictive) sufficient conditions than the currently available criteria whenever $\nabla_p H(\bar{p},\bar{x})$ is not surjective, i.e., the considered parameterization is not ample at $(\bar{p},\bar{x})$.

By the continuous differentiability of H one has that for M given in (8) and any $(q,u) \in \mathbb{R}^l \times \mathbb{R}^n$

$$DM(\bar{p},\bar{x},0)(q,u) =$$
$$\nabla_p H(\bar{p},\bar{x})q + \nabla_x H(\bar{p},\bar{x})u + D\hat{N}_\Gamma(\bar{x},-H(\bar{p},\bar{x}))(u,-\nabla_p H(\bar{p},\bar{x})q - \nabla_x H(\bar{p},\bar{x})u), \tag{9}$$

cf. [16, Exercise 10.43]. Likewise, for any $v^* \in \mathbb{R}^n$,

$$D^*M((\bar{p},\bar{x},0);(q,u,0))(v^*) =$$
$$\begin{bmatrix} \nabla_p H(\bar{p},\bar{x})^T v^* \\ \nabla_p H(\bar{x},\bar{x})^T v^* + D^*\hat{N}_\Gamma((\bar{x},-H(\bar{p},\bar{x}));(u,-\nabla_p H(\bar{p},\bar{x})q - \nabla_x H(\bar{p},\bar{x})u))(v^*) \end{bmatrix}, \tag{10}$$

cf. [8, Theorem 2.10]. The application of Theorem 1 requires thus the computation of $D\hat{N}_\Gamma(\bar{x},-H(\bar{p},\bar{x}))(\cdot,\cdot)$ and $D^*\hat{N}_\Gamma((\bar{x},-H(\bar{p},\bar{x}));(\cdot,\cdot))(v^*)$ for directions generated by the vectors q,u. This problem will be tackled in the next section.

3 Graphical derivatives and directional limiting coderivatives of $\hat{N}_\Gamma$

Throughout this section we will impose a weakened version of the *reducibility* and the *nondegeneracy* conditions introduced in [2]. Concretely, in what follows we will assume that

(A1): There exists a closed set $\Theta \subset \mathbb{R}^d$ along with a twice continuously differentiable mapping $h : \mathbb{R}^s \to \mathbb{R}^d$ and a neighborhood $\mathscr{V}$ of $g(\bar{x})$ such that $\nabla h(g(\bar{x}))$ is surjective and
$$D \cap \mathscr{V} = \{z \in \mathscr{V} \,|\, h(z) \in \Theta\};$$

(A2):
$$\operatorname{rge} \nabla g(\bar{x}) + \ker \nabla h(g(\bar{x})) = \mathbb{R}^l. \tag{11}$$

Note that conditions (A1), (A2) amount to the reducibility of D to Θ at $g(\bar{x})$ and the nondegeneracy of $\bar{x}$ with respect to Γ and the mapping h in the sense of [2] provided the sets D, Θ are convex. The assumptions (A1), (A2) have the following important impact on the representation of Γ and $\hat{N}_\Gamma$ near $\bar{x}$.

Proposition 1 *Let $b := h \circ g$. Then there exists neighborhoods $\mathscr{U}$ of $\bar{x}$ and $\mathscr{W} \supset g(\mathscr{U})$ of $g(\bar{x})$ such that*
$$\Gamma \cap \mathscr{U} = \{x \in \mathscr{U} \,|\, b(x) \in \Theta\}, \tag{12}$$
$\nabla b(x)$ *is surjective for every $x \in \mathscr{U}$, $\nabla h(y)$ is surjective for every $y \in \mathscr{W}$ and*
$$\hat{N}_D(y) = \nabla h(y)^T \hat{N}_\Theta(h(y)), y \in \mathscr{W}, \tag{13}$$
$$\hat{N}_\Gamma(x) = \nabla b(x)^T \hat{N}_\Theta(b(x)) = \nabla g(x)^T \hat{N}_D(g(x)), x \in \mathscr{U}. \tag{14}$$

Proof First we show that (11) is equivalent with the surjectivity of $\nabla b(\bar{x}) = \nabla h(g(\bar{x})) \nabla g(\bar{x})$. Indeed, $\nabla b(\bar{x})$ is surjective if and only if
$$\{0\} = \ker \nabla b(\bar{x})^T = \ker(\nabla g(\bar{x})^T \nabla h(g(\bar{x}))^T),$$
which, by the assumed surjectivity of $\nabla h(g(\bar{x}))$, in turn holds if and only if
$$\{0\} = \ker \nabla g(\bar{x})^T \cap \operatorname{rge} \nabla h(g(\bar{x}))^T = \left((\ker \nabla g(\bar{x})^T)^\perp + (\operatorname{rge} \nabla h(g(\bar{x}))^T)^\perp \right)^\perp$$
$$= (\operatorname{rge} \nabla g(\bar{x}) + \ker \nabla h(g(\bar{x})))^\perp$$

and this is clearly equivalent with (11). Hence $\nabla b(\bar{x})$ is surjective and we can find open neighborhoods $\mathscr{W} \subset \mathscr{V}$ and $\mathscr{U} \subset g^{-1}(\mathscr{W})$ of $\bar{x}$ such that $\nabla b(x)$ is surjective for all $x \in \mathscr{U}$ and $\nabla h(y)$ is surjective for all $y \in \mathscr{W}$, where $\mathscr{V}$ is given by assumption (A1). Hence for every $x \in \mathscr{U}$ we have $g(x) \in \mathscr{V}$ and (12) follows from (A1). The descriptions of the regular normal cones (13), (14) result from [16, Exercise 6.7]. $\square$

Remark 1 Note that, given a vector $x^* \in \widehat{N}_\Gamma(x)$ with $x \in \Gamma \cap \mathscr{U}$, there is a unique $\lambda \in N_D(g(x))$ satisfying
$$x^* = \nabla g(x)^T \lambda. \tag{15}$$
Indeed, from (14) it follows that there is a unique $\mu \in \widehat{N}_\Theta(b(x))$ such that $x^* = \nabla b(x)^T \mu$ thanks to the surjectivity of $\nabla b(x)$. Since $\lambda = \nabla h(g(x))^T \mu$, we are done.

The rest of this section is divided to two subsections devoted to the graphical derivatives and the directional limiting coderivatives of $\hat{N}_\Gamma$, respectively.

3.1 Graphical derivatives of $\hat{N}_\Gamma$

The computation of graphical derivatives of $\hat{N}_\Gamma$ has been considered in numerous works, see [16] and the references therein. Recently, in [12] and [13] the authors have derived two different formulas for $D\hat{N}_\Gamma$ by using a strengthened variant of (A1), (A2) together with some additional assumptions. They include either the convexity of Γ or a special *projection derivation condition* (PDC) defined next.

Definition 4 A convex set Ξ satisfies the *projection derivation condition* (PDC) at the point $\bar{z} \in \Xi$ if we have

$$P_\Xi(\bar{z}+b;h) = P_{K(\bar{z},b)}(h) \text{ for all } b \in N_\Xi(\bar{z}) \text{ and } h \in \mathbb{R}^s,$$

where $K(\bar{z},b) := T_\Xi(\bar{z}) \cap \{b\}^\perp$.

In our case the PDC condition is automatically fulfilled provided D is convex polyhedral. Throughout sections 3.1. and 3.2 it is enough to assume, however, the weakened reducibility and nondegeneracy assumptions (A1), (A2) and we obtain new workable formulas without any additional requirements.

Theorem 2 *Let assumptions (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying*

$$\bar{\lambda} \in \hat{N}_D(g(\bar{x})), \ \nabla g(\bar{x})^T \bar{\lambda} = \bar{x}^*. \tag{16}$$

Then

$$T_{\mathrm{gph}\,\hat{N}_\Gamma}(\bar{x},\bar{x}^*) = \{(u,u^*)|\exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda}), u^* = \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g \rangle(\bar{x})u\}. \tag{17}$$

Proof Let $(u,u^*) \in T_{\mathrm{gph}\,\hat{N}_\Gamma}(\bar{x},\bar{x}^*)$ and consider sequences $t_k \searrow 0$ and $(u_k,u_k^*) \to (u,u^*)$ with $x_k^* := \bar{x}^* + t_k u_k^* \in \hat{N}_\Gamma(x_k)$, where $x_k := \bar{x} + t_k u_k$. We can assume that $x_k \in \mathscr{U}$ and that $\nabla b(x_k)$ is surjective for all k, where b and $\mathscr{U}$ are given by Proposition 1. Hence we can find multipliers $\mu^k \in \hat{N}_\Theta(b(x_k))$ such that $x_k^* = \nabla b(x_k)^T \mu^k$. The sequence μ^k is bounded and, after passing to some subsequence, converges to some $\bar{\mu} \in \hat{N}_\Theta(h(g(\bar{x})))$ with $\bar{x}^* = \nabla b(\bar{x})^T \bar{\mu}$. Further, by (13) we have $\bar{\lambda} = \nabla h(g(\bar{x}))^T \mu$ for some $\mu \in \hat{N}_\Theta(h(g(\bar{x})))$ implying $\bar{x}^* = \nabla b(\bar{x})^T \mu$ and $\bar{\mu} = \mu$ follows from the surjectivity of $\nabla b(\bar{x})$.
Since

$$t_k u_k^* = x_k^* - \bar{x}^* = \nabla b(x_k)^T \mu_k - \nabla b(\bar{x})^T \bar{\mu} = t_k \nabla^2 \langle \bar{\mu}, b \rangle(\bar{x})u_k + \nabla b(\bar{x})^T(\mu^k - \bar{\mu}) + o(t_k),$$

we obtain that

$$\nabla b(\bar{x})^T \frac{\mu^k - \bar{\mu}}{t_k} = u^* - \nabla^2 \langle \bar{\mu}, b \rangle(\bar{x})u + o(t_k)/t_k.$$

By the surjectivity of $\nabla b(\bar{x})$ we obtain that the sequence $\eta^k := (\mu^k - \bar{\mu})/t_k$ is bounded and, after passing to some subsequence, η^k converges to some η fulfilling

$$\nabla b(\bar{x})^T \eta = u^* - \nabla^2 \langle \bar{\mu}, b \rangle(\bar{x})u.$$

Denoting $\lambda^k = \nabla h(g(x_k))\mu^k$ we obtain $\lambda^k \in \widehat{N}_D(g(x_k))$ by (13) and

$$\lambda^k - \bar{\lambda} =$$

$$\nabla h(g(x_k))^T \mu_k - \nabla h(g(\bar{x}))^T \bar{\mu} = \nabla^2 \langle \bar{\mu}, h \rangle (g(\bar{x})) \nabla g(\bar{x})(t_k u_k) + \nabla h(g(\bar{x}))^T (\mu_k - \bar{\mu}) + o(t_k),$$

implying that $(\lambda^k - \bar{\lambda})/t_k$ converges to

$$\xi := \nabla^2 \langle \bar{\mu}, h \rangle (g(\bar{x})) \nabla g(\bar{x}) u + \nabla h(g(\bar{x}))^T \eta. \tag{18}$$

We conclude $(\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\widehat{N}_D}(g(\bar{x}), \bar{\lambda})$ and

$$u^* = \nabla b(\bar{x})^T \eta + \nabla^2 \langle \bar{\mu}, b \rangle (\bar{x}) u$$
$$= \nabla g(\bar{x})^T \nabla h(g(\bar{x}))^T \eta + \nabla g(\bar{x})^T \nabla^2 \langle \bar{\mu}, h \rangle (g(\bar{x})) \nabla g(\bar{x}) u + \nabla^2 \langle \nabla h(g(\bar{x})^T \bar{\mu}, g \rangle (\bar{x}) u$$
$$= \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) u$$

showing

$$(u, u^*) \in$$
$$\mathscr{T} := \{(u, u^*) \mid \exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\widehat{N}_D}(g(\bar{x}), \lambda), \ u^* = \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) u\}$$

Thus $T_{\mathrm{gph}\widehat{N}_\Gamma}(\bar{x}, \bar{x}^*) \subset \mathscr{T}$ holds.

In order to show the reverse inclusion $T_{\mathrm{gph}\widehat{N}_\Gamma}(\bar{x}, \bar{x}^*) \supset \mathscr{T}$, consider $(u, u^*) \in \mathscr{T}$ together with some corresponding ξ. Then there are sequences $t_k \searrow 0$, $v_k \to \nabla g(\bar{x})u$ and $\xi^k \to \xi$ such that $\bar{\lambda} + t_k \xi^k \in \widehat{N}_D(g(\bar{x}) + t_k v_k)$ and thus $h(g(\bar{x}) + t_k v_k) \in \Theta$ and $\bar{\lambda} + t_k \xi^k = \nabla h(g(\bar{x}) + t_k v_k))^T \mu^k$ with $\mu^k \in \widehat{N}_\Theta(h(g(\bar{x}) + t_k v_k))$ for all k sufficiently large. Further, the sequence μ^k is bounded. Since

$$b(\bar{x} + t_k u_k) - h(g(\bar{x}) + t_k v_k) = \nabla b(\bar{x})(t_k u_k) - \nabla h(g(\bar{x}))(t_k v_k) + o(t_k)$$
$$= t_k \nabla h(g(\bar{x}))(\nabla g(\bar{x})u - v_k) + o(t_k) = o(t_k)$$

and $\nabla b(\bar{x})$ is surjective, we can find for each k sufficiently large some x_k with $b(x_k) = h(g(\bar{x}) + t_k v_k) \in \Theta$ and $x_k - (\bar{x} + t_k u_k) = o(t_k)$. It follows that

$$\nabla b(x_k)^T \mu^k = \nabla g(x_k)^T \nabla h(g(x_k))^T \mu^k \in \widehat{N}_\Gamma(x_k)$$

and

$$\nabla b(x_k)^T \mu^k - \bar{x}^* = \nabla g(x_k)^T \nabla h(g(x_k))^T \mu^k - \bar{x}^*$$
$$= \nabla g(x_k)^T (\nabla h(g(x_k))^T \mu^k - \bar{\lambda}) + \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x})(x_k - \bar{x}) + o(t_k)$$
$$= \nabla g(x_k)^T (\nabla h(g(\bar{x}) + t_k v_k)^T \mu^k - \bar{\lambda} + o(t_k)) + t_k \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) u + o(t_k)$$
$$= t_k \nabla g(x_k)^T \xi^k + t_k \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) u + o(t_k)$$
$$= t_k (\nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) u) + o(t_k)$$

showing $(u, u^*) \in T_{\mathrm{gph}\widehat{N}_\Gamma}(\bar{x}, \bar{x}^*)$. $\qquad\square$

Remark 2 Everything remains true if we replace $\widehat{N}_\Gamma$, $\widehat{N}_D$, $\widehat{N}_\Theta$ by N_Γ, N_D, N_Θ,

Remark 3 Note that to each pair $(u,u^*) \in T_{\mathrm{gph}\widehat{N}_\Gamma}(\bar{x},\bar{x}^*)$ there is a unique ξ satisfying the relations on the right-hand side of (17). Its existence has been shown in the first part of the proof and its uniqueness follows from (18) and the uniqueness of η implied by the surjectivity of $\nabla b(\bar{x})$.

From (17) one can relatively easily derive the formulas from [12] and [13] by imposing appropriate additional assumptions. Indeed, let us suppose that, in addition to (A1), (A2), D is convex and the (single-valued) operator P_D is directionally differentiable at $g(\bar{x})$. Then one has the relationship

$$T_{\mathrm{gph}N_D}(g(\bar{x}),\bar{\lambda}) = \left\{ (v,w) \,\middle|\, \begin{bmatrix} v+w \\ v \end{bmatrix} \in T_{\mathrm{gph}P_D}(g(\bar{x})+\bar{\lambda},g(\bar{x})) \right\} =$$
$$\{(v,w)\,|\,v = P'_D(g(\bar{x})+\bar{\lambda};v+w)\},$$

which implies that under the posed additional assumptions the relation

$$(\nabla g(\bar{x})u,\xi) \in T_{\mathrm{gph}N_D}(g(\bar{x}),\bar{\lambda}) \tag{19}$$

amounts to the equation

$$\nabla g(\bar{x})u = P'_D(g(\bar{x})+\bar{\lambda};\nabla g(\bar{x})u+\xi). \tag{20}$$

Formula (17) attains thus exactly the form from [12, Theorem 3.3]. Note that in this way it was not necessary to assume the convexity of Γ like in [12]. Thanks to this, upon imposing the PDC condition on D at $g(\bar{x})$, one gets from (20) that

$$\nabla g(\bar{x})u = P_K(\nabla g(\bar{x})u+\xi), \tag{21}$$

where K stands for the critical cone to D at $g(\bar{x})$ with respect to $\bar{\lambda}$, i..e., $K = T_D(g(\bar{x})) \cap [\bar{\lambda}]^\perp$. From (21) we easily deduce that

$$\xi \in N_K(\nabla g(\bar{x})u)$$

and relation (17) thus simplifies to

$$T_{\mathrm{gph}\widehat{N}_\Gamma}(\bar{x},\bar{x}^*) = \{(u,u^*)\,|\,u^* \in \nabla^2\langle\bar{\lambda},g\rangle(\bar{x})u + \nabla g(\bar{x})^T N_K(\nabla g(\bar{x})u)\}. \tag{22}$$

We have recovered the formula from [13, Theorem 5.2]. This enormous simplification of the way how this result has been derived is due to Theorem 2 and the equivalence of relations (19), (20) (under the posed additional assumptions).

As mentioned above, the PDC condition automatically holds whenever D is a convex polyhedral set. Thus, for instance, in case of variational systems with Γ given by NLP constraints, one can compute $DM(\bar{p},\bar{x},0)(q,u)$ by the workable formula

$$DM(\bar{p},\bar{x},0)(q,u) = \nabla_p H(\bar{p},\bar{x})q + \nabla_x\mathscr{L}(\bar{p},\bar{x},\bar{\lambda})u + \nabla g(\bar{x})^T N_K(\nabla g(\bar{x})u), \tag{23}$$

where

$$\mathscr{L}(p,x,\lambda) := H(p,x) + \nabla g(x)^T\lambda$$

is the *Lagrangian* associated with the considered variational system.

3.2 Regular and directional limiting coderivatives of $\hat{N}_\Gamma$

Theorem 3 *Let assumptions (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying* (16). *Then*

$$\hat{N}_{\mathrm{gph}\,\hat{N}_\Gamma}(\bar{x},\bar{x}^*) = \left\{ (w^*,w)\,|\,\exists v^* : \begin{array}{l} (v^*,\nabla g(\bar{x})w) \in \hat{N}_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda}), \\ w^* = -\nabla^2\langle\bar{\lambda},g\rangle(\bar{x})w + \nabla g(\bar{x})^T v^* \end{array} \right\}. \qquad (24)$$

Proof First we justify (24) in the case when the derivative operator $\nabla g(\bar{x}) : \mathbb{R}^n \to \mathbb{R}^s$ is surjective. By the definition we have $(w^*,w) \in \hat{N}_{\mathrm{gph}\,\hat{N}_\Gamma}(\bar{x},\bar{x}^*)$ if and only if $\langle w^*,u\rangle + \langle w,u^*\rangle \leq 0 \; \forall (u,u^*) \in T_{\mathrm{gph}\,\hat{N}_\Gamma}(\bar{x},\bar{x}^*)$, which by virtue of Theorem 2 is equivalent to the statement that $(0,0)$ is a global solution of the problem

$$\max_{u,\xi} \; \gamma(u,\xi) := \langle w^*,u\rangle + \langle w, \nabla g(\bar{x})^T\xi + \nabla^2\langle\bar{\lambda},g\rangle(\bar{x})u\rangle$$

$$\text{subject to} \quad (\nabla g(\bar{x})u,\xi) \in T_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda}).$$

Since the objective can be rewritten as $\gamma(u,\xi) = \langle w^* + \nabla^2\langle\bar{\lambda},g\rangle(\bar{x})w,u\rangle + \langle \nabla g(\bar{x})w,\xi\rangle$, this is in turn equivalent to the statement

$$(w^* + \nabla^2\langle\bar{\lambda},g\rangle(\bar{x})w, \nabla g(\bar{x})w) \in C^\circ$$

where $C := \{(u,\xi)\,|\,(\nabla g(\bar{x})u,\xi) \in T_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda})\}$. By surjectivity of $\nabla g(\bar{x})$ the linear mapping $(u,\xi) \to (\nabla g(\bar{x})u,\xi)$ is surjective as well and we can apply [16, Exercise 6.7] to obtain

$$C^\circ = \hat{N}_C(0,0) = \{(\nabla g(\bar{x})^T v^*,v)\,|\,(v^*,v) \in \hat{N}_{T_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda})}(0,0)\}$$

$$= \{(\nabla g(\bar{x})^T v^*,v)\,|\,(v^*,v) \in \hat{N}_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda})\}.$$

Now formula (24) follows.

It remains to replace the surjectivity of $\nabla g(\bar{x})$ by the weaker nondegeneracy assumption from (A2). To proceed, we employ the local representation of D provided by its reducibility at $g(\bar{x})$, see assumption (A1). By Proposition 1 we have $\Gamma \cap \mathcal{U} = \{x \in \mathcal{U}\,|\,b(x) \in \Theta\}$ and by assumption (A1) we have $D \cap \mathcal{V} = \{z \in \mathcal{V}\,|\,h(z) \in \Theta\}$, where $\mathcal{U}$ and $\mathcal{V}$ denote neighborhoods of $\bar{x}$ and $g(\bar{x})$, respectively. Since both $\nabla b(\bar{x})$ and $\nabla h(g(\bar{x}))$ are surjective, we can apply (24) twice to obtain

$$\hat{N}_{\mathrm{gph}\,\hat{N}_\Gamma}(\bar{x},\bar{x}^*) = \left\{ (w^*,w)\,|\,\exists z^* : \begin{array}{l} (z^*,\nabla b(\bar{x})w) \in \hat{N}_{\mathrm{gph}\,\hat{N}_\Theta}(b(\bar{x}),\bar{\mu}), \\ w^* = -\nabla^2\langle\bar{\mu},b\rangle(\bar{x})w + \nabla b(\bar{x})^T z^* \end{array} \right\} \qquad (25)$$

and

$$\hat{N}_{\mathrm{gph}\,\hat{N}_D}(g(\bar{x}),\bar{\lambda}) = \left\{ (v^*,v)\,|\,\exists z^* : \begin{array}{l} (z^*,\nabla h(g(\bar{x}))v) \in \hat{N}_{\mathrm{gph}\,\hat{N}_\Theta}(h(g(\bar{x})),\bar{\mu}), \\ v^* = -\nabla^2\langle\bar{\mu},h\rangle(g(\bar{x}))v + \nabla h(g(\bar{x}))^T z^* \end{array} \right\}, \qquad (26)$$

where $\bar{\mu}$ is the unique multiplier satisfying $\bar{\lambda} = \nabla h(g(\bar{x}))^T\bar{\mu}$. By the classical chain rule we have $\nabla b(\bar{x}) = \nabla h(g(\bar{x}))\nabla g(\bar{x})$ and

$$\nabla^2\langle\bar{\mu},b\rangle(\bar{x}) = \nabla g(\bar{x})^T\nabla^2\langle\bar{\mu},h\rangle(g(\bar{x}))\nabla g(\bar{x}) + \nabla^2\langle\nabla h(g(\bar{x}))^T\bar{\mu},g\rangle(\bar{x})$$

$$= \nabla g(\bar{x})^T\nabla^2\langle\bar{\mu},h\rangle(g(\bar{x}))\nabla g(\bar{x}) + \nabla^2\langle\bar{\lambda},g\rangle(\bar{x}).$$

Now consider $(w^*, w) \in \hat{N}_{\text{gph}\hat{N}_\Gamma}(\bar{x}, \bar{x}^*)$ and let z^* be chosen such that $(z^*, \nabla b(\bar{x})w) \in \hat{N}_{\text{gph}\hat{N}_\Theta}(b(\bar{x}), \bar{\mu})$ and $w^* = -\nabla^2 \langle \bar{\mu}, b \rangle (\bar{x})w + \nabla b(\bar{x})^T z^*$. By substituting $v := \nabla g(\bar{x})w$, $v^* := -\nabla^2 \langle \bar{\mu}, h \rangle (g(\bar{x}))q + \nabla h(g(\bar{x}))^T z^*$ we obtain $(z^*, \nabla h(g(\bar{x}))v) \in \hat{N}_{\text{gph}\hat{N}_\Theta}(h(g(\bar{x})), \bar{\mu})$ implying $(v^*, v) = (v^*, \nabla g(\bar{x})w) \in \hat{N}_{\text{gph}\hat{N}_D}(g(\bar{x}), \bar{\lambda})$ by (26) and

$$
\begin{aligned}
w^* &= -\nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x})w + \nabla g(\bar{x})^T \left(-\nabla^2 \langle \bar{\mu}, h \rangle (g(\bar{x}))\nabla g(\bar{x})w + \nabla h(g(\bar{x}))^T z^* \right) \\
&= -\nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x})w + \nabla g(\bar{x})^T v^*.
\end{aligned}
$$

Thus

$$
(w^*, w) \in \mathscr{N} := \left\{ (w^*, w) \,\middle|\, \exists v^* : \begin{array}{l} (v^*, \nabla g(\bar{x})w) \in \hat{N}_{\text{gph}\hat{N}_D}(g(\bar{x}), \bar{\lambda}), \\ w^* = -\nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x})w + \nabla g(\bar{x})^T v^* \end{array} \right\}
$$

establishing the inclusion $\hat{N}_{\text{gph}\hat{N}_\Gamma}(\bar{x}, \bar{x}^*) \subset \mathscr{N}$. To establish the reverse inclusion consider $(w^*, w) \in \mathscr{N}$ together with the corresponding element v^*. By (26) we can find some z^* such that $(z^*, \nabla h(g(\bar{x}))\nabla g(\bar{x})w) = (z^*, \nabla b(\bar{x})w) \in \hat{N}_{\text{gph}\hat{N}_\Theta}(h(g(\bar{x})), \bar{\mu})$ and $v^* = -\nabla^2 \langle \bar{\mu}, h \rangle (g(\bar{x}))\nabla g(\bar{x})w + \nabla h(g(\bar{x}))^T z^*$. Hence

$$
\begin{aligned}
w^* &= -\nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x})w + \nabla g(\bar{x})^T v^* \\
&= -(\nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) + \nabla g(\bar{x})^T \nabla^2 \langle \bar{\mu}, h \rangle (g(\bar{x}))\nabla g(\bar{x}))w + \nabla g(\bar{x})^T \nabla h(g(\bar{x}))^T z^* \\
&= -\nabla^2 \langle \bar{\mu}, b \rangle (\bar{x})w + \nabla b(\bar{x})^T z^*
\end{aligned}
$$

and we conclude $(w^*, w) \in \hat{N}_{\text{gph}\hat{N}_\Gamma}(\bar{x}, \bar{x}^*)$ by (25). Hence $\hat{N}_{\text{gph}\hat{N}_\Gamma}(\bar{x}, \bar{x}^*) = \mathscr{N}$ and this finishes the proof. □

By the definition of the regular coderivative we obtain the following Corollary.

Corollary 1 *Under the assumptions of Theorem 3 one has*

$$
\hat{D}^* \hat{N}_\Gamma(\bar{x}, \bar{x}^*)(w) = \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x})w + \nabla g(\bar{x})^T \hat{D}^* \hat{N}_D(g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x})w), \quad w \in \mathbb{R}^n. \quad (27)
$$

In order to show the following result on the directional limiting coderivative note that assumptions (A1) and (A2) hold for all $x \in \Gamma$ near $\bar{x}$. In fact, by taking into account Proposition 1 and its proof, we have that $\nabla h(g(x))$ and $\nabla b(x)$ are surjective for all x near $\bar{x}$ and the latter is equivalent with validity of the condition $\text{rge} \nabla g(x) + \ker \nabla h(g(x)) = \mathbb{R}^n$ for those x.

Theorem 4 *Let assumptions (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying (16). Further we are given a pair of directions $(u, u^*) \in T_{\text{gph}\hat{N}_\Gamma}(\bar{x}, \bar{x}^*)$. Then for any $w \in \mathbb{R}^n$*

$$
\begin{aligned}
&D^* \hat{N}_\Gamma((\bar{x}, \bar{x}^*); (u, u^*))(w) \\
&= \nabla^2 \langle \bar{\lambda}, g \rangle (x)w + \nabla g(\bar{x})^T D^* \hat{N}_D((g(\bar{x}), \bar{\lambda}); (\nabla g(\bar{x})u, \bar{\xi}))(\nabla g(\bar{x})w),
\end{aligned} \quad (28)
$$

where $\bar{\xi} \in \mathbb{R}^s$ is the (unique) vector satisfying the relations

$$
(\nabla g(\bar{x})u, \bar{\xi}) \in T_{\text{gph}\hat{N}_D}(g(\bar{x}), \bar{\lambda}), \quad u^* = \nabla g(\bar{x})^T \bar{\xi} + \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x})u. \quad (29)
$$

Proof In the first step we observe that for arbitrary sequences $\vartheta_k \searrow 0, u_k \to u, u_k^* \to u^*$ and $w_k \to w$ such that $(x_k, x_k^*) := (\bar{x} + \vartheta_k u_k, \bar{x}_k^* + \vartheta_k u_k^*) \in \mathrm{gph}\,\hat{N}_\Gamma$ and k sufficiently large one has

$$\hat{D}^*\hat{N}_\Gamma(x_k, x_k^*)(w_k) = \nabla^2\langle\lambda_k, g\rangle(x_k)w_k + \nabla g(x_k)^T\hat{D}^*\hat{N}_D(g(x_k), \lambda_k)(\nabla g(x_k)w_k),$$

where λ_k is the (unique) multiplier satisfying the relations

$$\nabla g(x_k)^T\lambda_k = x_k^*, \ \lambda_k \in \hat{N}_D(g(x_k)). \tag{30}$$

Indeed, this follows immediately from Corollary 1 due to the mentioned robustness of assumptions (A1), (A2). Moreover, we know that $\lambda_k \to \bar{\lambda}$ which is the unique multiplier satisfying (16). Next we observe that

$$g(x_k) = g(\bar{x}) + \vartheta_k h_k \text{ with } h_k = \frac{g(x_k) - g(\bar{x})}{\vartheta_k} \to \nabla g(\bar{x})u$$

and

$$\lambda_k = \bar{\lambda} + \vartheta_k\xi_k \text{ with } \xi_k = \frac{\lambda_k - \bar{\lambda}}{\vartheta_k}.$$

It follows that

$$\hat{D}^*\hat{N}_\Gamma(\bar{x} + \vartheta_k u_k, \bar{x}^* + \vartheta_k u_k^*)(w_k) \tag{31}$$
$$= \nabla^2\langle\lambda_k, g\rangle(x_k)w_k + \nabla g(x_k)^T\hat{D}^*\hat{N}_D(g(\bar{x}) + \vartheta_k h_k, \bar{\lambda} + \vartheta_k\xi_k)(\nabla g(x_k)w_k).$$

We may now use the argumentation from the proof of Theorem 2 to show that ξ_k converges to the unique $\bar{\xi}$ satisfying (29). Taking now the outer set limits for $k \to \infty$ on both sides of (31), we obtain that $w^* \in D^*\hat{N}_\Gamma((\bar{x}, \bar{x}^*); (u, u^*))(w)$ if and only if it admits the representation

$$w^* \in \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})w + \nabla g(\bar{x})^T D^*\hat{N}_D((g(\bar{x}), \bar{\lambda}); (\nabla g(\bar{x})u, \bar{\xi}))(\nabla g(\bar{x})w)$$

with $\bar{\lambda}$ and $\bar{\xi}$ specified above. $\qquad\qquad\square$

Remark 4 Setting $(u, u^*) = (0, 0)$, we recover in this way the formula

$$D^*\hat{N}_\Gamma(\bar{x}, \bar{x}^*)(w) = \nabla^2\langle\bar{\lambda}, g\rangle(x)w + \nabla g(\bar{x})^T D^*\hat{N}_D(g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x})w),$$

which has been derived in [14] under the standard reducibility and nondegeneracy assumptions from [2]. This formula thus holds also under the weakened assumptions (A1), (A2).

Under the additional assumptions, mentioned in Section 3.1, relations (29) can be simplified. In particular, under the PDC condition at $g(\bar{x})$, the first relation from (29) reduces to (21) (with ξ replaced by $\bar{\xi}$).

4 Main results

On the basis of Theorems 1, 2 and 4 we may now state our main result - a new criterion for the Aubin of the solution map of a variational system, given by (3), (8) around a specified reference point.

Theorem 5 *Let $0 \in M(\bar{p},\bar{x})$ with M specified by (8), the assumptions (A1), (A2) be fulfilled and let $\bar{\lambda}$ be the (unique) multiplier satisfying (16) with $\bar{x}^* = -H(\bar{p},\bar{x})$. Further assume that*

(i) for any $q \in \mathbb{R}^l$ the variational system

$$0 = \nabla_p H(\bar{p},\bar{x})q + \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})u + \nabla g(\bar{x})^T \xi$$
$$(\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\hat{N}_D}(g(\bar{x}),\bar{\lambda}) \tag{32}$$

has a solution $(u,\xi) \in \mathbb{R}^n \times \mathbb{R}^l$;
(ii) M is metrically subregular at $(\bar{p},\bar{x})$, and
(iii) for any nonzero (q,u) satisfying (with a corresponding unique ξ) relations (32) one has the implication

$$0 \in \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})^T v^* + \nabla g(\bar{x})^T D^* \hat{N}_D((g(\bar{x}),\bar{\lambda});(\nabla g(\bar{x}),\xi))(\nabla g(\bar{x})v^*)$$
$$\Rightarrow v^* \in \ker \nabla_p H(\bar{p},\bar{x})^T. \tag{33}$$

Then the respective S has the Aubin property around $(\bar{p},\bar{x})$ and for any $q \in \mathbb{R}^l$

$$DS(\bar{p},\bar{x})(q) = \{u | \exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\hat{N}_D}(g(\bar{x}),\bar{\lambda}),$$
$$0 = \nabla_p H(\bar{p},\bar{x})q + \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})^T u + \nabla g(\bar{x})^T \xi\}. \tag{34}$$

The above assertions remain true provided assumptions (ii), (iii) are replaced by

(iv) for any nonzero (q,u) satisfying (with a corresponding unique ξ) relations (32) one has the implication

$$0 \in \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})^T v^* + \nabla g(\bar{x})^T D^* \hat{N}_D((g(\bar{x}),\bar{\lambda});(\nabla g(\bar{x})u, \xi))(\nabla g(\bar{x})v^*)$$
$$\Rightarrow v^* = 0. \tag{35}$$

The proof follows easily from Theorems 1, 2 and 4 and relations (9), (10). By imposing the additional assumptions, mentioned in Section 3.1, formulas (32) and (34) can be appropriately simplified. In particular, when D is convex polyhedral, then (32) attains the form of the generalized equation (GE)

$$0 = \nabla_p H(\bar{p},\bar{x})q + \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})u + \nabla g(\bar{x})^T \xi, \; \xi \in N_K(\nabla g(\bar{x})u). \tag{36}$$

Denoting now $w := (q,u)$ and $\Lambda := \mathbb{R}^l \times (\nabla g(\bar{x}))^{-1}K$, (36) amounts to the homogenous affine variational inequality

$$0 \in \begin{bmatrix} 0 & 0 \\ \nabla_p H(\bar{p},\bar{x}), & \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda}) \end{bmatrix} w + N_\Lambda(w). \tag{37}$$

Indeed, thanks to the polyhedrality of D, K is also polyhedral and

$$N_\Lambda(w) = N_{\mathbb{R}^l}(q) \times \nabla g(\bar{x})^T N_K(\nabla g(\bar{x})u)$$

without any qualification conditions. For the solution of (37) various methods are available, cf. [6]. This case will now be illustrated by an academic example.

Example 1 Consider the solution map $S : \mathbb{R} \rightrightarrows \mathbb{R}^2$ of the GE

$$0 \in M(p,x) = \begin{bmatrix} x_1 - p \\ -x_2 + x_2^2 \end{bmatrix} + \hat{N}_\Gamma(x) \tag{38}$$

with Γ given by $D = \mathbb{R}^2_-$ and

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} 0.5x_1 - 0.5x_1^2 - x_2 \\ 0.5x_1 - 0.5x_1^2 + x_2 \end{bmatrix}.$$

Clearly, Γ is a nonconvex set depicted in Fig,1. Let $(\bar{p},\bar{x}) = (0,(0,0))$ be the reference point. Since Γ fulfills LICQ at $\bar{x}$, we conclude that assumptions (A1), (A2) are fulfilled. Clearly, $x^* = -H(\bar{p},\bar{x}) = (0,0)$ and $\bar{\lambda} = (0,0)$ as well. By virtue of the polyhedrality of D the variational system (32) attains the form (36). In our case it amounts to

$$0 = \begin{bmatrix} -q \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix} \xi, \ \xi \in N_{\mathbb{R}^2_-}\left(\begin{bmatrix} 0.5u_1 - u_2 \\ 0.5u_1 + u_2 \end{bmatrix} \right), \tag{39}$$

because $K = T_D(g(\bar{x})) \cap [\bar{\lambda}]^\perp = D$.

It is not difficult to compute that for $q \leq 0$ one has three different solutions (u,ξ) of (39), namely

$$u_1 = q, \ u_2 = 0, \ \xi_1 = 0, \ \xi_2 = 0 \tag{40}$$

$$u_1 = \frac{4}{3}q, \ u_2 = -\frac{2}{3}q, \ \xi_1 = 0, \ \xi_2 = -\frac{2}{3}q \tag{41}$$

$$u_1 = \frac{4}{3}q, \ u_2 = \frac{2}{3}q, \ \xi_1 = -\frac{2}{3}q, \ \xi_2 = 0, \tag{42}$$

and for $q \geq 0$ we have the unique solution

$$u_1 = u_2 = 0, \ \xi_1 = \xi_2 = q. \tag{43}$$

So, assumption (i) of Theorem 5 is fulfilled and we know the critical directions $(q,u) \neq 0$ for which the implication (35) will be examined. Starting with (40), one has $\nabla g(\bar{x})u = (0.5q, 0.5q)$ and

$$D^* N_{\mathbb{R}^2_-}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right); \left(\begin{bmatrix} 0.5q \\ 0.5q \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \left(\begin{bmatrix} 0.5v_1^* - v_2^* \\ 0.5v_1^* + v_2^* \end{bmatrix} \right) =$$

$$\begin{bmatrix} D^* N_{\mathbb{R}_-}((0,0);(0.5q,0))(0.5v_1^* - v_2^*) \\ D^* N_{\mathbb{R}_-}((0,0);(0.5q,0))(0.5v_1^* + v_2^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
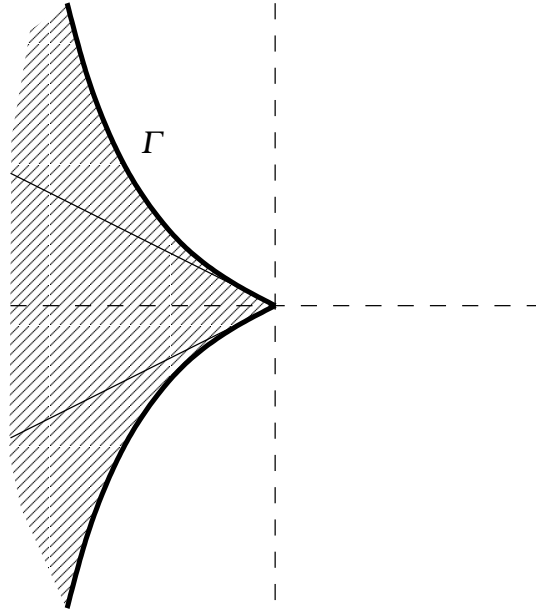
Fig. 1 Set Γ.

by virtue of the definition and [16, Proposition 6.41]. The left-hand side of (35) reduces to the linear system in variables $(v^*, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$

$$0 = \begin{bmatrix} v_1^* \\ -v_2^* \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix} \eta, \ \eta = 0,$$

verifying the validity of implication (35). In the case (41), $\nabla g(\bar{x})u = (\frac{4}{3}q, 0)$ and

$$D^* N_{\mathbb{R}^2_-} \left(\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right); \left(\begin{bmatrix} \frac{4}{3}q \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{3}q \end{bmatrix} \right) \right) \left(\begin{bmatrix} 0.5v_1^* - v_2^* \\ 0.5v_1^* + v_2^* \end{bmatrix} \right) = \{0\} \times \mathbb{R}$$

provided $v_2^* = -0.5v_1^*$. The respective linear system in variables (v^*, η) reduces to

$$0 = \begin{bmatrix} v_1^* \\ 0.5v_1^* \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \eta \end{bmatrix},$$

verifying again the validity of (35). In the same way we compute that in the case (42) one has $\nabla g(\bar{x})u = (0, \frac{4}{3}q)^T$ and

$$D^* N_{\mathbb{R}^2_-} \left(\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right); \left(\begin{bmatrix} 0 \\ \frac{4}{3}q \end{bmatrix}, \begin{bmatrix} -\frac{2}{3}q \\ 0 \end{bmatrix} \right) \right) \left(\begin{bmatrix} 0.5v_1^* - v_2^* \\ 0.5v_1^* + v_2^* \end{bmatrix} \right) = \mathbb{R} \times \{0\}$$

provided $v_2^* = 0.5v_1^*$. Taking this into account, we arrive at the linear system

$$0 = \begin{bmatrix} v_1^* \\ -0.5v_1^* \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \eta \\ 0 \end{bmatrix},$$

showing that $v^* = 0$. Finally, concerning the last case (43), $\nabla g(\bar{x})u = (0,0)$ and

$$D^* N_{\mathbb{R}^2_-} \left(\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right); \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} q \\ q \end{bmatrix} \right) \right) \left(\begin{bmatrix} 0.5 v_1^* - v_2^* \\ 0.5 v_1^* + v_2^* \end{bmatrix} \right) = \mathbb{R} \times \mathbb{R},$$

provided $v_1^* = 0.5 v_2^*$ and, at the same time, $v_1^* = -0.5 v_2^*$. This imediately implies that $v^* = 0$ and we are done. On the basis of Theorem 5 we have shown that the implicit multifunction S generated by (38) has the Aubin property around $(0,0)$ and, for a given q, $DS(0,0)(q)$ is the set of solutions to (39).

Next we show that this result cannot be obtained via the Mordukhovich criterion and the standard calculus, which amounts to proving that the "standard" adjoint GE (cf.[10, Corollary 4.61]) possesses only the trivial solution. Indeed, this GE amounts in our case to

$$0 \in \begin{bmatrix} v_1^* \\ -v_2^* \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix} D^* N_{\mathbb{R}^2_-} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \left(\begin{bmatrix} 0.5 v_1^* - v_2^* \\ 0.5 v_1^* + v_2^* \end{bmatrix} \right) \tag{44}$$

and it is easy to check that, e.g., $v^* = (-0.5, 1)^T$ is a solution of (44). Consequently, the Aubin property of S cannot be detected in this way. $\triangle$

5 Variational systems with conic constraint sets

In this concluding section we will consider a variant of Theorem 5 under the additional assumption that D is a closed convex cone with vertex at 0 and $P_D(\cdot)$ is directionally differentiable over $\mathbb{R}^s$. As implied by (20), the variational system (32) attains then the form

$$\begin{aligned} 0 &= \nabla_p H(\bar{p}, \bar{x})q + \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda})u + \nabla g(\bar{x})^T \xi \\ \nabla g(\bar{x})u &= P_D'(g(\bar{x}) + \bar{\lambda}; \nabla g(\bar{x})u + \xi) \end{aligned} \tag{45}$$

which, under the PDC condition at $g(\bar{x})$, further simplifies to the form (36). If D is the Carthesian product of Lorentz cones or the Löwner cone ([2]), then we dispose with an efficient formula for $P_D'(\cdot; \cdot)$ which depends on the position of $(g(\bar{x}), \bar{\lambda})$ in $\mathrm{gph} N_D$, cf. [15, Lemma 2] and [17, Theorem 4.7].

Concerning the GE on the left-hand side of (33) or (35), it is advantageous to rewrite it in terms of P_D (instead of N_D). Let $(\bar{a}, \bar{b}) \in \mathrm{gph} N_D$. Since

$$\mathrm{gph} N_D = \left\{ (a,b) \in \mathbb{R}^s \times \mathbb{R}^s \, \middle| \, \begin{pmatrix} a+b \\ a \end{pmatrix} \in \mathrm{gph} P_D \right\},$$

one has, by virtue of [10, Theorem 1.17], that

$$p \in \hat{D}^* N_D(a,b)(q) \iff -q \in \hat{D}^* P_D(a+b,a)(-q-p)$$

for any $(p,q) \in \mathbb{R}^s \times \mathbb{R}^s$. It follows that the GE on the left-hand side of (33) can be equivalently written down as the system

$$0 = \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda})^T v^* + \nabla g(\bar{x})^T (d - \nabla g(\bar{x})v^*) \tag{46}$$

$$-\nabla g(\bar{x})v^* \in D^* P_D((g(\bar{x}) + \bar{\lambda}, g(\bar{x})); (\nabla g(\bar{x})u + \xi, \nabla g(\bar{x})u))(-d) \tag{47}$$

in variables $(v^*, d) \in \mathbb{R}^n \times \mathbb{R}^s$. If D is the Carthesian product of Lorentz cones or the Löwner cone, then the directional limiting coderivative of P_D can be computed by using Definition 2(ii) and the formulas for regular coderivatives of P_D in [15] and [3], respectively. For illustration consider the case when D amounts to just one Lorentz cone in $\mathbb{R}^s$, i.e.,

$$D = \mathscr{K} := \{(z_0, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{s-1} | z_0 \geq \|\bar{z}\|\}.$$

We will analyze here only the most difficult situation when $g(\bar{x}) = 0$ and $\bar{\lambda} = 0$ and provide formulas for the directional limiting coderivatives of $P_{\mathscr{K}}$ at $(0,0)$ for all possible nonzero directions from

$$T_{\text{gph} P_{\mathscr{K}}}(0,0) = \{(h,k) | k \in P_{\mathscr{K}}(h)\}, \tag{48}$$

see [15, Lemma 2(iv)]. We have thus to distinguish between the following five situations:

- $h \in \text{int}\,\mathscr{K},\ k = h;$ (49)
- $h \in \text{int}\,\mathscr{K}^\circ,\ k = 0;$ (50)
- $h \notin \mathscr{K} \cup \mathscr{K}^\circ, k = P_{\mathscr{K}}(h);$ (51)
- $h \in \text{bd}\,\mathscr{K},\ k = h;$ (52)
- $h \in \text{bd}\,\mathscr{K}^\circ,\ k = 0.$ (53)

In the cases (49), (50) we get immediately from [15, Lemma 1(iv)] the formulas

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = u^*, \tag{54}$$
$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = 0, \tag{55}$$

respectively. Likewise, in the case (51) one has

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = \{C(w,\alpha)u^* | w \in \mathbb{S}_{n-1}, \alpha \in [0,1]\}, \tag{56}$$

where

$$C(w,\alpha) = \frac{1}{2} \begin{bmatrix} 2\alpha I + (1-2\alpha)ww^T & w \\ w^T & 1 \end{bmatrix}.$$

Concerning the case (52), by passing to subsequences if necessary, one may have sequences $(h_i, k_i) \xrightarrow{\text{gph} P_{\mathscr{K}}} (h,k), \lambda_i \searrow 0$ such that for i sufficiently large one of the following three situations occurs:

* $h_i \notin \mathscr{K} \cup \mathscr{K}^0\ (k_i = P_{\mathscr{K}}(h_i));$
* $h_i \in \text{int}\,\mathscr{K}\ (k_i = h_i);$
* $h_i \in \text{bd}\,\mathscr{K}\ (k_i = h_i).$

Correspondingly, we obtain from [15, Lemma 1(iv) and Theorem 4], that

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = \{C(w,\alpha)u^* | w \in \mathbb{S}_{n-1}, \alpha \in [0,1]\} \cup \bigcup_{A \in \mathscr{A}(u^*)} \text{conv}\{u^*, Au^*\},$$

$$\tag{57}$$

where

$$\mathscr{A}(u^*) := \left\{ I + \frac{1}{2} \begin{bmatrix} -ww^T & w \\ w^T & -1 \end{bmatrix} \middle| w \in \mathbb{S}_{n-1}, \left\langle \begin{bmatrix} -w \\ 1 \end{bmatrix}, u^* \right\rangle \geq 0 \right\}.$$

Analogously, in the case (53), by passing to subsequences if necessary, one may have sequences $(h_i, k_i) \xrightarrow{\mathrm{gph}P_{\mathscr{K}}} (h,k), \lambda_i \searrow 0$ such that for i sufficiently large one of the following three situations occurs:

* $h_i \notin \mathscr{K} \cup \mathscr{K}^0$ $(k_i = P_{\mathscr{K}}(h_i))$;
* $h_i \in \mathrm{int}\, \mathscr{K}^0$ $(k_i = 0)$;
* $h_i \in \mathrm{bd}\, \mathscr{K}_0$ $(k_i = 0)$.

Correspondingly, we obtain from [15, Lemma 1(iv) and Theorem 4] that

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = \{C(w,\alpha)u^* \mid w \in \mathbb{S}_{n-1}, \alpha \in [0,1]\} \cup \bigcup_{B \in \mathscr{B}(u^*)} \mathrm{conv}\{u^*, Bu^*\},$$

(58)

where

$$\mathscr{B}(u^*) := \left\{ \frac{1}{2} \begin{bmatrix} ww^T & w \\ w^T & 1 \end{bmatrix} \middle| w \in \mathbb{S}_{n-1}, \left\langle \begin{bmatrix} w \\ 1 \end{bmatrix}, u^* \right\rangle \geq 0 \right\}.$$

Next we illustrate the above described procedure via a conic reformulation of [8, Example 5].

Example 2 Consider the solution map $S : \mathbb{R} \rightrightarrows \mathbb{R}^2$ of the GE given by (3), (8) with

$$H(p,x) = \begin{bmatrix} x_1 - p \\ -x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}$$

and $D = \mathscr{K}$ being the Lorentz cone in $\mathbb{R}^2$. Let $(\bar{p}, \bar{x}) = (0, (0,0))$ be the reference point so that $\bar{\lambda} = (0,0)$. It is easy to see that assumptions (A1), (A2) are fulfilled and, since the Lorentz cone in $\mathbb{R}^2$ is a polyhedral set, instead of (45) we can compute the "critical" directions via (36). The variational system (36) attains the form

$$0 = \begin{bmatrix} -q \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \xi, \ \xi \in N_{\mathscr{K}}\left(\begin{bmatrix} 2u_2 \\ -u_1 \end{bmatrix} \right).$$

(59)

It is not difficult to compute that for $q \leq 0$ one has three different solutions (u, ξ) of (59), namely

$$u_1 = q, \ u_2 = 0, \ \xi_1 = 0, \ \xi_2 = 0 \tag{60}$$

$$u_1 = \frac{4}{3}q, \ u_2 = -\frac{2}{3}q, \ \xi_1 = -\frac{1}{3}q, \ \xi_2 = \frac{1}{3}q \tag{61}$$

$$u_1 = \frac{4}{3}q, \ u_2 = \frac{2}{3}q, \ \xi_1 = \frac{1}{3}q, \ \xi_2 = \frac{1}{3}q \tag{62}$$

and for $q \geq 0$ one has the unique solution

$$u_1 = u_2 = 0, \ \xi_1 = 0, \ \xi_2 = -q. \tag{63}$$

So, assumption (i) of Theorem 5 is fulfilled and we will check assumption (iv). Starting with (60), system (46), (47) attains the form

$$0 = \begin{bmatrix} v_1^* \\ -v_2^* \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} v^* = \begin{bmatrix} -d_2 \\ -5v_2^* + 2d_1 \end{bmatrix} \tag{64}$$

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} \in D^* P_{\mathscr{K}} \left((0,0); \left(\begin{bmatrix} 0 \\ -q \end{bmatrix}, \begin{bmatrix} 0 \\ -q \end{bmatrix} \right) \right) (-d). \tag{65}$$

By virtue of formula (54) this system reduces to the equations

$$d_2 = 0, \ d_1 = \frac{5}{2} v_2^*, \ v_1^* = 0, \ 2v_2^* = d_1,$$

verifying that $v^* = 0$. In the case (61), one arrives at the equation (64) together with the relation

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} \in D^* P_{\mathscr{K}} \left((0,0); \left(\begin{bmatrix} -\frac{5}{3}q \\ -q \end{bmatrix}, \begin{bmatrix} -\frac{4}{3}q \\ -\frac{4}{3}q \end{bmatrix} \right) \right) (-d). \tag{66}$$

Now we have to employ formula (56). For $w = -1$ one obtains from (66) the equation

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} = - \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} d$$

which, together with (64), implies that $v^* = 0$. For $w = 1$ one obtains from (66) the equation

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} = - \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} d$$

that again implies that $v^* = 0$. Thus the case (61) is completed. Likewise, in the remaining cases (62), (63) we apply the formulas (56) and (55), respectively, and verify again that in all solutions of the respective system (46), (47) one has $v^* = 0$. The examined solution map S has thus the Aubin property around $(\bar{p}, \bar{x})$. Note that, as in Example 1, this conclusion cannot be made on the basis of the standard criteria.

<div align="right">△</div>

6 Concluding remarks

The formulas provided in the second part of Section 5 for D being the Lorentz cone could easily be extended to the case when D amounts to the Carthesian product of several Lorentz cones. Further, on the basis of [3] one could compute the directional limiting coderivatives of the projection mapping onto the Löwner cone which would enable us to apply the presented theory also to parameterized semidefinite programs. Finally, one could think of variational systems, not having the (relatively simple) structure (8). For example, p could arise also in the constraints or one could consider implicit constraints like in quasi-variational inequalities [11]. All these situations offer an interesting topic for a future research.

Acknowledgements

References

1. J. P. AUBIN, Lipschitz behavior of solutions to convex minimization problems. Math. Oper. Res. 9 (1984), 87-111.
2. J.F. BONNANS, A. SHAPIRO, *Perturbation Analysis of Optimization Problems*. Springer, New York, 2000.
3. C. DING, D. SUN, J.J. YE, First-order optimality conditions for mathematical programs with semidefinite cone complementarity constraints. Math. Prog., Ser. A 147 (2014), 539-579.
4. A. L. DONTCHEV, R. T. ROCKAFELLAR, Ample parameterization of variational inclusions. SIAM J. Optimization 12 (2001), 170-187.
5. A. L. DONTCHEV, R. T. ROCKAFELLAR, *Implicit Functions and Solution Mappings*. Springer, Heidelberg, 2014.
6. F. FACCHINEI, J.-S. PANG, *Finite-Dimensional Variational Inequalities and Complementarity Problems II*, Springer, New York, 2003.
7. H. GFRERER, *On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs*, Set-Valued Var. Anal., 21 (2013), 151–176.
8. H. GFRERER, J.V. OUTRATA, On Lipschitzian properties of implicit multifunctions. To appear in SIAM J. Optimization.
9. R. HENRION, A. Y. KRUGER, J. V. OUTRATA, Some remarks on stability of generalized equations. J. Optim. Theory Appl. 159 (2013), 681-697.
10. B.S. MORDUKHOVICH, *Variational Analysis and Generalized Differentiation I*, Springer, Heidelberg, 2006.
11. B.S. MORDUKHOVICH, J.V. OUTRATA, Coderivative analysis of quasi-variational inequalities with applications to stability and optimization. SIAM J. Optim. 18 (2007), 389-412.
12. B.S. MORDUKHOVICH, J.V. OUTRATA, H. RAMIRÉZ C., Second-order variational analysis in conic programming with application to optimality and stability, SIAM J. Optimization 25 (2015), 76-101.
13. B.S. MORDUKHOVICH, J.V. OUTRATA, H. RAMIRÉZ C., Graphical derivative and stability analysis for parameterized equilibria with conic constraints, Set-Valued and Variational Analysis 23 (2015), 687-704.
14. J.V. OUTRATA, H. RAMIRÉZ C., On Aubin property of critical points to perturbed second-order cone programs. SIAM J. Optimization 21 (2011), 798-823.
15. J.V. OUTRATA, D. F. SUN, On the coderivative of the projection operator on the second-order cone. Set-Valued Analysis 16 (2008), 999-1014.
16. R. T. ROCKAFELLAR, R. J-B. WETS, *Variational Analysis*, Springer, Berlin, 1998.
17. D.F. SUN, J. SUN, Strong semismoothness of eigenvalues of symmetric matrices and its applications in inverse eigenvalue problems, SIAM J. Numer. Anal. 40 (2003), 2352-2367.