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OPTIMAL REES MATRIX CONSTRUCTIONS FOR ANALYSIS OF DATA

A.V. KELAREV, J.L. YEARWOOD and LIFANG ZI

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Abstract

Rees matrix semigroups and max-plus algebras are well known in the literature and have many useful applications. The present article introduces a novel construction defined by the Rees matrix semigroups and max-plus algebras. This new construction turns out very convenient for generating sets of centroids motivated by their applications in analysis of data for the design of centroid-based classifiers and clusterers, as well as for the design of multiple classifiers and clusterers combining several individual initial classifiers and clusterers. Our article gives a complete description of all optimal sets of centroids for all Rees matrix semigroups without any restrictions on the sandwich-matrices.

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1. Introduction

Rees matrix semigroups and max-plus algebras have many useful applications and are well known, see [1] and [9]. The present paper introduces a new construction combining Rees matrix semigroups and max-plus algebras. This construction turns out very convenient for generating the sets of centroids motivated by their applications in analysis of data for the design of centroid-based classifiers or clusterers, as well as for the design of multiple classifiers and clusterers combining several individual initial classifiers and clusterers. In this paper we have managed to describe all optimal sets of centroids in the general case of arbitrary Rees matrix semigroups without any restrictions on the sandwich-matrices.

The paper is organised as follows. Necessary background information is included in Section 2. An overview of applications of the Rees matrix

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constructions in classification and clustering for analysis of data is provided in Section 3 as a motivation for research. The main result of this paper is Theorem 4.1 in Section 4, which completely describes all optimal sets of centroids. Proofs are given in Section 5.

2. Preliminaries

Let us begin with concise preliminaries on Rees matrix semigroups required for our main theorem. Rees matrix semigroups and associated notions of completely 0-simple semigroups and Rees quotients are very well known in semigroup theory and play crucial roles in describing the structure of semigroups and in proofs, see [9]. For examples of recent results, let us also refer to [4, 10, 11, 12].

Suppose that G is a group, I and Λ are nonempty sets, and e is the identity of G. As usual, we denote by $G^0 = G \cup \{\theta\}$ the group G with zero θ adjoined in a standard fashion. Let $P = [p_{\lambda i}]$ be a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in G^0$, for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix semigroup* $M^0(G; I, \Lambda; P)$ over G with *sandwich-matrix* P consists of all triples $(g; i, \lambda)$, where $i \in I$, $\lambda \in \Lambda$, and $g \in G^0$, where all triples $(\theta; i, \lambda)$ are identified with θ, and where multiplication is defined by the rule

$$(g_1; i_1, \lambda_1)(g_2; i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2; i_1, \lambda_2). \tag{1}$$

If G is a group, $M = M^0(G; I, \Lambda; P)$, and $i \in I$, $\lambda \in \Lambda$, then we use standard notation for the following sets

$$\begin{aligned} G_{*\lambda} &= \{(g; i, \lambda) \; : \; g \in G, i \in I\}, \\ G_{i*} &= \{(g; i, \lambda) \; : \; g \in G, \lambda \in \Lambda\}, \\ G_{i\lambda} &= \{(g; i, \lambda) \; : \; g \in G\}. \end{aligned}$$

Further, let S be a subset of the Rees matrix semigroup $M^0(G; I, \Lambda; P)$. The following notation will be used. For any $i, \lambda \in I$, set

$$\begin{aligned} S_{i\lambda} &= S \cap G_{i\lambda}, \\ S_{*\lambda} &= S \cap G_{*\lambda}, \\ S_{i*} &= S \cap G_{i*}. \end{aligned}$$

Also, for any subsets $X \subseteq I$, $Y \subseteq \Lambda$, we put

$$\begin{aligned} S_{X*} &= \cup_{i \in X} S_{i*}, \\ S_{*Y} &= \cup_{\lambda \in Y} S_{*\lambda}. \end{aligned}$$

We assume that $S_{\emptyset *} = S_{*\emptyset} = \emptyset$. Notice that θ never belongs to any of these sets above. For any subset X of T, we put $X^0 = X \cup \{\theta\}$.

The *max-plus algebra* is the set $\mathbb{R} \cup \{-\infty\}$ with two binary operations, max and $+$. It is very important in the investigation of discrete event systems, see [1]. The max-plus algebra is also sometimes called the *schedule algebra*, see [8]. Our main results remain valid in the more general case of all idempotent semifields, and so we record them in this setting.

A *semiring* is a set F with two binary operations, addition $+$ and multiplication $\cdot$, such that the following conditions are satisfied:

(S1) $(F, +)$ is a commutative semigroup with zero 0,

(S2) $(F, \cdot)$ is a semigroup,

(S3) multiplication distributes over addition,

(S4) zero 0 annihilates F, i.e., $0 \cdot F = F \cdot 0 = 0$.

It is also often assumed that every semiring satisfies the additional property

(S5) $(F, \cdot)$ has an identity element 1.

In this paper we consider more general semirings, which do not have to satisfy (S5), since such more general terminology adds the convenience of allowing us to consider more general subsets as subsemirings without assuming that all subsemirings contain the identity element. In analogy with a similar situation in ring theory, we then call every semiring satisfying (S5) a *semiring with identity element*. Both terminologies are essentially equivalent, since it is always easy to adjoin an identity element in a standard fashion to every semiring that does not have one. Originally, our investigation of semirings was motivated by the development of methods useful for duality theory see [3, 6].

A semiring F is said to be *idempotent* if $x + x = x$ for all $x \in F$. If the set of nonzero elements of a semiring F forms a group with respect to multiplication, then F is called a *semifield*.

Let F be a semiring, and let S be a semigroup. The *semigroup semiring* is denoted by $F[S]$ and is defined as the set

$$F[S] = \left\{ \sum_{i=1}^{n} f_i s_i \;\middle|\; f_i \in F, s_i \in S, n \in \mathbb{N} \right\}$$

where $\mathbb{N}$ stands for the set of all positive integers, and where addition and multiplication are defined by the associative and distributive laws and the rules

$$\sum_{s \in S} f_s s + \sum_{s \in S} f'_s s = \sum_{s \in S} (f_s + f'_s) s, \tag{2}$$

$$\left(\sum_{s \in S} r_s s \right) \left(\sum_{t \in S} r'_t t \right) = \sum_{s,t \in S} (r_s r'_t) st. \tag{3}$$

If S has a zero θ, then a *contracted semigroup semiring* is denoted by $F_0[S]$ and is defined as the quotient semiring of $F[S]$ modulo the ideal $F\theta$. Notice that if S has no zero, then S^0 stands for the semigroup $S \cup \{\theta\}$ with zero θ adjoined; and $F[S]$ is isomorphic to $F_0[S^0]$. If S is a semigroup without zero, then we also let $F_0[S] = F_0[S^0] \cong F[S]$. Contracted semigroup semirings enable us to formulate main results more concisely. We refer to [7, 13, 14, 15, 16, 17, 18, 19] for examples of results using these constructions and other areas where they are used.

3. Motivation

The design of efficient classifiers and clusterers is very important in data mining, see [20]. Rees matrix semigroups can be used in order to generate convenient sets of centroids for centroid-based clusterers and to design combined multiple clusterers capable of correcting the errors of individual initial clusterers.

The clustering process begins with feature extraction and representation of data in a standard vector space F^n, where $n \in \mathbb{N}$ and where F can be regarded as a semifield. Every centroid-based clusterer selects special elements $c_1, \ldots, c_k$ in F^n, called *centroids* (see, for example, [2]). For $i = 1, \ldots, k$, each centroid c_i defines its cluster $N(c_i)$ consisting of all vectors v such that c_i is the nearest centroid of v. Every vector is assigned to the cluster of its nearest centroid.

On the other hand, multiple classifiers and clusterers are often used in analysis of data to combine individual initial classifiers or clusterers (see, for example, [5, 21]). A well-known method for the design of multiple clusterers consists in designing several simpler initial or individual clusterers, and then combining them into one multiple clustering scheme with several clusters. This method is very effective, and is often recommended for various applications, see [20], Section 7.5. The main advantage of using combined multiple clusterers is in their ability to correct errors of individual clusterers and produce correct clusterings despite individual clustering errors.

Denote the number of initial clusterers being combined by n. Every clusterer outputs a symbol that indicates the cluster of the current instance. Without loss of generality we may assume that all these outputs belong to the same semifield F, because it is possible to extend the semifield and replace it with a larger one whenever necessary. If $o_1, \ldots, o_n$ are the outputs of the initial clusterers, then the sequence $(o_1, \ldots, o_n)$ is called a *vector of outputs* of the initial clusterers. In order to define the multiple clusterer and enable correction of errors of the initial clusterers, a set of centroids $c_1, \ldots, c_k$ is again selected in F^n. For $i = 1, \ldots, k$, the cluster $N(c_i)$ of the centroid c_i is again defined as the set of all observations with the vector outputs of the initial clusterers having c_i as its nearest centroid.

The design of multiple clusterers by combining individual binary clusterers is quite common in the literature. We refer to [18] and [20] for a list of properties required of the sets of centroids. In particular, it is essential to find sets of centroids with large weights and small numbers of generators. The *weight* $\mathrm{wt}(v)$ of $v \in F^n$ is the number of nonzero components v. The *weight* of a set $C \subseteq F^n$ is the minimum weight of a nonzero element in C. For additional references and discussion of experimental research related to these properties we refer the readers to [18], which handled constructions with certain restrictions on the sandwich-matrices.

Suppose that S is a finite semigroup with n nonzero elements. Then the additive semigroup of $F_0[S]$ is isomorphic to F^n and we can introduce multiplication in F^n by identifying it with $F_0[S]$. Accordingly, further we consider sets of centroids as subsets generated in $F_0[S]$. Every set of elements $g_1, \dots, g_k \in F_0[S]$ generates the set of all sums of multiples of these elements

$$C(g_1, \dots, g_k) = \tag{4}$$
$$= \left\{ \sum_{j=1}^{m_1} \ell_{1,j} g_1 r_{1,j} + \cdots + \sum_{j=1}^{m_k} \ell_{k,j} g_k r_{k,j} \;\middle|\; \ell_{i,j}, r_{i,j} \in F_0[S] \cup \{1\} \right\}.$$

The set $C(g_1, \dots, g_k)$ is also often called an *ideal* generated by $g_1, \dots, g_k$.

4. Main Results

Let S be a subsemigroup of a Rees matrix semigroup $M^0(G; I, \Lambda; P)$ over a group G with sandwich-matrix P. Consider the sets

$$L = L(S) = \left\{ \lambda \in \Lambda \;\middle|\; S_{*\lambda} = \emptyset \text{ or } \bigcup_{i \in I} p_{\lambda i} S_{i*} \subseteq \{\theta\} \right\} \text{ and} \tag{5}$$

$$R = R(S) = \left\{ i \in I \;\middle|\; S_{i*} = \emptyset \text{ or } \bigcup_{\lambda \in \Lambda} S_{*\lambda} p_{\lambda i} \subseteq \{\theta\} \right\}. \tag{6}$$

Here $p_{\lambda i} S_{i*} = \{(p_{\lambda i} g; i, \mu) : (g; i, \mu) \in S_{i*}\}$, and so $p_{\lambda i} S_{i*} \subseteq \{\theta\}$ means that $p_{\lambda i} = \theta$ or $S_{i*} = \emptyset$. Likewise, $S_{*\lambda} p_{\lambda i} = \{(g p_{\lambda i}; i, \mu) : (g; i, \mu) \in S_{i*}\}$, and so $S_{*\lambda} p_{\lambda i} \subseteq \{\theta\}$ means that $p_{\lambda i} = \theta$ or $S_{*\lambda} = \emptyset$.

Let us define the following numbers

$$M_Z = |S_{R*} \cap S_{*L}|, \tag{7}$$
$$M_L = \max\{|S_{i*} \cap S_{*L}| : \text{ for all } i \notin L\}, \tag{8}$$
$$M_R = \max\{|S_{*\lambda} \cap S_{R*}| : \text{ for all } \lambda \notin R\}, \tag{9}$$
$$M_G = \max\{|S_{i\lambda}| : \text{ for all } i \notin L, \lambda \notin R\}. \tag{10}$$

Denote by $\mathcal{G}_Z$ the set of all elements $r = \sum_{s \in S_{R*} \cap S_{*L}} r_s s$ such that $0 \neq r_s \in F$ for all $s \in S_{R*} \cap S_{*L}$. If $|M_Z| \geq 1$, then it is easily seen that the set $\mathcal{G}_Z$ is nonempty and contains only nonzero elements.

Let $\mathcal{G}_L$ be the set of all elements $r = \sum_{s \in S_{i*} \cap S_{*L}} r_s s$, such that $0 \neq r_s \in F$ for all $s \in S_{i*} \cap S_{*L}$, where i runs over all elements of $I \setminus L$ such that $|S_{i*} \cap S_{*L}| = M_L$. If $|M_L| \geq 1$, then the set $\mathcal{G}_L$ is nonempty and contains only nonzero elements.

Denote by $\mathcal{G}_R$ the set of all elements $r = \sum_{s \in S_{*\lambda} \cap S_{R*}} r_s s$, such that $0 \neq r_s \in F$ for all $s \in S_{*\lambda} \cap S_{R*}$, where λ runs over all elements of $\Lambda \setminus R$ such that $|S_{*\lambda} \cap S_{R*}| = M_R$. If $|M_R| \geq 1$, then the set $\mathcal{G}_R$ is nonempty and contains only nonzero elements.

Let $\mathcal{G}_G$ be the set of all elements $r = \sum_{s \in S_{i\lambda}} r_s s$, such that $0 \neq r_s \in F$ for all $s \in S_{i\lambda}$, where i runs over all elements of $I \setminus L$ and λ runs over all elements of $\Lambda \setminus R$ such that $|S_{i\lambda}| = M_G$. If $|M_G| \geq 1$, then the set $\mathcal{G}_G$ is nonempty and contains only nonzero elements.

Our main theorem completely describes all sets $C(g_1, \ldots, g_k)$ with the largest weight in $F_0[S]$. Notice that the results of [18] did not use max-plus algebras and involved a restriction on the sandwich-matrix of the underlying Rees matrix semigroup. Examples show that is impossible to drop this restriction from the results of [18]. Our new construction introduced in the present article with the use of max-plus algebras turns out so convenient that the main theorem of this paper completely describes all optimal sets of centroids in the general case of arbitrary Rees matrix semigroups without any restrictions on the sandwich-matrices.

THEOREM 4.1. *Let $C = C(g_1, \ldots, g_k)$ be a centroid set with the largest weight in $F_0[S]$, where F is an idempotent semifield, $T = M^0(G; I, \Lambda; P)$ is a Rees matrix semigroup over a group G with sandwich-matrix P, and S is a finite subsemigroup of T. Then the following conditions are satisfied:*

(i) $\mathrm{wt}(C) = \max\{M_Z, M_L, M_R, M_G\}$;

(ii) *C contains an element of weight $\mathrm{wt}(C)$ belonging to the union of $\mathcal{G}_Z$, $\mathcal{G}_L$, $\mathcal{G}_R$ and $\mathcal{G}_G$;*

(iii) $\mathrm{wt}(C(r)) = \mathrm{wt}(r) = M_Z$, *for all $r \in \mathcal{G}_Z$;*

(iv) $\mathrm{wt}(C(r)) = \mathrm{wt}(r) = M_L$, *for all $r \in \mathcal{G}_L$;*

(v) $\mathrm{wt}(C(r)) = \mathrm{wt}(r) = M_R$, *for all $r \in \mathcal{G}_R$;*

(vi) $\mathrm{wt}(C(r)) = \mathrm{wt}(r) = M_G$, *for all $r \in \mathcal{G}_G$.*

5. Proofs

For completeness and convenience of the readers, we begin with a few easy and useful lemmas.

LEMMA 5.1. *Let F be an idempotent semiring, and let $x_1, \dots, x_n \in F$. Then*

$$x_1 + \cdots + x_n = 0 \Longleftrightarrow x_1 = \cdots = x_n = 0. \tag{11}$$

PROOF. Suppose that $x_1 + \cdots + x_n = 0$. Then the laws of addition and multiplication in the definition of an idempotent semiring imply that $x_i = x_i + 0 = x_i + (x_1 + \cdots + x_n) = x_1 + \cdots + x_n = 0$, for all $i = 1, \dots, n$. This completes the proof, since the reversed implication is clear. □

Every semiring satisfying (11) is said to be *zerosumfree*. Thus, Lemma 5.1 tells us that all idempotent semirings are zerosumfree.

LEMMA 5.2. *Let F be an idempotent semiring, S a semigroup with zero θ, and let $x_1, \dots, x_n \in F$, $s_1, \dots, s_n \in S$. Then*

$$0 = \sum_{i=1}^{n} x_i s_i \in F_0[S] \Longleftrightarrow x_i = 0 \textit{ for all } s_i \neq \theta. \tag{12}$$

PROOF. Let us assume that $0 = \sum_{i=1}^{n} x_i s_i \in F_0[S]$. Combining the like terms, we see that $\sum_{i=1}^{n} x_i s_i = \sum_{s\in S} \left(\sum_{s_i=s} x_i\right) s$. Hence, fixing any $\theta \neq s \in S$, we get $\sum_{s_i=s} x_i = 0$. Lemma 5.1 shows that $x_i = 0$ for all $s_i = s$. It follows that $x_i = 0$ for all $s_i \neq \theta$, as required. This completes the proof, since the reversed implication is clear. □

For $x = \sum_{s\in S} r_s s \in F_0[S]$, the set $\operatorname{supp}(x) = \{s \in S \,:\, r_s \neq 0\}$ is called the *support* of x. Evidently, $\operatorname{wt}(x) = |\operatorname{supp}(x)|$.

LEMMA 5.3. *Let F be an idempotent semiring, S a semigroup, and let $x, y \in F_0[S]$. Then $\operatorname{supp}(x + y) = \operatorname{supp}(x) \cup \operatorname{supp}(y)$.*

PROOF follows from (2) and Lemma 5.1. □

Let S be a semigroup with zero θ. The *left annihilator* of S is the set

$$\operatorname{Ann}_\ell(S) = \{x \in S \,:\, xS = \theta\}, \tag{13}$$

and the *right annihilator* of S is the set

$$\operatorname{Ann}_r(S) = \{x \in S \,:\, Sx = \theta\}. \tag{14}$$

LEMMA 5.4. *Let $T = M^0(G; I, \Lambda; P)$ be a Rees matrix semigroup over a group G with sandwich-matrix P, and let S be a subsemigroup of T. Then*

$$\operatorname{Ann}_r(S) = S_{R*} \cup \{\theta\}, \tag{15}$$

$$\operatorname{Ann}_\ell(S) = S_{*L} \cup \{\theta\}. \tag{16}$$

Proof. We are going to prove only equality (15), since the proof of (16) is dual. Clearly, θ belongs to both sides of equality (15). Besides, (1) and (6) imply that $\operatorname{Ann}_r(S) \supseteq S_{R*}$. To prove the reversed inclusion, let us suppose to the contrary that $\operatorname{Ann}_r(S)$ is not contained in S_{R*}.

Then we can choose $i \in I \setminus R$ and pick $x = (g; i, \mu) \in \operatorname{Ann}_r(S)$, where $g \in G$, $\mu \in \Lambda$. Since $x \in S$, we get $S_{i*} \not\subseteq \theta$. Therefore (6) shows that $S_{*\lambda} p_{\lambda i} \not\subseteq \theta$ for some $\lambda \in \Lambda$. Hence there exist $h \in G$ and $j \in I$ such that $(h; j, \lambda)x = (hp_{\lambda i}g; j, \mu) \neq \theta$. This contradicts the choice of x in $\operatorname{Ann}_r(S)$ and completes our proof. $\square$

For any semiring F, the *left annihilator* of F is the set

$$\operatorname{Ann}_\ell(F) = \{x \in F \,:\, xF = 0\}, \tag{17}$$

and the *right annihilator* of F is the set

$$\operatorname{Ann}_r(F) = \{x \in F \,:\, Fx = 0\}. \tag{18}$$

Lemma 5.5. *Let F be an idempotent semifield, and let S be a semigroup with zero θ. Then*

$$\operatorname{Ann}_r(F_0[S]) = F_0[\operatorname{Ann}_r(S)], \tag{19}$$

$$\operatorname{Ann}_\ell(F_0[S]) = F_0[\operatorname{Ann}_\ell(S)]. \tag{20}$$

Proof. Take any two elements $x = \sum_{i=1}^n x_i s_i$ and $y = \sum_{j=1}^m y_j t_j$ in $F_0[S]$, where $0 \neq x_i \in F$ for all $i = 1, \ldots, n$, and $0 \neq y_j \in F$ for all $j = 1, \ldots, m$. The product $x_i y_j$ is nonzero for any i, j, because F is a semifield. Therefore Lemma 5.2 shows that

$$xy = 0 \Longleftrightarrow s_i t_j = \theta \text{ for all } i, j. \tag{21}$$

Equalities (19) and (20) follow from equivalence (21). $\square$

Let S be a subsemigroup of a Rees matrix semigroup $M^0(G; I, \Lambda; P)$. It is clear that $S^0_{R*} = S_{R*} \cup \{\theta\}$ and $S^0_{*L} = S_{*L} \cup \{\theta\}$ are subsemigroups of S.

Lemma 5.6. *Let F be an idempotent semiring, and let S be a subsemigroup of a Rees matrix semigroup $M^0(G; I, \Lambda; P)$ over a group G with sandwich-matrix P. Then*

$$\operatorname{Ann}_r(F_0[S]) = F_0[S^0_{R*}], \tag{22}$$

$$\operatorname{Ann}_\ell(F_0[S]) = F_0[S^0_{*L}]. \tag{23}$$

Proof follows from Lemmas 5.4 and 5.5. $\square$

Proof of Theorem 4.1. If S does not contain θ, then we can replace S with $S^0 = S \cup \{\theta\}$ in the statement of the theorem. This will not change the sets L, R and numbers M_Z, M_L, M_R and M_G. Therefore, further we assume that S contains θ.

Let us first prove condition (iii). Take any element $r \in \mathcal{G}_Z$. By definition, we know that $r = \sum_{s\in S_{R*}\cap S_{*L}} r_s s$, where $0 \neq r_s \in F$ for all $s \in S_{R*} \cap S_{*L}$. Hence $\mathrm{wt}(r) = M_Z$. It follows from equality (22) of Lemma 5.6 that $r \in \mathrm{Ann}_r(F_0[S])$. Equality (23) demonstrates that $r \in \mathrm{Ann}_\ell(F_0[S])$. Since F is a semifield, it follows that $C(r)$ coincides with the subsemiring

$$\{cr \,:\, c \in \mathbb{N}\}$$

generated by r in $F_0[S]$. All elements of this set have the same weights equal to the weight of r. Hence $\mathrm{wt}(C(r)) = \mathrm{wt}(r)$ in this case, and so condition (iii) holds.

Next, we are going to prove condition (iv). Choose any element $r \in \mathcal{G}_L$. There exist $i \in I \setminus L$ such that $r = \sum_{s\in S_{i*}\cap S_{*L}} r_s s$, where $0 \neq r_s \in F$ for all $s \in S_{i*} \cap S_{*L}$, and $|S_{i*} \cap S_{*L}| = M_L$. Therefore $\mathrm{wt}(r) = M_L$.

To prove that $\mathrm{wt}(C(r)) = \mathrm{wt}(r)$, let us pick any element x in $C(r)$. We claim that $\mathrm{wt}(x) \geq \mathrm{wt}(r)$.

By (4), we get $x = \sum_{j=1}^{k} a_j r b_j$, for some $a_j, b_j \in F_0[S]^1 \cup \{1\}$. Since every nonzero element of $F_0[S]$ is equal to the sum of some elements from the set

$$FS = \{fs \,:\, 0 \neq f \in F, \theta \neq s \in S\},$$

the distributive law allows us to assume that $a_j, b_j \in FS \cup \{1\}$. We may assume that all summands $a_j r b_j$ are nonzero.

Suppose that $b_j \neq 1$ for some j. Since $\mathrm{supp}\,(r) \subseteq S_{*L}$, equality 16 in Lemma 5.4 shows that $rb_j = 0$, and so $a_j r b_j = 0$. Therefore, further we may assume that $b_j = 1$ for all $j = 1, \ldots, k$.

In view of Lemma 5.3 it remains to verify that $\mathrm{wt}(a_j r) \geq \mathrm{wt}(r)$, for all $j = 1, \ldots, k$.

Consider a product $a_j r$, where $a_j \in FS$, i.e. $a_j = fs$ for $f \in F$, $s \in S$. Since F is a semifield, we get $\mathrm{wt}(fsr) = \mathrm{wt}(sr)$. We can write $s = (g; j', \mu)$ for some $j' \in I$ and $\mu \in \Lambda$. Since $sr \neq 0$, it follows that $p_{\mu i} \neq \theta$. Hence it follows from (1) that $\mathrm{supp}\,(sr) = S_{j'*} \cap S_{*L}$. Therefore $|\mathrm{supp}\,(sr)| = |S_{j'*} \cap S_{*L}| = |S_{i*} \cap S_{*L}| = \mathrm{supp}\,(r)$.

Thus $\mathrm{wt}(a_j r) \geq \mathrm{wt}(r)$, and so $\mathrm{wt}(x) \geq \mathrm{wt}(r)$ by Lemma 5.3. It follows that $\mathrm{wt}(C(r)) = \mathrm{wt}(r)$, which means that condition (iv) holds.

The proof of condition (v) is dual to that of condition (iv) and we omit it.

Let us now prove condition (vi). Take any element $r \in \mathcal{G}_G$. There exist $i \in I \setminus L$ and $\lambda \in \Lambda \setminus R$ such that $r = \sum_{s\in S_{i\lambda}} r_s s$ and $|S_{i\lambda}| = M_G$. Therefore $\mathrm{supp}\,(r) = |M_G|$. It remains to prove that $\mathrm{wt}(C(r)) = \mathrm{wt}(r)$. To this end let us pick any element x in $C(r)$. By (4), it can be written as $x = \sum_{j=1}^{k} a_j r b_j$, for some $a_j, b_j \in F_0[S] \cup \{1\}$, where the distributive law allows us to assume that $a_j, b_j \in FS \cup \{1\}$, and where we may assume that all summands $a_j r b_j$ are nonzero.

We claim that $\mathrm{wt}(x) \geq \mathrm{wt}(r)$. Keeping in mind Lemma 5.3, it suffices to verify that $\mathrm{wt}(a_j r b_j) \geq \mathrm{wt}(r)$, for all $j = 1, \ldots, k$.

Suppose that $a_j = f_a(g_a; i_a, \lambda_a)$ and $b_j = f_b(g_b; i_b, \lambda_b)$ for some $g_a, g_b \in G$, $i_a, i_b \in I$, $\lambda_a, \lambda_b \in \Lambda$) and $f_a, f_b \in F$. Since $\mathrm{supp}\,(r) \subseteq S_{i\lambda}$ and $a_j r b_j \neq 0$, it follows from (1) that $p_{\lambda_a i}, p_{\lambda i_b} \neq \theta$. Therefore (1) implies that $|\,\mathrm{supp}\,(a_j r b_j)| = |\,\mathrm{supp}\,(r)|$, because

$$\mathrm{supp}\,(a_j r b_j) = (g_a; i_a, \lambda_a)\,\mathrm{supp}\,(r)(g_b; i_b, \lambda_b).$$

Thus $\mathrm{wt}(a_j r b_j) = \mathrm{wt}(r)$ in this case.

The cases where $a_j = 1$ or $b_j = 1$ are similar, and are even simpler. In these cases it follows too that $\mathrm{wt}(a_j r b_j) = \mathrm{wt}(r)$. Thus, Lemma 5.3 shows that $\mathrm{wt}(C(r)) = \mathrm{wt}(r)$, as required. This means that condition (vi) holds.

Now we are going to prove condition (ii). Choose a nonzero element r of minimal weight in C and consider several possible cases.

Case 5.1. $r \in \mathrm{Ann}_\ell(F_0[S]) \cap \mathrm{Ann}_r(F_0[S])$. By Lemma 5.6, we get $r \in F_0[(S_{R*} \cap S_{*L})^0]$, and so $\mathrm{supp}\,(r) \subseteq S_{R*} \cap S_{*L}$. It follows from the maximality of $\mathrm{wt}(C)$ and condition (iii), which we have already proved above, that $|\,\mathrm{supp}\,(r)| = M_Z$. Therefore $\mathrm{supp}\,(r) = S_{R*} \cap S_{*L}$; whence $r \in \mathcal{G}_Z$. Since $\mathrm{wt}(r) = \mathrm{wt}(C)$, this means that condition (ii) holds in this case.

Case 5.2. $r \in \mathrm{Ann}_\ell(F_0[S]) \setminus \mathrm{Ann}_r(F_0[S])$. Equality (19) of Lemma 5.5 shows that $\mathrm{supp}\,(r) \not\subseteq \mathrm{Ann}_r(S)$. Hence there exists $b \in S$ such that $b\,\mathrm{supp}\,(r) \neq \theta$. We can write it down as $b = (g_b; i_b, \lambda_b)$, for some $g_b \in G$, $i_b \in I$, $\lambda_b \in \Lambda$. Here $i_b \notin R$, because of equality (15) of Lemma 5.4. It follows from (1) that $\mathrm{supp}\,(br) \subseteq S_{i_b*}$. Since $r \in \mathrm{Ann}_\ell(F_0[S]) = S_{*L}$, we get $\mathrm{supp}\,(br) \subseteq S_{i_b*} \cap S_{*L}$.

Condition (iv) proved above shows that $F_0[S]$ contains a set $C(g_1, \ldots, g_n)$ of weight M_L. Since $M_L \geq |S_{i_b*} \cap S_{*L}|$, the maximality of the weight of C ensures that every nonzero element in C has weight at least M_L. It follows that $\mathrm{wt}(br) = M_L = |S_{i_b*} \cap S_{*L}|$ and $\mathrm{supp}\,(br) = S_{i_b*} \cap S_{*L}$. This means that $br \in \mathcal{G}_L$.

Since r has minimal weight in C and $0 \neq br \in C$, we get $\mathrm{wt}(br) = \mathrm{wt}(r) = \mathrm{wt}(C)$. Thus, condition (ii) holds in this case, too.

Case 5.3. $r \in \mathrm{Ann}_r(F_0[S]) \setminus \mathrm{Ann}_\ell(F_0[S])$. The proof in this case is dual to the one in Case 2, and so we omit it.

Case 5.4. $r \notin \mathrm{Ann}_r(F_0[S]) \cup \mathrm{Ann}_\ell(F_0[S])$. Lemma 5.5 shows that $r \notin F_0[\,\mathrm{Ann}_r(S)] \cup F_0[\,\mathrm{Ann}_\ell(S)]$. Hence there exists $a, b \in S$ such that $a\,\mathrm{supp}\,(r)$, $\mathrm{supp}\,(r)b \neq \theta$. Hence (1) shows that $a\,\mathrm{supp}\,(r)b \neq \theta$. Since r has minimum weight in C and $0 \neq arb \in C$, we get $\mathrm{wt}(arb) = \mathrm{wt}(r) = \mathrm{wt}(C)$.

Consider the representations $a = (g_a; i_a, \lambda_a)$ and $b = (g_b; i_b, \lambda_b)$, where $g_a, g_b \in G$, $i_a, i_b \in I$, $\lambda_a, \lambda_b \in \Lambda$. By (1), we see that $\mathrm{supp}\,(arb) \subseteq S_{i_a \lambda_b}$; whence $\mathrm{supp}\,(arb) \le M_G$ in view of the maximality of M_G. Condition (vi) proved above tells us that $F_0[S]$ contains a set of the form $C(g_1, \ldots, g_n)$ with weight M_G. By the maximality of $\mathrm{wt}(C)$, we get $\mathrm{wt}(arb) \ge M_G$. Hence $\mathrm{wt}(arb) = M_G$. Therefore $arb \in \mathcal{G}_G$, which means that condition (ii) holds in this case, too.

Clearly, condition (ii) implies that

$$\mathrm{wt}(C) \le \max\{M_Z, M_L, M_R, M_G\}.$$

On the other hand, the maximality of $\mathrm{wt}(C)$ and conditions (iii), (iv), (v), (vi) show that $\mathrm{wt}(C) \ge M_Z, M_L, M_R, M_G$. Therefore condition (i) is satisfied. This completes our proof. □

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School of Science, Information Technology and Engineering
University of Ballarat
P.O. Box 663, Ballarat
Victoria 3353, Australia
{a.kelarev,j.yearwood,l.zi}@ballarat.edu.au