CONICAL DECOMPOSITION AND VECTOR LATTICES WITH RESPECT TO SEVERAL PREORDERS

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Abstract. The decomposition set-valued mapping in a Banach space E with cones K_i, $i = 1, \dots, n$ describes all decompositions of a given element on addends, such that addend i belongs to the i-th cone. We examine the decomposition mapping and its dual.

We study conditions that provide the additivity of the decomposition mapping. For this purpose we introduce and study the Riesz interpolation property and lattice properties of spaces with respect to several preorders. The notion of 2-vector lattice is introduced and studied. Theorems that establish the relationship between the Riesz interpolation property and lattice properties of the dual spaces are given.

1. Introduction

1. The goal of this paper is to study general cone decomposition. Let us explain the matter of the problem.

Consider n convex cones $K_1, \dots, K_n$ in a vector space E with $n \geq 2$. It is possible that some of these cones coincide. Let $L = \sum_{i=1}^{n} K_i$ be the Minkowski sum of these cones. A collection of elements $x_i \in K_i,\ i = 1, \dots, n$ is called the decomposition of an element $x \in L$ with respect to the collection of cones $(K_i)_{i=1}^{n}$ if $x = x_1 + x_2 + \ldots + x_n$. We are mainly interested in the totality of all possible decompositions for all vectors $x \in L$. In other words we shall study the set-valued mapping σ defined on L by

$$\sigma(x) = \{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i = x,\, x_i \in K_i,\, i = 1, \dots, n\}.$$

The mapping σ is called the *decomposition mapping* with respect to cones $K_1, \ldots,$ K_n. We can describe this mapping in the following way. Consider the space E^n and the operator of summation $A : E^n \to E$ defined by

$$A(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i.$$

Let $K = K_1 \times \ldots \times K_n \subset E^n$ and let A_K be the restriction of A to K. Then A_K is the linear operator defined on K and mapping onto $L = \sum_{i=1}^{k} K_i$. It is clear that σ coincides with the set-valued mapping A_K^{-1} inverse to A_K.

2. Decomposition mapping arises in different fields of mathematics and its applications. The situation when all cones $K_1, \ldots, K_n$ coincide was mainly investigated. An important field of application of the decomposition mapping is mathematical economics. Assume that we have an economy with n agents and m products. Let $E = \mathbb{R}^m$ and $K_1, \ldots, K_n$ coincide with the cone $\mathbb{R}^m_+ \subset \mathbb{R}^m$ of vectors with nonnegative coordinates. A vector $x = (x^1, \ldots, x^m) \in \mathbb{R}^m_+$ describes a certain collection of products (x^j is the quantity of the product j in this collection.) Having vector x, agents need to distribute it between themselves, that is to find vectors $x_1, \ldots, x_n \in \mathbb{R}^m_+$ such that $\sum_{i=1}^{n} x_i = x$. The totality of all such distributions coincides with the set $\sigma(x)$. The decomposition mapping plays an important role in the study of some models of economic equilibrium and economic dynamics (see [3] for details). From an economical point of view it is interesting to consider efficient decompositions of a given element x, that is, decompositions than are better (in a certain sense) that the other decompositions of this element. A cone decomposition theory based on efficiency has been developed by J.E. Martinez Legaz and A. Seeger in [4].

3. We use methods of convex analysis for examination of the decomposition mapping. Let $K_1, \ldots, K_n$ be convex cones in the space E and let σ be the corresponding decomposition mapping. Then the graph

$$\operatorname{gr} \sigma = \{((x_1, \ldots, x_n), y) \in E^n \times E : y \in \sigma(x_1, \ldots, x_n)\}$$

is a convex cone, hence σ is a convex process (see [5]). In another terminology (see [6, 2, 3]) σ is a superlinear set-valued mapping. The dual theory of superlinear mappings is well developed. We give an explicit description of dual mapping to the decomposition mapping and describe its properties. This approach allows us to discover some interesting properties of the decomposition mapping itself.

4. An important question related to decomposition mapping is to find conditions that guarantee its additivity. In the simplest case when all cones K_i coincide with a cone K, this property is equivalent to the following: the space E with the order relation generated by K possesses the Riesz interpolation property. It is of interest to extend this result to the case of two or more cones. To this end, we introduce a space and objects defined with respect to several cones which can be viewed as generalizations of such classical notions as vector lattice, exact upper and lower bounds, Riesz interpolation property and Riesz decomposition property, Double Partition Lemma etc. On the whole, the problem on additive decomposition can be solved in such spaces. We establish the relationship between Riesz interpolation property with respect to several cones, and lattice properties of the dual space w.r.t. the corresponding dual cones.

The lattices with respect to several cones are quite natural from the point of view of applications to mathematical economics (see **2**). Indeed, it is quite natural to assume that each agent i is interested only in the products with the numbers from a certain subset J_i of the set of indices $\{1, \dots, n\}$. This observation leads to decomposition mapping with respect to a system of cones $K_1, \dots, K_n$, where K_i is a face of the cone $\mathbb{R}^m_+$. It can be shown that decomposition mapping with respect to such systems is additive.

5. Next, we summarize the structure of the paper. Some definitions and results related to superlinear mappings are given in Section 2. Decomposition mapping and dual to decomposition mapping are described in Section 3. Properties of the support function of the decomposition mapping are discussed in Section 4. Section 5 provides different characterizations of space with several cones that are equivalent to the additivity of the decomposition mapping. Vector lattices with respect to several pre-orders are examined in Section 6. Kantorovich- Riesz type theorems in spaces with two cones are studied in Section 7.

2. Superlinear Set-valued Mappings (Convex Processes)

Let E_1, E_2 be Banach spaces. A set-valued mapping $a : E_1 \to 2^{E_2}$ is called a convex process ([5]), if its graph $\operatorname{gr} a = \{(x, y) \in E_1 \times E_2 : y \in a(x)\}$ is a cone in $E_1 \times E_2$ and $(0, 0) \in \operatorname{gr} \varphi$.

Sometimes (see, for example, [2, 6, 3]) convex processes are called *superlinear set-valued mappings*. It is more convenient for us to use this terminology. A superlinear mapping a is called bounded if

$$\|a\| := \sup\{\|y\| : y \in a(x),\, x \in \operatorname{dom} a,\, \|x\| \le 1\} < +\infty.$$

Here $\operatorname{dom} a = \{x : a(x) \neq \emptyset\}$.

Let E be a Banach space. A cone $K \subset E$ is called locally compact if each bounded subset of K is compact. The following result is well-known and can be easily proved.

Theorem 2.1. *Let $a : E_1 \to 2^{E_2}$ be a closed positively homogeneous mapping, the cone $K :=$ dom a be locally compact and $a(0) = \{0\}$, then a is bounded.*

Definition 2.1. The set-valued mapping $a^* : E_2' \to E_1'$ is called *dual* to a superlinear mapping $a : E_1 \to 2^{E_2}$, if

$$a^*(g) = \{f \in E_1' : [f, x] \leq [g, y],\ \forall x \in \text{dom}\, a,\ y \in a(x)\}.$$

It is well-known and easy to check that the dual mapping a^* is superlinear for an arbitrary mapping a. The following duality theorem holds:

Theorem 2.1. (see ([2, 6]) *Let a be a superlinear mapping. Then the sublinear function $p_g(x) = \inf\{[g, y] \ : \ y \in a(x)\}$ is sublinear. If p_g is lower semicontinuous for all $g \in E'$, then for all $x \in$ dom $a, g \in$ dom a^* the following holds*

$$\sup\{[f, x] \ : \ f \in a^*(g)\} = \inf\{[g, y] \ : \ y \in a(x)\}, \quad \text{and} \quad \partial p_g(0) = a^*(g).$$

Here $\partial p(x)$ is the subdifferential of a sublinear function p at a point x.

3. Decomposition Mapping and Its Dual

3.1. Decomposition Mapping

Let E be a Banach space and let $E^n = E \times E \ldots \times E$ be the cartesian product of its n copies. We assume that E^n is equipped with the sum-norm: if $X = (x_1, \ldots, x_n) \in E^n$ then $\|X\| = \sum_{i=1}^n \|x_i\|$. By $E', (E^n)'$ we will denote the dual spaces to E and E^n, respectively. Note that $(E^n)' = (E')^n$. For $f \in E'$ we have $\|f\| = \sup_{\|x\| \leq 1} |f(x)|$. If $F = (f_1, \ldots, f_n) \in (E^n)'$ then $\|F\| = \max_{i=1,\ldots,n} \|f_i\|$. In particular, if $f_1 = \ldots = f_n := f$ then $\|F\| = \|f\|$.

In the space E let us consider a collection of convex closed cones $K_1, K_2, \ldots, K_n$, and in the space E^n consider their cartesian product $K = K_1 \times K_2 \times \cdots \times K_n$. The dual cones to $K_1, K_2, \ldots, K_n$ and K will be denoted by $K_1^*, K_2^*, \ldots, K_n^*$ and K^*, respectively. It is clear that $K^* = K_1^* \times K_2^* \times \cdots \times K_n^*$. We also use the following notation:

$$L = K_1 + \ldots + K_n.$$

It is well-known and easy to check that $L^* = \bigcap_{i=1}^n K_i^*$

Definition 3.1. A set-valued mapping $\sigma_{K_1,\ldots,K_n} : E \to 2^{E^n}$, defined by

$$\sigma_{K_1,\ldots,K_n}(x) := \begin{cases} \{X = (x_1, \ldots, x_n) \in K \ : \ \sum_{i=1}^n x_i = x\} & x \in L \\ \emptyset & x \notin L \end{cases}$$

is called *decomposition mapping* with respect to cones $K_1, \dots, K_n$, and the elements of the set $\sigma_{K_1,\dots,K_n}$ are called the *decompositions* of x.

For the sake of simplicity we denote $\sigma_{K_1,\dots,K_n}$ by σ if it does not lead to confusion. It is clear that dom $\sigma = L := \sum_{i=1}^n K_i$. The decomposition mapping is closed. Moreover, this mapping possesses a stronger property than the property to be closed. Indeed, if $X^k \to X$ then $x_i^k \to x_i$ for all i and hence $\sum_i^k x_i^k \to \sum_i^k x_i$. Thus the following holds: if $X^k \to X$ and $X^k \in \sigma(x^k)$ then there exists $\lim x^k = x$ and $X \in \sigma(x)$.

3.2. The Description of the Mapping σ^*

In this subsection we give an explicit description of the mapping σ^* dual to the decomposition mapping $\sigma_{K_1,\dots,K_n} \equiv \sigma$. Let

$$\mathcal{K} = \text{dom } \sigma^*. \tag{3.1}$$

It follows from the superlinearity of σ^* that the set $\mathcal{K}$ is a convex cone.

The following theorem allows one to get an explicit form of the mapping σ^* dual to σ.

Theorem 3.1. *The equality* $\sigma^*(G) = \bigcap_{i=1}^n (g_i - K_i^*)$ *holds for all* $G = (g_1, \dots, g_n) \in \mathcal{K}$.

Proof. Let $f \in \sigma^*(G)$ $(G \in \mathcal{K})$, then by the definition of σ^* we have

$$[f, x] \le [G, X] \quad \forall x \in \text{dom}\, \sigma,\ X \in \sigma(x). \tag{3.2}$$

For every $i = 1, 2, \dots, n$, and any $x_i \in K_i$ put $X_{x_i} = (0, \dots, 0, x_i, 0, \dots, 0) \in E^n$. It is clear that $X_{x_i} \in \sigma(x_i)$, and (3.2) implies that $[f, x_i] \le [G, X_{x_i}]$ for all $x_i \in K_i,\ i = 1, 2, \dots, n$, or $[f, x_i] \le [g_i, x_i]$ for all $x_i \in K_i,\ i = 1, 2, \dots, n$, i.e. $[f - g_i, x_i] \le 0$ for all $x_i \in K_i,\ i = 1, 2, \dots, n$. It follows from the definition of the conjugate cone that $f \in g_i - K_i^*,\ i = 1, 2, \dots, n$. This means that $f \in \bigcap_{i=1}^n (g_i - K_i^*)$.

Conversely, let the last inclusion hold for an element f. Then $g_i - f \in K_i^*,\ i = 1, 2, \dots, n$, hence for all $x_i \in K_i,\ i = 1, 2, \dots, n$, we have $[f, x_i] \le [g_i, x_i]$. Summing over i from 1 to n we get after simple calculations that

$$[f, \sum_{i=1}^n x_i] \le \sum_{k=1}^n [g_k, x_k] \quad \text{for all} \quad x_i \in K_i,\ i = 1, 2, \dots, n. \tag{3.3}$$

Let $x \in E$ and $X = (x_1, \dots, x_n) \in \sigma(x)$. Then $\sum_{i=1}^n x_i = x$. Applying (3.3) we get $[f, x] \le [G, X]\ \forall x,\ X \in \sigma(x)$, which is equivalent to the inclusion $f \in \sigma^*(G)$. ■

3.3 Domain of the Mapping σ^*

It will be shown in this subsection that the cone $\mathcal{K} = \operatorname{dom}\sigma^*$ is the sum of two summands, one of which is described in the following assertion.

Proposition 3.1. *The equality $K^* = (\sigma^*)^{-1}(0)$ is valid. (Recall that $K = K_1 \times \ldots \times K_n$.)*

Proof. In the view of Theorem 3.1 we have that $G \in (\sigma^*)^{-1}(0)$ if and only if $0 \in \bigcap_{i=1}^n (g_i - K_i^*)$ which is equivalent $g_i \in K_i^*$ for all i. ∎

Corollary 3.1. *The inclusion $K^* \subset \mathcal{K}$ holds.*

Consider the set

$$M = \{X \in E^n \ : \ \sum_{i=1}^n x_i = 0\}.$$

Let M^* be the orthogonal to M subspace: $M^* = \{G \in (E^n)^* \ : \ [G, X] = 0 \quad \forall X \in M\}$. Consider also the diagonal $D = \{G = (g, g, \ldots, g) \ : \ g \in E'\}$ of the space $(E')^n$. It is clear that D is w^*-closed in $(E^n)' = (E')^n$. In the sequel an element $(g, g, \ldots, g) \in D$ will be denoted by $g^\wedge$.

Proposition 3.2. *The subspaces M^* and D of the dual space $(E^n)'$ coincide.*

Proof. Let $G = g^\wedge \in D$, then for every $X \in M$ we have $[G, X] = \sum_{i=1}^n [g, x_i] = [g, \sum_{i=1}^n x_i] = 0$, i.e. $G \in M^*$, and hence $D \subset M^*$. Now let us prove the opposite inclusion. Suppose, there exists an element $\overline{G} \in (E')^n$ such that $\overline{G} \in M^* \setminus D$. Since D is w^*-closed and convex we can apply the separation theorem which implies the existence of $\overline{X} = (\bar{x}_i) \in E^n$ such that

$$[\overline{G}, \overline{X}] > \sup_{g \in E'} [g^\wedge, \overline{X}] = \sup_{g \in E'} \sum_i [g, \bar{x}_i] = \sup_{g \in E'} \left[g, \sum_i \bar{x}_i \right]. \tag{3.4}$$

The following cases are possible:

1. if $\overline{X} \in M$, then the right-hand side of the last inequality is equal to zero, and $[\overline{G}, \overline{X}] > 0$. On the other hand, $[\overline{G}, \overline{X}] = 0$, since $\overline{G} \in M^*$;
2. if $\overline{X} \notin M$, then $\sum_i \bar{x}_i \neq 0$ hence $\sup_{g \in E'} [g^\wedge, \overline{X}] = +\infty$ and we have $[\overline{G}, \overline{X}] > +\infty$,

therefore the both cases lead us to a contradiction. ∎

Proposition 3.3. *For every $g^\wedge \in M^*$ the equality $\sigma^*(g^\wedge) = g - \bigcap_{i=1}^n K_i^*$ is valid.*

Proof. Since the equality $[g, x] = [g, \sum_{i=1}^n x_i] = \sum_{i=1}^n [g, x_i] = [g^\wedge, X]$ holds for all $x \in \text{dom}\, \sigma$, $X = (x_1, \ldots, x_n) \in \sigma(x)$ and every $g \in E'$, then $g \in \sigma^*(g^\wedge)$, $\forall g \in E'$. From Theorem 3.1 it follows that $\sigma^*(0) = -\bigcap_{i=1}^n K_i^*$, then using the superlinearity of the dual mapping σ^* we obtain the relations $\sigma^*(g^\wedge) = \sigma^*(g^\wedge + 0) \supset \sigma^*(g^\wedge) + \sigma^*(0) \supset g - \bigcap_{i=1}^n K_i^*$. These inclusions imply that $\sigma^*(g^\wedge) \neq \emptyset$ for every $g^\wedge \in M^*$. If $f \in \sigma^*(g^\wedge)$ then (see Theorem 3.1) $g - f \in K_i^*$, $i = 1, \ldots, n$, and hence $f \in g - \bigcap_{i=1}^n K_i^*$. ■

Corollary 3.2. $M^* \subset \mathcal{K}$.

Indeed, if $G \in M^* = D$ then there exists g such that $G = g^\wedge$. Since $\sigma^*(g^\wedge)$ is nonempty it follows that $G \in \text{dom}\, \sigma^* = \mathcal{K}$.

Corollary 3.3. *If* $g^\wedge \in M^*$, $G \in K^*$ *then* $\sigma^*(g^\wedge + G) = g + \sigma^*(G)$.

Proof. As $G \in K^*$ then Corollary 3.1 yields $\sigma^*(G) \neq \emptyset$. Since $g \in \sigma^*(g^\wedge)$ and the mapping σ^* is superlinear then $\sigma^*(g^\wedge + G) \supset \sigma^*(g^\wedge) + \sigma^*(G) \supset g + \sigma^*(G)$. We now prove the opposite inclusion. If $f \in g + \sigma^*(G)$, then $f - g \in \bigcap_{i=1}^n (g_i - K_i^*)$. The last inclusion is equivalent to the following: $f \in g + g_i - K_i^*$, $i = 1, 2, \ldots, n$, i.e. $f \in \bigcap_{i=1}^n (g + g_i - K_i^*) = \sigma^*(g^\wedge + G)$. ■

The following theorem provides us with the explicit form of the effective domain of the dual mapping σ^*.

Theorem 3.2. *The cone* $\mathcal{K} = \text{dom}\, \sigma^*$ *has the form* $\mathcal{K} = K^* + M^*$.

Proof. From Corollaries 3.1 and 3.2 it follows that $K^* \subset \mathcal{K}$ and $M^* \subset \mathcal{K}$. Since $\mathcal{K}$ is a convex cone, then $K^* + M^* \subset \mathcal{K}$. Conversely, let an element $G = (g_1, \ldots, g_n) \in \mathcal{K}$ and let $f \in \sigma^*(G) = \bigcap_{i=1}^n (g_i - K_i^*)$. Then $f \in g_i - K_i^*$, $i = 1, 2, \ldots, n$, hence

$$g_i \in f + K_i^*, \qquad i = 1, \ldots, n. \tag{3.5}$$

Due to Proposition 3.2, an element $f^\wedge = (f, f, \ldots, f)$ belongs to M^*. Then it follows from (3.5) that $G \in f^\wedge + K^* \subset M^* + K^*$. ■

3.4. Closedness of $\mathcal{K}$ for $n = 2$

The cone $\mathcal{K}$ is not necessarily closed. We describe conditions which guarantee that $\mathcal{K}$ is closed only for $n = 2$. We need the following Lemma.

Lemma 3.1. *Let* $n = 2$. *Then*

$$\mathcal{K} = \{(h_1, h_2) : h_1 - h_2 \in K_1^* - K_2^*\}.$$

Proof. Let $\mathcal{K}_0 = \{(h_1, h_2) : h_1 - h_2 \in K_1^* - K_2^*\}$. First we show that $\mathcal{K} \subset \mathcal{K}_0$. Let $(h_1, h_2) \in \mathcal{K}$. Since $\mathcal{K} = M^* + K^* = D + (K_1^* \times K_2^*)$ it follows that there exist $f \in E'$ and $l_i \in K_i^*$, $i = 1, 2$ such that $h_1 = f + l_1$, $h_2 = f + l_2$. We have $h_1 - h_2 = l_1 - l_2 \in K_1^* - K_2^*$, hence $(h_1, h_2) \in \mathcal{K}_0$. We have proved that $\mathcal{K} \subset \mathcal{K}_0$. We now prove the opposite inclusion. Let $(h_1, h_2) \in \mathcal{K}_0$. Then there exist $l_1 \in K_1$ and $l_2 \in K_2$ such that $h_1 - h_2 = l_1 - l_2$. Let $f := h_1 - l_1 = h_2 - l_2$. Then $h_1 = f + l_1$, $h_2 = f + l_2$, hence $(h_1, h_2) = (f, f) + (l_1, l_2) \in D + (K_1^* \times K_2^*) = \mathcal{K}$. ■

Theorem 3.3. *Let $n = 2$. Then the cone $\mathcal{K}$ is closed if and only if the cone $K_1^* - K_2^*$ is closed.*

Proof. Let $K_1^* - K_2^*$ be closed. Let $(h_1^k, h_2^k) \in \mathcal{K}$, $k = 1, \ldots$ and let $(h_1^k, h_2^k) \to (h_1, h_2)$. It follows from Lemma **??** that $h_1^k - h_k^2 \in K_1^* - K_2^*$. Hence $\lim_k h_1^k - h_k^2 = h_1 - h_2 \in K_1^* - K_2^*$. Applying again Lemma 3.1 we conclude that $(h_1, h_2) \in \mathcal{K}$.

Now assume that $K_1^* - K_2^*$ is not closed. Then we can find a sequence $l^k \in K_1^* - K_2^*$, such that there exists $l := \lim_k l^k$ and $l \notin K_1^* - K_2^*$. Let $g_i^k \in K_i^*$, $i = 1, 2$ be sequences such that $\lim_k g_i^k = 0$. Consider sequences $h_1^k = g_1^k + l^k$ and $h_2^k = g_2^k$, $k = 1, \ldots$. Since $g_1^k - g_2^k \in K_1^* - K_2^*$, $l^k \in K_1^* - K_2^*$ and $K_1^* - K_2^*$ is a cone it follows that $h_1^k - h_k^2 = g_1^k - g_2^k + l_k \in K_1^* - K_2^*$. Hence $(h_1^k, h_2^k) \in \mathcal{K}_0 = \mathcal{K}$. We have $(h_1^k, h_k^2) \to (l, 0)$. Since $l - 0 = l \notin K_1^* - K_2^*$ it follows that $(l, 0) \notin \mathcal{K}_0 = \mathcal{K}$. Hence $\mathcal{K}$ is not closed. ■

3.5. Dual to the decomposition mapping in the case when the cone L is normal

Recall the following well-known definition (see, for example, [7]): A cone $K \subset E$ is called normal if there exists $m > 0$ such that $0 \leq_K x \leq_K y$ implies $\|x\| \leq m\|y\|$. It is well known that if K is a normal cone then K^* is generating: $K^* - K^* = E'$ (see, for example, [7]).

Theorem 3.4. *If the cones $K_1, K_2, \ldots, K_n$ in E are such that $\sum_{i=1}^n K_i = L$ is a normal cone, then*

$$K^* + M^* = (E^n)'.$$

Proof. Take an arbitrary element $G = (g_1, \ldots, g_n) \in (E^n)'$. Since L is normal, then the conjugate cone L^* is a generating cone. It is follows from this that each finite subset of E' is bounded from below. In particular, for the set $\{g_1, \ldots, g_n\} \subset E'$ there exists an element $h \in E'$ such that $g_i \geq_{L^*} h$, $i = 1, 2, \ldots, n$. In view of $L^* = \bigcap_{i=1}^n K_i^*$ we obtain $g_i - h \in K_i^*$ for all $i = 1, \ldots, n$ which is equivalent to $h \in \bigcap_{i=1}^n (g_i - K_i^*)$. In view of Theorem 3.1 we have $h \in \sigma(G)$. Therefore for every $G = (g_1, \ldots, g_n) \in (E^n)'$ the set $\sigma^*(G) \neq \emptyset$ and $\operatorname{dom} \sigma^* := \mathcal{K} = (E^n)'$, but $\mathcal{K} = K^* + M^*$, which completes the proof. ■

Proposition 3.4. *If* $\sum_{i=1}^n K_i = L$ *is a normal cone in* E *then the decomposition mapping* σ *is bounded, that is, there exists a constant* $C > 0$ *such that* $\|X\| \le C\|x\|$ *for each* $x \in L$ *and* $X \in \sigma(x)$.

Proof. Since L is a normal cone it follows that there exists a constant $m > 0$ such that the inequalities $x \ge_L y \ge_L 0$ imply $\|x\| \ge m\|y\|$. Let $x \in L$ and $X = (x_1, \dots, x_n) \in \sigma(x)$. For each $j = 1, \dots, n$ we have $\sum_{i \ne j} x_i \in \sum_{i \ne j} K_i \subset L$, hence $x - x_j \in L$. We also have $x_j \in K_j \subset L$. This means that $x \ge_L x_j \ge_L 0$, hence $\|x\| \ge m\|x_j\|$, $j = 1, \dots, n$. Since $X = \sum_{j=1}^n \|x_j\|$ we get $\|X\| = \sum_{j=1}^n \|x_j\| \le \frac{n}{m}\|x\| = C\|x\|$, where $C = n/m$. ■

4. A Support Function to the Decomposition Mapping σ

In this section we will study the properties of the decomposition mapping $\sigma_{K_1,\dots,K_n} \equiv \sigma$, using the methods of subdifferential calculus.

For every $G \in (E^n)'$ consider the function $p_G : E \to \bar{\mathbb{R}}$ defined by

$$p_G(x) = \inf_{X \in \sigma(x)} [G, X] \qquad (x \in E).$$

(We assume that the infimum of the empty set is equal to $+\infty$. We also assume that $+\infty + (-\infty) = +\infty$.)

The function p_G is called the support function to the decomposition mapping σ corresponding to the linear function G. Let

$$q_G(x) \equiv q_{G,K_1,\dots,K_n}(x) = \sup \left\{ \sum_{i=1}^n [g_i, x_i] : \sum_{i=1}^n x_i = x : x_i \in K_i,\ i = 1, \dots, n \right\}.$$

Then

$$q_{G,K_1,\dots,K_n}(x) = -p_{G,-K_1,\dots,-K_n}(-x).$$

It follows from this equality that we do not need to specially study the function q_G.

Proposition 4.1. *The function* p_G *is sublinear.*

Proof. Let $x, y \in \operatorname{dom} \sigma$. Then $x + y \in \operatorname{dom} \sigma$ also. Since the mapping σ is superlinear, we have

$$\begin{aligned} p_G(x+y) &= \inf_{Z \in \sigma(x+y)} [G, Z] \le \inf_{Z \in \sigma(x)+\sigma(y)} [G, Z] \\ &= \inf_{X \in \sigma(x), Y \in \sigma(y)} ([G, X] + [G, Y]) \\ &= \inf_{X \in \sigma(x)} \inf_{Y \in \sigma(y)} ([G, X] + [G, Y]) \\ &= \inf_{X \in \sigma(x)} [G, X] + \inf_{Y \in \sigma(y)} [G, Y] = p_G(x) + p_G(y). \end{aligned}$$

If at least one of the elements x, y does not belong to dom σ then $p_G(x)+p_G(y) = +\infty$, so $p_G(x+y) \le p_G(x)+p_G(y)$ in this case as well. Thus p is subadditive. It is easy to check that p is positively homogeneous. ■

Assume that $p_G(0) = -\infty$. Then for all $x \in \operatorname{dom}\sigma = \sum_{i=1}^n K_i$ we have $p_G(x) = p_G(x+0) \le p_G(x) + p_G(0) = -\infty$ so it is important to describe G such that $p_G(0) > -\infty$. For such G we have $p_G(0) = 0$.

Proposition 4.2. *The equality $p_G(0) = 0$ holds if and only if $G \in cl\,\mathcal{K}$.*

Proof. Since $p_G(0) = \inf_{X\in\sigma(0)}[G, X]$ it follows that $p_G(0) = 0$ if and only if $[G, X] \ge 0$ for all $X \in \sigma(0)$. The set

$$\sigma(0) = \{X = (x_1, \dots, x_n) : \sum_i x_i = 0,\ x_1 \in K_1, \dots, x_n \in K_n\}$$

coincides with the cone $M \cap K$, hence $p_G(0) = 0$ if and only if $G \in (M \cap K)^*$. However

$$(M \cap K)^* = \text{cl}\,(M^* + K^*) = \text{cl}\,(D + K^*) = \text{cl}\,\mathcal{K}.$$ ■

Proposition 4.2. *For every $G \in \mathcal{K}$ the equality dom σ = dom p_G holds.*

Proof. Since $G \in \mathcal{K}$ it follows that there exist $f \in E'$ and $l_i \in K_i^*$ such that $G = f^\wedge + (l_1, \dots, l_n)$. Let $x \in \operatorname{dom}\sigma = \sum_{i=1}^n K_i$ and $X = (x_1, \dots, x_n) \in \sigma(x)$ then

$$[G, X] = \sum_{i=1}^n [f, x_i] + \sum_{i=1}^n [l_i, x_i] = [f, x] + \sum_{i=1}^n [l_i, x_i].$$

Note that $[l_i, x_i] \ge 0$ for all i, therefore $[G, X] \ge [f, x]$. Hence

$$p_G(x) = \inf_{X\in\sigma(x)} [G, X] \ge f(x) > -\infty.$$

It is clear that $p_G(x) \le [G, X] < +\infty$. We have proved that dom $\sigma \subset$ dom p_G. If $x \notin \sum_{i=1}^n K_i = \operatorname{dom}\sigma$ then $p_G(x) = +\infty$ (because the infimum over the empty set is equal to zero). Hence dom σ = dom p_G. ■

5. The Additivity of the Decomposition Mapping

In this section we study conditions that provide the additivity of the decomposition mapping σ. In order to give a description of these conditions we need to extend many notions of the theory of ordered space for spaces that are equipped with several preorders.

5.1. Riesz interpolation property in a space with two cones

Consider an ordered Banach space with the cone of positive elements K. Consider now the family of cones $K_1, \dots K_n$ with an arbitrary $n > 1$ where $K_i = K$ for each $i = 1, \dots, n$. It can be shown that the decomposition mapping $\sigma_{K_1,\dots,K_n}$ is additive if and only if the space (E, K) possesses the Riesz interpolation property. (See Theorem 5.1, where a more general result is proved.) Our goal is to generalize this result for the case of different cones $K_1, \dots, K_n$. For this purpose we need to generalize the notions of vector lattice and Riesz interpolation property for a space with different cones. In the classical situation where a cone K can be repeated n times with an arbitrary n we have different equivalent definitions of vector lattice. One of them is given in terms of arbitrary finite sets and the other in terms of sets that contain only two elements. If we have different cones $K_1, \dots, K_n$ then the situation is different: we can consider the supremum and the infimum only finite sets that contain exactly n elements with the given n. A similar remark can be made with respect to the Riesz interpolation property, the Riesz decomposition property and the double partition lemma.

We will start with the Riesz interpolation property.

Let pointed cones $K_1, \dots, K_n$ in a vector space E be given. Each of them induces its own order relation $\geq_i$ $(i = 1, \dots, n)$ on E . The space E with cones $K_1, \dots K_n$ is denoted by $E = (E; K_1, \dots, K_n)$.

Remark 5.1. If the cones $K_1, \dots, K_n$ coincide and are equal to a cone K, we will use either notation $(E, K_1, \dots K_n)$ with $K_i = K$, $i = 1, \dots, n$ or notation (E, K) (if the latter is used, it is assumed that the number n is known).

For the sake of simplicity we consider the case $n = 2$. Then we will show how the definitions and results obtained can be extended for an arbitrary n.

Definition 5.1. Consider a space $(E; K_1, K_2)$ and let $L = K_1 + K_2$. We say that the space $(E; K_1, K_2)$ possesses the Riesz interpolation property if for for every four elements $x_1, x_2, y_2, y_2 \in E$, satisfying the inequalities

$$y_1 \geq_{K_1} x_1, \quad y_2 \geq_{K_2} x_2, \quad y_1 \geq_L x_2, \quad y_2 \geq_L x_1, \tag{5.1}$$

there exists an "intermediate" element $c \in E$ such that

$$y_1 \geq_{K_1} c \geq_{K_1} x_1, \quad \text{and} \quad y_2 \geq_{K_2} c \geq_{K_2} x_2, \tag{5.2}$$

We will also call this property "the Riesz interpolation property in E with respect to cones K_1, K_2".

Remark 5.2. It follows from (5.2) that $y_1 \geq_L c \geq_L x_2$ and $y_2 \geq_L c \geq_L x_1$. Indeed, if there exists an element $c \in E$ such that

$$y_1 \geq_{K_1} c \geq_{K_1} x_1 \quad \text{and} \quad y_2 \geq_{K_2} c \geq_{K_2} x_2,$$

then $c - x_1 \in K_1 \subset L,\ y_2 - c \in K_2 \subset L$. Since $c - x_2 \in K_2 \subset L,\ y_1 - c \in K_1 \subset L$ then $x_2 \leq_L c$ and $y_1 \geq_L c$.

Note that

$$K_1 + K_1 = K_1, \quad K_2 + K_2 = K_2, \quad K_1 + K_2 = L, \quad K_2 + K_1 = L.$$

Hence (5.1) can be expressed in the form

$$y_j - x_i \in K_i + K_j, \qquad i, j = 1, 2.$$

We will use the definition of an interval $\langle x, y\rangle_H$ with respect to a cone $H \subset E$. Recall that

$$\langle x, y\rangle_H = (x + H) \bigcap (y - H), \qquad (x, y \in E,\ y \geq_H x).$$

We can express Definition 5.1 in terms of intervals: if x_1, x_2, y_1, y_2 are four elements such that $y_j - x_i \in K_i + K_j,\ i, j = 1, 2$, then

$$\langle x_1, y_1\rangle_{K_1} \cap \langle x_2, y_2\rangle_{K_2} \neq \emptyset. \tag{5.3}$$

It follows from (5.3) and Remark 5.2 that

$$\bigcap_{i,j=1,2} \langle x_i, y_j\rangle_{K_i + K_j} \neq \emptyset.$$

Remark 5.3. To check the Riesz interpolation property with respect to the cones K_1, K_2 in the space $E = (E; K_1, K_2)$ it is sufficient to verify that an intermediate element exists under the additional hypothesis: $x_1, x_2 \in L$. Indeed, assume that the Riesz interpolation property holds for all four-tips $\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2$ such that $\tilde{y}_j - \tilde{x}_i \in K_i + K_j$ and $\tilde{x}_1, \tilde{x}_2 \in L$. Let $x_i, y_j \in E,\ i, j = 1, 2$ and $y_j - x_i \in K_i + K_j\ (i, j = 1, 2)$. Let $z = x_1 + x_2 - y_1$. Consider four elements $\tilde{x}_1 = x_1 - z,\ \tilde{x}_2 = x_2 - z,\ \tilde{y}_1 = y_1 - z,\ \tilde{y}_2 = y_2 - z$. We have

$$\tilde{x}_1 := x_1 - z = y_1 - x_2 \in L, \quad \tilde{x}_2 := x_2 - z = y_1 - x_1 \in K_1 \subset L.$$

Therefore the Riesz interpolation property holds for elements $x_i - z,\ y_j - z\ (i, j = 1, 2)$ so an element $\tilde{c}$ exists such that

$$\tilde{y}_1 \geq_{K_1} \tilde{c} \geq_{K_1} \tilde{x}_1, \quad \text{and} \quad \tilde{y}_2 \geq_{K_2} \tilde{c} \geq_{K_2} \tilde{x}_2.$$

Let $c = \tilde{c} + z$. Then

$$y_1 \geq_{K_1} c \geq_{K_1} x_1, \quad \text{and} \quad y_2 \geq_{K_2} c \geq_{K_2} x_2.$$

We have proved that the Riesz interpolation property holds in $(E; K_1, K_2)$.

5.2. Riesz decomposition property and double partition lemma in a space with two cones

Definition 5.2. We say that the space $E = (E; K_1, K_2)$ possesses the Riesz decomposition property if

$$\langle x_1 + x_2, y_1 + y_2 \rangle_{K_1+K_2} = \langle x_1, y_1 \rangle_{K_1} + \langle x_2, y_2 \rangle_{K_2}$$

for all $x_1, y_1 \in K_1,\ x_2, y_2 \in K_2$ such that $y_1 \geq_{K_1} x_1,\ y_2 \geq_{K_2} x_2$.

Consider a space $(E; K_1, K_2)$. Let $x_1, y_1 \in K_1, x_2, y_2 \in K_2$ and $y_1 \geq_{K_1} x_1$, $y_2 \geq_{K_2} x_2$. Then it is easy to check that

$$\langle x_1 + x_2, y_1 + y_2 \rangle_{K_1+K_2} \supset \langle x_1, y_1 \rangle_{K_1} + \langle x_2, y_2 \rangle_{K_2}. \tag{5.4}$$

In view of (5.4), the Riesz decomposition property is equivalent to the following:

$$\langle x_1 + x_2, y_1 + y_2 \rangle_{K_1+K_2} \subset \langle x_1, y_1 \rangle_{K_1} + \langle x_2, y_2 \rangle_{K_2}.$$

This means that each element z such that

$$x_1 + x_2 \leq_{K_1+K_2} z \leq_{K_1+K_2} y_1 + y_2$$

can be represented as the sum $z = z_1 + z_2$ with

$$x_1 \leq_{K_1} z_1 \leq_{K_1} y_1 \quad \text{and} \quad x_2 \leq_{K_2} z_2 \leq_{K_2} y_2.$$

Remark 5.4. It is easy to check that the Riesz decomposition property with respect to cones K_1, K_2 is equivalent to the fact that the equality

$$\langle 0, x + y \rangle_{K_1+K_2} = \langle 0, x \rangle_{K_1} + \langle 0, y \rangle_{K_2}$$

holds for all $x \in K_1,\ y \in K_2$.

Consider a space $(E; K_1, K_2)$ with two cones K_1 and K_2. Consider two arbitrary elements $y_1, z_1 \in K_1$ and two arbitrary elements $y_2, z_2 \in K_2$. Let

$$x_1 = y_1 + z_1,\ x_2 = y_2 + z_2 \quad \text{and} \quad y = y_1 + y_2,\ z = z_1 + z_2. \tag{5.5}$$

and $x = x_1 + x_2$. Then $x_1 \in K_1$, $x_2 \in K_2$ and $x \in L$. We can also represent x as the sum of two elements from L: $x = y + z$. We say that the *double partition lemma* holds in the space $E = (E; K_1, K_2)$, if the reverse assertion holds: if for an element $x \in L$ the following equalities hold:

$$x = x_1 + x_2, \quad \text{where} \quad x_1 \in K_1, x_2 \in K_2$$

and

$$x = y + z, \quad \text{where} \quad y, z \in L,$$

then elements $y_1, z_1 \in K_1,\ y_2, z_2 \in K_2$ exist such that each x_i $(i = 1, 2)$ can be represented in the form $x_i = y_i + z_i$ and also $y = y_1 + y_2$ and $z = z_1 + z_2$.

Remark 5.5. Let $K_1 = K_2 := K$. Then the Riesz interpolation property holds in the space $(E; K_1, K_2)$ if the ordered space possesses the "classical" Riesz interpolation property. The same conclusion can be made with respect to the Riesz decomposition property and the double partition lemma.

5.3. Additivity of the decomposition mapping

The decomposition mapping $\sigma_{K_1,K_2} = \sigma : E \to 2^{E^2}$ with respect to cones K_1 and K_2 in the space $E = (E; K_1, K_2)$ is expressed in the following way:

$$\sigma(x) = \{X = (x_1, x_2) \in K_1 \times K_2 : x_1 + x_2 = x\} \quad (x \in E).$$

Recall that $\operatorname{dom} \sigma = L := K_1 + K_2$. We are interested in conditions that guarantee the additivity of the decomposition mapping. The following theorem claims that all above definitions are equivalent and that each of them is equivalent to the required additivity.

Theorem 5.1. *The followings statements are equivalent:*

(1) *The space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property;*

(2) *The space $E = (E; K_1, K_2)$ possesses the Riesz decomposition property;*

(3) *The double partition Lemma takes place in the space $E = (E; K_1, K_2)$;*

(4) *The decomposition mapping $\sigma_{K_1,K_2} = \sigma : E \to 2^{E^2}$ is additive, i.e. if $x, y \in L$ then $\sigma(x + y) = \sigma(x) + \sigma(y)$. (Here $L = K_1 + K_2$.)*

Proof. $1 \implies 2$. In view of Remark 5.4 it is enough to show that

$$\langle 0, x_1 + x_2 \rangle_{K_1+K_2} = \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2}.$$

Let $x_1 \in K_1,\ x_2 \in K_2$ and $y \in L$ and let $x_1 + x_2 \geq_L y$. We can express these conditions in the following way:

$$y \geq_{K_1} y - x_1, \quad x_2 \geq_{K_2} 0, \quad y \geq_L 0, \quad x_2 \geq_L y - x_1.$$

Let us apply the Riesz interpolation property to these inequalities, and find an intermediate element, i.e. an element $c \in E$ such that

$$y \geq_{K_1} c \geq_{K_1} y - x_1, \quad x_2 \geq_{K_2} c \geq_{K_2} 0. \tag{5.6}$$

Let $y_1 = y - c$ and $y_2 = c$. Then (5.6) yields

$$y_1 \in K_1, \quad y_2 \in K_2, \quad x_1 \geq_{K_1} y_1, \quad x_2 \geq_{K_2} y_2.$$

We have also $y = y_1 + y_2$, i.e. y_1 and y_2 form the required decomposition and

$$y_1 \in \langle 0, x_1 \rangle_{K_1}, \quad y_2 \in \langle 0, x_2 \rangle_{K_2}.$$

$2 \implies 3$. Let an element $x \in E$ be such that $x = x_1 + x_2$, where $x_1 \in K_1,\ x_2 \in K_2$ and $x = y + z$, where $y, z \in L$. Then $x_1 + x_2 \geq_L y \geq_L 0$. By the Riesz decomposition property elements $y_1 \in K_1,\ y_2 \in K_2$ exist such that

$$x_1 \geq_{K_1} y_1,\ x_2 \geq_{K_2} y_2,\ y = y_1 + y_2.$$

Let $z_1 = x_1 - y_1,\ z_2 = x_2 - y_2$. We have $z_1 \in K_1,\ z_2 \in K_2,\ x_1 = y_1 + z_1,\ x_2 = y_2 + z_2$ and

$$z_1 + z_2 = x_1 + x_2 - (y_1 + y_2) = x - y = z.$$

Therefore the elements y_1, y_2, z_1, z_2 are as desired.

$3 \implies 4$. Let $y, z \in L$. Since the decomposition mapping σ is superlinear, then $\sigma(y + z) \supset \sigma(y) + \sigma(z)$. Let us prove the opposite inclusion. Let $X = (x_1, x_2) \in \sigma(y + z)$, then by the definition of the mapping σ we have

$$x_1 \in K_1,\ x_2 \in K_2 \quad \text{and} \quad x_1 + x_2 = y + z.$$

In view of the double partition Lemma there exist elements $y_1, z_1 \in K_1,\ y_2, z_2 \in K_2$, such that every $x_i\ (i = 1, 2)$ can be represented in the form $x_1 = y_1 + z_1,\ x_2 = y_2 + z_2$ and $y = y_1 + y_2,\ z = z_1 + z_2$. It means that

$$Y = (y_1, y_2) \in \sigma(y), \quad Z = (z_1, z_2) \in \sigma(z)$$

and $X = Y + Z$, i.e. $X \in \sigma(y) + \sigma(z)$.
$4 \implies 3$. It can be proved by an argument similar to that in the proof of $3 \implies 4$.

$3 \implies 1$. Let elements $a_1, a_2, b_1, b_2 \in E$ satisfy the inequalities

$$b_1 \geq_{K_1} a_1,\ b_1 \geq_L a_2,\ b_2 \geq_{K_2} a_2,\ b_2 \geq_L a_1.$$

Let $u_1 = b_1 - a_1 \in K_1,\ u_2 = b_2 - a_2 \in K_2,\ v_1 = b_2 - a_1 \in L,\ v_2 = b_1 - a_2 \in L.$ Then $u_1 + u_2 = v_1 + v_2$. From the double partition Lemma it follows that elements $y_1, z_1 \in K_1$ and $y_2, z_2 \in K_2$ exist such that

$$u_1 = y_1 + z_1 \in K_1,\ u_2 = y_2 + z_2 \in K_2 \quad \text{and} \quad v_1 = y_1 + y_2 \in L,\ v_2 = z_1 + z_2 \in L.$$

The element $c = a_1 + y_1$ is an intermediate between a_i and b_j $(i, j = 1, 2)$. Indeed, $u_1 = b_1 - a_1 \geq_{K_1} y_1$ yields $b_1 \geq_{K_1} a_1 + y_1 \geq_{K_1} a_1$,and $v_1 = b_2 - a_1 \geq_L y_1$ implies that $b_2 \geq_L a_1 + y_1 \geq_L a_1$. Since the equality $v_1 = y_1 + y_2$ yields $b_2 - y_2 = a_1 + b_1$, then from $u_2 = b_2 - a_2 \geq_L y_2$ and $u_1 = b_1 - a_1 \geq_L y_1$ we obtain $b_1 \geq_L a_1 + y_1 = b_2 - y_2 \geq_L a_2$. Finally, the inequality $u_2 = b_2 - a_2 \geq_{K_2} y_2$ yields $b_2 \geq_{K_2} b_2 - y_2 = a_1 + y_1 \geq_{K_2} a_2$. ■

5.4. Examples

First we will give an example of cones such that the decomposition mapping σ is nonadditive.

Example 5.1. Let the following cones be given in the space $E = \mathbb{R}^2$: the positive orthant and the ray passing through the point $T = (-1, 1) \in \mathbb{R}^2$, i.e.

$$K_1 = \{X = (u, v) \in \mathbb{R}^2 \ :\ u \geq 0, v \geq 0\}$$

$$K_2 = \{X = (u, v) \in \mathbb{R}^2 \ :\ u = -\lambda, v = \lambda, \lambda \geq 0\}.$$

Let $x = (1, 0) \in K_1$, $y = (-1, 1) \in K_2$, then $z := x + y = (0, 1)$. An easy calculation shows that

$$\begin{aligned} \sigma(x) &= \{(x, 0)\}, \qquad \sigma(y) = \{(0, y)\}. \\ \sigma(z) &= \{Z = ((\alpha, 1 - \alpha), (-\alpha, \alpha)) : \alpha \in [0, 1]\}. \\ \sigma(x) + \sigma(y) &= \{(x, 0) + (0, y)\} = \{(1, 0) + (-1, 1)\} = \{(0, 1)\} \\ &= \{(\alpha, 1 - \alpha), (-\alpha, \alpha)) : \alpha = 0)\}. \end{aligned}$$

Thus $\sigma(z) \neq \sigma(x) + \sigma(y)$.

Let K_1 be a cone and K_2 be a subcone of K_1. Recall, that K_2 is called a *face* of K_1, if the inclusions $x, y \in K_1$ and $x + y \in K_2$ imply $x, y \in K_2$.

Theorem 5.2. *Let the double partition Lemma take place in the space $E = (E, K_1)$ and let a cone K_2 be a face of the cone K_1. Then the double partition Lemma is valid in the space $E = (E; K_1, K_2)$.*

Proof. Let $z_1 + z_2 = x + y$, where $x, y \in K_1 + K_2,\ z_1 \in K_1,\ z_2 \in K_2$. Since the double partition Lemma takes place in the space $E = (E; K_1)$, then there exist elements $x_1, x_2, y_1, y_2 \in K_1$ such that $z_1 = x_1 + y_1, \quad z_2 = x_2 + y_2, \quad x = x_1 + x_2, \quad y = y_1 + y_2$. As $x_2, y_2 \in K_1$, $z_2 = x_2 + y_2 \in K_2$ and the cone K_2 is a face of the cone K_1, then $x_2, y_2 \in K_2$, i.e. the double partition Lemma holds in the space $E = (E; K_1; K_2)$ with respect to the cones K_1 and K_2. ■

Theorem 5.3. *Let the space $E = (E; H)$ possess the Riesz interpolation property, and let cones K_1, K_2 be faces of the cone H. Then the space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property.*

Proof. Let $L = K_1 + K_2$ and elements $x_1, x_2, y_1, y_2 \in E$ satisfy the following relations:

$$y_1 \geq_{K_1} x_1, \quad y_1 \geq_L x_2, \quad y_2 \geq_{K_2} x_2, \quad y_2 \geq_L x_1.$$

Since $K_1, K_2, L \subset H$ then $y_i \geq_H x_j,\ i, j = 1, 2$. As the space $E = (E; H)$ possesses the Riesz interpolation property, then there exists an element $c \in E$ such that

$$y_i \geq_H c \geq_H x_j,\ i, j = 1, 2,$$

i.e.

$$y_1 - c \in H, \quad y_2 - c \in H, \quad c - x_1 \in H, \quad c - x_2 \in H.$$

It follows from the inequality $y_1 \geq_{K_1} x_1$ that $y_1 - x_1 = (y_1 - c) + (c - x_1) \in K_1$. In the same manner the inequality $y_2 \geq_{K_2} x_2$ implies $y_2 - x_2 = (y_2 - c) + (c - x_2) \in K_2$. As K_1, K_2 are faces of the cone H, we have $y_1 - c,\ c - x_1 \in K_1, \quad y_2 - c,\ c - x_2 \in K_2$, i.e. $y_1 \geq_{K_1} c \geq_{K_1} x_1,\ y_2 \geq_{K_2} c \geq_{K_2} x_2$. Therefore, the space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property. ■

The definitions and results presented above can be easily extended to the case where the number of cones is greater than two. We will consider this only for the Riesz decomposition property. This property in the space $E = (E; K_1, \ldots, K_n)$ can be expressed in the following form: if $x_i \in K_i$ $(i = 1, \ldots, n)$ then

$$\langle 0, x_1 + x_2 + \cdots + x_n \rangle_{K_1 + K_2 + \cdots + K_n} = \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2} + \cdots + \langle 0, x_n \rangle_{K_n}.$$

Lemma 5.1. *Let the Riesz decomposition property hold for the space $(E; K_1, \ldots, K_{n-1})$. Let $K^{(1)} = K_1 + \ldots + K_{n-1}$ and let the Riesz decomposition property hold for the space $(E, K^{(1)}, K_n)$. Then this property also holds for the space $(E; K_1, \ldots, K_n)$.*

Proof. We have for an arbitrary $x_i \in K_i$ $i = 1, \ldots, n-1$:

$$\langle 0, x_1 + x_2 + \cdots + x_{n-1} \rangle_{K_1 + K_2 + \cdots + K_{n-1}} = \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2} + \cdots + \langle 0, x_{n-1} \rangle_{K_{n-1}}$$

and we also have for $y \in K^{(1)}$ and $x_n \in K_n$:

$$\langle 0, y + x_n \rangle_{K^{(1)}+K_n} = \langle 0, y \rangle_{K^{(1)}} + \langle 0, x_n \rangle_{K^n}.$$

Let $y = x_1 + x_2 + \cdots + x_{n-1} \in K^{(1)}$. Since$K^{(1)} + K_n = K_1 + \ldots + K_{n-1} + K_n$ it follows that $\langle 0, y + x_n \rangle_{K^{(1)}+K_n} = \langle 0, x_1 + x_2 + \cdots + x_n \rangle_{K_1+K_2+\cdots+K_n}$ and

$$\langle 0, y \rangle_{K^{(1)}} + \langle 0, x_n \rangle_{K^n} = \langle 0, x_1 + x_2 + \cdots + x_{n-1} \rangle_{K_1+K_2+\cdots+K_{n-1}} + \langle 0, x_n \rangle_{K_n}$$
$$= \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2} + \cdots + \langle 0, x_{n-1} \rangle_{K_{n-1}} + \langle 0, x_n \rangle_{K_n}.$$

Thus the result follows. ■

Using this lemma and induction we can easily extend all results that known for the Riesz decomposition property for the case of two cones, to the case of n cones. Definition of the Riesz interpolation property can be extended to the case of n-cones in a similar manner. We can also define in a similar way what it means for the double partition lemma to hold with respect to n cones and define the additivity of the decomposition mapping in this situation. Using induction it is easy to extend all results that were proved in this section for the case of two cones to the case of n cones.

6. A Vector Lattice with Respect to Several Preorders

6.1. Supremum and infimum in a space with two cones

Let cones $K_1, \ldots, K_n$ be given in a vector space E. Let us introduce a pre-order $\geq_{K_i},\ i = 1, \ldots, n$ on E by means of the cone K_i. As usual we denote this space by $E = (E; K_1, \ldots, K_n)$. Let us introduce the notions of supremum and infimum in the space $E = (E; K_1, \ldots, K_n)$. We will need these notions only for sets of n elements so we give a corresponding definitions only for such subsets of E. Let $(x_1, \ldots, x_n) \subset E = (E; K_1, \ldots, K_n)$.

Definition 6.1. An element $u \in E = (E; K_1, \ldots, K_n)$ is called an *infimum* of the set $\{x_1, \ldots, x_n\}$ with respect to $K_1, \ldots, K_n$, if

(i) $x_i \geq_{K_i} u$ for every $i = 1, 2, \ldots, n$;

(ii) if an element $z \in E$ is such that $x_i \geq_{K_i} z$ for every $i = 1, 2, \ldots, n$, then $u \geq_{K_i} z, \quad i = 1, 2, \ldots, n.$

We will denote an element with properties (i) and (ii) by $u = \text{Inf}\{x_1, \ldots, x_n\}$.

A supremum is defined in a similar way.

Definition 6.2. An element $v \in E = (E; K_1, \ldots, K_n)$ is called a *supremum* of the set $\{x_1, \ldots, x_n\}$ with respect to $K_1, \ldots, K_n$, if

(i) $v \geq_{K_i} x_i$ for every $i = 1, 2, \ldots, n$;

(ii) if an element $z \in E$ is such that $z \geq_{K_i} x_i$ for every $i = 1, 2, \ldots, n$, then $z \geq_{K_i} v,\ i = 1, 2, \ldots, n$.

We will denote an element with properties (i) and (ii) by $v = \mathrm{Sup}\{x_1, \ldots, x_n\}$.

Lemma 6.1. *If all $K_1, \ldots, K_n$ coincide, then the definitions of Inf and Sup coincide with the definitions of ordinary inf and sup for n elements.*

Let us study the properties of these new objects.

Proposition 6.1. *Let*

$$(\bigcap_{i=1}^{n} K_i) \cap (-\bigcap_{i=1}^{n} K_i) = \{0\}. \tag{6.1}$$

Then each set $\{x_1, \ldots, x_n\}$ cannot have more than one infimum and supremum with respect to $(K_1, \ldots, K_n)$.

Proof. Assume that elements u and $u' \neq u$ are infimums of a set $x_1, \ldots, x_n$ with respect to $K_1, \ldots, K_n$. Since $x_i \geq_{K_i} u$ for every i and $u' = \mathrm{Inf}\,(x_1, \ldots, x_n)$ we conclude that $u' \geq_{K_i} u$ for all i. Hence $u' - u \in K_i$ for all i. This means that $u' - u \in \bigcap_{i=1,\ldots,n} K_i$. The same argument shows that $u - u' \in \bigcap_{i=1,\ldots,n} K_i$, i.e. $u' - u \in -\bigcap_{i=1,\ldots,n} K_i$. Since $(\bigcap_{i=1}^{n} K_i) \cap (-\bigcap K_i)_{i=1}^{n} = \{0\}$ it follows that $u = u'$. The same argument shows that a set $(x_1, \ldots, x_n)$ cannot have more than one supremum with respect to $(K_1, \ldots, K_n)$. ■

It is easy to find an example that shows that if condition (6.1) does not hold then a set of n elements can have more than one infimum.

In the rest of the paper we always assume that we consider infimum and supremum only with respect to a system $(K_1, \ldots, K_n)$ of cones such that (6.1) holds. We will now present some simple properties of the infimum and supremum.

Proposition 6.2. *Let $x_i, y_i \in E = (E; K_1, \ldots, K_n),\ i = 1, 2, \ldots, n$ and let there exist Inf$\{x_i\}$ and Sup$\{x_i\}$, Inf$\{y_i\}$ and Sup$\{y_i\}$ with respect to cones $K_1, \ldots, K_n$. Then the following assertions are valid:*

(1) *Sup$\{x_i\} \geq_{K_l}$ Inf$\{x_i\}$, $(l = 1, 2, \ldots, n)$;*

(2) *there exist elements Sup$\{-x_i\}$ and Inf$\{-x_i\}$ and Inf$\{x_i\} = -$Sup$\{-x_i\}$, Sup$\{x_i\} = -$Inf$\{-x_i\}$;*

(3) *for every $z \in E$ there exist Sup$\{x_i + z\}$ and Inf$\{x_i + z\}$ and Inf$\{x_i\} + z =$ Inf$\{x_i + z\}$, Sup$\{x_i\} + z =$ Sup$\{x_i + z\}$;*

(4) *for every* $\lambda > 0$ *there exist elements Inf*$\{\lambda x_i\}$ *and Sup*$\{\lambda x_i\}$ *and* λ*Inf*$\{x_i\} =$ *Inf*$\{\lambda x_i\}$, λ*Sup*$\{x_i\} =$ *Sup*$\{\lambda x_i\}$;

(5) *for every* $\lambda \leq 0$ *there exist Sup*$\{\lambda x_i\}$ *and Inf*$\{\lambda x_i\}$ *and* λ*Inf*$\{x_i\} =$ *Sup*$\{\lambda x_i\}$, λ*Sup*$\{x_i\} =$ *Inf*$\{\lambda x_i\}$;

(6) *if* $x_i \geq_{K_i} y_i,\ i = 1, 2, \ldots, n$, *then Inf*$\{x_i\} \geq_{K_l}$ *Inf*$\{y_i\},\ l = 1, 2, \ldots, n$, *Sup*$\{x_i\} \geq_{K_l}$ *Sup*$\{y_i\},\ l = 1, 2, \ldots, n$.

We omit the simple proof of this proposition.

In general, the operation Inf and Sup do not commute in the sense that Inf(x_1, x_2) is not necessarily equal to Inf(x_2, x_1) and Sup(x_1, x_2) is not necessarily equal to Sup(x_2, x_1). An example can be found in Proposition 6.6.

The operation Inf and Sup with respect to a system of cones can be useful for the description of some objects. We now present an interesting example. Consider a space (E, K_1) where $E = \mathbb{R}^n$ and $K_1 = \mathbb{R}^n_+$. Let $x \in \mathbb{R}^n_{++} = \text{int}\,\mathbb{R}^n_+$. Consider the conic segment $\langle 0, x\rangle_{K_1}$. This is a parallelepiped with 2^n vertices. One of these vertices is zero and one more of the vertices is x. We cannot describe other vertices of $\langle 0, x\rangle$ in terms of the order relation generated by the cone K_1.

We will now show that Inf operation allows, by choosing appropriate cones to "catch" other vertices of the parallelepiped $\langle 0, x\rangle$. Moreover for each of the vertices x_j there exists a cone H_j such that $x_j =$ Inf $(x, 0)$ with respect to the pair of cones (K_1, H_j).

Let $E = \mathbb{R}^n$ be the Euclidian space and $K_1 = \mathbb{R}^n_+$ be the positive orthant. Let $x = (x^1, \ldots, x^n) \in E$ be an element with positive coordinates: $x^i > 0,\ i = 1, 2, \ldots, n)$. Then the set

$$\langle 0, x\rangle_{K_1} = \{y \in \mathbb{R}^n\ :\ x \geq_{K_1} y \geq_{K_1} 0\}$$

is an n-dimensional parallelepiped.

Let $k = 2^n$ be the number of vertices of $\langle 0, x\rangle_{K_1}$ and let $x_j = (x_j^1, \ldots, x_j^n)$ $(j = 1, 2, \ldots, k)$ be these vertices. Let us introduce the index sets $I = \{1, 2, \ldots, n\}$ and let $I_j = \{i \in I\ :\ x_j^i = 0\},\ j = 1, 2, \ldots, k$. Observe, that if $i \notin I_j$ then $x^i = x_j^i \quad (j \in I)$. Consider the cone:

$$H_j = \{(y^1, \ldots, y^n) \in \mathbb{R}^n\ :\ y^i \in \mathbb{R}_+, \quad i \in I_j\}.$$

The following assertion holds:

Proposition 6.3. *The vertex* $x_j\ (j = 1, \ldots, k)$ *of the parallelepiped* $\langle 0, x\rangle_{K_1}$ *can be calculated as Inf*$\{x; 0\}$ *in the space* $(E; K_1, H_j)\ \ j = 1, \ldots, k$.

Proof. Since $x_j = (x_j^1, \ldots, x_j^n) \in \langle 0, x\rangle_{K_1}\ (j = 1, 2, \ldots, k)$ it follows that $x \geq_{K_1} x_j,\ j = 1, 2, \ldots, k$. From the construction of the set I_j and the cone H_j it is easy to see that $-x_j \in H_j$, i.e. $0 \geq_{H_j} x_j$.

Now let an element $z = (z^1, \dots, z^n) \in E$ be such that $x \geq_{K_1} z, \qquad 0 \geq_{H_j} z$. Then $x^i \geq z^i,\ i \in I$ and $z^i \in -\mathbb{R}_+,\ i \in I_j$. Since $x^i = x^i_j$ for $i \notin I_j$ and $x^i_j \geq 0\ (i \in I)$ then $x^i_j \geq z^i\ (i \in N)$, i.e. $x_j \geq_{K_1} z$. As $x^i_j - z^i \geq 0\ (i \in I_j)$, then $x_j - z = (x_j - z, x_j - z, \dots, x_j - z) \in H_j$. Thus we have proved that $x_j \geq_{K_1} z,\ x_j \geq_{H_j} z$. This means that $x_j = \operatorname{Inf}\{x; 0\}$ (with respect to the pair of cones K_1, H_j). ■

In the following, unless otherwise indicated, we will consider the case where the number of cones is equal to two.

Let cones K_1 and K_2 be given in a space E. We say that a pair of cones K_1 and K_2 *generates* a space E, if $E = K_1 - K_2$. It is clear that $E = K_1 - K_2$ if and only if $E = K_2 - K_1$. A set $\Omega \subset E = (E; K_1, K_2)$ is called *bounded from above (below)* if an element $u \in E$ exists such that $u \geq_{K_i} x$ ($x \geq_{K_i} u$, respectively), $i = 1, 2$ for all $x \in \Omega$.

Observe that the following simple proposition holds.

Proposition 6.4.

(1) *If for each $x \in E$ the two-element subset $\{0, x\}$ is bounded from below then a pair of cones K_1 and $-K_2$ generates the space E.*

(2) *If for each $x \in E$ the subset $\{0, x\}$ is bounded from above then a pair of cones K_1, K_2 generates the space E.*

Proof. We prove only the first part of proposition. Let $x \in E$. If the two-element set $\{x, 0\}$ is bounded from below, then there exists $u \in E$ such that $x \geq_{K_1} u, \qquad 0 \geq_{K_2} u$, i.e. $x - u \in K_1,\ u \in -K_2$. Then the element x can be represented in the form $x = (x-u) + u \in K_1 + K_2 = K_1 - (-K_2)$ and since x is an arbitrary element, we obtain $E = K_1 - (-K_2)$. ■

Proposition 6.5. *Assume that the cone $H := K_1 \cap K_2$ is generating. Then for each $x, y \in E$ the set $\{x, y\}$ is bounded from above and from below.*

Proof. Let $x, y \in E$. Since $E = H - H$ it follows that there exists $x_1, y_1 \in H,\ x_2, y_2 \in H$ such that $x = x_1 - x_2,\ y = y_1 - y_2$. This means that $x \leq_H x_1,\ y \leq_H y_1$. We have $x \leq_H x_1 \leq_H x_1 + y_1$ and $y \leq_H y_1 \leq_H x_1 + y_1$. Since $K_1 \supset H,\ K_2 \supset H$ it follows that $x \leq_{K_1} x_1 + y_1,\ y \leq_{K_2} x_1 + y_1$. Thus $\{x, y\}$ is bounded from above. A similar argument shows that this set is bounded from below. ■

6.2. 2-Vector lattices

Definition 6.3. A space $E = (E; K_1, K_2)$ is called a *2-lower (upper) vector semi-lattice,* if for any two elements $x_1, x_2 \in E$ there exists $\operatorname{Inf}\{x_1, x_2\}$ ($\operatorname{Sup}\{x_1, x_2\}$, respectively) in the space $E = (E; K_1, K_2)$.

Definition 6.4. A space $E = (E; K_1, K_2)$ is called a *2-vector lattice*, if for any two elements $x_1, x_2 \in\subset E$ there exist $\mathrm{Inf}\{x_1, x_2\}$ and $\mathrm{Sup}\{x_1, x_2\}$ in the space $E = (E; K_1, K_2)$.

We will now present some examples of a 2-vector lattice.

Proposition 6.6. *Let (S, Σ, μ) be a measure space and $E = L^p(S, \Sigma, \mu)$ with $1 \leq p \leq +\infty$. Assume that E is equipped with the natural order relation ($x \geq y \iff x(s) \geq y(s)$ a.e.). Let K_1 be the cone of nonnegative on S functions $x \in E$. Let $B \in \Sigma$ and $K_2 = \{x \in E : x(s) \geq 0, s \in B\}$ be the cone of nonnegative on B functions. Then*

(1) *the space (E, K_1, K_2) is a 2-vector lattice; if $x, y \in E$ then Sup $(x, y) = v$ and $Inf(x, y) = u$, where*

$$v(s) = \begin{cases} \sup(x(s), y(s)) & s \in B \\ x(s) & s \in S \setminus B; \end{cases} \tag{6.2}$$

$$u(s) = \begin{cases} \inf(x(s), y(s)) & s \in B \\ x(s) & s \in S \setminus B. \end{cases} \tag{6.3}$$

(2) *the space (E, K_2, K_1) is a 2-vector lattice; if $x, y \in E$ then Sup $(x, y) = v'$ and $Inf(x, y) = u'$, where*

$$v'(s) = \begin{cases} \sup(x(s), y(s)) & s \in B \\ y(s) & s \in S \setminus B; \end{cases} \tag{6.4}$$

$$u'(s) = \begin{cases} \inf(x(s), y(s)) & s \in B \\ y(s) & s \in S \setminus B. \end{cases} \tag{6.5}$$

Proof. (1) Let $x, y \in E$. We will prove that v defined by (6.2) coincides with Sup (x, y) in (E, K_1, K_2). First we will show that $v \geq_{K_1} x$. Indeed, $v(s) \geq x(s)$ for $s \in B$ and $v(s) = x(s)$ for $s \in S \setminus B$, hence $v \geq_{K_1} x$. Since $v(s) \geq y(s)$ for $s \in B$, it follows that $v \geq_{K_2} y$. Now let $z \geq_{K_1} x$ and $z \geq_{K_2} y$. Then $z(s) \geq x(s)$ for all $s \in S$ and $z(s) \geq y(s)$ for $s \in B$, hence $z \geq_{K_1} v$ and $z \geq_{K_2} y$.

The same argument shows that the function u defined by (6.3) is equal to Inf (x, y) in (E, K_1, K_2).

(2) Let $x, y \in E$ and let v' be defined by (6.4). Then $v'(s) \geq x(s)$ for $s \in B$ and $v'(s) \geq y(s)$ for all $s \in S$, hence $v' \geq_{K_2} x$ and $v' \geq_{K_1} y$. It is easy to check that ($z \geq_{K_2} x, z \geq_{K_1} y$) $\implies$ ($z \geq_{K_2} v', \ z \geq_{K_1}$), so $v' =$ Sup (x, y) in (E, K_2, K_1). The same argument shows that $u' =$ Inf (x, y) in (E, K_2, K_1). ■

Proposition 6.7. *Let (S, Σ, μ) be a measure space and $E = L^p(S, \Sigma, \mu)$ with $1 \leq p \leq +\infty$. Let $B_1 \in \Sigma$ and $B_2 = S \setminus B_1$. Consider the cones*

$$K_1 = \{x \in E : x(s) \geq 0,\ s \in B_1\}, \quad K_2 = \{x \in E : x(s) \geq 0,\ s \in B_2\}.$$

Then $(E; K_1, K_2)$ is 2 vector lattice and for each $x, y \in E$ we have

$$\text{Sup}\,(x, y) = \text{Inf}\,(x, y) = \begin{cases} x(s) & s \in B_1 \\ y(s) & s \in B_2. \end{cases}$$

Proof. The proof follows immediately from the definitions of Sup and Inf. ■

It follows from Proposition 6.7 that in 2- vector lattices the equality Inf $(x, y) =$ Sup (x, y) can be valid for $x \neq y$. Of course this is impossible in classical lattices.

Theorem 6.1. *Let $E = (E; K_1, K_2)$ be a 2-vector lattice. Then for any $x_1, x_2 \in E$ the equalities*

$$x_1 + x_2 = \text{Inf}\,\{x_1; x_2\} + \text{Sup}\,\{x_2; x_1\} = \text{Inf}\,\{x_2; x_1\} + \text{Sup}\,\{x_1; x_2\}$$

hold.

Proof. Let $x_1, x_2 \in E$, then Item 3. of Theorem 6.2 yields

$$\text{Sup}\,\{x_2; x_1\} - x_1 - x_2 = \text{Sup}\,\{x_2 - x_1 - x_2; x_1 - x_1 - x_2\} = \text{Sup}\,\{-x_1; -x_2\}.$$

Item 2. of the same theorem implies that

$$\text{Sup}\,\{-x_1; -x_2\} = -\text{Inf}\,\{x_1; x_2\}.$$

Hence, $x_1 + x_2 = \text{Inf}\,\{x_1; x_2\} + \text{Sup}\,\{x_2; x_1\}$.

Similarly, since $\text{Sup}\,\{x_1; x_2\} - x_2 - x_1 = \text{Sup}\,\{x_1 - x_2 - x_1; x_2 - x_2 - x_1\} = \text{Sup}\,\{-x_2; -x_1\}$ and $\text{Sup}\,\{-x_2; -x_1\} = -\text{Inf}\,\{x_2; x_1\}$ then $x_1 + x_2 = \text{Inf}\,\{x_2; x_1\} + \text{Sup}\,\{x_1; x_2\}$. ■

Let a space $E = (E; K_1, K_2)$ be a 2-vector lattice.

Definition 6.5. The elements

$$x'_+ = \text{Sup}\{0; x\}, \quad x'_- = -\text{Inf}\{x; 0\}$$

are called *the positive and the negative parts* of an element $x \in E = (E; K_1, K_2)$ with respect to a pair of cones (K_1, K_2).

It follows from the definition of Sup and Inf that $x'_+ \geq_{K_1} 0$ and $-x'_- \leq_{K_2} 0$, hence $x'_+ \in K_1$ and $x'_- \in K_2$.

Definition 6.6. The elements

$$x''_+ = \text{Sup}\{x; 0\} \in K_2, \quad x''_- = -\text{Inf}\{0; x\} \in K_1$$

are called *the positive and the negative parts* of an element $x \in E = (E; K_1, K_2)$ with respect to a pair of cones (K_2, K_1).

Put

$$|x|' = x'_+ + x'_-, \quad |x|'' = x''_+ + x''_-.$$

We have $|x|' \in L,\ |x|'' \in L$, where $L = K_1 + K_2$.

6.3. Modulus in 2-vector lattices

Definition 6.7. The quantity

$$|x| = \frac{|x|' + |x|''}{2}$$

is called *the modulus* of an element $x \in E = (E; K_1, K_2)$ in a 2-vector lattice.

Example 6.1. Let (S, Σ, μ) be a measure space and let $E = L^p(S, \Sigma, \mu)$. Consider the space (E, K_1, K_2) where $K_1 = \{x \in E : x(s) \geq 0, \text{a.e } s \in S\}$, $K_2 = \{x \in E : x(s) \geq 0 : x(s) \geq 0, \text{a.e. } s \in B\}$, where $B \in \Sigma$. Let $u \in E$. Then

$$|u|(s) = \begin{cases} |u(s)| & s \in B \\ 0 & s \notin B. \end{cases}$$

This equality easily follows from Proposition 6.6.

Example 6.2. Let $E = L^p(S, \Sigma, \mu)$, $B_1 \in \Sigma, B_2 = S \setminus \Sigma$, $K_1 = \{x \in E : x(s) \geq 0,\ s \in B_1\}$, $K_2 = \{x \in E : x(s) \geq 0,\ s \in B_2\}$. Applying Proposition 6.7, we can obtain that $|x| = 0$ for all $x \in E$.

Now let us study the properties of the modulus.

Theorem 6.2. *Let $E = (E; K_1, K_2)$ be a 2-vector lattice, and let $x, y \in E$. Then*

1. $x = x'_+ - x'_- = x''_+ - x''_-$;
2. $|x|' = \text{Sup}\{-x; x\},\ |x|'' = \text{Sup}\{x; -x\}$,

$$
(6.7)\qquad \begin{aligned} |x| &= \text{Sup}\{0;x\} + \text{Sup}\,\{0;-x\} \\ &= \text{Sup}\,\{x;0\} + \text{Sup}\,\{-x;0\} \ \textit{and}\ |x| \geq_{K_i} 0,\ i=1,2; \end{aligned}
$$

3. $|-x| = |x|$.

   If at least one of the cones K_1, K_2 is a pointed cone then

4. $\text{Inf}\{x'_-; x'_+\} = 0,\ \text{Inf}\{x''_-; x''_+\} = 0;$
5. $|x|' = \text{Sup}\{x'_-; x'_+\},\ |x|'' = \text{Sup}\{x''_+; x''_-\},$

   If both K_1 and K_2 are pointed then

6. $|x| = 0$ *if and only if* $x = 0$;

Proof.

1. By substituting $x_2 = 0$ in (6.6), we obtain the required result.
2. Taking into account Item 1. of the current theorem and Item 2. of Theorem 6.2 we have

$$
|x|' = x'_+ + x'_- = (x'_+ - x'_-) + (x'_- + x'_-) = x + 2x'_- =
$$
$$
= x - \text{Inf}\{2x;0\} = x + \text{Sup}\{-2x;0\} = \text{Sup}\{-x;x\}.
$$

Similarly

$$
|x|'' = x''_+ + x''_- = (x''_+ - x''_-) + (x''_- + x''_-) = x + 2x''_- =
$$
$$
= x - \text{Inf}\{0;2x\} = x + \text{Sup}\{0;-2x\} = \text{Sup}\{x;-x\}.
$$

Thus

$$
\begin{aligned}
|x| &= \frac{|x|'+|x|''}{2} = \frac{\text{Sup}\{-x;x\}+\text{Sup}\{x;-x\}}{2} \\
&= \frac{(\text{Sup}\{-x;x\}+x)+(-x+\text{Sup}\{x;-x\})}{2} = \frac{\text{Sup}\{0;2x\}+\text{Sup}\{0;-2x\}}{2} \\
&= \text{Sup}\{0;x\}+\text{Sup}\{0;-x\} \geq_{K_1} 0,
\end{aligned}
$$

and

$$
\begin{aligned}
|x| &= \frac{(\text{Sup}\{-x;x\} - x) + (x + \text{Sup}\{x;-x\})}{2} \\
&= \frac{\text{Sup}\{-2x;0\} + \text{Sup}\{2x;0\}}{2} \\
&= \text{Sup}\{-x;0\} + \text{Sup}\{x;0\} \geq_{K_2} 0.
\end{aligned}
$$

3. It is obvious.

4. Let $u = \text{Inf}\{x'_+; x'_-\}$. Since $x'_+ \geq_{K_1} 0,\ x'_- \geq_{K_2} 0$, then $u \geq_{K_i} 0,\ i = 1, 2$. Let $z_1 = x'_+ - u,\ z_2 = x'_- - u$. It follows from the definition of Inf that $z_1 \geq_{K_1} 0$ and $z_2 \geq_{K_2} 0$. Item 1. of the current Theorem yields $x = z_1 - z_2$, and therefore $z_1 \geq_{K_2} x$. Since also $z_1 \geq_{K_1} 0$ we have $z_1 \geq_{K_i} \text{Sup}\{0; x\} = x'_+,\ i = 1, 2$.

   The latter inequality implies that $u = x'_+ - z_1 \in -K_i,\ i = 1, 2$. Since at least one of the cones K_1 and K_2 is pointed it follows that $u = 0$.

   The second equality can be deduced by similar reasoning.

5. Theorem 6.1 and Item 4. of the current theorem yield

$$|x|' = x'_+ + x'_- = \text{Sup}\{x'_-; x'_+\} + \text{Inf}\{x'_+; x'_-\} = \text{Sup}\{x'_-; x'_+\},$$

$$|x|'' = x''_+ + x''_- = \text{Sup}\{x''_+; x''_-\} + \text{Inf}\{x''_+; x''_-\} = \text{Sup}\{x''_+; x''_-\}.$$

6. Let $|x| = 0$. Applying (6.7) we have $\text{Sup}\{x; 0\} + \text{Sup}\{-x; 0\} = 0$, therefore $\text{Inf}\,(x, 0) = -\text{Sup}\{-x; 0\} = \text{Sup}\{x; 0\}$. We have

$$x \geq_{K_1} \text{Inf}\{x; 0\} = \text{Sup}\{x; 0\} \geq_{K_1} x.$$

   Since K_1 is a pointed cone, then $x = \text{Inf}\{x; 0\} = \text{Sup}\{x; 0\}$. It follows from this that $x \leq_{K_2} 0$ and $x \geq_{K_2} 0$. Since K_2 is a pointed cone, we have $x = 0$. The proof of assertion $x = 0 \implies |x| = 0$ is trivial. ∎

Proposition 6.8. *Consider a space* (E, K_1, K_2) *such that the cone* $H = K_1 \cap K_2$ *is a generating cone. Assume that for each* $h_1, h_2 \in H$ *there exists* $\text{Inf}\,(h_1, h_2)$ *and* $\text{Sup}\,(h_1, h_2)$ in the space of (E, K_1, K_2). Then (E, K_1, K_2) is a 2-vector lattice.

Proof. Let $x, y \in E$. Since H is a generating cone, then the set $\{x, y\} \subset (E, H)$ is bounded from below, i.e. there exists an element $z \in E$ such that $x, y \geq_H z$. This means that $x - z \in H,\ y - z \in H$ so there exists $u = \text{Inf}\,(x - z, y - z)$ in the space (E, K_1, K_2) . The result follows now from Theorem 6.2, Item 3. A similar argument shows that there exists $\text{Sup}\,(x, y)$. ∎

Now we consider Inf and Sup in a 2-vector lattice $E = (E; K_1, K_2)$ as operators acting from the space E^2 to the space E.

We need the following definitions. Let G be a vector space. An operator $A : G \to (E, K_1, K_2)$ is called sublinear if A is positively homogeneous $(A(\lambda x) = \lambda A(x)$ for all $x \in G$ and $\lambda > 0)$ and subadditive: for each $x_1, x_2 \in G$ it holds:

$$A(x_1 + x_2) \leq_{K_i} A(x_1) + A(x_2), \qquad i = 1, 2.$$

An operator $A : G \to (E, K_1, K_2)$ is called superlinear if A is positively homogeneous and superadditive: if for each $x_1, x_2 \in G$ it holds:

$$A(x_1 + x_2) \geq_{K_i} A(x_1) + A(x_2), \qquad i = 1, 2.$$

Theorem 6.3. *Consider operators $P : E^2 \to E$ and $Q : E^2 \to E$, where*

$$P(X) = \text{Inf}\{x_1; x_2\},\ Q(X) = \text{Sup}\{x_1; x_2\},\ where\ X = (x_1, x_2) \in E^2.$$

Then P is a superlinear operator and Q is a sublinear one.

Proof. We will only prove that P is superlinear. Since P is positively homogeneous (see Theorem 6.2, Item 4.), we need only to prove that P is superadditive. We will start with P. Let $X^1 = (x_1^1, x_2^1) \in E^2,\ X^2 = (x_1^2, x_2^2) \in E^2$. Then

$$P(X^1) = \text{Inf}\{x_1^1; x_2^1\}, \quad P(X^2) = \text{Inf}\{x_1^2; x_2^2\},$$

$$P(X^1 + X^2) = \text{Inf}\{x_1^1 + x_1^2; x_2^1 + x_2^2\}.$$

By the definition of Inf we have

$$x_1^1 \geq_{K_1} P(X^1), \quad x_2^1 \geq_{K_2} P(X^1), \quad x_1^2 \geq_{K_1} P(X^2), \quad x_2^2 \geq_{K_2} P(X^2).$$

Therefore $x_1^2 + x_1^2 \geq_{K_1} P(X^1) + P(X^2),\ x_2^1 + x_2^2 \geq_{K_2} P(X^1) + P(X^2)$. Then

$$\text{Inf}\{x_1^1 + x_1^2; x_2^1 + x_2^2\} \geq_{K_i} P(X^1) + P(X^2),\ i = 1, 2,$$

or $P(X^1 + X^2) \geq_{K_i} P(X^1) + P(X^2),\ i = 1, 2.$ ∎

The following theorem states that $x'_+,\ x'_-,\ x''_+,\ x''_-$ are sublinear projections. First, we will prove the following lemma.

Lemma 6.1. *Let $E = (E; K_1, K_2)$ be a 2-vector lattice with the pointed cones K_1 and K_2. Then for every $x \in E$ the relations*

$$\text{Sup}\{0; \text{Sup}\{0; x\}\} = \text{Sup}\{0; x\}, \quad \text{Sup}\{\text{Sup}\{x; 0\}; 0\} = \text{Sup}\{x; 0\}$$

and

$$\text{Inf}\{0; \text{Inf}\{0; x\}\} = \text{Inf}\{0; x\}, \quad \text{Inf}\{\text{Inf}\{x; 0\}; 0\} = \text{Inf}\{x; 0\}$$

are valid.

Proof. We will prove only the first equality. Other assertions can be proved by similar reasoning. Let $U = \text{Sup}\{0; x\}$ and $V = \text{Sup}\{0; U\}$. We have $U \geq 0_{K_1}$, $U \geq_{K_2} U$, hence $U \geq_{K_i} \text{Sup}\,(0, U) = V,\ i = 1, 2$. Conversely, $V \geq_{K_1} 0,\ V \geq_{K_2}$

U yield $V \geq_{K_i} \text{Sup}\{0; U\} \geq_{K_2} U$. Since K_2 is a pointed cone then $U \geq_{K_2} V$ and $V \geq_{K_2} U$ imply $U = V$. ■

An operator $A : E \to E$ is called a projector if $A^2 = A$.

Theorem 6.4. *Let a space $E = (E; K_1, K_2)$ be a 2-vector lattice. Assume that the cones K_1 and K_2 are pointed. Consider the operators T'_+, T'_-, T''_+, T''_- defined on E by*

$$T'_+(x) = x'_+, \quad T'_-(x) = x'_-, \quad T''_+(x) = x''_+, \quad T''_-(x) = x''_-.$$

Then these operators, acting from E to E are sublinear projectors, besides $T'_+(E) \subset K_1,\ T''_-(E) \subset K_2$ and $T'_-(E) \subset K_1, T''_+(E) \subset K_2$.

Proof. Let $x \in E = (E; K_1, K_2)$. Let

$$T'_+(x) = x'_+, \quad T'_-(x) = x'_-, \quad T''_+(x) = x''_+, \quad T''_-(x) = x''_-.$$

Consider the vectors $Y_x = (x, 0) \in E^2,\ Z_x = (0, x) \in E^2$. Then

$$T'_+(x) = Q(Z_x), \quad T'_-(x) = -P(Y_x), \quad T''_+(x) = Q(Y_x), \quad T''_-(x) = -P(Z_x),$$

where the operators P and Q are the same as in Theorem 6.3. Then Theorem 6.3 implies the sublinearity of T'_+, T'_-, T''_+, T''_-.

The definitions of x'_+, x'_-, x''_+, x''_- yield

$$T'_+(x),\ T''_-(x) \in K_1 \quad \text{and} \quad T'_-(x),\ T''_+(x) \in K_2$$

for all $x \in E$.

Finally, let us show that $(T'_+)^2 = T'_+,\ (T'_-)^2 = T'_-,\ (T''_+)^2 = T''_+,\ (T''_-)^2 = T''_-$. It can easily be obtained by means of Lemma 6.1:

$$(T'_+)^2(x) = T'_+(T'_+(x)) = T'_+(x'_+) = \text{Sup}\{0; \text{Sup}\{0; x\}\} = \text{Sup}\{0; x\} = T'_+(x),$$

where $x \in E$. By acting analogously with T'_-, T''_+, T''_- the required assertion can be proved. ■

7. Kantorovich-Riesz Type Theorems

Let $E = (E; K_1, K_2)$ be a space with two cones K_1, K_2. Consider the space $E' = (E'; K_1^*, K_1^*)$ with the cones K_1^*, K_2^*, where E' is the dual space to E and K_i^* are the conjugate cones to K_i $(i = 1, 2)$. We consider the relation between the Riesz interpolation property in $E = (E; K_1, K_2)$ and the property of $E' = (E'; K_1^*, K_1^*)$ to be a 2-vector lattice. As above, let

$$\sigma \equiv \sigma_{K_1,K_2}(x) = \{X = (x_1, x_2) \in K_1 \times K_2 \ : \ x_1 + x_2 = x\} \quad (x \in K_1 + K_2),$$

be the decomposition mapping with respect to the cones K_1, K_2 and let

$$p_G(x) = \inf_{Y \in \sigma_{K_1,K_2}(x)} [G, Y] \qquad (x \in E,\ G \in \text{dom } \sigma^*)$$

be the support function of σ corresponding to a linear function G (p_G was defined and studied in Section 4). First, we will prove the following assertion.

Proposition 7.1. *Let cones K_1, K_2 be given in the space E and let $L = K_1 + K_2$. If the decomposition mapping $\sigma \equiv \sigma_{K_1,K_2} : E \to 2^{E^2}$ is additive on the cone L, then p_G is a positive additive on L function for every $G \in K^* = K_1^* \times K_2^*$.*

Proof. Let $G \in K^*$ and $x, y \in L$. Since σ is additive, then $\sigma(x + y) = \sigma(x) + \sigma(y)$. Thus,

$$\begin{aligned} p_G(x+y) &= \inf_{Z \in \sigma(x+y)} [G, Z] = \inf_{Z \in \sigma(x)+\sigma(y)} [G, Z] \\ &= \inf_{Z' \in \sigma(x),\ Z'' \in \sigma(y)} [G, Z' + Z''] = \inf_{Z' \in \sigma(x)} [G, Z'] + \inf_{Z'' \in \sigma(y)} [G, Z''] \\ &= p_G(x) + p_G(y). \end{aligned}$$

We proved that p_G is additive on L. Now let us show that p_G $(G \in K^*)$ is positive on the cone L, i.e. if $x \in L$ then $p_G(x) \geq 0$. Indeed, it follows from the fact that $\sigma(x) \subset K = K_1 \times K_2$ and $G \in K^* = K_1^* \times K_2^*$. ∎

We also need the following assertion.

Proposition 7.2. *Assume that the cone $L = K_1 + K_2$ from Proposition 7.1 is generating and closed. Consider a function l_G define on E by*

$$l_G(x) = p_G(x_1) - p_G(x_2), \qquad x = x_1 - x_2, \quad x_1, x_2 \in L. \tag{7.1}$$

Then l_G is well defined and $l_G \in E'$.

Proof. First we show that l_G is well-defined. Let $x = x_1 - x_2 = y_1 - y_2$. Since $x_1 + y_2 = y_1 + x_2$ and p_G is additive it follows that $p_G(x_1) + p_G(y_2) = p_G(y_1) + p_G(x_2)$, therefore $p_G(x_1) - p_G(y_2) = p_G(y_1) - p_G(y_2)$. This means that the number $l_G(x)$ does not depend on the presentation of x as the difference of two elements from L. It is clear that l_G is an additive function. Since p_G is sublinear it follows that p_G is positive homogeneous. Let $x = x_1 - x_2$. Then $-x = x_2 - x_1$, hence $l_G(-x) = p_G(x_2) - p_G(x_1) = -l_G(x)$. Thus p_G is homogeneous. Since the cone L is generating and closed it follows that each positive on L linear function is continuous, hence $l_G \in E'$. ■

Let

$$q_G(x) = \sup_{Y \in \sigma_{K_1,K_2}(x)} [G, Y] \qquad (x \in E,\ G \in \text{dom } \sigma^*). \tag{7.2}$$

The links between q_G and p_G were discussed at the beginning of Section 4. Assume that the mapping σ is additive. Then the function q_G is additive. Assume that the cone L is generating and closed. Then the function

$$m_G(x) = q_G(x_1) - q_G(x_2), \qquad x = x_1 - x_2 \tag{7.3}$$

is well defined. This function is a linear continuous function defined on E. These results can be proved in the same manner as the corresponding results for the function p_G.

Proposition 7.3. *Let a be a superlinear mapping defined on a cone $L \subset E_1$ and mapping into E_2 with weakly compact images. Let for all $g \in E_2'$ the function p_g defined by $p_g(x) = \sup_{y \in a(x)}[g, y]$ be linear. Then a is an additive mapping: $a(x_1 + x_2) = a(x_1) + a(x_2)$ for all $x_1, x_2 \in L$.*

Proof. Assume, on the contrary, that there exist vectors $x_1, x_2 \in L$ such that $a(x_1 + x_2) \neq a(x_1) + a(x_2)$. Since a is superlinear we have $a(x_1 + x_2) \supset a(x_1) + a(x_2)$. Hence there exists $y \in a(x_1 + x_2)$ such that $y \notin a(x_1) + a(x_2)$. The set $a(x_1) + a(x_2)$ is convex and weakly closed. Then there exists $g \in E'$ such that

$$\begin{aligned}
[g, y] &> \sup\{[g, z] : z \in a(x_1) + a(x_2)\} \\
&= \sup\{[g, z] : z \in a(x_1)\} + \sup\{[g, z] : z \in a(x_2)\|\} \\
&= p_g(x_1) + p_g(x_2).
\end{aligned}$$

It follows from this that

$$p_g(x_1 + x_2) = \sup\{[g, z] : z \in a(x_1 + x_2)\} \geq [g, y] > p_g(x_1) + p_g(x_2).$$

This contradicts the linearity of p_g. ■

The following statement is a version of L. V. Kantorovich-F. Riesz Theorem (see, for example, [7, 8]) for spaces with two cones.

Theorem 7.1. *Let E be a Banach ordered space with the closed cones K_1, K_2 and let the cone $L = K_1 + K_2$ be closed and normal. If the space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property with respect to the cones K_1, K_2 then the dual space $E' = (E'; K_1^*, K_2^*)$ is a 2-vector lattice with respect to the conjugate cones K_1^*, K_2^*.*

Proof. Since $L = K_1 + K_2$ it follows that $L^* = K_1^* \cap K_2^*$. Since L is normal it follows that L^* is a generating cone. In view of Proposition 3.4 it is enough to show that Inf (g_1, g_2) and Sup (g_1, g_2) exist for elements $G = (g_1, g_2)$ with $g_1, g_2 \in L^*$. We will prove only the existence of Inf (x_1, x_2). The existence of Sup (x_1, x_2) can be proved by a similar argument.

Theorem 5.1 shows that the Riesz interpolation property with respect to the cones K_1, K_2 in the space $E = (E; K_1, K_2)$ is equivalent to the additivity of the decomposition mapping, so applying Proposition 7.2 we conclude that the function l_G defined by (7.1) is a positive linear continuous function.

We will prove that $l_G = \text{Inf}\{g_1; g_2\} \in E' = (E'; K_1^*, K_2^*)$. Evidently for all $x_1 \in K_1, \ x_2 \in K_2$ the following inequalities hold:

$$l_G(x_1) = p_G(x_1) \leq g_1(x_1), \quad l_G(x_2) = p_G(x_2) \leq g_2(x_2).$$

By the definition of the conjugate cone we obtain $l_G \leq_{K_1^*} g_1, \quad l_G \leq_{K_2^*} g_2$.

Let an element $h \in E'$ be such that $h \leq_{K_1^*} g_1, \quad h \leq_{K_2^*} g_2$. Let $x \in L$ and elements $x_1 \in K_1$ and $x_2 \in K_2$ be such that $x = x_1 + x_2$. (In other words, $(x_1, x_2) \in \sigma_{K_1,K_2}(x)$.) Since $x_1 \in K_1$ and $x_2 \in K_2$ we have $[h, x_1] \leq [g_1, x_1]$, $[h, x_2] \leq [g_2, x_2]$. Hence

$$[h, x] \leq [g_1, x_1] + [g_2, x_2] \quad \text{for all} \quad (x_1, x_2) \in \sigma_{K_1,K_2}(x).$$

This yields

$$[h, x] \leq \inf_{(x_1,x_2) \in \sigma_{K_1,K_2}(x)} \{[g_1, x_1] + [g_2, x_2]\} = p_G(x) = l_G(x) \quad (x \in L).$$

Therefore $[h, x] \leq [l_G, x]$ $(x \in L)$, that is $h \leq_{L^*} l_G$. Since $L^* = K_1^* \bigcap K_2^*$ it follows that $h \leq_{K_1^*} f, \quad h \leq_{K_2^*} f$. This means that l_G is the infimum of the elements g_1, g_2 with respect to the cones K_1^*, K_2^*. We have proved that for each $g_1, g_2 \in L^*$ the infimum with respect to K_1^*, K_2^* exists. The existence of Sup (g_1, g_2) can be proved by the same argument using functions q_G defined by (7.2) instead of p_G and functions m_G defined by (7.3) instead of l_G. ■

It is interesting to find conditions that guarantee that the inverse to the statement in Theorem 7.1 holds. We will demonstrate that this statement is valid if E is a reflexive space. Actually we will prove the following stronger result.

Theorem 7.2. *Let $E = (E; K_1, K_2)$ be a reflexive Banach space with cones K_1 and K_2. Assume that the cone $L = K_1 + K_2$ is closed, normal and generating. Assume that the space $(E'; K_1^*, K_2^*)$ is a 2-vector lower semilattice. Then (E, K_1, K_2) possesses the Riesz interpolation property.*

We need the following assertion.

Lemma 7.1. *Let the the space E be reflexive and let the cone $L = \sum_{i=1}^n K_i$ be normal. Then the function p_G is lower semicontinuous for all $G \in \mathcal{K}$.*

Proof. Since the cone L is normal it follows that (see Theorem 3.4) $\mathcal{K} = (E^n)'$ and (see Proposition 3.4) the mapping σ is bounded. Let $x \in L$ and let r be a number such that $\|x\| < r$. Let $B = \{x' \in E : \|x'\| \leq r\}$. Then the set $\sigma(B)$ is contained in the ball $B_1 = \{X \in E^n : \|X\| \leq r\|\sigma\|\}$. The set $B \times B_1$ is weakly compact and the mapping σ is weakly closed. Hence this mapping is weakly upper semicontinuous on B. We will now show that the function p_G is weakly lower semicontinuous at x. Indeed, let

$$\lambda < p_G(x) = \inf_{X \in \sigma(x)} G(X).$$

Consider the set $A = \{Y \in E^n : [G, Y] > \lambda\}$. Then the set $\sigma(x)$ is contained in the open set A. Since σ is weakly upper semicontinuous then there exists a weak neighborhood V of x such that $\sigma(V) \subset A$. If $y \in V$ then $p_G(y) = \inf_{Y \in \sigma(y)}[G, Y] \geq \lambda$. Hence p_G is weakly lower semicontinuous. Since p_G is convex, this function is also strongly lower continuous. ∎

We now turn to the proof of Theorem 7.2.

Proof. For the sake of definiteness we assume that $(E'; K_1^*, K_2^*)$ is a 2-vector lower semilattice. We will check that the decomposition mapping $\sigma \equiv \sigma_{K_1,K_2}$ is additive, this implies the Riesz interpolation property. We will show that for all $G = (g_1, g_2) \in (E')^2$ the support function p_G of the decomposition mapping $\sigma \equiv \sigma_{K_1,K_2}$ coincides with the restriction of a certain linear function on L. Recall that p_G is sublinear and (see Theorem 7.1) is lower semicontinuous for all $G \in (E^2)'$.

Let $U = \{h \in E' : h \leq_{K_1^*} g_1,\ h \leq_{K_2^*} g_2\}$. We have $[h, x_1] \leq [g_1, x_1]$, $([h, x_2] \leq [g_2, x_2])$ for each $h \in U$, $x \in L$ and $X = (x_1, x_2) \in \sigma(x)$. Therefore

$$[h, x] = [h, x_1] + [h, x_2] \leq \inf_{X=(x_1,x_2)\in\sigma(x)} ([g_1, x_1] + [g_2, x_2]) = p_G(x).$$

We have demonstrated that $h \in \partial p_G$, so $U \subset \partial p_G$.

Now let $h \in \partial p_G$ and let $x_1 \in K_1$. Then

$$[h, x_1] \leq p_G(x_1) = \inf_{X=(x_1',x_2')\in\sigma(x_1)} [g_1, x_1'] + [g_2, x_2'] \leq [g_1, x_1] + [g_2, 0] = [g_1, x_1].$$

Thus $h \le_{K_1^*} g_1$. In the same manner we can show that $h \le_{K_2^*} g_2$. It follows from this that $\partial p_G \subset U$. We have proved that $\partial p_G = U$. Let $h_G = \text{Inf}\,(g_1, g_2)$. Then $h_g \in U$ and $h_G \ge_{K_1^*} h$, $h_G \ge_{K_2^*} h$ for all $h \in U$. Since p_G is lower semicontinuous we have

$$p_G(x) = \sup_{h \in \partial p_G} [h, x] = \sup_{h \in U} [h, x].$$

Since $h_G \ge_{K_i^*} h$ for all $h \in U$ we have that $h_G(x_i) \ge h(x_i)$ for all $h \in U$ and $x_i \in K_i$, $(i = 1, 2)$ hence $p_G(x_1) = \sup_{h \in U}[h, x_1] = [h_G, x_1], \qquad x \in K_1$ and

$$p_G(x_2) = \sup_{h \in U}[h, x_2] = [h_G, x_2], \qquad x \in K_2.$$

Now let $x \in L$ and $X = (x_1, x_2) \in \sigma(x)$. Since p_G is sublinear and $h_G \in \partial p_G$ we have

$$[h_G, x] \le p_G(x) \le p_G(x_1) + p_G(x_2) = [h_G, x_1] + [h_G, x_2] = [h_G, x].$$

Thus $p_G(x) = [h_G, x]$ for all $x \in L$. Hence we can consider p_G as the restriction of a function $h_G \in E'$ to the cone L.

Applying Proposition **??** we conclude that the decomposition mapping is bounded, therefore sets $\sigma(x)$ are bounded for all $x \in K$. Since the space E is reflexive it follows that these sets are weakly compact. We now can apply Proposition **??** that show that σ is an additive mapping. ∎

A similar result can be proved for 2-vector upper semilattices.

Theorem 7.3. *Let $E = (E; K_1, K_2)$ be a reflexive Banach space with cones K_1 and K_2. Assume that the cone $L = K_1 + K_2$ is closed, normal and generating. Assume that the space $(E'; K_1^*, K_2^*)$ is a 2-vector upper semilattice. Then (E, K_1, K_2) possesses the Riesz interpolation property.*

The proof is similar to that of Theorem 7.2. We need to consider the superlinear function q_G, where $g_G(x) = \sup_{X=(x_1,x_2)\in\sigma(x)}([g_1, x_1] + [g_2, x_2]$ and repeat the proof of Theorem 7.2 with obvious changes.

Corollary 7.1. *Let $E = (E; K_1, K_2)$ be a reflexive Banach space with cones K_1 and K_2. Assume that the cone $L = K_1 + K_2$ is closed, normal and generating. If the space $(E'; K_1^*, K_2^*)$ is either a 2-vector lower semilattice or 2-vector upper semilattice then this space is a vector lattice.*

Indeed, applying either Theorem 7.2 or Theorem 7.3 we conclude that $(E; K_1, K_2)$ possesses Riesz interpolation property. Combining this with Theorem 7.1 we obtain the desired result.

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