Necessary and Sufficient Conditions for Stable Conjugate Duality*

R. S. Burachik[†], V. Jeyakumar [‡] and Z-Y. Wu[§]

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Abstract

The conjugate duality, which states that $\inf_{x\in X}\phi(x,0) = \max_{v\in Y'} -\phi^*(0,v)$, whenever a regularity condition on ϕ is satisfied, is a key result in convex analysis and optimization, where $\phi : X\times Y \to \mathbb{R}\cup\{+\infty\}$ is a convex function, X and Y are Banach spaces, Y' is the continuous dual space of Y and ϕ^* is the Fenchel-Moreau conjugate of ϕ. In this paper, we establish a necessary and sufficient condition for the stable conjugate duality,

$$\inf_{x\in X}\{\phi(x,0)+x^*(x)\} = \max_{v\in Y'}\{-\phi^*(-x^*,v)\},\ \forall x^*\in X',$$

and obtain a new global dual regularity condition, which is much more general than the popularly known interior-point type conditions, for the conjugate duality. As a consequence we present an epigraph closure condition which is necessary and sufficient for a stable Fenchel-Rockafellar duality theorem. In the case where one of the functions involved in the duality is a polyhedral convex function, we also provide generalized interior-point conditions for the epigraph closure condition. Moreover, we show that a stable Fenchel's duality for sublinear functions holds whenever a subdifferential sum formula for the functions holds. As applications, we give general sufficient conditions for a minimax theorem, a subdifferential composition formula and for duality results of convex programming problems.

Keywords: conjugate duality, constraint qualifications, convex programming, polyhedral functions, sublinear functions.

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[†]Engenharia de Sistemas e Computacao, COPPE-UFRJ CP 68511, Rio de Janeiro-RJ, CEP 21945-970, Brazil. E-mail: regi@cos.ufrj.br

[‡]School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia. E-mail: jeya@maths.unsw.edu.au

[§]Department of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, P. R. China; School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia. E-mail: zhiyouwu@Maths.unsw.edu.au

1 Introduction

A fundamental duality scheme for studying the convex minimization problem

$$\min f(x), \quad x \in X, \tag{1.1}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$, is by a *representation function* $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(x, 0) = f(x)$. In which case the dual problem associated to (1.1) is given by

$$\max -\phi^*(0, v), \quad v \in Y', \tag{1.2}$$

where X and Y are Banach spaces with the duals X' and Y' respectively, and ϕ^* is the Fenchel-Moreau conjugate of ϕ. A conjugate duality states that

$$\inf_{x \in X} \phi(x, 0) = \max_{v \in Y'} -\phi^*(0, v) \tag{1.3}$$

whenever a regularity condition on ϕ is satisfied. In other words, under the regularity condition, there is no duality gap and the dual problem has a solution. The conjugate duality, which is a key to the study of convex optimization, constrained best approximation and interpolation (see [7, 8, 9]), enables, for instance, one to find the optimal solution by solving the corresponding dual optimization problem. A central question in convex analysis and optimization is to find a general regularity condition for the conjugate duality. From the point of view of applications, it is vital to find conditions on ϕ, which characterize the stable conjugate duality:

$$\inf_{x \in X} \{\phi(x, 0) + x^*(x)\} = \max_{v \in Y'} \{-\phi^*(-x^*, v)\}, \ \forall x^* \in X'.$$

Various (primal) regularity conditions for the duality have been given in the literature (see [14, 20] and other reference therein). However, these regularity conditions are either (global) interior-point type conditions [1, 16, 18] which frequently restrict applications or are based on local conditions. In recent years, it has been shown in [5, 6, 4] that Fenchel's duality holds under a dual epigraph condition, which is strictly weaker than the usual interior-point conditions [1, 12], and the bounded linear regularity condition [2], which have played important roles in convex analysis and optimization.

The purpose of this paper is to establish a necessary and sufficient condition for the stable conjugate duality and derive new global dual regularity conditions, which are much more general than the popularly known interior-point type conditions, for conjugate duality results. We also present an epigraph closure condition that is necessary and sufficient for a stable Fenchel-Rockafellar duality theorem. In the case where one of the functions involved in the duality is a polyhedral convex function, we provide generalized interior-point conditions for the epigraph closure condition. Moreover, we show that a stable Fenchel duality for sublinear functions holds whenever a subdifferential sum formula for the functions holds. As applications, we give general sufficient conditions for a minimax theorem, a subdifferential composition formula and for duality results of convex programming problems involving polyhedral constraints.

2 Preliminaries: Epigraphs of Conjugate Functions

We begin by fixing some definitions and notations. We assume throughout that X and Y are Banach spaces. The continuous dual space of X will be denoted by X' and will be endowed with the weak* topology. For the set $D \subset X$ the **closure** of D will be denoted $\mathrm{cl}\, D$. If a set $A \subset X'$, the expression $\mathrm{cl}\, A$ will stand for the weak* closure. The **indicator function** δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The **support function** σ_D is defined by $\sigma_D(u) = \sup_{x\in D} u(x)$. The **normal cone** of D is given by $N_D(x) := \{v \in X' \ : \ \sigma_D(v) = v(x)\} = \{v \in X' : v(y-x) \leq 0, \forall y \in D\}$ when $x \in D$, and $N_D(x) := \emptyset$ when $x \notin D$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the **conjugate** function of f, $f^* : X' \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \ \mathrm{dom}\, f\}$$

where the *domain* of f, $\mathrm{dom}\, f$, is given by $\mathrm{dom}\, f = \{x \in X \mid f(x) < +\infty\}$. The *epigraph of* f, $\mathrm{Epi}\, f$, is defined by

$$\mathrm{Epi}\, f = \{(x,r) \in X \times \mathbb{R} \mid x \in \mathrm{dom}\, f,\ f(x) \leq r\}.$$

The *subdifferential of* f, $\partial f : X \rightrightarrows X'$ is defined as

$$\partial f(x) = \{v \in X' \mid f(y) \geq f(x) + v(y-x), \forall\, y \in X\}.$$

Note also that $\partial \delta_D = N_D$. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous *sublinear* function, i.e., f is convex and positively homogeneous ($f(0) = 0$, and $f(\lambda x) = \lambda f(x), \forall x \in X, \forall \lambda \in (0,\infty)$), then $\partial f(0)$ is non-empty and for each $x \in \mathrm{dom}\, f$,

$$\partial f(x) = \{v \in \partial f(0) \mid v(x) = f(x)\}.$$

Applying the well-known Moreau-Rockafellar Theorem (see e.g. [18, Theorem 3.2]) we obtain the conclusion in the lemma below.

Lemma 2.1 *[18] Let X be a Banach space. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous convex functions such that* $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. *Then* $\mathrm{Epi}\,(f+g)^* = \mathrm{cl}\,(\mathrm{Epi}\ f^* + \mathrm{Epi}\ g^*)$.

If both f and g are proper lower semi-continuous and sublinear functions then it easily follows from Lemma 2.1 that

$$\partial(f+g)(0) = \mathrm{cl}\,(\partial f(0) + \partial g(0)),$$

For the details see [20].

3 Stable Conjugate Duality

In this section we establish characterizations of the conjugate duality and derive a global dual condition for conjugate duality. We begin by establishing a necessary and sufficient condition for the conjugate duality.

Proposition 3.1 *Let $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Let $\alpha := \inf_{x\in X} \phi(x,0)$ be finite. Then the following statements are equivalent.*

(i) $\inf_{x\in X} \phi(x,0) = \max_{v\in Y'} -\phi^*(0,v)$.

(ii) $(0,0,-\alpha) \in \operatorname{Epi}\phi^* + (\{0\} \times Y' \times \mathbb{R}_+)$.

Proof. $[(i) \Longrightarrow (ii)]$ By assumption (i), there exists $v \in Y'$ such that $-\phi^*(0,v) = \alpha \in \mathbb{R}$. Since $(0,v,\phi^*(0,v)) \in \operatorname{Epi}\phi^*$ and $(0,-v,0) \in \{0\} \times Y' \times \mathbb{R}_+$ we can write

$$(0,0,-\alpha) = (0,0,\phi^*(0,v)) = (0,v,\phi^*(0,v)) + (0,-v,0) \in \operatorname{Epi}\phi^* + (\{0\} \times Y' \times \mathbb{R}_+).$$

$[(ii) \Longrightarrow (i)]$ From (ii), there exist $(u,v,\delta) \in \operatorname{Epi}\phi^*, w \in Y'$ and $\varepsilon \in \mathbb{R}_+$ such that

$$(0,0,-\alpha) = (u,v,\delta) + (0,w,\varepsilon) = (u, v+w, \delta+\varepsilon).$$

Then $u = 0, v = -w$ and $\delta + \varepsilon + \alpha = 0$. Now, by Fenchel's inequality, for each $x \in X$ and $v' \in Y'$, $\phi(x,0) + \phi^*(0,v') \geq 0$. Using this for $v' := v$ we get

$$\begin{aligned} \inf_{x\in X} \phi(x,0) &\geq -\phi^*(0,v) \\ &\geq -\delta \\ &= \varepsilon + \alpha \\ &\geq \alpha = \inf_{x\in X} \phi(x,0). \end{aligned}$$

Thus, $\inf_{x\in X} \phi(x,0) = \max_{y'\in Y'} -\phi^*(0,y') = -\phi^*(0,v)$, and hence (i) holds. □

We now establish a necessary and sufficient condition for the stable conjugate duality.

Theorem 3.1 ***(Stable Conjugate Duality)*** *Let $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Suppose that $\alpha := \inf_{x\in X}\{\phi(x,0)\} < +\infty$. Then the following statements are equivalent.*

(i) $\inf_{x\in X}\{\phi(x,0) + x^*(x)\} = \max_{v\in Y'}\{-\phi^*(-x^*,v)\}, \;\; \forall x^* \in X'$.

(ii) $\operatorname{Epi}\phi^* + (\{0\} \times Y' \times \mathbb{R}_+)$ *is weak* closed.*

Proof. [(i) $\Longrightarrow$ (ii)]. Let $M = \{(x,y) \in X \times Y \mid y = 0\}$. Then, $\operatorname{Epi}\delta_M^* = \{0\} \times Y' \times \mathbb{R}_+$. Since $\alpha = \inf_{x\in X} \phi(x,0) < +\infty$ we have that $\operatorname{dom}\phi \cap M \neq \emptyset$. So we can apply Lemma 2.1 to write

$$\operatorname{cl}(\operatorname{Epi}\phi^* + \{0\} \times \mathrm{Y}' \times \mathbb{R}_+) = \operatorname{cl}(\operatorname{Epi}\phi^* + \operatorname{Epi}\delta_M{}^*) = \operatorname{Epi}(\phi + \delta_M)^*.$$

Fix now an element $(x^*, y^*, r) \in \operatorname{cl}(\operatorname{Epi}\phi^* + \{0\} \times \mathrm{Y}' \times \mathbb{R}_+)$. By the above equality we have that $(\phi + \delta_M)^*(x^*, y^*) \leq r$. So,

$$r \geq (\phi + \delta_M)^*(x^*, y^*) \geq -\inf_{x\in X}\{\phi(x,0) - x^*(x)\}.$$

By assumption (i) we get

$$\inf_{x\in X}\{\phi(x,0)-x^*(x)\}=\max_{v\in Y'}\{-\phi^*(x^*,v)\}.$$

Hence,

$$r\geq(\phi+\delta_M)^*(x^*,y^*)\geq\phi^*(x^*,v),$$

for some $v\in Y'$. This implies that $(x^*,v,r)\in\operatorname{Epi}\phi^*$. Therefore, $(x^*,y^*,r)=(x^*,v,r)+(0,y^*-v,0)\in\operatorname{Epi}\phi^*+\{0\}\times \mathrm{Y}'\times\mathbb{R}_+$ and (ii) holds. [(ii) $\Longrightarrow$ (i)]. Note that the assumption $\alpha=\inf_{x\in X}\phi(x,0)<+\infty$ yields $\alpha(x^*):=\inf_{x\in X}\phi(x,0)+x^*(x)<+\infty$ for all $x^*\in X'$. If $x^*\in X'$ is such that $\alpha(x^*)=-\infty$, then Fenchel-Young inequality yields the equality in (i) for this choice of x^*. Now assume that $x^*\in X'$ is such that $\alpha(x^*)>-\infty$, so our basic assumption yields $\alpha(x^*)\in\mathbb{R}$. We can write

$$\alpha(x^*)=\inf_{x\in X}\phi(x,0)+x^*(x)=-(\phi+\delta_M)^*(-x^*,0)\in\mathbb{R}. \tag{3.4}$$

Note also that $\alpha(0)=\inf_{x\in X}\phi(x,0)<+\infty$ and hence $M\cap\operatorname{dom}\phi\neq\emptyset$, which gives

$$\begin{aligned}(-x^*,0,-\alpha(x^*)) &\in \operatorname{Epi}(\phi+\delta_M)^*=\operatorname{cl}(\operatorname{Epi}\phi^*+\{0\}\times Y'\times\mathbb{R}_+)\\ &= \operatorname{Epi}\phi^*+\{0\}\times Y'\times\mathbb{R}_+,\end{aligned}$$

where we used (3.4) in the inclusion, Lemma 2.1 in the first equality and assumption (ii) in the last one. The above inclusion implies that $(-x^*,0,-\alpha(x^*))=(u,v,\gamma)+(0,w,r)$, with $\phi^*(u,v)\leq\gamma$ and $r\geq 0$. Therefore,

$$\alpha(x^*)=-\gamma-r\leq-\gamma\leq-\phi^*(-x^*,v)\leq\max_{z\in Y'}-\phi^*(-x^*,z)\leq\alpha(x^*),$$

where we used Fenchel-Young inequality in the last inequality. The above expression yields the equality in condition (i) for this choice of x^*. □

It is worth noting that Theorem 3.1 gives a *necessary and sufficient condition* for the equality

$$\phi(\cdot\,,0)^*(-x^*)=\min_{v\in Y'}\phi^*(x^*,v),\ \ \forall x^*\in X',$$

which is equivalent to the equality (i) of Theorem 3.1. For numerous sufficient conditions for this equality, see Theorem 2.7.1 and Corollary 2.7.3 of [20].

Corollary 3.1 ***(Generalized Conjugate Duality)*** *Let $\phi:X\times Y\to\mathbb{R}\cup\{+\infty\}$ be a proper and lower semi-continuous convex function such that $\alpha:=\inf_{x\in X}\phi(x,0)<+\infty$. If $\operatorname{Epi}\phi^*+(\{0\}\times Y'\times\mathbb{R}_+)$ is weak* closed then*

$$\inf_{x\in X}\phi(x,0)=\max_{v\in Y'}-\phi^*(0,v).$$

Proof. The conclusion follows from Theorem 3.1 by taking $x^*=0$. □

Let us see how the epigraph closure condition in the conjugate duality results can be expressed in the particular case where ϕ is described in terms of the sum of two convex functions. Define the continuous linear projection map, $P_{X'\times\mathbf{R}}:X'\times Y'\times\mathbb{R}\to X'\times\mathbb{R}$, by $P_{X'\times\mathbf{R}}(u,v,r)=(u,r)$. The inverse of the projection map is denoted by $P^{-1}_{X'\times\mathbf{R}}$. Recall that for a continuous linear map $A:X\to Y$ the *adjoint operator* of A, denoted by $A^\star$, is the unique linear application from Y' to X' with the property $(A^\star v)(x) = v(Ax)$ for all $v \in Y', x \in X$. Associated to the linear mapping A, we consider the mapping $A^\star \times I : Y' \times \mathbb{R} \to X' \times \mathbb{R}$ defined as $(A^\star \times I)(w, \alpha) = (A^\star w, \alpha)$. Note that the image of $\operatorname{Epi} g^*$ through this application is $(A^\star \times I)(\operatorname{Epi} g^*) = \{(A^\star w, \alpha) | (w, \alpha) \in \operatorname{Epi} g^*\} = \{(A^\star w, \alpha) | g^*(w) \leq \alpha\}$. Let $(\operatorname{Epi} g^*)_A := (A^\star \times I)(\operatorname{Epi} g^*)$.

Lemma 3.1 *Let X and Y be Banach spaces and take $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ two proper functions. Let $A : X \to Y$ be a continuous linear mapping. If $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is given by $\phi(x, y) := f(x) + g(Ax + y)$, then*

$$\operatorname{Epi} \phi^* + (\{0\} \times Y' \times \mathbb{R}_+) = P^{-1}_{X' \times \mathbf{R}}(\operatorname{Epi} f^* + (\operatorname{Epi} g^*)_A)$$

Proof. Call $E := P^{-1}_{X' \times \mathbf{R}}(\operatorname{Epi} f^* + (\operatorname{Epi} g^*)_A)$ and $F := \operatorname{Epi} \phi^* + (\{0\} \times Y' \times \mathbb{R}_+)$. Let $(u, v, r) \in E$. Then, $(u, r) \in \operatorname{Epi} f^* + (\operatorname{Epi} g^*)_A$. So, there exist $(u_1, r_1) \in \operatorname{Epi} f^*$ and $(u_2, r_2) \in \operatorname{Epi} g^*$ such that

$$(u, r) = (u_1 + A^* u_2, r_1 + r_2).$$

Since $\phi^*(u_1, u_2) = f^*(u_1 - A^* u_2) + g^*(u_2)$, $\phi^*(u_1 + A^* u_2, u_2) = f^*(u_1) + g^*(u_2) \leq r_1 + r_2$, and so, $(u_1 + A^* u_2, u_2, r_1 + r_2) \in \operatorname{Epi} \phi^*$. Hence,

$$(u, v, r) = (u_1 + A^* u_2, u_2, r_1 + r_2) + (0, v - u_2, 0) \in F;$$

thus, $E \subset F$.

Conversely, let $(u, v, r) \in F$. Then, $(u, v, r) = (u_1, u_2, r_1) + (0, v, r_2)$, where $\phi^*(u_1, u_2) = f^*(u_1 - A^* u_2) + g^*(u_2) \leq r_1$, $v \in Y'$ and $r_2 \in \mathbb{R}_+$. So, $(u_1 - A^* u_{2,} r_1 - g^*(u_2)) \in \operatorname{Epi} f^*$ and $(u_2, g^*(u_2) + r_2) \in \operatorname{Epi} g^*$. Hence,

$$(u, r) = (u_1 - A^* u_2 + A^* u_2, r_1 - g^*(u_2) + g^*(u_2) + r_2) \in (Epi\, f^* + (Epi\, g^*)_A);$$

thus, $F \subset E$. □

Note that if $\phi(x, y) := f(x) + g(x + y)$, then

$$\operatorname{Epi} \phi^* + (\{0\} \times Y' \times \mathbb{R}_+) = P^{-1}_{X' \times \mathbf{R}}(\operatorname{Epi} f^* + \operatorname{Epi} g^*).$$

Remark 3.1 *Let $C \subset X' \times \mathbb{R}$ be an arbitrary set. Then it is easy to check that the set $P^{-1}_{X' \times \mathbf{R}}(C)$ can be identified with $C \times Y'$. Therefore, C is weak* closed if and only if $P^{-1}_{X' \times \mathbf{R}}(C)$ is weak* closed.*

Theorem 3.2 ***(Stable Fenchel-Rockafellar Duality)*** *Let $A : X \to Y$ be a continuous linear mapping. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semi-continuous convex functions such that $A(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. Then the following statements are equivalent:*

(i) $\inf_{x \in X} \{f(x) + g(Ax) + x^*(x)\} = \max_{v \in X'} \{-f^*(A^* v - x^*) - g^*(-v)\}, \forall x^* \in X'$.

(ii) $\operatorname{Epi} f^* + (\operatorname{Epi} g^*)_A$ *is weak* closed.*

Proof. Define $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by $\phi(x, y) := f(x) + g(Ax + y)$ for each $(x, y) \in X \times Y$. Then, $\phi^*(u, v) = f^*(u - A^\star v) + g^*(v)$. So that condition (i) becomes

$$\inf_{x \in X} \phi(x, 0) + x^*(x) = \max_{v \in Y'} \{-\phi^*(-x^*, -v)\} \; \forall x^* \in X'. \tag{3.5}$$

Since $A(\mathrm{dom}\, f) \cap \mathrm{dom}\, g \neq \emptyset$, there exists $y_0 \in \mathrm{dom}\, g$ such that $y_0 = Ax_0$ with $x_0 \in \mathrm{dom}\, f$. Hence, $\inf_{x \in X} \phi(x, 0) \leq f(x_0) + g(Ax_0) < +\infty$. So we can apply Theorem 3.1, and conclude that (3.5) is equivalent to the weak* closedness of $\mathrm{Epi}\, \phi^* + \{0\} \times Y' \times \mathbb{R}_+$. Now the result follows from Lemma 3.1 and Remark 3.1. □

Observe that, in the case where $X = Y$ and $A = I$, Theorem 3.2 shows that $\mathrm{Epi}\, f^* + \mathrm{Epi}\, g^*$ is weak* closed if and only if

$$(f + g)^*(x^*) = \min\{f^*(u) + g^*(v) \mid u + v = x^*\}, \;\; \forall x^* \in X',$$

whenever $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. For related results, see [5, 12, 19].

Corollary 3.2 ***(Stable Duality for Convex Programs)*** *Let $A : X \to \mathrm{Y}$ be a continuous linear mapping, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Let $C \subset \mathrm{dom}\, f$ be a closed convex set, $b \in \mathrm{Y}$ and let $K \subset \mathrm{Y}$ be a closed convex cone. Assume that $A(\mathrm{C}) \cap (K + b) \neq \emptyset$. Then the following statements are equivalent.*

(i) $\displaystyle\inf_{\substack{x \in C \\ Ax - b \in K}} \{f(x) + x^*(x)\} = \max_{v \in K^0} -(f + \delta_C)^*(A^* v - x^*) + v(b), \;\; \forall x^* \in X'$

(ii) $\mathrm{Epi}\, (f + \delta_C)^* + (\mathrm{Epi}\, \delta^*_{b+K})_A$ *is weak* closed,*

where $K^0 = \{v \in Y' \mid v(y) \leq 0, \; \forall y \in K\}$ is the polar cone of K.

Proof. By Theorem 3.2, applied to the functions $f + \delta_C$ and $g := \delta_{b+K}$, we readily see the equivalence between (i) and (ii). □

Corollary 3.3 ***(Generalized Fenchel-Rockafellar Duality)*** *Let $A : X \to Y$ be a continuous linear mapping. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semi-continuous convex functions such that $A(\mathrm{dom}\, f) \cap \mathrm{dom}\, g \neq \emptyset$. If the set $\mathrm{Epi}\, f^* + (\mathrm{Epi}\, g^*)_A$ is weak* closed, then*

$$\inf_{x \in X} \{f(x) + g(Ax)\} = \max_{v \in Y'} \{-f^*(A^\star v) - g^*(-v)\}. \tag{3.6}$$

Proof. The conclusion follows from Theorem 3.2 by taking $x^* = 0$. □

When the functions f and g are sublinear, we have that stable Fenchel duality is equivalent to the subdifferential sum formula.

Corollary 3.4 ***(Subdifferential Sum Formula)****Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous sublinear functions with $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. Then the following statements are equivalent:*

(i) $\inf_{x \in X} \{f(x) + g(x) + x^*(x)\} = \max_{v \in X'} \{-f^*(v - x^*) - g^*(-v)\}, \forall x^* \in X'$.

(ii) $\partial(f + g)(x) = \partial f(x) + \partial g(x), \;\; \forall x \in dom\, f \cap dom\, g$.

(iii) $\mathrm{Epi}\, f^* + \mathrm{Epi}\, g^*$ *is weak* closed.*

Proof. [(i)$\Longrightarrow$ (ii)]. This implication is well known and holds for arbitrary proper convex and lower semicontinuous functions, see for instance Theorem 2.1 of [13]. [(ii)$\Longrightarrow$ (iii)]. If (ii) holds then $\partial(f+g)(0) = \partial f(0) + \partial g(0)$. Since f and g are proper lower semi-continuous sublinear functions, $\partial(f+g)(0) = \mathrm{cl}\,(\partial f(0) + \partial g(0))$. So, $\mathrm{cl}\,(\partial f(0) + \partial g(0)) = \partial f(0) + \partial g(0)$; thus, $\partial f(0) + \partial g(0)$ is weak*closed. Hence,

$$\mathrm{Epi}\, f^* + \mathrm{Epi}\, g^* = \partial f(0) \times \mathbb{R}_+ + \partial g(0) \times \mathbb{R}_+ = (\partial f(0) + \partial g(0)) \times \mathbb{R}_+$$

is weak*closed. [(iii)$\Longrightarrow$ (i)]. This follows from Theorem 3.2, where $X = Y$ and $A = I$. □

Corollary 3.5 ***(Subdifferential Composition Formula)*** *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *and* $g : Y \to \mathbb{R} \cup \{+\infty\}$ *be proper and lower semi-continuous convex functions such that* $A(\mathrm{dom}\, f) \cap \mathrm{dom}\, g \neq \emptyset$. *Let* $A : X \to Y$ *be a continuous linear mapping. If the set* $\mathrm{Epi}\, f^* + (\mathrm{Epi}\, g^*)_A$ *is weak* closed then for each* $x_0 \in X$,

$$\partial(f + (g \circ A))(x_0) = \partial f(x_0) + A^* \partial g(Ax_0).$$

Proof. Let $v \in \partial(f + (g \circ A))(x_0)$. Then, for each $x \in X$, $(f + g \circ A)(x) - (f + (g \circ A))(x_0) \geq v(x - x_0)$, and so, it follows from Corollary 3.3 that there exits $y_0 \in Y'$ such that

$$\begin{aligned} (f - v)(x_0) + (g \circ A)(x_0) &= \inf\{(f - v)(x) + g(Ax) \mid x \in X\} \\ &= -(f - v)^*(A^* y_0) - g^*(-y_0). \end{aligned}$$

Now, it is easy to show that $A^* y_0 \in \partial(f - v)(x_0) = \partial f(x_0) - \{v\}$, and $-y_0 \in \partial g(Ax_0)$. Indeed, the above equality yields

$$\begin{aligned} -(f - v)(x_0) - (g \circ A)(x_0) &= (f - v)^*(A^* y_0) + g^*(-y_0) \\ &\geq [(A^* y_0)(x) - (f - v)(x)] + [(-y_0)(y) - g(y)], \end{aligned}$$

For all $x \in X$ and $y \in Y$. Taking $y = Ax_0$, we see that $A^* y_0 \in \partial(f - v)(x_0) = \partial f(x_0) - \{v\}$, and choosing $x = x_0$, we obtain $-y_0 \in \partial g(Ax_0)$. Thus, $v \in \partial f(x_0) + A^* \partial g(Ax_0)$. Hence, $\partial(f + (g \circ A))(x_0) \subset \partial f(x_0) + A^* \partial g(Ax_0)$. The required equality now follows as the reverse inclusion easily holds. □

We now show how a stable minimax theorem can be derived from the stable Fenchel-Rockafellar conjugate duality theorem. For related general stable minimax theorems see [10, 11] and other references therein.

Corollary 3.6 ***(Stable Minimax Theorem)*** *Let* $A : X \to Y$ *be a continuous linear mapping. Let* C *and* D *be closed convex subsets of* X *and* Y' *respectively. Suppose that* $A(C) \cap \mathrm{dom}\, \sigma_D \neq \emptyset$. *Then the set* $\mathrm{Epi}\, \sigma_C + A^*(D) \times \mathbb{R}_+$ *is weak* closed if and only if*

$$\inf_{x \in C} \sup_{v \in D} v(Ax) + x^*(x) = \max_{v \in D} \inf_{x \in C} v(Ax) + x^*(x), \quad \forall x^* \in X'.$$

Proof. Let $f = \delta_C$ and $g = \delta_D^* = \sigma_D$ in Theorem 3.2. Then $A(\operatorname{dom} f) \cap \operatorname{dom} g = A(C) \cap \operatorname{dom} \sigma_D \neq \emptyset$, and it is easy to check that $(\operatorname{Epi} \delta_D)_A = A^\star(D) \times \mathbb{R}_+$. Now, it follows from Theorem 3.2 that $\operatorname{Epi} \sigma_C + A^*(D) \times \mathbb{R}_+$ is weak* closed if and only if

$$\inf_{x \in X} \{\delta_C(x) + \sigma_D(Ax) + x^*(x)\} \quad = \max_{u \in Y'} \{-\sigma_C(A^\star u - x^*) - \delta_D(-u)\},$$

which is equivalent to the equalities

$$\begin{aligned} \inf_{x \in C} \sup_{v \in D} v(Ax) + x^*(x) \quad &= \max_{-u \in D} \{-\sigma_C(A^\star u - x^*)\} \\ &= \max_{-u \in D} \inf_{x \in C} [x^*(x) + (A^\star(-u))(x)] \\ &= \max_{v \in D} \inf_{x \in C} (x^* + A^\star v)(x) \\ &= \max_{v \in D} \inf_{x \in C} v(Ax) + x^*(x). \end{aligned}$$

□

4 Duality and Polyhedral Convex Functions

This section studies the case in which one of the convex functions is a polyhedral convex function, i.e., a function which has as epigraph a polyhedral convex set. Recall that, for $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex and lower-semicontinuous functions, the classical Fenchel's Duality Theorem [17, Theorem 31.1] states that one has

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{v \in X'} \{-f^*(v) - g^*(-v)\}, \tag{4.7}$$

when $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$. However, if one of these functions, say g, is polyhedral, the latter condition can be weakened to $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. Moreover, when both functions are polyhedral, then the relative interiors can be replaced by the domains of the functions. This fundamental result has been recently extended to arbitrary Banach spaces in [16, Theorem 3.3], where the concept of relative interior has been replaced by the *strong quasi relative interior* (see [14, 15, 20]), which is denoted by $\operatorname{sqri}(C)$, and defined as

$$\operatorname{sqri}(C) = \{x \in C \mid \operatorname{cone}(C - x) \text{ is a closed subspace}\}.$$

Note that if the set C is contained in a finite dimensional space, then $\operatorname{ri}(C) = \operatorname{sqri}(C)$.

In the case in which both functions f and g are lower semi-continuous polyhedral convex functions on $\mathbb{R}^n$, then it is well known that both sets $\operatorname{Epi} f^*$ and $\operatorname{Epi} g^*$ are closed polyhedrons, and their sum $\operatorname{Epi} f^* + \operatorname{Epi} g^*$, being also a polyhedral, is also closed.

We now show that, when g is polyhedral, our closure condition holds whenever $\operatorname{sqri}(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. This condition is used in [16, Theorem 3.3], which is stated below.

Theorem 4.1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be convex proper and lower semicontinuous functions, and let $A : X \to Y$ be a linear and continuous operator. Suppose that* $\operatorname{sqri}(A(\operatorname{dom} f)) \cap \operatorname{dom} g \neq \emptyset$ *and that g is polyhedral. Then*

$$\inf_{x \in X} \{f(x) + g(Ax)\} = \max_{v \in X'} \{-f^*(A^\star v) - g^*(-v)\}. \tag{4.8}$$

Theorem 4.2 *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *and* $g : Y \to \mathbb{R} \cup \{+\infty\}$ *be convex proper and lower semicontinuous functions, and let* $A : X \to Y$ *be a linear and continuous operator. Suppose that* g *is polyhedral and that* $\operatorname{dom} g \cap \operatorname{sqri}(A(\operatorname{dom} f)) \neq \emptyset$*. Then* $\operatorname{Epi} f^* + (\operatorname{Epi} g^*)_A$ *is weak* closed.*

Proof. Let $x^* \in X'$ and define $\hat{f}(x) := f(x) + x^*(x)$. Then, $\hat{f}$ is also a proper convex function and $\operatorname{dom} \hat{f} = \operatorname{dom} f$, so that we still have $\operatorname{dom} g \cap \operatorname{sqri}(A(\operatorname{dom} \hat{f})) \neq \emptyset$. Moreover, for each $v \in X'$, $\hat{f}^*(v) = f^*(v - x^*)$. By Theorem 4.1, we know that if g is polyhedral and $\operatorname{dom} g \cap \operatorname{sqri}(A(\operatorname{dom} \hat{f})) \neq \emptyset$, then

$$\inf_{x \in X} (\hat{f}(x) + g(Ax)) = \max_{v \in X^*} (-\hat{f}^*(A^\star v) - g^*(-v)).$$

Re-writing the expression above we get

$$\inf_{x \in X} (f(x) + g(Ax) + x^*(x)) = \max_{v \in X^*} (-f^*(A^\star v - x^*) - g^*(-v)).$$

Having shown the above equality for arbitrary $x^* \in X'$, Theorem 3.2 applies and we can conclude that $\operatorname{Epi} f^* + (\operatorname{Epi} g^*)_A$ is weak* closed. □

A corollary of the above result is an application to convex programs with polyhedral constraints.

Corollary 4.1 ***(Duality for Convex Programs with Polyhedral Constraints)*** *Let* $A : X \to Y$ *be a continuous linear mapping,* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper and lower semi-continuous convex function. Let* C *be a closed convex subset of* $\operatorname{dom} f$*,* $b \in Y$ *and let* $K \subset Y$ *be a polyhedral cone. Assume that* $A(C) \cap (K + b) \neq \emptyset$*. Under one of the following conditions:*

(i) $Y = \mathbb{R}^n$ *and* $\operatorname{ri}(A(C)) \cap (b + K) \neq \emptyset$*,*

(ii) $\operatorname{sqri}(A(C)) \cap (b + K) \neq \emptyset$*,*

(iii) $\operatorname{Epi}(f + \delta_C)^* + (\operatorname{Epi} \delta^*_{b+K})_A$ *is weak* closed,*

one has

$$\inf \{f(x) \mid x \in C,\ Ax - b \in K\} = \max \left\{-(f + \delta_C)^*(A^* v) + v(b) \mid v \in K^0\right\}. \quad (4.9)$$

Proof. By Corollary 3.2, we have that condition (iii) above implies (4.9), which is condition (i) of Corollary 3.2 for the choice $x^* = 0$. Using now Theorem 4.2 applied to the functions $f + \delta_C$ and $g := \delta_{b+K}$, we have that both (ii) and its finite dimensional version (i) are stronger than (iii). □

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