On graphs of defect at most 2

Ramiro Feria-Purón[1,*] Mirka Miller[1,2,3,4,†] Guillermo Pineda-Villavicencio[5,‡]

[1] *School of Electrical Engineering and Computer Science*
The University of Newcastle, Australia

[2] *Department of Mathematics*
University of West Bohemia, Czech Republic

[3] *Department of Computer Science*
King's College London, UK

[4] *Department of Mathematics*
ITB Bandung, Indonesia

[5] *Centre for Informatics and Applied Optimization*
University of Ballarat, Australia

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Abstract

In this paper we consider the *degree/diameter problem*, namely, given natural numbers $\Delta \geq 2$ and $D \geq 1$, find the maximum number $\mathrm{N}(\Delta, D)$ of vertices in a graph of maximum degree Δ and diameter D. In this context, the Moore bound $\mathrm{M}(\Delta, D)$ represents an upper bound for $\mathrm{N}(\Delta, D)$.

Graphs of maximum degree Δ, diameter D and order $\mathrm{M}(\Delta, D)$, called *Moore graphs*, turned out to be very rare. Therefore, it is very interesting to investigate graphs of maximum degree $\Delta \geq 2$, diameter $D \geq 1$ and order $\mathrm{M}(\Delta, D) - \epsilon$ with small $\epsilon > 0$, that is, $(\Delta, D, -\epsilon)$-graphs. The parameter ϵ is called the *defect*.

Graphs of defect 1 exist only for $\Delta = 2$. When $\epsilon > 1$, $(\Delta, D, -\epsilon)$-graphs represent a wide unexplored area. This paper focuses on graphs of defect 2. Building on the approaches developed in [11] we obtain several new important results on this family of graphs.

First, we prove that the girth of a $(\Delta, D, -2)$-graph with $\Delta \geq 4$ and $D \geq 4$ is $2D$. Second, and most important, we prove the non-existence of $(\Delta, D, -2)$-graphs with even $\Delta \geq 4$ and $D \geq 4$; this outcome, together with a proof on the non-existence of $(4, 3, -2)$-graphs (also provided in the paper), allows us to complete the catalogue of $(4, D, -\epsilon)$-graphs with $D \geq 2$ and $0 \leq \epsilon \leq 2$. Such a catalogue is only the second census of $(\Delta, D, -2)$-graphs known at present, the first being the one of $(3, D, -\epsilon)$-graphs with $D \geq 2$ and $0 \leq \epsilon \leq 2$ [14].

[*] ramiro.feria.puron@gmail.com
[†] mirka.miller@newcastle.edu.au
[‡] work@guillermo.com.au (Corresponding author)

Other results of this paper include necessary conditions for the existence of $(\Delta, D, -2)$-graphs with odd $\Delta \geq 5$ and $D \geq 4$, and the non-existence of $(\Delta, D, -2)$-graphs with odd $\Delta \geq 5$ and $D \geq 5$ such that $\Delta \equiv 0, 2 \pmod{D}$.

Finally, we conjecture that there are no $(\Delta, D, -2)$-graphs with $\Delta \geq 4$ and $D \geq 4$, and comment on some implications of our results for the upper bounds of $\mathrm{N}(\Delta, D)$.

Keywords: Moore bound, Moore graph, degree/diameter problem, defect, repeat.

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1 Introduction

Due to the diverse features and applications of interconnection networks, it is possible to find many interpretations about network "optimality" in the literature. Here we are concerned with the following; see [12, pp. 168].

> *An optimal network contains the maximum possible number of nodes, given a limit on the number of connections attached to a node and a limit on the distance between any two nodes of the network.*

In graph-theoretical terms, this interpretation leads to the *degree/diameter problem* [19], which can be stated as follows:

> *Degree/diameter problem*: Given natural numbers $\Delta \geq 2$ and $D \geq 1$, find the largest possible number $\mathrm{N}(\Delta, D)$ of vertices in a graph of maximum degree Δ and diameter D.

Note that $\mathrm{N}(\Delta, D)$ is well defined for $\Delta \geq 2$ and $D \geq 1$. An upper bound for $\mathrm{N}(\Delta, D)$ is given by the *Moore bound* $\mathrm{M}(\Delta, D)$,

$$\mathrm{M}(\Delta, D) = 1 + \Delta \left(1 + (\Delta - 1) + \cdots + (\Delta - 1)^{D-1}\right).$$

Graphs of degree Δ, diameter D and order $\mathrm{M}(\Delta, D)$ are called *Moore graphs*.

Only a few values of $\mathrm{N}(\Delta, D)$ are known at present. With the exception of $\mathrm{N}(4, 2) = \mathrm{M}(4, 2) - 2$ (see [3]), $\mathrm{N}(5, 2) = \mathrm{M}(5, 2) - 2$ (see [21]), $\mathrm{N}(6, 2) = \mathrm{M}(6, 2) - 5$ (see [20]), $\mathrm{N}(3, 3) = \mathrm{M}(3, 3) - 2$ (see [14]) and $\mathrm{N}(3, 4) = \mathrm{M}(3, 4) - 8$ (see [4]), the other known values of $\mathrm{N}(\Delta, D)$ are those for which there exists a Moore graph.

Moore graphs are very rare. For $\Delta = 2$ and $D \geq 1$ they are the cycles on $2D + 1$ vertices, whereas for $D = 1$ and $\Delta \geq 2$ they are the complete graphs on $\Delta + 1$ vertices. If $D = 2$ and $\Delta \geq 3$, Moore graphs exist for $\Delta = 3, 7$ and possibly 57, but not for any other degree; see [13, 19]. When $\Delta \geq 3$ and $D \geq 3$, there are no Moore graphs ([1, 6, 19]).

Therefore, we are interested in studying the existence or otherwise of graphs of given maximum degree Δ, diameter D and order $\mathrm{M}(\Delta, D) - \epsilon$ for small $\epsilon > 0$, that is, $(\Delta, D, -\epsilon)$-graphs, where the parameter ϵ is called the *defect*.

The family of graphs of defect $\epsilon = 1$ has been fully characterized; see [2, 10, 15]. For $\Delta = 2$ and each $D \geq 2$, the cycle on $2D$ vertices is the only $(2, D, -1)$-graph. For other values of Δ and D there are no $(\Delta, D, -1)$-graphs.

Graphs of defect $\epsilon = 2$ represent a wide unexplored area. The catalogue of $(3, D, -2)$ was completed by Jørgensen in [14]. So far there have been several partial results achieved on the existence or otherwise of $(\Delta, D, -2)$-graphs with $\Delta \geq 4$ and $D \geq 2$; see [3, 5, 9, 14, 16, 21] for $D = 2$ and [18, 22] for $\Delta = 4, 5$. While the paper [18] claimed to have proved the non-existence of $(4, D, -2)$-graphs for $D \geq 3$, it turns out that the proof contained a mistake, so that only structural properties of $(4, D, -2)$-graphs were obtained. As a consequence, for $(\Delta, D, -2)$-graphs with $\Delta \geq 4$ and $D \geq 2$ there has not been any definitive catalogue of any subfamily of such graphs until now.

For the sake of completeness we mention that, in the case of graphs with defect $\epsilon \geq 3$, the only known work is the complete catalogue of $(3, D, -4)$-graphs provided in [17].

In this paper we consider $(\Delta, D, -2)$-graphs with $\Delta \geq 4$ and $D \geq 4$, and advance considerably the aforementioned question of the existence or otherwise of such graphs. To obtain our results we rely on combinatorial approaches which are inspired by those developed in [11].

Our first result is a proof that the girth of a $(\Delta, D, -2)$-graph with $\Delta \geq 4$ and $D \geq 4$ cannot be $2D - 1$ and therefore must be $2D$. Subsequently, we offer a non-existence proof of $(\Delta, D, -2)$-graphs with even $\Delta \geq 4$ and $D \geq 4$. After ruling out the existence of $(4, 3, -2)$-graphs, we provide the first catalogue of $(\Delta, D, -\epsilon)$-graphs for $\Delta \geq 4$, $D \geq 2$ and $0 \leq \epsilon \leq 2$, namely, the one of $(4, D, -\epsilon)$-graphs.

Other results of the paper include structural properties and necessary conditions for the existence of $(\Delta, D, -2)$-graphs with odd $\Delta \geq 5$ and $D \geq 4$, and the non-existence of $(\Delta, D, -2)$-graphs with odd $\Delta \geq 5$ and $D \geq 5$ such that $\Delta \equiv 0, 2 \pmod{D}$.

Finally, we conjecture there are no $(\Delta, D, -2)$-graphs with $\Delta \geq 4$ and $D \geq 4$, and comment on the implications of our results for the upper bounds for $\mathrm{N}(\Delta, D)$.

2 Known $(\Delta, D, -2)$-graphs

When $\Delta = 2$ or $D = 1$, there are no graphs of defect 2.

For $D = 2$ there is a unique $(2, 2, -2)$-graph (the path of length 2), exactly two non-isomorphic $(3, 2, -2)$-graphs, a unique $(4, 2, -2)$-graph, and a unique $(5, 2, -2)$-graph. All these graphs are depicted in Fig. 1. The non-existence of $(\Delta, 2, -2)$-graphs with $\Delta \geq 6$, has been conjectured but not yet proved in spite of the partial support given in [5, 16].

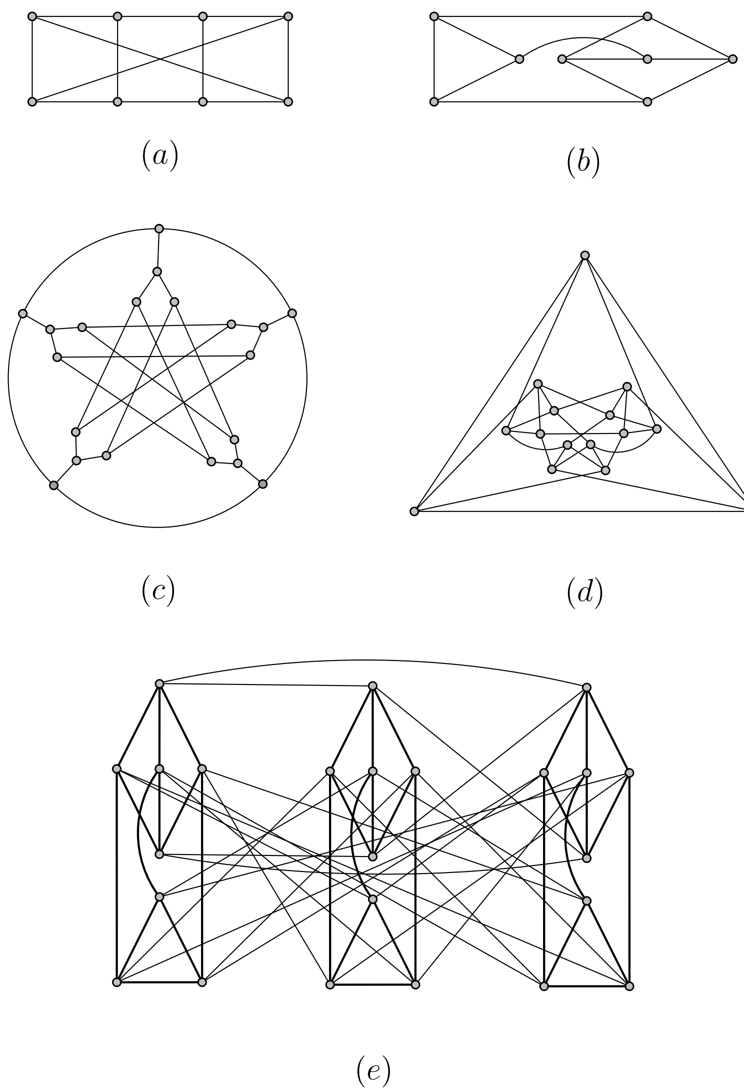

Figure 1: Known graphs of defect 2. (a) and (b) the two $(3, 2, -2)$-graphs, (c) the unique $(3, 3, -2)$-graph, (d) the unique $(4, 2, -2)$-graph and (e) the unique $(5, 2, -2)$ graph (note that this graph is formed by connecting appropriately 3 copies of the graph (b)).

For $\Delta = 3$ and $D \geq 3$ there is a unique $(3, 3, -2)$-graph, which is depicted in Fig. 1 (c). This graph, together with the two aforementioned $(3, 2, -2)$-graphs, comprise the complete catalogue of cubic graphs of defect 2; see [14].

3 Notation and Terminology

The terminology and notation used in this paper is standard and consistent with that used in [8], so only those concepts that can vary from texts to texts will be defined.

All graphs considered in this paper are simple. The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. For an edge $e = \{x, y\}$, we write $x \sim y$. The set of neighbors of a vertex x in

Γ is denoted by $N(x)$.

A path of length k is called a *k-path*. A path from a vertex x to a vertex y is denoted by $x - y$. We use the following notation for subpaths of a path $P = x_0x_1 \dots x_k$: $x_iPx_j = x_i \dots x_j$, where $0 \le i \le j \le k$. A cycle of length k is called a *k-cycle*. The *girth* of Γ, denoted g=g(Γ), is the length of the shortest cycle in Γ.

The union of three independent paths of length D with common endvertices is denoted by Θ_D. In a graph Γ, a vertex of degree at least 3 is called a *branch vertex* of Γ.

4 Preliminary Results

We begin this section with a known condition for the regularity of a $(\Delta, D, -\epsilon)$-graph, which can be easily deduced by considering the existence of a vertex of degree at most $\Delta - 1$ in such a graph.

Proposition 4.1 *For $\epsilon < 1 + (\Delta - 1) + (\Delta - 1)^2 + \ldots + (\Delta - 1)^{D-1}$, $\Delta \ge 3$ and $D \ge 2$, a $(\Delta, D, -\epsilon)$-graph is regular.*

By Proposition 4.1, a $(\Delta, D, -2)$-graph Γ with $\Delta \ge 3$ and $D \ge 2$ must be regular; we therefore use the symbol d rather than Δ to denote the degree of Γ, as is customary. We call a cycle of length at most $2D$ in Γ a *short cycle*.

Proposition 4.2 (Lemma 2 from [14]) *Let Γ be a $(d, D, -2)$-graph with $d \ge 3$ and $D \ge 2$. Then $2D - 1 \le \mathrm{g}(\Gamma) \le 2D$. Furthermore, if x is a vertex in Γ then either*

(*i*) *x is contained in one $(2D - 1)$-cycle and no other short cycle; or*

(*ii*) *x is contained in one Θ_D, and every short cycle containing x is contained in this Θ_D; or*

(*iii*) *x is contained in exactly two $2D$-cycles whose intersection is a ℓ-path with $0 \le \ell \le D - 1$, and no other short cycle.*

Each case is considered as a type. For instance, a vertex satisfying case (i) is called a vertex of Type (i).

While the statements of Proposition 4.2 and [14, Lemma 2] slightly differ, both assertions are clearly equivalent. However, the statement of Proposition 4.2 is more consistent with the presentation of our results and allows us to make the following observation, which will be used implicitly throughout the paper.

Observation 4.1 *Let Γ be a $(d, D, -2)$-graph with $d \ge 3$ and $D \ge 2$, and C a short cycle in Γ. Then all vertices in C are of the same type.*

In view of Proposition 4.2, we define the following concepts:

We say that the vertex x' is a *repeat* of the vertex x with *multiplicity* $\mathrm{m}_x(x')$ $(1 \leq \mathrm{m}_x(x') \leq 2)$ if there are exactly $\mathrm{m}_x(x') + 1$ different paths of length at most D from x to x'. For vertices x and x' lying on a short cycle C, we denote the vertex x' by $\mathrm{rep}^C(x)$ if x and x' are repeats.

A vertex x is called *saturated* if the set S_x of short cycles containing x have been completely identified. As a consequence no further short cycle outside S_x can contain x. If two $2D$-cycles C^1 and C^2 are non-disjoint, we say that C^1 and C^2 are *neighbor cycles.*

From now on, whenever we refer to paths we mean shortest paths. As in [11], we extend the concept of repeat to paths. For a path $P = x - y$ of length at most $D - 1$ contained in a $2D$-cycle C, we denote by $\mathrm{rep}^C(P)$ the path $P' \subset C$ defined as $\mathrm{rep}^C(x) - \mathrm{rep}^C(y)$. We say that P' is the *repeat* of P in C and vice versa, or simply that P and P' are *repeats* in C.

Often our arguments revolve around the identification of the set of short cycles containing a given vertex x, we call this process "saturating the vertex x". Next we present a couple of lemmas that will help us in this cycle identification.

Lemma 4.1 (Odd Saturating Lemma) *Let Γ be a $(d, D, -2)$-graph with $d \geq 4$ and $D \geq 2$, and $\mathcal{C}$ a $(2D-1)$-cycle in Γ. Let α be a vertex in $\mathcal{C}$ with repeat vertices α'_1, α'_2 in $\mathcal{C}$, γ a neighbor of α not contained in $\mathcal{C}$, and $\mu_1, \mu_2, \ldots, \mu_{d-2}$ the neighbors of α'_2 not contained in $\mathcal{C}$.*

Then there is in Γ a vertex $\mu \in \{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$ and a $2D$-cycle $\mathcal{C}^1$ such that γ and μ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$.

Proof. Let α'_3 be the neighbor of α'_2 in $\mathcal{C}$ other than α'_1. For $1 \leq i \leq d-2$, consider the path $P^i = \gamma - \mu_i$. Since all vertices in $\mathcal{C}$ are saturated, P^i cannot go through $\mathcal{C}$ and must be a D-path, so $P^i \cap \mathcal{C} = \emptyset$. Also, it follows that $V(P^i \cap P^j) = \{\gamma\}$ for any $1 \leq i < j \leq d-2$; otherwise either $\mathrm{g}(\Gamma) < 2D - 1$ or the vertex α'_2 would belong to an additional short cycle, both contradictions to Proposition 4.2. See Fig. 2 (a).

Let ρ be a neighbor of γ other than α, not contained in any of the paths $P^1, P^2, \ldots, P^{d-2}$ (there is exactly one such vertex). Consider a path $P = \rho - \alpha'_2$. P cannot go through α'_3; otherwise there would be a second short cycle $\rho P \alpha'_3 \mathcal{C} \alpha \gamma \rho$ in Γ containing α. Similarly, P cannot go through α'_1 and consequently, it must go through a vertex $\mu_k \in \{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$. Finally note that, since all vertices in $\mathcal{C}$ are saturated and $2D - 1 \leq \mathrm{g}(\Gamma) \leq 2D$, P must be a D-path, $V(P \cap P^k) = \{\mu_k\}$ and $V(P \cap \mathcal{C}) = \{\alpha'_2\}$.

This way, we ascertain that there is a vertex $\mu = \mu_k$ and a $2D$-cycle $\mathcal{C}^1 = \gamma \rho P \mu P^k \gamma$ such that γ and μ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$ (Fig. 2 (b)). □

Lemma 4.2 (Even Saturating Lemma) *Let Γ be a $(d, D, -2)$-graph with $d \geq 4$ and $D \geq 2$, and $\mathcal{C}$ a $2D$-cycle in Γ. Let α, α' be two vertices in $\mathcal{C}$ such that $\alpha' = \mathrm{rep}^{\mathcal{C}}(\alpha)$, γ a neighbor of α not*

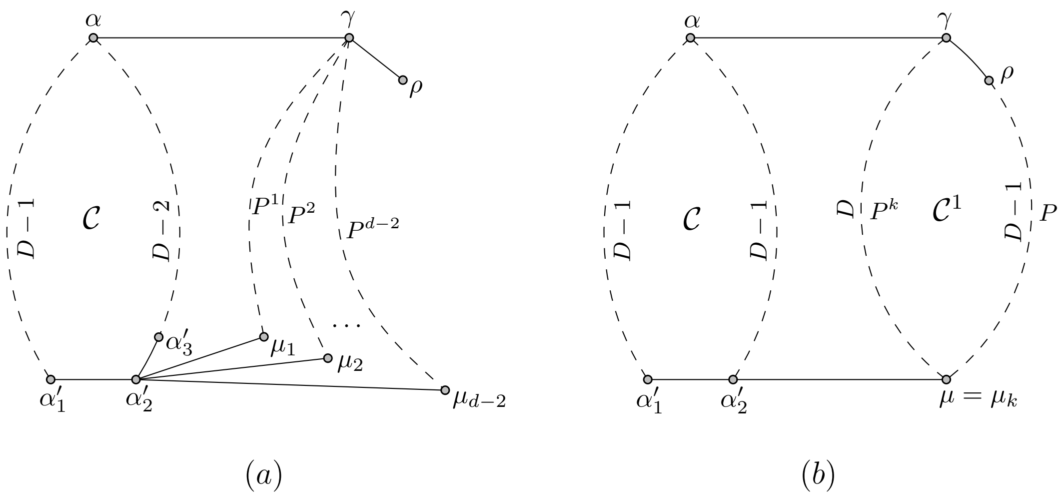

Figure 2: Auxiliary figure for Lemma 4.1

contained in $\mathcal{C}$, and $\mu_1, \mu_2, \ldots, \mu_{d-2}$ the neighbors of α' not contained in $\mathcal{C}$. Suppose there is no short cycle in Γ containing the edge $\alpha \sim \gamma$ and intersecting $\mathcal{C}$ at a path of length greater than $D-2$.

Then there is in Γ a vertex $\mu \in \{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$ and a short cycle $\mathcal{C}^1$ such that γ and μ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$.

Proof. Let α'_1, α'_2 be the neighbors of α' contained in $\mathcal{C}$. First, consider a path $P = \gamma - \alpha'$. Since there is no short cycle in Γ containing the edge $\alpha \sim \gamma$ and intersecting $\mathcal{C}$ at a path of length greater than $D-2$, P must be a D-path and cannot go through α'_1 or α'_2. Therefore, the path P must go through one of the neighbors of α not contained in $\mathcal{C}$ (say μ_1). In addition, we have that $V(P \cap \mathcal{C}) = \{\alpha'\}$. See Fig. 3 (a).

Let $\rho_1, \rho_2, \ldots, \rho_{d-2}$ be the neighbors of γ other than α, not contained in P. For $1 \leq i \leq d-2$, consider the path $P^i = \rho_i - \alpha'$. Since there is no short cycle in Γ containing the edge $\alpha \sim \gamma$ and intersecting $\mathcal{C}$ at a path of length greater than $D-2$, P^i must have length at least $D-1$ and cannot contain any of the vertices in $\{\alpha'_1, \alpha'_2, \gamma\}$. Consequently, P^i must go through one of the vertices in $\{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$. Note also that $V(P^i \cap \mathcal{C}) = \{\alpha'\}$ and that $V(P^i \cap P^j) \subseteq \{\alpha'\} \cup \{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$, for any $1 \leq i < j \leq d-2$.

If, for some j $(1 \leq j \leq d-2)$, the path P^j goes through μ_1 then P^j must be a D-path and there is a $(2D-1)$-cycle $\mathcal{C}^1 = \gamma P \mu_1 P^j \rho_j \gamma$ in Γ such that γ and $\mu = \mu_1$ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$. This case is depicted in Fig. 3 (b).

If, on the other hand, there is no j $(1 \leq j \leq d-2)$ such that P^j goes through μ_1 then there must exist a vertex μ_k $(2 \leq k \leq d-2)$ and paths P^r, P^s $(1 \leq r < s \leq d-2)$ such that both P^r and P^s go through μ_k. Since $\mathrm{g}(\Gamma) \geq 2D-1$, at most one of the paths P^r, P^s has length $D-1$. If one of these paths (say P^r) has length $D-1$ then there is a $(2D-1)$-cycle $\mathcal{C}^1 = \gamma \rho_r P^r \mu_k P^s \rho_s \gamma$ in Γ such that γ and $\mu = \mu_k$ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$ (as in Fig. 3 (c)). If both P^r and P^s are D-paths then there is a $2D$-cycle $\mathcal{C}^1 = \gamma\rho_r P^r \mu_k P^s \rho_s \gamma$ in Γ such that γ and $\mu = \mu_k$ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$ (as in Fig. 3 (d)). $\square$

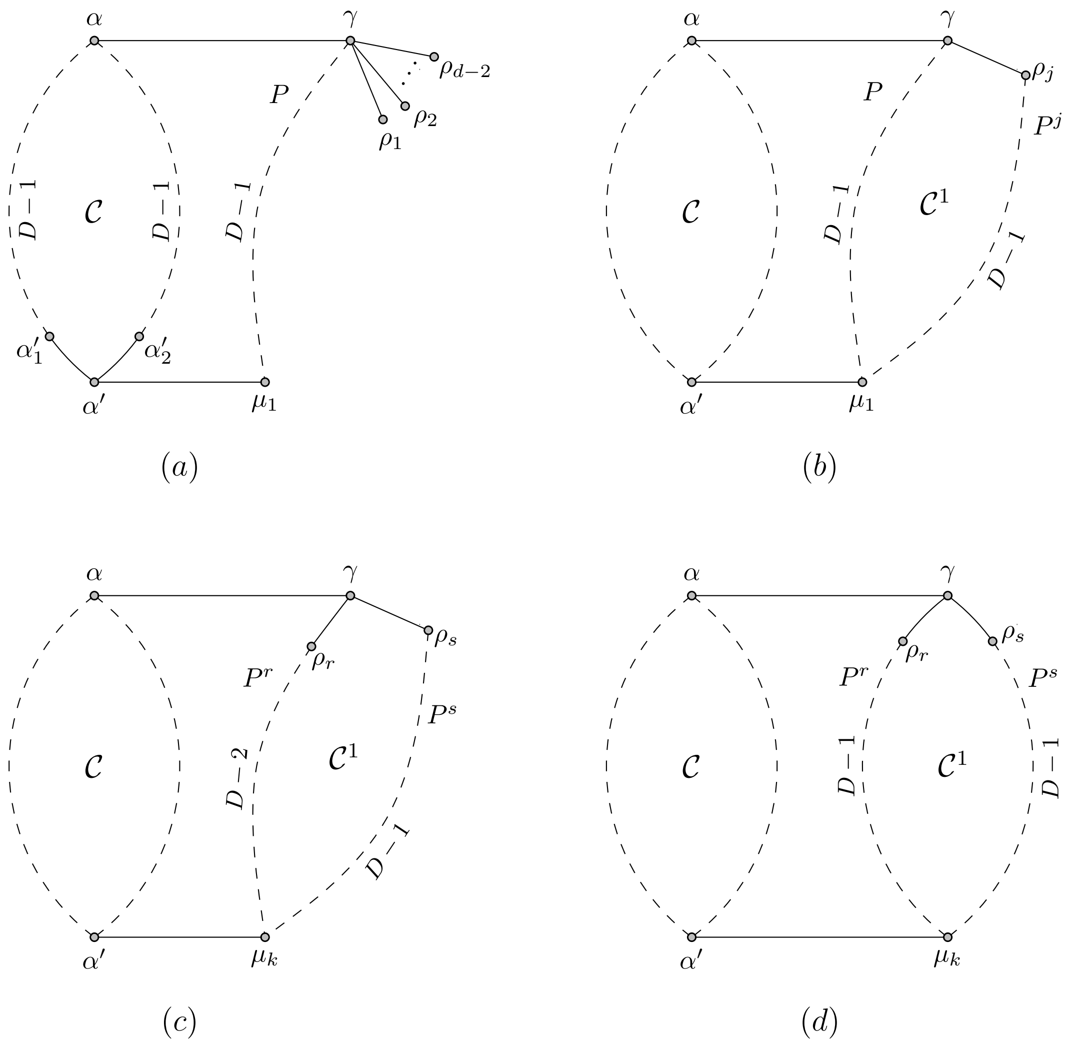

Figure 3: Auxiliary figure for Lemma 4.2

4.1 Repeats of Cycles

The extension of the concept of repeat to short cycles was introduced in [11] in the context of bipartite graphs missing the bipartite Moore bound by 4 vertices. Here, inspired by the ideas put forward in [11], we extend the concept of repeat to $2D$-cycles of graphs of defect 2; see the Repeat Cycle Lemma.

Lemma 4.3 (Repeat Cycle Lemma) *Let Γ be a $(d, D, -2)$-graph with $d \geq 4$ and $D \geq 2$, and C a $2D$-cycle in Γ. Let $\{C^1, C^2, \ldots, C^k\}$ be the set of neighbor cycles of C, and $I_i = C^i \cap C$ for $1 \leq i \leq k$. Suppose at least one I_j, for $j \in \{1, \ldots, k\}$, is a path of length smaller than $D - 1$. Then there is an additional $2D$-cycle C' in Γ intersecting C^i at $I'_i = \mathrm{rep}^{C^i}(I_i)$, where $1 \leq i \leq k$.*

Proof. We denote the neighbors of C by $C^1, C^2, \dots C^k$ and their corresponding intersection paths with C by $I_1 = x_1 - y_1, I_2 = x_2 - y_2, \dots, I_k = x_k - y_k$ in such a way that $C = x_1 I_1 y_1 x_2 I_2 y_2 \dots x_k I_k y_k x_1$. For $1 \leq i \leq k$, we also denote the repeats of I_i by $I'_i = x'_i - y'_i$, where $x'_i = \text{rep}^{C^i}(x_i)$ and $y'_i = \text{rep}^{C^i}(y_i)$ (see Fig. 4 (a)).

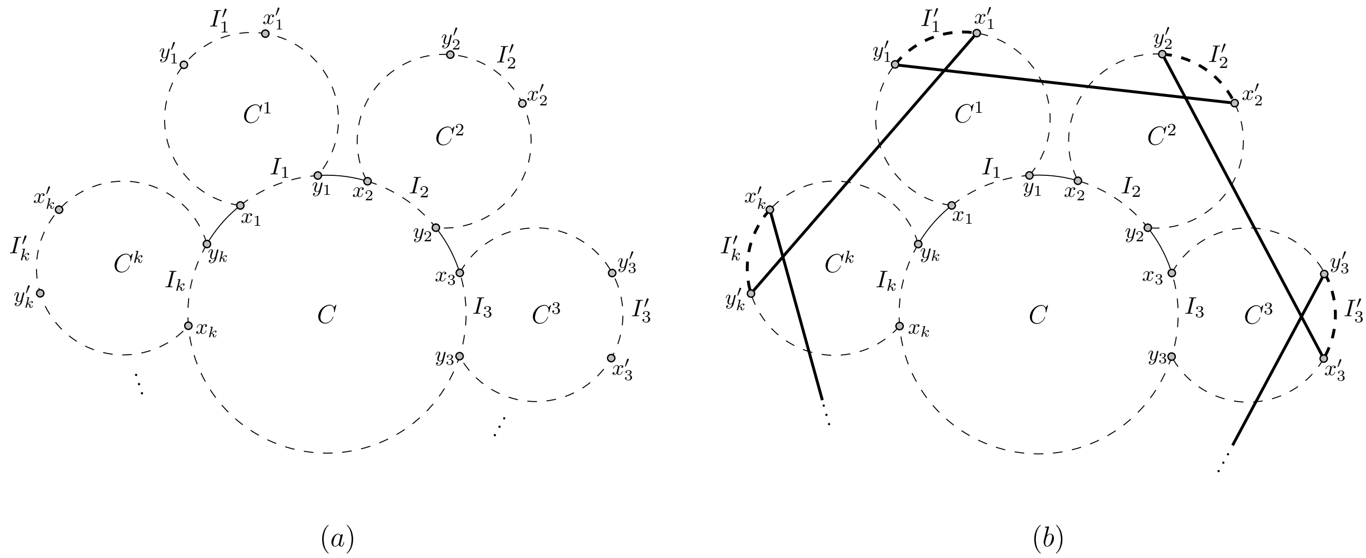

Figure 4: Auxiliary figure for Lemma 4.3

For $1 \leq i \leq k$, consider the cycles C^i and $C^{(i \bmod k)+1}$.

Suppose that I_i is a path of length smaller than $D-1$. Since y_i is saturated, there cannot be a short cycle in Γ, other than C, containing the edge $y_i \sim x_{(i \bmod k)+1}$. Since I_i is a path of length smaller than $D-1$, we apply the Even Saturating Lemma (mapping C^i to $\mathcal{C}$, y_i to α, y'_i to α' and $x_{(i \bmod k)+1}$ to γ) and obtain an additional short cycle $\mathcal{C}^1$ in Γ such that $x_{(i \bmod k)+1}$ is a repeat in $\mathcal{C}^1$ of a neighbor $\mu \notin C^i$ of y'_i, and $\mathcal{C}^1 \cap C^i = \emptyset$. Since $x_{(i \bmod k)+1}$ is saturated, we have that necessarily $\mathcal{C}^1 = C^{(i \bmod k)+1}$, which, in turn, implies $\mu = x'_{(i \bmod k)+1}$. In other words, it follows that $y'_i \sim x'_{(i \bmod k)+1} \in E(\Gamma)$.

If, instead, I_i is a $(D-1)$-path then $I_{(i \bmod k)+1}$ must be a path of length smaller than $D-1$, otherwise there would not exist a path I_j $(1 \leq j \leq k)$ of length smaller than $D-1$, contrary to our assumptions. Therefore, we can apply the above reasoning and deduce that $x'_{(i \bmod k)+1} \sim y'_i \in E(\Gamma)$.

This way we obtain a subgraph $\Upsilon = \bigcup_{i=1}^{k} \left(I'_i \cup y'_i \sim x'_{(i \bmod k)+1} \right) = x'_1 I'_1 y'_1 x'_2 I'_2 y'_2 \dots x'_k I'_k y'_k x'_1$ intersecting C^i at I'_i for $1 \leq i \leq k$ (see Fig. 4(b), where part of the subgraph Υ is highlighted in bold).

We next show that Υ must be indeed a cycle.

Claim 1. Υ is a $2D$-cycle.

Proof of Claim 1. First note that Υ is connected and that $|\Upsilon| \leq 2D$. By Proposition 4.2, unless Υ is a $2D$-cycle, Υ contains no short cycle. If the neighbors of C are pairwise disjoint then Υ is a

$2D$-cycle. Suppose that some neighbors of C are non-disjoint and that Υ is not a cycle, then Υ is a tree.

Let $z \in C^\ell$ be an arbitrary leaf in Υ. If the repeat path $I'_\ell = x'_\ell - y'_\ell$ had length greater than 0, then z would have at least two neighbors in Υ. Therefore, $I_\ell = C \cap C^\ell$ contains exactly one vertex, and thus, $x_\ell = y_\ell$ and $z = x'_\ell = y'_\ell$.

Recall we perform addition modulo k on the subscripts of the vertices and the superscripts of the cycles.

Since $x'_\ell \sim y'_{\ell-1}$ and $x'_\ell \sim x'_{\ell+1}$ are edges in Υ, it holds that $y'_{\ell-1}$ and $x'_{\ell+1}$ denote the same vertex. Let $u'_{\ell-1}, v'_{\ell-1}$ be the neighbors of $y'_{\ell-1}$ in $C^{\ell-1}$; $u'_{\ell+1}, v'_{\ell+1}$ the neighbors of $x'_{\ell+1}$ in $C^{\ell+1}$; and u_ℓ, v_ℓ the neighbors of x_ℓ in C^ℓ. We have that $V(C^{\ell-1} \cap C^{\ell+1}) = \{y'_{\ell-1}\}$, otherwise there would be a third short cycle in Γ containing x_ℓ. In particular, the vertices in $\{u'_{\ell-1}, v'_{\ell-1}, u'_{\ell+1}, v'_{\ell+1}, x'_\ell\}$ are pairwise distinct and $d \geq 5$. See Fig. 5 (a) and (b) for two drawings of this situation.

Now consider a path $P = x_\ell - y'_{\ell-1}$. Since x_ℓ cannot be contained in a further short cycle, we have that P must be a D-path and go through a neighbor $w'_{\ell-1}$ of $y'_{\ell-1}$ not contained in $\{u'_{\ell-1}, v'_{\ell-1}, u'_{\ell+1}, v'_{\ell+1}, x'_\ell\}$, which implies $d \geq 6$. By similar arguments, we obtain that P must go through a neighbor w_ℓ of x_ℓ not contained in $\{y_{\ell-1}, x_{\ell+1}, u_\ell, v_\ell\}$.

Finally, let $t_1, t_2, \ldots, t_{d-5}$ denote the vertices in $N(x_\ell) - \{y_{\ell-1}, x_{\ell+1}, u_\ell, v_\ell, w_\ell\}$; see Fig. 5 (c). Consider a path $Q^i = t_i - y'_{\ell-1}$. Since x_ℓ cannot be contained in a further short cycle, Q^i must be a D-path and go through a neighbor of $y'_{\ell-1}$ not contained in $\{u'_{\ell-1}, v'_{\ell-1}, u'_{\ell+1}, v'_{\ell+1}, x'_\ell, w'_{\ell-1}\}$. Therefore, we have that $d \geq 7$ and, by the pigeonhole principle, that there are two paths Q^r and Q^s containing a common neighbor of $y'_{\ell-1}$. This way, x_ℓ would be contained in a third short cycle, a contradiction.

As a result, we conclude that the repeat graph Υ of C is indeed a $2D$-cycle C' as claimed. This completes the proof of Claim 1, and thus, of the lemma. □

We call the aforementioned cycle C' the *repeat* of the cycle C in Γ, and denote it by $\text{rep}(C)$. Some simple consequences of the Repeat Cycle Lemma follow next.

Corollary 4.1 (Repeat Cycle Uniqueness) *If a $2D$-cycle C has a repeat cycle C' then C' is unique.*

Corollary 4.2 (Repeat Cycle Symmetry) *If $C' = \text{rep}(C)$ then $C = \text{rep}(C')$.*

Corollary 4.3 *Let Γ be a $(d, D, -2)$-graph with $d \geq 4$ and $D \geq 2$. Let C, C^1 be two $2D$-cycles in Γ which intersect at a path I of length smaller than $D - 1$, and set $I' = \text{rep}^{C^1}(I)$. Then the repeat cycle of C intersects C^1 at I'.*

Corollary 4.4 (Handy Corollary) *Let Γ be a $(d, D, -2)$-graph with $d \geq 4$ and $D \geq 2$, $\mathcal{C}$ a $2D$-cycle in Γ, and x, x' repeat vertices in $\mathcal{C}$. Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be $2D$-cycles other than $\mathcal{C}$ containing x*

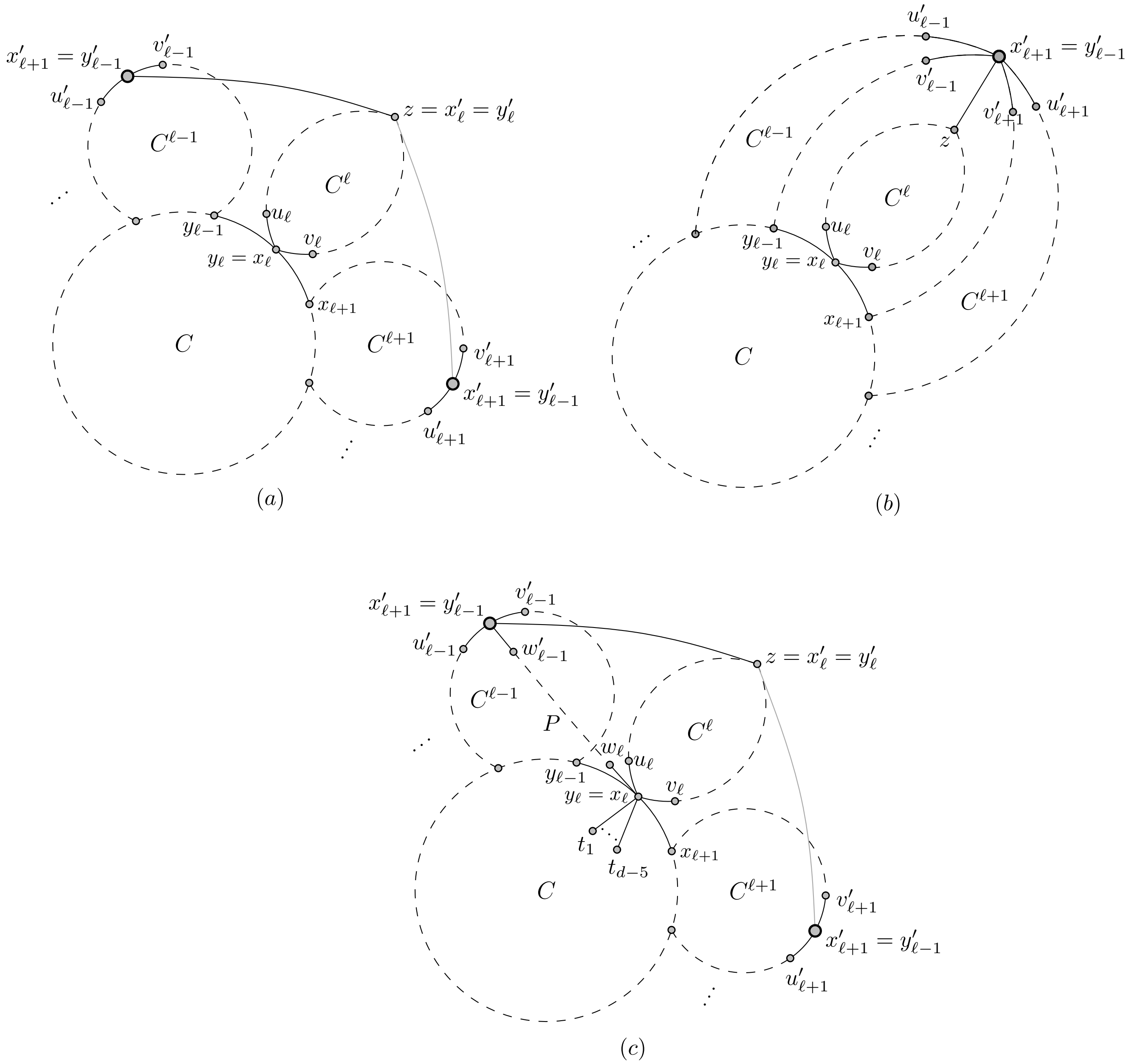

Figure 5: Auxiliary figure for Claim 1 of Lemma 4.3.

and x', respectively. Suppose that $I = \mathcal{C}^1 \cap \mathcal{C}$ is a path of length smaller than $D-1$. Then, setting $y = \mathrm{rep}^{\mathcal{C}^1}(x)$ and $y' = \mathrm{rep}^{\mathcal{C}^2}(x')$, we have that y and y' are repeat vertices in the repeat cycle of $\mathcal{C}$.

Proof. We denote the k neighbor cycles of $\mathcal{C}$ by $E^1, E^2, \dots E^k$ and their respective intersection paths with $\mathcal{C}$ by $I_1 = x_1 - y_1, I_2 = x_2 - y_2, \dots, I_k = x_k - y_k$ in such a way that $\mathcal{C} = x_1 I_1 y_1 x_2 I_2 y_2 \dots x_k I_k y_k x_1$. For $1 \le j \le k$, we also denote $I'_j = x'_j - y'_j$, where $x'_j = \mathrm{rep}^{E^j}(x_j)$ and $y'_j = \mathrm{rep}^{E^j}(y_j)$.

Obviously, for some r, s $(1 \le r, s \le k)$ we have that $\mathcal{C}^1 = E^r$, $\mathcal{C}^2 = E^s$, $x \in I_r$, $x' \in I_s$, $y \in I'_r$, and $y' \in I'_s$. We may assume $r < s$. By the Repeat Cycle Lemma, the vertices y and y' belong to the repeat cycle $\mathcal{C}'$ of $\mathcal{C}$. Then the paths $x I_r y_r x_{r+1} I_{r+1} y_{r+1} \dots x_{s-1} I_{s-1} y_{s-1} x_s I_s x' \subset \mathcal{C}$ and $y I'_r y'_r x'_{r+1} I'_{r+1} y'_{r+1} \dots x'_{s-1} I'_{s-1} y'_{s-1} x'_s I'_s y' \subset \mathcal{C}'$ are both D-paths in Γ, and the corollary follows. $\square$

5 Main Results

5.1 On the girth of $(d, D, -2)$-graphs

Proposition 5.1 *A $(d, D, -2)$-graph Γ with $d \geq 4$ and $D \geq 4$ does not contain $(2D-1)$-cycles.*

Proof. Suppose, by way of contradiction, that there is a $(2D-1)$-cycle C in Γ.

Let p_1, p_2 be two repeat vertices in C, and q_1 a neighbor of p_1 not contained in C. According to the Odd Saturating Lemma, there are both a neighbor q_2 of p_2 not contained in C and a $2D$-cycle D^1, such that q_1 and q_2 are repeats in D^1 (see Fig. 6 (a)).

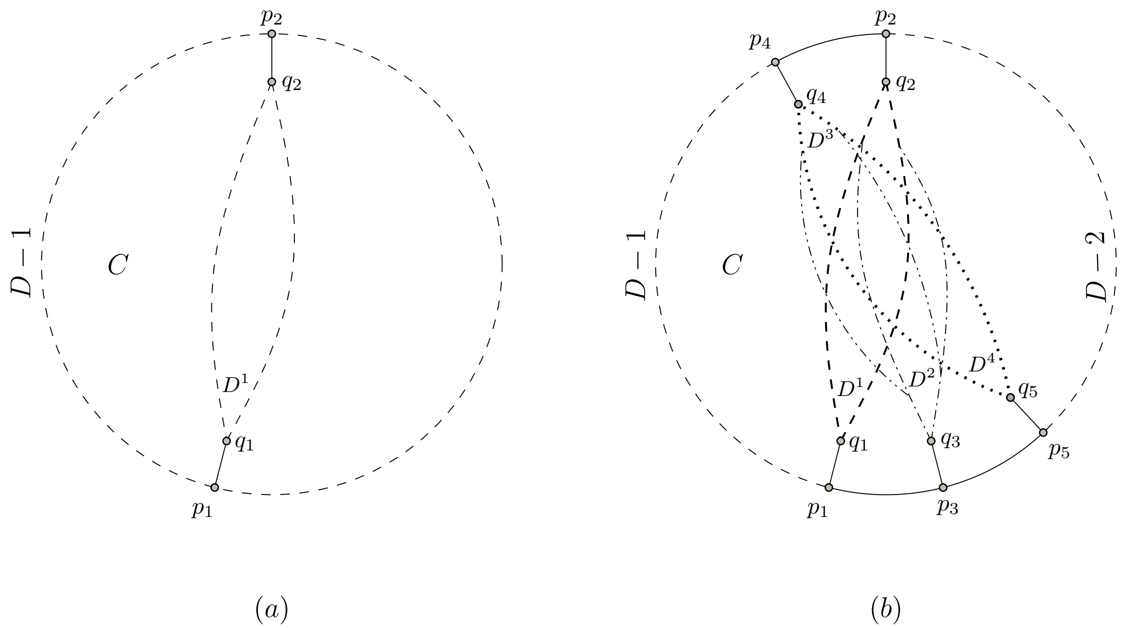

Figure 6: Auxiliary figure for Proposition 5.1

For $1 \leq i \leq 3$, denote by p_{i+2} the repeat of p_{i+1} in C other than p_i. We now apply the Odd Saturating Lemma (mapping C to $\mathcal{C}$, p_2 to α, p_3 to α_2', q_2 to γ) and ascertain the existence of a $2D$-cycle D^2 and a neighbor q_3 of p_3 not contained in C, such that q_2 and q_3 are repeats in D^2. For $i = 3, 4$ by repeatedly applying the Odd Saturating Lemma (mapping C to $\mathcal{C}$, p_i to α, p_{i+1} to α_2', q_i to γ) we ensure the existence of a $2D$-cycle D^i and a neighbor q_{i+1} of p_{i+1} not contained in C, such that q_i and q_{i+1} are repeats in D^i. See Fig. 6 (b).

Note that $D^1 \cap D^2$ is a path of length at most $2 < D-1$; otherwise for some vertex $t \in D^1 \cap D^2$ the cycle $tD^1q_1p_1p_3q_3D^2t$ would have length at most $2D-1$, a contradiction. Similarly, $D^2 \cap D^3$ and $D^3 \cap D^4$ are paths of length at most 2.

We now apply the Handy Corollary. By mapping the cycle D^2 to $\mathcal{C}$, the vertex q_2 to x, the vertex q_3 to x', the cycle D^1 to $\mathcal{C}^1$, the cycle D^3 to $\mathcal{C}^2$, the vertex q_1 to y and the vertex q_4 to y', we obtain that q_1 and q_4 are repeat vertices in the repeat cycle of D^2. Therefore, since $q_4 \in D^4$, it follows that

D^2 and D^4 are repeat cycles and $q_1 = q_5$. As a consequence, there is in Γ a cycle $q_1p_1p_3p_5q_5$ of length $4 < 2D - 1$, a contradiction. □

From Propositions 4.2 and 5.1, it follows immediately that

Theorem 5.1 *The girth of a $(d, D, -2)$-graph Γ with $d \geq 4$ and $D \geq 4$ is $2D$.*

5.2 Non-existence of subgraphs isomorphic to Θ_D

Proposition 5.2 *A $(d, D, -2)$-graph Γ with $d \geq 4$ and $D \geq 4$ does not contain a subgraph isomorphic to Θ_D.*

Proof. In this proof our reasoning resembles that of Proposition 5.1.

Suppose that Γ contains a subgraph Θ isomorphic to Θ_D, with branch vertices a and b. Let p_1, p_2, p_3, p_4 and p_5 be as in Fig. 7 (a), and let q_1 be one of the neighbors of p_1 not contained in Θ.

Since all vertices of Θ are saturated, there cannot be a short cycle in Γ containing any of the incident edges of p_1, p_2, p_3, p_4 or p_5 which are not contained in Θ. According to this and by applying the Even Saturating Lemma, there is an additional $2D$-cycle D^1 in Γ such that q_1 and one of the neighbors of p_2 not contained in Θ (say q_2) are repeats in D^1. Also, it follows that $D^1 \cap \Theta = \emptyset$. Analogously, by repeatedly applying the Even Saturating Lemma, for $2 \leq i \leq 4$ we obtain that there is an additional $2D$-cycle D^i such that q_i and one of the neighbors of p_{i+1} not contained in Θ (say q_{i+1}) are repeats in D^i. Also, we have that $D^i \cap \Theta = \emptyset$ (see Fig. 7 (b)).

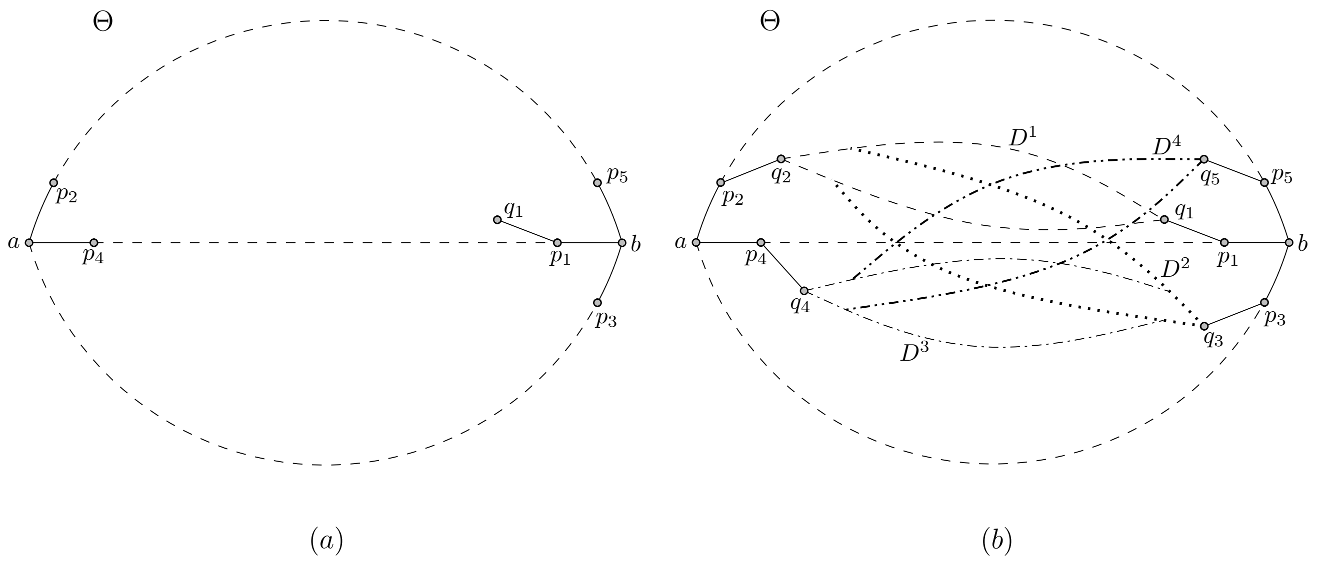

Figure 7: Auxiliary figure for Proposition 5.2

Note that $D^1 \cap D^2$ is a path of length at most $2 < D - 1$; otherwise for some vertex $t \in D^1 \cap D^2$ there would be a cycle $tD^1q_1p_1bp_3q_3D^2t$ of length at most $2D$ to which the vertex b would belong, a contradiction. For similar reasons, the intersection paths $D^2 \cap D^3$ and $D^3 \cap D^4$ have both length at most 2.

We now apply the Handy Corollary. By mapping the cycle D^2 to $\mathcal{C}$, the vertex q_2 to x, the vertex q_3 to x', the cycle D^1 to $\mathcal{C}^1$, the cycle D^3 to $\mathcal{C}^2$, the vertex q_1 to y and the vertex q_4 to y', we obtain that q_1 and q_4 are repeat vertices in the repeat cycle of D^2. Therefore, since $q_4 \in D^4$, it follows that D^2 and D^4 are repeat cycles and $q_1 = q_5$; but then there is a cycle $q_1p_1bp_5q_5$ in Γ of length $4 < 2D$, a contradiction. □

Corollary 5.1 *Every vertex in a $(d, D, -2)$-graph Γ with $d \geq 4$ and $D \geq 4$ is of Type (iii).*

5.3 Non-existence results on $(d, D, -2)$-graphs

In view of Corollary 5.1, the following corollary, which was obtained in [7], follows immediately.

Corollary 5.2 (Corollary 2.3 from [7]) *The feasible values of d for $(d, D, -2)$-graphs are restricted according to the following conditions.*

When D is even, d is odd.

When D is a power of an odd prime, $d - 1$ is a multiple of D.

When $D \geq 4$ is a power of 2, $d - 1$ is a multiple of $D/2$.

Proposition 5.3 *The number N_{2D} of $2D$-cycles in a $(d, D, -2)$-graph Γ with $d \geq 4$ and $D \geq 4$ is given by the expression $N_{2D} = \frac{n}{D} = \frac{d\left(1+(d-1)+\ldots+(d-1)^{D-1}\right)-1}{D}$, where n is the order of Γ.*

Proof. According to Proposition 4.2 and Corollary 5.1, every vertex of Γ is contained in exactly two $2D$-cycles. We then count the number N_{2D} of $2D$-cycles of Γ . Since the order of Γ is $n = 1 + d\big(1 + (d-1) + \ldots + (d-1)^{D-1}\big) - 2$, we have that

$$N_{2D} = \frac{2\times\left(1+d\left(1+(d-1)+\ldots+(d-1)^{D-1}\right)-2\right)}{2D} = \frac{d\left(1+(d-1)+\ldots+(d-1)^{D-1}\right)-1}{D},$$

and the proposition follows. □

Lemma 5.1 *Every two non-disjoint $2D$-cycles in a $(d, D, -2)$-graph Γ with $d \geq 4$ and $D \geq 4$ intersect at a path of length at most $D - 2$.*

Proof. We follow a strategy very similar to the one used in the proof of [11, Lemma 5.1].

Since Γ does not contain a graph isomorphic to Θ_D, it is only necessary to prove here that any two non-disjoint $2D$-cycles in Γ cannot intersect at a path of length $D-1$. Suppose, by way of contradiction, that there are two $2D$-cycles C^1 and C^2 in Γ intersecting at a path I_1 of length $D-1$.

Let v be an arbitrary vertex on I_1, and $v' = \text{rep}^{C^2}(v)$. Let C^3 be the other $2D$-cycle containing v', and $I_2 = C^2 \cap C^3$. If I_2 were a path of length smaller than $D-1$ then, by Corollary 4.3, the repeat cycle of C^3 would intersect C^2 at a proper subpath of I_1 containing v. This is a clear contradiction to the fact that v is already saturated. Consequently, I_2 must be a $(D-1)$-path and C^2 is intersected by exactly two $2D$-cycles, namely C^1 and C^3, at two independent $(D-1)$-paths.

By repeatedly applying this reasoning and considering that Γ is finite, we obtain a maximal length sequence $C^1, C^2, C^3, \ldots, C^m$ of pairwise distinct $2D$-cycles in Γ such that C^i intersects C^{i+1} at a path I_i of length $D-1$ ($1 \leq i \leq m-1$). Furthermore, it follows that $C^j \cap C^k = \emptyset$ for any $j,k \in \{1,\ldots,m\}$ such that $2 \leq |i-j| \leq m-2$. Let us denote the paths $I_1 = x_1 - y_1, \ldots, I_{m-1} = x_{m-1} - y_{m-1}$ in such a way that, for $1 \leq i \leq m-2$, $x_i \sim x_{i+1}$ and $y_i \sim y_{i+1}$ are edges in Γ. Also, let $x_0 \in N(x_1) \cap (C^1 - I_1)$, $y_0 \in N(y_1) \cap (C^1 - I_1)$, $x_m \in N(x_{m-1}) \cap (C^m - I_{m-1})$, and $y_m \in N(y_{m-1}) \cap (C^m - I_{m-1})$. Figure 8 (a) shows this configuration. Set $I_0 = x_0 - y_0$ and $I_m = x_m - y_m$. Since the sequence $C^1, C^2, C^3, \ldots, C^m$ is maximal and all the vertices in $I_1, \ldots, I_{m-1}$ are saturated, it follows that $I_0 = I_m$, and we have either $x_0 = x_m$ and $y_0 = y_m$ (as in Fig. 8 (b)), or $x_0 = y_m$ and $y_0 = x_m$ (as in Fig. 8 (c)).

If $x_0 = x_m$ and $y_0 = y_m$ then $m > 2D$; otherwise the cycle $x_1 x_2 \ldots x_m x_1$ would have length at most $2D$, contradicting the saturation of x_1. If, conversely, $x_0 = y_m$ and $y_0 = x_m$ then $m > D$; otherwise the cycle $x_1 x_2 \ldots x_m y_1 y_2 \ldots y_m x_1$ containing x_1 would have length at most $2D$, a contradiction. For our purposes, it is enough to state $m > D \geq 4$ in any case.

We now proceed with the second part of the proof.

Let q_1 a neighbor of y_1 not contained in $\cup_{i=1}^{5} C^i$ (see Fig. 9 (a)).

Since y_1 is saturated, the edge $q_1 \sim y_1$ cannot be contained in a further short cycle. We apply the Even Saturating Lemma (by mapping C^2 to $\mathcal{C}$, y_1 to α, x_2 to α', and q_1 to γ), and obtain in Γ an additional $2D$-cycle D^1 such that q_1 and one of the neighbors of x_2 not contained in $\cup_{i=1}^{5} C^i$ (say q_2) are repeats in D^1, and $D^1 \cap C^2 = \emptyset$. Analogously, there exists an additional $2D$-cycle D^2 such that q_2 and a neighbor of y_3 not contained in $\cup_{i=1}^{5} C^i$ (say q_3) are repeats in D^2, and $D^2 \cap C^3 = \emptyset$; an additional $2D$-cycle D^3 such that q_3 and a neighbor of x_4 not contained in $\cup_{i=1}^{5} C^i$ (say q_4) are repeats in D^3, and $D^3 \cap C^4 = \emptyset$; and an additional $2D$-cycle D^4 such that q_4 and a neighbor of y_5 not contained in $\cup_{i=1}^{5} C^i$ (say q_5) are repeats in D^4, and $D^4 \cap C^5 = \emptyset$. See Fig. 9 (b).

Note that $D^1 \cap D^2$ cannot be a $(D-1)$-path; otherwise for some vertex $t \in D^1 \cap D^2$ there would be a cycle $tD^1 q_1 y_1 y_2 y_3 q_3 D^2 t$ of length at most $4 + D - 2 + D - 2$ (since $D - 1 \geq 3$), a contradiction to the fact that y_1 is saturated and $\text{g}(\Gamma) = 2D$. Analogously, $D^i \cap D^{i+1}$ cannot be a $(D-1)$-path for $i = 2, 3$.

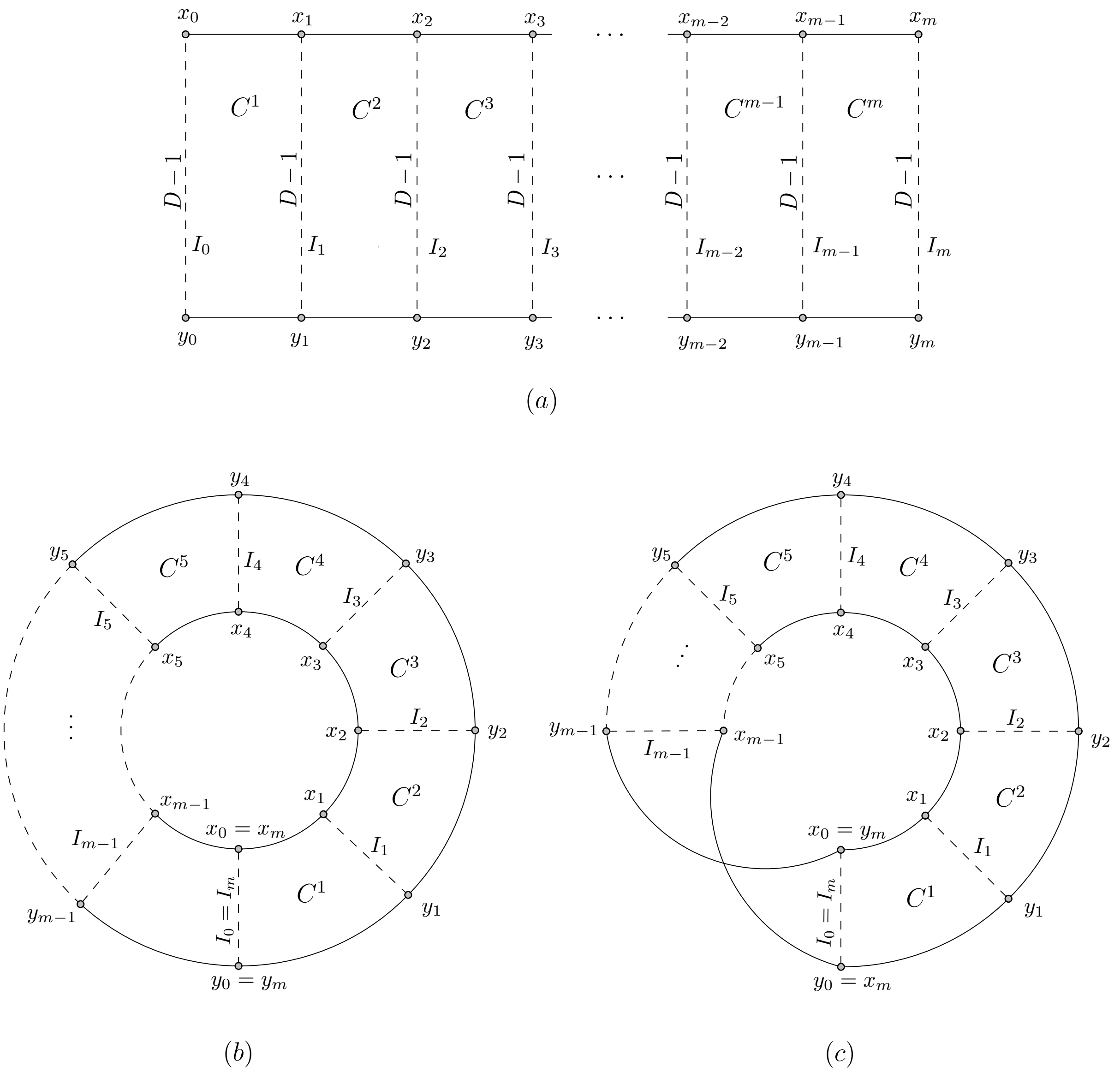

Figure 8: Auxiliary figure for Lemma 5.1

We now apply the Handy Corollary as in the proofs of the previous theorems. By mapping the cycles D^2 to $\mathcal{C}$, D^1 to $\mathcal{C}^1$ and D^3 to $\mathcal{C}^2$, and the vertices q_2 to x, q_3 to x', q_1 to y, and q_4 to y', it follows that the vertices q_1 and q_4 are repeat vertices in the repeat cycle of D^2. Since $q_4 \in D^4$, we have that D^2 and D^4 are repeat cycles and that $q_5 = q_1$. This way, we obtain a cycle $q_1y_1y_2y_3y_4y_5q_5$ in Γ of length $6 < 2D$, a contradiction.

This completes the proof of the lemma. □

We are now in a position to prove our second main result.

Theorem 5.2 *There are no $(d, D, -2)$-graphs with even $d \geq 4$ and $D \geq 4$.*

Proof. Suppose there is a $(d, D, -2)$-graph Γ with even $d \geq 4$ and $D \geq 4$.

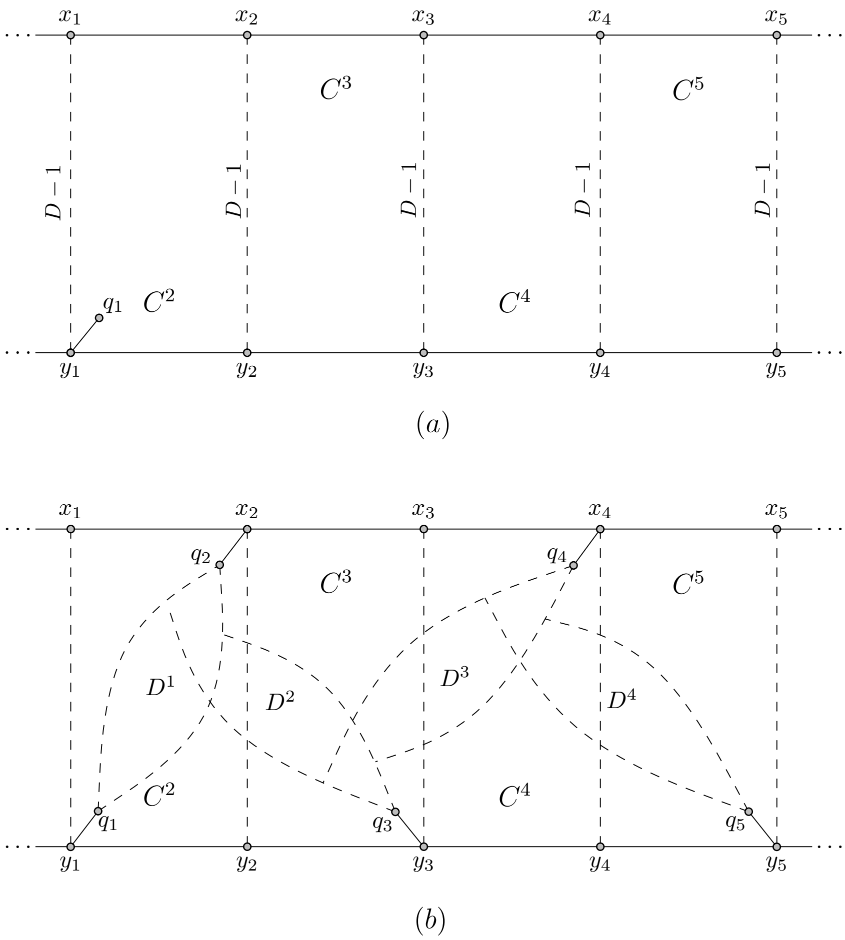

Figure 9: Auxiliary figure for Lemma 5.1

According to Lemma 5.1, any two non-disjoint $2D$-cycles in Γ intersect at a path of length smaller than $D-1$, which means that every $2D$-cycle C in Γ has a repeat cycle C' (by the Repeat Cycle Lemma). Because of the uniqueness and symmetry of repeat cycles, the number N_{2D} of $2D$-cycles in Γ must be even.

However, since d is even, the number $N_{2D} = \frac{d\left(1+(d-1)+\ldots+(d-1)^{D-1}\right)-1}{D}$ of $2D$-cycles in Γ is odd, a contradiction. $\square$

Note that Theorem 5.2 contains, as a special case, the result of the non-existence of $(4, D, -2)$-graphs for $D \geq 4$, which was claimed prematurely in [18].

From Proposition 5.3 we easily derive the following results:

Theorem 5.3 *There are no $(d, D, -2)$-graphs with odd $d \geq 5$, $D \geq 4$ and order $n = d\left(1+(d-1)+\ldots+(d-1)^{D-1}\right) - 1 \not\equiv 0 \pmod{D}$.*

Corollary 5.3 *There are no $(d, D, -2)$-graphs with odd $d \geq 5$ and $D \geq 5$ such that $d \equiv 0, 2 \pmod{D}$.*

Furthermore, for a particular value of $D \geq 5$ it is possible to rule out the existence of $(d, D, -2)$-graphs with odd $d \geq 5$ for many other values of d, by considering the set of all possible residues of d in the division by D. If, for some $r \in \{0, 1, \ldots, D-1\}$, we have $d \equiv r \pmod{D}$ implies $d\big(1 + (d-1) + \ldots + (d-1)^{D-1}\big) - 1 \not\equiv 0 \pmod{D}$, then there are no $(d, D, -2)$-graphs with odd $d \geq 5$ such that $d \equiv r \pmod{D}$.

Accordingly, the following table shows all values of $4 \leq D \leq 16$ and odd $d \geq 5$ for which a $(d, D, -2)$-graph might still exist.

D	d
4	$d \equiv 1, 3 \pmod{4}$
5	$d \equiv 1 \pmod{10}$
6	$d \equiv 1 \pmod{6}$
7	$d \equiv 1 \pmod{14}$
8	$d \equiv 1, 5 \pmod{8}$
9	$d \equiv 1 \pmod{18}$
10	$d \equiv 1, 9 \pmod{10}$
11	$d \equiv 1 \pmod{22}$
12	$d \equiv 1, 7 \pmod{12}$
13	$d \equiv 1 \pmod{26}$
14	$d \equiv 1, 13 \pmod{14}$
15	$d \equiv 1, 13 \pmod{30}$
16	$d \equiv 1, 9 \pmod{16}$

6 Non-existence of $(4, 3, -2)$-graphs

In this section we prove the non-existence of $(4, 3, -2)$-graphs (see Theorem 6.1), which will allow us to provide the full catalogue of $(4, D, -2)$-graphs with $D \geq 2$.

Proposition 5.2 asserts the non-existence of a subgraph isomorphic to Θ_D in a $(d, D, -2)$-graph Γ with $d \geq 4$ and $D \geq 4$. We next give an alternative proof for $d = 4$ that covers also the case $D = 3$.

Proposition 6.1 *A $(4, D, -2)$-graph Γ with $D \geq 3$ does not contain a subgraph isomorphic to Θ_D.*

Proof. Suppose that Γ contains a subgraph Θ isomorphic to Θ_D, where α and α' are its branch vertices. Let α'_1, α'_2, α'_3, γ and μ be as in Figure 10 (a).

First consider a path $P = \gamma - \alpha'$. As α cannot belong to any further short cycle, P must go through μ and be a D-path. Let ρ_1 and ρ_2 be the neighbors of γ other than α and not contained in P. Consider a path $P^1 = \rho_1 - \alpha'$. As α is saturated, P^1 cannot go through α'_1, α'_2 or α'_3, so it must go through μ and be a D-path. This way, γ is contained in a $(2D-1)$-cycle $C = \gamma P \mu P^1 \rho_1 \gamma$, and γ

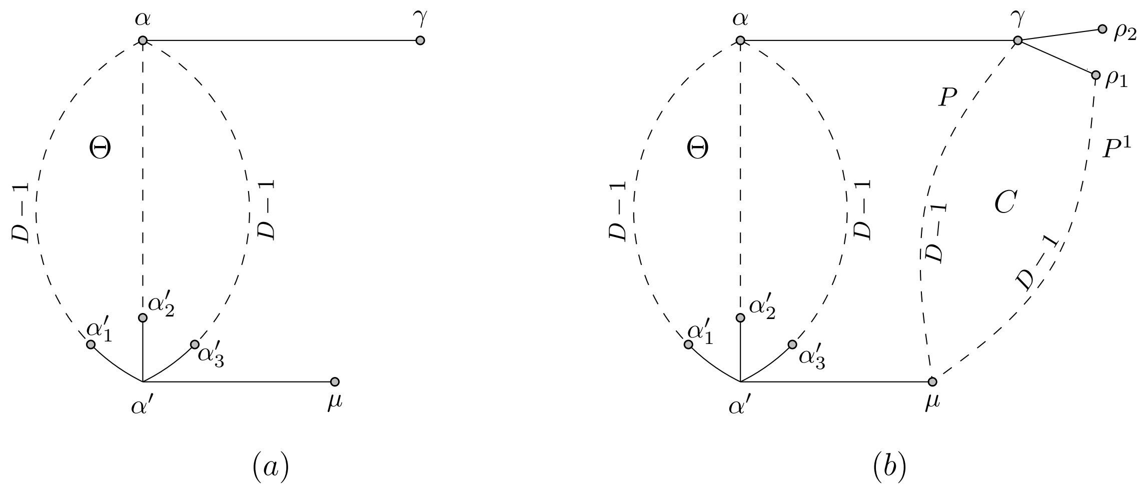

Figure 10: Auxiliary figure for Proposition 6.1.

becomes saturated. Analogously, a path $P^2 = \rho_2 - \alpha'$ must go through μ, causing the formation of another short cycle containing μ, a contradiction to Proposition 4.2 (ii). See Figure 10 (b). □

Next we prove that the girth of a $(4, 3, -2)$-graph must be 6 by ruling out the existence of 5-cycles.

Proposition 6.2 *A* $(4, 3, -2)$*-graph* Γ *has girth* 6.

Proof. We proceed by contradiction, supposing there is a 5-cycle C in Γ. In view of Proposition 4.2, the graph Γ contains the subgraph G of Fig. 11, where T_i denotes the enclosed set of 6 vertices at distance 2 from x_i, for $1 \leq i \leq 5$.

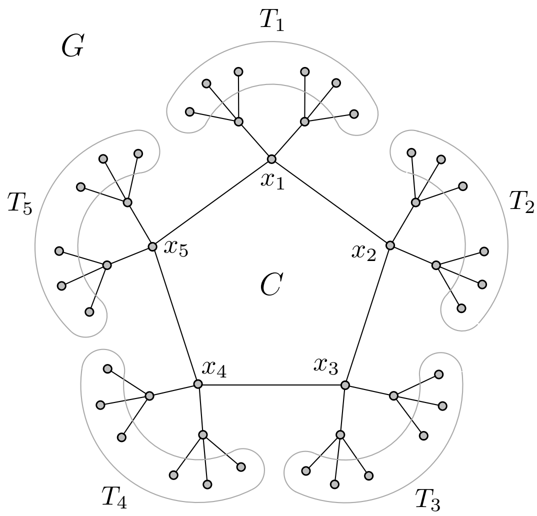

Figure 11: Auxiliary figure for Proposition 6.2.

Since $|\Gamma| = 51$ and $|G| = 45$, there is a set $X \subset V(\Gamma)$ such that $|X| = 6$ and $X \cap V(G) = \emptyset$. Any vertex $x \in X$ must be adjacent to a vertex in T_i, for $1 \leq i \leq 5$, in order to reach x_i in at most 3 steps. However, this is clearly impossible since Γ has degree 4. □

In view of Propositions 4.2, 6.1, 5.1 and 6.2, it follows that every vertex in a $(4, D, -2)$-graph Γ with $D \geq 3$ is contained in exactly two short cycles, namely, two $2D$-cycles.

Proposition 6.3 *The number N_{2D+1} of $(2D+1)$-cycles in a $(4, D, -2)$-graph Γ with $D \geq 3$ is given by $N_{2D+1} = \frac{2\times 3^D(2\times 3^D-3)}{2D+1}$.*

Proof. The number of $(2D+1)$-cycles in Γ is closely related to the number of edges involving only vertices at distance D from any vertex x in Γ. The number of vertices at level D is $4 \times 3^{D-1} - 2$, and the number of elements in the set F of edges involving only vertices at distance D from x is

$$|F| = \frac{2\times 2 + 3(4\times 3^{D-1} - 4)}{2} = 2\times 3^D - 4,$$

since x is contained in exactly two $2D$-cycles C^1 and C^2.

Denote by y_1 and y_2 the vertices at distance D from x on C^1 and C^2, respectively. Before proceeding to count, we prove that $y_1 \sim y_2 \notin E(\Gamma)$.

Claim 1. $y_1 \sim y_2 \notin E(\Gamma)$.

Proof of Claim 1. Suppose, by way of contradiction, that $y_1 \sim y_2 \in E(\Gamma)$. Since $\mathrm{g}(\Gamma) = 2D$, it holds that $V(C^1 \cap C^2) = \{x\}$; see Fig. 12. By Corollary 4.3, the repeat cycle C' of C^1 intersects C^2 exactly at y_2; consequently, C' contains the edge $y_1 \sim y_2$. However, this contradicts the fact that C^1 and its repeat cycle C' must be disjoint cycles. □

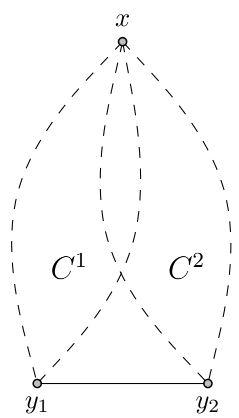

Figure 12: Auxiliary figure for Proposition 6.3.

Accordingly, we partition the set F into F_1, F_2 and F_3, where F_1 and F_2 are the sets of edges in F adjacent to the vertices y_1 and y_2, respectively, and F_3 contains the remaining edges in F.

Each edge in F_1 or F_2 determines two $(2D+1)$-cycles containing x, while each edge from F_3 determines only one $(2D+1)$-cycle containing x. Therefore, given that $|F_1| = |F_2| = 2$, we have that the number of $(2D+1)$-cycles passing through the vertex x is

$$2|F_1| + 2|F_2| + |F_3| = 4 + 4 + 2\times 3^D - 8 = 2\times 3^D.$$

Thus, the total number of $(2D+1)$-cycles in Γ is given by the expression

$$N_{2D+1} = \frac{2 \times 3^D(2 \times 3^D - 3)}{2D+1},$$

and the proposition follows. □

Now we can readily prove Theorem 6.1.

Theorem 6.1 *There is no* $(4, 3, -2)$*-graph.*

Proof. By Proposition 6.3, the number of 7-cycles in a $(4, 3, -2)$-graph is $2\times3^3(2\times3^3-3)/7 = 2754/7$, which is a contradiction. □

Theorems 5.2 and 6.1 tell us that the $(4, 2, -2)$-graph of Fig. 1 (d) is the only $(4, D, -2)$-graph for $D \geq 2$. Thus, we have successfully completed the census of all $(4, D, -2)$-graphs.

7 Conclusions

In this paper, by exploiting the idea of extending the concept of repeats to paths and cycles, put forward in [11], we obtained the results summarized below.

First, we proved that the girth of a $(d, D, -2)$-graph with $d \geq 4$ and $D \geq 4$ is $2D$. By obtaining necessary conditions for the existence of $(d, D, -2)$-graphs with $d \geq 4$ and $D \geq 4$, we proved the non-existence of $(d, D, -2)$-graphs with even $d \geq 4$ and $D \geq 4$. This outcome, together with a non-existence proof of $(4, 3, -2)$-graphs, completed the catalogue of $(4, D, -\epsilon)$-graphs with $D \geq 2$ and $0 \leq \epsilon \leq 2$.

Catalogue of $(4, D, 0)$**-graphs with** $D \geq 2$**.** There is no Moore graph of degree 4 and diameter $D \geq 2$.

Catalogue of $(4, D, -1)$**-graphs with** $D \geq 2$**.** There is no $(4, D, -1)$-graph for $D \geq 2$.

Catalogue of $(4, D, -2)$**-graphs with** $D \geq 2$**.** There is a unique $(4, 2, -2)$-graph, shown in Fig. 1 (d).

We proved the non-existence of $(d, D, -2)$-graphs with odd $d \geq 5$ and $D \geq 5$ such that $d \equiv 0, 2 \pmod{D}$. Furthermore, our new necessary conditions allow us also to rule out the existence of graphs of defect 2 for many other values of d and D using a simple approach.

7.1 Remarks on the upper bound for $\mathrm{N}(\Delta, D)$

Our results improve the upper bound on $N(\Delta, D)$ for many combinations of Δ and D.

Proposition 7.1 *For even $\Delta \geq 4$ and $D \geq 4$, $\mathrm{N}(\Delta, D) \leq \mathrm{M}(\Delta, D) - 3$.*

In the particular case of $\Delta = 4$, we have that $\mathrm{N}(4, 2) = \mathrm{M}(4, 2) - 2$ and $\mathrm{N}(4, D) \leq \mathrm{M}(4, D) - 3$ for $D \geq 3$.

According to Proposition 4.1, a $(\Delta, D, -3)$-graph Γ must be regular; consequently, $(\Delta, D, -3)$-graphs with odd $\Delta \geq 5$ and $D \geq 4$ do not exist.

Proposition 7.2 *For odd $\Delta \geq 5$ and $D \geq 4$ such that $\Delta\big(1 + (\Delta - 1) + \ldots + (\Delta - 1)^{D-1}\big) - 1 \not\equiv 0 \pmod{D}$, $\mathrm{N}(\Delta, D) \leq \mathrm{M}(\Delta, D) - 4$.*

Corollary 7.1 *For odd $\Delta \geq 5$ and $D \geq 5$ such that $\Delta \equiv 0, 2 \pmod{D}$, $\mathrm{N}(\Delta, D) \leq \mathrm{M}(\Delta, D) - 4$.*

Finally, we feel that the following conjectures also hold.

Conjecture 7.1 *There are no $(\Delta, D, -2)$-graph with $\Delta \geq 4$ and $D \geq 4$.*

Conjecture 7.2 *For odd $\Delta \geq 5$ and $D \geq 4$, $\mathrm{N}(\Delta, D) \leq \mathrm{M}(\Delta, D) - 4$.*

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