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XI. *On the Theory of Stellar Scintillation.* *By* Lord RAYLEIGH, *Sec. R.S.**

ARAGO'S theory of this phenomenon is still perhaps the most familiar, although I believe it may be regarded as abandoned by the best authorities. According to it the momentary disappearance of the light of the star is due to accidental interference between the rays which pass the two halves of the pupil of the eye or the object-glass of the telescope. When the relative retardation amounts to an odd multiple of the half wave-length of any kind of light, such light, it is argued, vanishes from the spectrum of the star. But this theory is based upon a complete misconception. "It is as far as possible from being true that a body emitting homogeneous light would disappear on merely covering half the aperture of vision with a half wave-plate. Such a conclusion would be in the face of the principle of energy, which teaches plainly that the retardation in question would leave the aggregate brightness unaltered" †. It follows indeed from the principle of interference that there will be darkness at the precise point which before the introduction of the half wave-plate formed the centre of the image, but the light missing there is to be found in a slightly displaced position ‡.

The older view that scintillation is due to the actual diversion of light from the aperture of vision by atmospheric irregularities was powerfully supported by Montigny §, to

* Communicated by the Author.

† *Enc. Brit.*, "Wave Theory,' p. 441.

‡ Since the remarks in the text were written I have read the version of Arago's theory given by Mascart (*Traité d'Optique*, t. iii. p. 348). From this some of the most objectionable features have been eliminated. But there can be no doubt as to Arago's meaning. "Supposons que les rayons qui tombent à gauche du centre de l'objectif aient rencontré, depuis les limites supérieures de l'atmosphère, des couches qui, à cause de leur densité, de leur température, ou de leur état hygrométrique, étaient douées d'une réfringence différente de celle que possédaient les couches traversées par les rayons de droite; il pourra arrivér, qu'à raison de cette différence de réfringence, les rayons rouges de droite détruisent *en totalité* les rayons rouges de gauche, et que le fòyer passe du blanc, son état normal, au vert; . . . . . . Voilà donc le résultat théorique parfaitement d'accord avec les observations; voilà le phénomène de la scintillation dans une lunette rattaché d'une manière intime à la doctrine des interférences" (*l'Annuaire du Bureau des Longitudes pour* 1852, pp. 423, 425).

That the difference between Arago's theory and that followed in the present paper is fundamental will be recognized when it is noticed that, according to the former, the colour effects of scintillation would be nearly independent of atmospheric *dispersion.* Arago gives an interesting summary of the views held by early writers.

§ *Mém. de l'Acad. d. Bruxelles,* t. xxviii. (1856).

whom we owe also a leading feature of the true theory, that is, the explanation of the chromatic effects by reference to the different paths pursued by rays of different colours in virtue of *regular* atmospheric dispersion. The path of the violet ray lies higher than that of the red ray which reaches the eye of the observer from the same star, and the separation may be sufficient to allow the one to escape the influence of an atmospheric irregularity which operates upon the other. In Montigny's view the diversion of the light is caused by total reflexion at strata of varying density.

But the most important work upon this subject is undoubtedly that of Respighi *, who, following in the steps of Montigny and Wolf, applied the spectroscope to the investigation of stellar scintillation. The results of these observations are summed up under thirteen heads, which it will be convenient to give almost at full length.

(I.) In spectra of stars near the horizon we may observe dark or bright bands, transversal or perpendicular to the length of the spectrum, which more or less quickly travel from the red to the violet or from the violet to the red, or oscillate from one to the other colour; and this however the spectrum may be directed from the horizontal to the vertical.

(II.) In normal atmospheric conditions the motion of the bands proceeds regularly from red to violet for stars in the west, and from violet to red for stars in the east; while in the neighbourhood of the meridian the movement is usually oscillatory, or even limited to one part of the spectrum.

(III.) In observing the horizontal spectra of stars more and more elevated above the horizon, the bands are seen sensibly parallel to one another, but more or less inclined to the axis of the spectrum, passing from red to violet or reversely according as the star is in the west or the east.

(IV.) The inclination of the bands, or the angle formed by them with the axis (? transversal) of the spectrum depends upon the height of the star; it reduces to 0° at the horizon and increases rapidly with the altitude so as to reach 90° at an elevation of 30° or 40°, so that at this elevation the bands become longitudinal.

(V.) The inclination of the bands, reckoned downwards, is towards the more refrangible end of the spectrum.

(VI.) The bands are most marked and distinct when the altitude of the star is *least*. At an altitude of more than 40° the longitudinal bands are reduced to mere shaded streaks, and

* *Roma, Atti Nuovi Lincei,* xxi. (1868); *Assoc. Française, Compt. Rend.* i. (1872) p. 169.

often can only be observed upon the spectrum as slight general variations of brightness.

(VII.) As the altitude increases, the movement of the bands becomes quicker and less regular.

(VIII.) As the prism is turned so as to bring the spectrum from the horizontal to the vertical position, the inclination of the bands to the transversal of the spectrum continually diminishes until it becomes zero when the spectrum is nearly vertical; but the bands then become less marked, retaining, however, the movement in the direction indicated above (III.).

(IX.) Luminous bands are less frequent and less regular than dark bands, and occur well marked only in the spectra of stars near the horizon.

(X.) In the midst of this general and violent movement of bright and dark masses in the spectra of stars, the black spectral lines proper to the light of each star remain sensibly quiescent or undergo very slight oscillations.

(XI.) Under abnormal atmospheric conditions the bands are fainter and less regular in shape and movement.

(XII.) When strong winds prevail the bands are usually rather faint and ill defined, and then the spectrum exhibits mere changes of brightness, even in the case of stars near the horizon.

(XIII.) Good definition and regular movement of the bands seems to be a sign of the probable continuance of fine weather, and, on the other hand, irregularity in these phenomena indicates probable change.

These results show plainly that the changes of intensity and colour in the images of stars are produced by a momentary real diversion of the luminous rays from the object-glass of the telescope; that in the neighbourhood of the horizon rays of different colours are affected separately and successively, and that all the rays of a given colour are momentarily withdrawn from the whole of the object-glass.

Most of his conclusions from observation were readily explained by Respighi as due to irregular refractions, not necessarily or usually amounting (as Montigny supposed) to total reflexions, taking place at a sufficient distance from the observer. The progress of the bands in one direction along the spectrum (II.) is attributed to the diurnal motion. In the case of a setting star, for instance, the blue rays by which it is seen, pursuing a higher course through the atmosphere, encounter an obstacle somewhat later than do the red rays. Hence the band travels towards the violet end of the spectrum. In the neighbourhood of the meridian this cause of a progressive movement ceases to operate.

The observations recorded in (III.) are of special interest as establishing a connexion between the rates with which various parts of the object-glass and of the spectrum are affected. Since the spectrum is horizontal, various parts of its width correspond to various horizontal sections of the objective, and the existence of bands at a definite inclination shows that at the moment when the shadow of the obstacle thrown by blue rays reaches the bottom of the glass the shadow at the top is that thrown by green, yellow, or red rays of less refrangibility. When the altitude of the star reaches 30° or 40°, the difference of path due to atmospheric dispersion is insufficient to differentiate the various parts of the spectrum. The bands then appear longitudinal.

The *definite* obliquity of the bands at moderate altitudes, reported by Respighi, leads to a conclusion of some interest, which does not appear to have been noticed. In the case of a given star, observed at a given altitude, the linear separation at the telescope of the shadows of the same obstacle thrown by rays of various colours will of necessity depend upon the distance of the obstacle. But the definiteness of the obliquity of the bands requires that this separation shall not vary, and therefore that the obstacles to which the effects are due are sensibly at one distance only. It would seem to follow from this that, under "normal atmospheric conditions," scintillation depends upon irregularities limited to a comparatively narrow horizontal stratum situated overhead. A further consequence will be that the distance of the obstacles increases as the altitude of the star diminishes, and this according to a definite law.

The principal object of the present communication is to exhibit some of the consequences of the theory of scintillation in a definite mathematical form. The investigation may be conducted by simple methods, if, as suffices for most purposes, we regard the whole refraction as small, and neglect the influence of the earth's curvature. When the object is to calculate with accuracy the refraction itself, further approximations are necessary, but even in this case the required result can be obtained with more ease than is generally supposed.

The foundation upon which it is most convenient to build is the idea of James Thomson *, which establishes instantaneously the connexion between the curvature of a ray travelling in a medium of varying optical constitution and the rate at which the index changes at the point in question. The following is from Everett's memoir :—

* Brit. Assoc. Rep. 1872. Everett, Phil. Mag. March 1873.

"Draw normal planes to a ray at two consecutive points of its path. Then the distance of their intersection from either point will be ρ, the radius of curvature. But these normal planes are tangential to the wave-front in its two consecutive positions. Hence it is easily shown by similar triangles that a very short line $d\mathrm{N}$ drawn from either of the points towards the centre of curvature is to the whole length ρ, of which it forms part, as dv the difference of the velocities of light at its two ends is to v the velocity at either end. That is

$$d\mathrm{N}/\rho = -dv/v,$$

the negative sign being used because the velocity diminishes in approaching the centre of curvature. But, since v varies inversely as μ, we have

$$-dv/v = d\mu/\mu.$$

Hence the curvature $1/\rho$ is given by any of the four following expressions:—

$$\frac{1}{\rho} = -\frac{1}{v}\frac{dv}{d\mathrm{N}} = -\frac{d\log v}{d\mathrm{N}} = \frac{1}{\mu}\frac{d\mu}{d\mathrm{N}} = \frac{d\log\mu}{d\mathrm{N}}. \quad . \quad (1)$$

"The curvatures of different rays at the same point are directly as the rates of increase of μ in travelling along their respective normals." If θ denote the angle which the ray makes with the direction of most rapid increase of index, the curvatures will be directly as the values of $\sin\theta$. In fact, if $d\mu/dr$ denote the rate at which μ increases in a direction normal to the surfaces of equal index, we have

$$\frac{d\mu}{d\mathrm{N}} = \frac{d\mu}{dr}\sin\theta,$$

and therefore

$$\pm\frac{1}{\rho} = \frac{1}{\mu}\frac{d\mu}{dr}\sin\theta = \frac{d\log\mu}{dr}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Everett shows how the well-known equation

$$\mu p = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

can be deduced from (2), p being the perpendicular upon the ray from the centre of *spherical* surfaces of equal index. In general,

$$\frac{1}{\rho} = \frac{1}{r}\frac{dp}{dr}, \quad \sin\theta = \frac{p}{r},$$

and thus

$$-\frac{1}{r}\frac{dp}{dr} = \frac{p}{r}\frac{d\log\mu}{dr},$$

giving (3) on integration.

At a first application of (2) we may find by means of it a first approximation to the law of atmospheric refraction, on the supposition that the whole refraction is small and that the curvature of the earth may be neglected. Under these limitations θ in (2) may be treated as constant along the whole path of the ray; and if $d\psi$ be the angle through which the ray turns in describing the element of arc ds, we have

$$d\psi = \frac{d\log\mu}{dr}\sin\theta\, ds = \tan\theta\; d\log\mu.$$

If we integrate this along the whole course of the ray through the atmosphere, that is from $\mu=1$ to $\mu=\mu_0$, we get, as the whole refraction,

$$\psi = \log\mu_0\tan\theta = (\mu_0-1)\tan\theta, \quad . \quad . \quad . \quad (4)$$

for to the order of approximation in question $\log\mu_0$ may be identified with (μ_0-1).

If $\delta\psi$ denote the chromatic variation of ψ corresponding to $\delta\mu_0$, we have from (4)

$$\delta\psi/\psi = \delta\mu_0/(\mu_0-1). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

According to Mascart * the value of the right-hand member of (5) in the case of air and of the lines B and H is

$$\delta\mu_0/(\mu_0-1) = \cdot 024. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We will now take a step further and calculate the linear deviation of a ray from a straight course, still upon the supposition that the whole refraction is small. If η denote the linear deviation (reckoned perpendicularly) at any point defined by the length s measured along the ray θ, we have

$$\frac{d^2\eta}{ds^2} = \frac{1}{\rho} = \tan\theta\frac{d\log\mu}{ds},$$

so that

$$\frac{d\eta}{ds} = \int\tan\theta\; d\log\mu = \tan\theta(\mu-1)+\alpha,$$

α being a constant of integration. A second integration now gives

$$\eta = \tan\theta\int(\mu-1)ds + \alpha s + \beta, \quad . \quad . \quad . \quad . \quad (7)$$

which determines the path of the ray. If y be the height of any point above the surface of the earth, $ds = dy\sec\theta$; so that (7) may also be written

$$\eta = \frac{\sin\theta}{\cos^2\theta}\int(\mu-1)dy + \alpha s + \beta. \quad . \quad . \quad . \quad . \quad (8)$$

* Everett's C.G.S. System of Units.

The origin of s is arbitrary, but we may conveniently take it at the point (A) where the ray strikes the earth's surface.

We will now consider also a second ray, of another colour, deviating from the line θ by the distance $\eta+\delta\eta$, and corresponding to a change of μ to $\mu+\delta\mu$. The distance between the two rays at any point y is

$$\delta\eta=\frac{\sin\theta}{\cos^2\theta}\int_0^y \delta\mu\, dy+d\alpha\,.\,s+\delta\beta. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In this equation $\delta\beta$ denotes the separation of the rays at A, where $y=0$, $s=0$. And $\delta\alpha$ denotes the angle between the rays when outside the atmosphere.

Equation (9) may be applied at once to Montigny's problem, that is to determine the separation of two rays of different colours, both coming from the same star, and both arriving at the same point A. The first condition gives $\delta\alpha\doteq 0$, and the second gives $\delta\beta=0$. Accordingly,

$$\delta\eta=\frac{\sin\theta}{\cos^2\theta}\int_0^y \delta\mu\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

is the solution of the question.

The integral in (10) may be otherwise expressed by means of the principle that $(\mu-1)$ and $\delta\mu$ are proportional to the density. Thus, if l denote the "height of the homogeneous atmosphere," and h the elevation in such an atmosphere determined by the condition that there shall be as much air below it as actually exists below y,

$$\int_0^y \delta\mu\, dy=\delta\mu_0 h, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$\delta\mu_0$ being the value of $\delta\mu$ at the surface of the earth. Equation (10) thus becomes

$$\delta\eta=\frac{\delta\mu_0 h\sin\theta}{\cos^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

At the limits of the atmosphere and beyond, $h=l$, and the separation there is

$$\delta\eta=\frac{\delta\mu_0 l\sin\theta}{\cos^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These results are applicable to all altitudes higher than about 10°.

The formulæ given by Montigny (*loc. cit.*) are quite different from the above. That corresponding to (13) is

$$\delta\eta=\delta\mu_0\, a\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

a being the radius of the earth! The substitution of a for l increases the calculated result some 800 times. But this is in a large measure compensated by the factor $\sec^2\theta$ in (13), for at low altitudes $\sec\theta$ is large. According to Montigny the separation at moderately low altitudes would be nearly independent of the altitude, a conclusion entirely wide of the truth.

The value of (μ_0-1) for air at 0° and 760 millim. at Paris is ·0002927, so that $\delta\mu_0$ (for the lines B and H) is ·000007025. The height of the homogeneous atmosphere is $7{\cdot}990\times 10^5$ centim., and thus $\delta\eta$ reckoned in centim. is

$$\delta\eta=5{\cdot}612\frac{h\sin\theta}{l\cos^2\theta} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

The following are a few corresponding values of θ and $\sin\theta/\cos^2\theta$:—

θ.	$\sin\theta/\cos^2\theta$.	θ.	$\sin\theta/\cos^2\theta$.
0°	0·000	60°	3·46
20	0·387	70	8·03
40	1·095	80	32·66

Thus at the limit of the atmosphere the separation of rays which reach the observer at an apparent altitude of 10° is 185 centim. Nearer the horizon the separation would be still greater, but its value cannot well be found from (15). Although these estimates are considerably less than those of Montigny, the separation near the horizon seems to be sufficient to explain the vertical position of the bands in the spectrum, recorded by Respighi (I.). The fact that the margin is not very great suggests that the obstacles to which scintillation is due may often be situated at a considerable elevation.

We have now to consider the effect of an obstacle situated at a given point B at level y on the course of the ray. And the first desideratum will be the estimation of the separation at A, the object-glass of the telescope, of rays of various colours coming from the same star, which all pass through the given point B. It will appear at once that no fresh question is raised. For, since the rays come from the same star at the same time, $\delta\alpha=0$, and thus by (9) $\delta\eta_A=\delta\beta$. The value of $\delta\beta$ is given at once by the condition that $\delta\eta_B=0$. Thus

$$-\delta\eta_A=\frac{\sin\theta}{\cos^2\theta}\int_0^y \delta\mu\, dy=\frac{\delta\mu_0\, h\sin\theta}{\cos^2\theta}, \quad . \quad . \quad . \quad (16)$$

as before. The discussion, already given of (15), is thus immediately applicable.

Equation (16) solves the problem of determining the inclination of the bands seen in the spectra of stars not very low (III.). It is only necessary to equate $-\delta\eta_A$ to the aperture of the telescope. $\delta\mu_0$ then gives the range of refrangibility covered by the bands as inclined. In practice h would not be known beforehand; but from the observed inclination of the bands it would be possible to determine it.

In a given state of the atmosphere h, so far as it is definite, must be constant and then $\delta\mu_0$ must be proportional to $\cos^2\theta/\sin\theta$. This gives the relation between the altitude of the star and the inclination of the bands.

When θ is small, $\delta\mu_0$ is large; that is, the bands become longitudinal.

As a numerical example, let us suppose that the aperture of the telescope is 10 centim., and that at an altitude of 10° the obliquity of the bands is such that the vertical diameter of the object-glass corresponds to the entire range from B to H. In this case (15) gives

$$h=\frac{10\,l}{185}=\cdot054\,l,$$

indicating that the obstacles to which the bands are due are situated at such a level that about $\frac{1}{20}$ of the whole mass of the atmosphere is below them.

The next question to which (9) may be applied is to find the angle $\delta\alpha$ outside the atmosphere between two rays of different colours which pass through the *two* points A and B. Here $\delta\eta_A=0$, and thus $\delta\beta=0$. And further, since $\delta\eta_B=0$, we get

$$-\delta\alpha=\frac{\sin\theta}{s\cos^2\theta}\int_0^y \delta\mu\, dy=\frac{\delta\mu_0\, h\tan\theta}{y}. \quad . \quad . \quad . \quad (17)$$

If the height of the obstacle above the ground be so small that the density of the air below it is sensibly uniform, then $h=y$, and

$$-\delta\alpha=\delta\mu_0\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In this case the angle is the same as that of the spectrum of the star observed at A, as appears from (4) and (5). In general, y is greater than h, so that $\delta\alpha$ is somewhat less than the value given by (18).

The interest of (18) lies in the application of it to find the time occupied by a band in traversing the spectrum in virtue of the diurnal motion, according to Respighi's observation (II.). The time required is that necessary for the star to rise or fall through the angle of its dispersion-spectrum at the altitude in question. At an altitude of 10°, this angle will be 8″, being always about $\frac{1}{40}$ of the whole refraction. The rate at which a star rises or falls depends of course upon the declination of the star and upon the latitude of the observer, and may vary from zero to 15° per hour. At the latter maximum rate the star would describe 8″ in about one half of a second, which would therefore be the time occupied by a band in crossing the spectrum under the circumstances supposed. In the case of a star quite close to the horizon, the progress of the band would be a good deal slower.

The fact that the larger planets scintillate but little, even under favourable conditions, is readily explained by their sensible apparent magnitude. The separation of rays of one colour thus arising during their passage through the atmosphere is usually far greater than the already calculated separation, due to chromatic dispersion; so that if a fixed star of no apparent magnitude scintillates in colours, the different parts of the area of a planet must *à fortiori* scintillate independently. But under these circumstances the eye perceives only an average effect, and there is no scintillation visible.

The non-scintillation of small stars situated near the horizon may be referred to the failure of the eye to appreciate colour when the light is faint.

In the case of stars higher up the whole spectrum is affected simultaneously. A momentary accession of illumination, due to the passage of an atmospheric irregularity, may thus render visible a star which on account of its faintness could not be steadily seen through an undisturbed atmosphere*.

In the preceding discussion the refracting obstacles have for the sake of brevity been spoken of as throwing sharp shadows. This of course cannot happen, if only in consequence of diffraction; and it is of some interest to inquire into the magnitude of the necessary diffusion. The theory of diffraction shows that even in the case of an opaque screen with a definite straight boundary, the transition of illumination at the edge of the shadow occupies a space such

* The theory of Arago leads him to a directly opposite conclusion (*loc. cit.* p. 381).

as $\sqrt{(b\lambda)}$, where λ is the wave-length of the light, and b is the distance across which the shadow is thrown. We may take λ at 6×10^{-5} centim., and if b be reckoned in kilometres, we have as the space of transition, $\sqrt{(6b)}$. Thus if b were 4 kilometres, the space of transition would amount to about 5 centim. The inference is that the various parts of the aperture of a small telescope cannot be very differently affected unless the obstacles to which the scintillation is due are at a less distance than 4 kilometres.

One of the principal outstanding difficulties in the theory of scintillation is to see how the transition from one index to another in an atmospheric irregularity can be sufficiently sudden. The fact that the various parts of a not too small object-glass are diversely affected seems to prove that the transitions in question do not occupy many centimetres. Now, whether the irregularity be due to temperature or to moisture, we should expect that a transition, however abrupt at first, would after a few minutes or hours be eased off to a greater degree than would accord with the above estimate. Perhaps the abruptness of transition is, as it were, continually renewed by the coming into contact of fresh portions of light and dense air as the ascending and descending streams proceed in their courses. The speculations and experiments of Jevons on the Cirrus form of Cloud* may find some application here. A preliminary question requiring attention is as to the origin of the irregularities which cause scintillation. Is it always at the ground, and mainly under the influence of sunshine? Or may irregular absorption of solar heat in the atmosphere, due to varying proportions of moisture, give rise to transitions of the necessary abruptness? Again, we may ask how many obstacles are to be supposed operative upon the same ray? Is the ultimate effect only a small residue from many causes in the main neutralizing one another? It does not appear that in the present state of meteorological science satisfactory answers can be given to these questions.

A complete investigation of atmospheric refraction can only be made upon the basis of some hypothesis as to the distribution of temperature; but, as has already been hinted, a second approximation to the value of the refraction can be obtained independently of such knowledge and without difficulty. In Laplace's elaborate investigation it is very insufficiently recognized, if indeed it be recognized at all,

* Phil. Mag. xiv. p. 22, 1857. For a mathematical investigation, by the author, see Math. Soc. Proc. xiv. April 1883.

that the whole difficulty of the problem depends upon the curvature of the earth. If this be neglected, that is if the strata are supposed to be plane, the desired result' ollows at once from the law of refraction, without the necessity of knowing anything more than the conditions of affairs at the surface. For in virtue of the law of refraction,

$$\mu \sin \theta = \text{constant};$$

so that if θ be the apparent zenith distance of a star seen at the earth's surface, and $\delta\theta$ the refraction, we have at once

$$\mu_0 \sin \theta = \sin (\theta + \delta\theta), \quad . \quad . \quad . \quad . \quad . \quad (19)$$

from which the refraction can be rigorously calculated. If an expansion be desired,

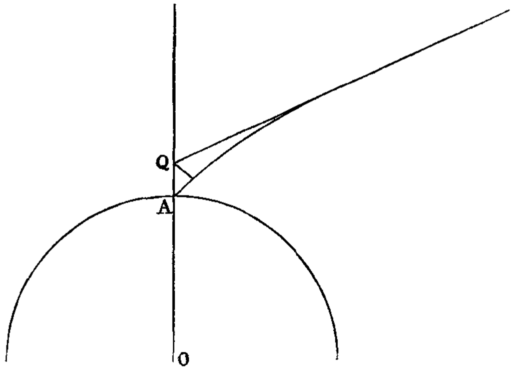
$$\delta\theta = \sin \delta\theta = \tan \theta\, (\mu_0 - \cos \delta\theta)$$
$$= (\mu_0 - 1) \tan \theta \{1 + \tfrac{1}{2}(\mu_0 - 1) \tan^2 \theta\} \quad . \quad . \quad (20)$$

is the second approximation.

When the curvature of the earth is retained, so that the atmospheric strata are supposed to be spheres described round O the centre of the earth, the appropriate form of the law of refraction is

$$\mu p = \text{constant}.$$

Thus, if A be the point of observation at the earth's surface



where the apparent zenith distance is θ, and if the original direction of the ray outside the atmosphere meet the vertical OA at the point Q,

$$\mu_0 \,.\, \text{OA} \,.\, \sin \theta = \text{OQ} \,.\, \sin (\theta + \delta\theta);$$

or if $\text{OA} = a$, $\text{AQ} = c$,

$$\mu_0 a \sin \theta = (a + c) \sin (\theta + \delta\theta). \quad . \quad . \quad . \quad (21)$$

If c be neglected altogether, we fall back upon the former equations (19), (20). For the purposes of a second approximation c, though it cannot be neglected, may be calculated as if the refraction were small, and the curvature of the strata negligible. If η be the whole linear deviation of the ray due to the refraction,

$$c=\eta/\sin\theta, \quad \ldots \ldots \ldots \quad (22)$$

and, as in (16),

$$\eta=(\mu_0-1)\, l \sin\theta/\cos^2\theta, \quad \ldots \ldots \quad (23)$$

so that

$$c=\frac{(\mu_0-1)l}{\cos^2\theta}. \quad \ldots \ldots \ldots \quad (24)$$

By equations (21), (24) the value of $\delta\theta$ may be calculated from the trigonometrical tables without further approximation.

To obtain an expansion, we have

$$\begin{aligned}\delta\theta &= \sin\delta\theta = \frac{\mu_0 \tan\theta}{1+c/a} - \tan\theta\cos\delta\theta \\ &= \tan\theta\left\{\frac{\mu_0}{1+c/a} - 1 + \tfrac{1}{2}(\delta\theta)^2\right\} \\ &= (\mu_0-1)\tan\theta\left\{1 - \frac{\mu_0 c}{(\mu_0-1)a} + \tfrac{1}{2}(\mu_0-1)\tan^2\theta\right\} \\ &= (\mu_0-1)\left(1-\frac{l}{a}\right)\tan\theta \\ &\qquad -(\mu_0-1)\left(\frac{l}{a}-\frac{\mu_0-1}{2}\right)\tan^3\theta. \quad \ldots \quad (25)\end{aligned}$$

To this order of approximation the refraction can be expressed in terms of the condition of things at the earth's surface, and (25) is equivalent to an expression deduced at great length by Laplace.

From the value of l already quoted, and $a=6{\cdot}3709\times 10^8$ centim., we get

$$l/a=\cdot 0012541. \quad \ldots \ldots \ldots \quad (26)$$

If further we take as the value under standard conditions for the line D

$$\mu_0-1=\cdot 0002927, \quad \ldots \ldots \ldots \quad (27)$$

we find as the refraction expressed in seconds of arc

$$\delta\theta=60''{\cdot}29\tan\theta-0''{\cdot}06688\tan^3\theta. \quad \ldots \quad (28)$$

In (28) θ is the apparent zenith distance, and it should be understood that the application of the formula must not be pushed too close to the horizon. If the density of the air at the surface of the earth differ from the standard density (0° and 760 millim.) the numbers in (28) must be altered proportionally. It will be observed that the result has been deduced entirely *à priori* on the basis of data obtained in laboratory experiments.

It may be convenient for reference to give a few values calculated from (28) of the refraction, and of the dispersion, reckoned at $\frac{1}{40}$ of the refraction.

Apparent zenith distance.	Refraction.	Dispersion (B to H).
° 0	″ 0·0	″ 0·0
20	21·9	·5
40	′ 50·5	1·3
45	1 0·2	1·5
60	1 40·1	2·5
70	2 44·2	4·1
75	3 41·5	5·5
80	5 29·7	8·2
85	9 49·2	14·7

The results of the formula (28) agree with the best tables up to a zenith distance of 75°, at which point the value of the second term is 3″·5. For 85° the number usually given is about 10′ 16″, and for 90° about 36′; but at these low altitudes the refraction is necessarily uncertain on account of irregularities such as those concerned in the production of mirage.

XII. *Notices respecting New Books.*

A Memorial of Joseph Henry. *Published by Order of Congress.* Washington: Government Printing Office. 1880.

On the death of Prof. Henry, more than fourteen years ago, the Senate and House of Representatives of the United States of America resolved concurrently that their members, with the Vice-President of the Republic, should take part in a memorial service. It was also resolved that the exercises and other memorial addresses should be printed at public expense, and this book of 528 pages is the result. It contains prayers, sermons, and speeches by eminent clergymen, statesmen, and scientific men; and some of the discourses, notably that of W. B. Taylor, which is given in 221 pages, give an account of Prof. Henry's services to Physical Science and