corresponding algebraic equation is of deficiency 1, and the corresponding integrals elliptic. Hence the hyperelliptic integrals to which the group in question leads, are those which can be formed by a quadratic substitution from elliptic integrals. But these are of known form, and hence in this particular case a definite relation is obtained, without further calculation, between the nature of the group and the nature of the algebraic equation to which it leads. As a further verification, it may be very easily shown in this case, from the relation

$$\theta(z, J_p) = \theta'(z, J'_p),$$

proved on the last page, that

$$a_{12} = \tfrac{1}{2}a_{11},$$

which relation between the periods of the hyperelliptic integral involves the possibility of expressing it in terms of elliptic integrals by a quadratic substitution.

Note on Approximate Evolution. By H. W. Lloyd Tanner, M.A., F.R.A.S.

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In a paper published in the *Proceedings* of the Society (Vol. XVIII., pp. 171–178), Professor Hill has pointed out that the rule (Todhunter's *Algebra*, Art. 246) for contracting the process of finding the square and cube roots of a number, is incorrect in some cases. It is desirable to have a practical test for distinguishing the cases in which the rule is available from those in which it fails. Such a test is obtained by a slight modification of Todhunter's discussion (*loc. cit.*), which enables us also to state two limits between which the required root must lie.

Square Root.

It is convenient to take the decimal point in N, the number whose square root is to be found, so that the integral part of $\sqrt{N}$ may consist of $2n+1$ figures. Of these the first $n+1$ are supposed to have been found, giving an approximate value, a, of the root. If x be the rest of the root, we have

$$N = (a+x)^2,$$

and, putting

$$q = (N-a^2)/2a,$$

$$q = x + \frac{x^2}{2a}.$$

Hence, providing that

$$\delta > \frac{x^2}{2a},$$

$$x+\delta > q > x;$$

or, what is the same thing,

$$q > x > q-\delta.$$

Now, from the meanings of a, x,

$$a \geqq 10^{2n}, \quad x < 10^n;$$

therefore

$$\frac{x^2}{2a} < \frac{1}{2},$$

and we may take $\frac{1}{2}$ as a value of δ. It follows that

$$q > x > q-\tfrac{1}{2}.$$

Hence it is clear that, if q and $q-\frac{1}{2}$ have the same integral part, this is also the integral part of x. If, on the other hand, there be an integer between q and $q-\frac{1}{2}$, it is the nearest integer to x, but the integral part of x is not determined.

Since $q > x$, $q^2/2a > x^2/2a$. We may therefore take $\delta = q^2/2a$, a value which will usually, but not always, be less than the value assigned above. Hence

$$q > x > q - \frac{q^2}{2a}.$$

The limits for $\sqrt{N}$ thus obtained,

$$a+q > \sqrt{N} > a+q-\frac{q^2}{2a},$$

are the first two, or three, terms of the expansion of $(a^2+2aq)^{\frac{1}{2}}$.

Cube Root.

Suppose $\sqrt[3]{N}$ to have $2n+2$ figures in its integral part, of which $n+2$ have been determined, giving an approximate value a of the root. Then, writing q for $(N-a^3)/3a^2$, and putting

$$N = (a+x)^3,$$

we have

$$q = x + \frac{x^2}{a} + \frac{x^3}{3a^2};$$

therefore

$$q > x > q-\delta,$$

if

$$\delta > \frac{x^2}{a} + \frac{x^3}{3a^2}.$$

Now

$$a \geqq 10^{2n+1}, \quad x < 10^n;$$

therefore

$$\frac{x^2}{a} + \frac{x^3}{3a^2} < \frac{1}{10} + \frac{1}{3} \cdot \frac{1}{10^{n+2}}$$

$$< \frac{1}{10} + \frac{1}{3} \cdot \frac{1}{10^3},$$

so that we may take

$$\delta = \cdot 100\dot{3}.$$

Hence, if the decimal part of q be less than $\cdot 100\dot{3}$, we cannot tell the integral part of x, but the nearest integer to x is known, being the integral part of q.

If the decimal part of q lie between $\cdot 600\dot{3}$ and $\cdot 5$ we cannot tell the nearest integer to x, but the integral part of x is the same as the integral part of q.

In all other cases we can tell both the greatest integer in x and its nearest integer.

The limits found for $\sqrt[3]{N}$, viz. $a+q$ and $a+q-\delta$, become respectively the first two and the first three terms of the expansion of $(a^3+3qa^2)^{\frac{1}{3}}$ by the binomial theorem, if q^2/a be put for δ. This is a proper value for δ; for, since

$$q > x + x^2/a$$

$$q^2/a > x^2/a + 2x^3/a^2 + x^4/a^3$$

$$> \frac{x^2}{a} + \frac{x^3}{3a^2}.$$