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LVI. On the distribution of pressure around spheres in a viscous fluid

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Table IV. is a comparison of the results of other experimenters with those given in this paper.

TABLE IV.

Material.	Value of Poisson's Ratio.		
	Bauschinger.	Stromeyer.	From Table II.
Mild Steel.........	·29	·273 to ·300	·271 to ·281
Wrought Iron ...	·26 to ·31	·279 to ·301	·270 to ·289
Brass Rod.........	....	·283 to ·357	·320 to ·351
Copper Rod	....	·325	·310 to ·340
Cast Iron	·16 to ·19 (Tension) ·32 to ·38 (Compression)	·148 to ·269	·228 to ·270

In conclusion I must acknowledge my indebtedness to my colleague, Mr. E. L. Watkin, M.A., for his valuable assistance in carrying out these experiments, which were made in the Engineering Laboratory at University College, Bristol.

LVI. *On the Distribution of Pressure around Spheres in a Viscous Fluid.* By S. R. COOK, *M.S., A.M., Former Fellow in Physics in the University of Nebraska, Instructor in Physics, Case School of Applied Science**.

[Plate XVIII.]

1. THE motion of a sphere in an incompressible frictionless fluid, at rest at infinity, has been discussed by Poisson, Stokes, Rayleigh, Kelvin, Koenig, and others. The solution in its present form was first given by Stokes in his celebrated paper "On Some Cases of Fluid Motion," read before the Cambridge Philosophical Society in 1843†.

On the principle that the mutual force acting between two adjacent elements of a fluid is normal to the surface which separates them, Stokes finds that the kinetic energy T of a sphere moving in an incompressible frictionless fluid at rest

* Communicated by Prof. D. B. Brace: read before the American Association for the Advancement of Sciences at Washington, January 1, 1903.

† Camb. Trans. vol. viii. p. 184; Math. Papers, vol. i. p. 41.

at infinity is increased by an inertia term equivalent to one-half the mass of the fluid displaced times the square of the velocity:

$$2T = -\rho \iint \phi \frac{d\phi}{dr} ds = \tfrac{2}{3}\pi\rho a^3 u^2. \quad . \quad . \quad . \quad . \quad (1)$$

When the sphere moves in a straight line, its motion being accelerated, and there are no external forces acting on the fluid, the resultant pressure is equivalent to a force

$$-\tfrac{2}{3}\pi\rho a^3 \frac{du}{dT}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in the direction of motion. If the velocity of the sphere is constant, there being no external force, the force acting on the sphere is zero, the pressure is symmetrical with respect to any axis, and the sphere will move with uniform velocity through the fluid.

The problem of the motion of two spheres in a perfect fluid was discussed by Stokes in the paper already referred to, and a method for obtaining the solution was suggested. Later a solution was obtained by W. M. Hicks and presented to the Royal Society in 1879 in his paper "On the Motion of Two Spheres in a Fluid"*.

Hicks finds that the kinetic energy T of two spheres moving in a perfect fluid may be expressed as a very simple function of their relative velocities u_1, u_2:

$$2T = A_1 u_1^2 - 2B u_1 u_2 + A_2 u_2^2; \quad . \quad . \quad . \quad . \quad (3)$$

and that the rate of change of the distance between the centres of gravity of the two spheres is given by the expression

$$\frac{\partial r}{\partial T} = \pm\sqrt{\frac{2T\rho - a^2}{A_1 A_2 - B}}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the positive or negative sign being taken according as the spheres are separating or approaching one another. The spheres will therefore move as though they repelled or attracted one another according as

$$\frac{\partial}{\partial r}\left\{\frac{2T\rho - a}{A_1 A_2 - B}\right\}$$

is positive or negative. This condition does not depend on the relative motion of the two spheres at any time, but only on their distance apart and the ratio of the constant energy

* Phil. Trans. p. 455 (1880).

to the constant momentum. Since $K=\frac{a^2}{2T}$ is always positive K is always positive, and $\frac{\partial r}{\partial T}$ tends to decrease, *i. e.* when moving in line of centres, the spheres tend to repel each other. When the two spheres are moving perpendicular to the line of their centres, Hicks finds that for a perfect fluid the spheres tend to attract each other.

Koenig* solving the same problem finds that the mutual forces between two spheres moving in a perfect fluid are

$$X=-\frac{\frac{3}{2}\pi\rho a^3b^3u^2}{c^4}\sin\theta(1-5\cos^2\theta) \quad . \quad . \quad . \quad (5)$$

$$Z=-\frac{\frac{3}{2}\pi\rho a^3b^3u^2}{c^4}\cos\theta(3-5\cos^2\theta) \quad . \quad . \quad . \quad (6)$$

$$Y=0,$$

where a and b are the radii, c the distance apart, and θ the angle which the line of centres makes with the direction of motion, Y vanishing on account of symmetry.

When $\theta=\frac{n\pi}{2}$, n being an integer,

$$X=-\frac{\frac{3}{2}\pi\rho a^3b^3u^2}{c^4}; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

when $\theta=n\pi$,

$$Z=\frac{3\pi\rho a^3b^3u^2}{c^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

giving repulsion parallel and attraction perpendicular to the stream-lines.

As these results have been obtained on the assumption that the medium is a perfect fluid, it is not possible to obtain experimental data to test their validity. All known fluids are susceptible to changes of density, and possess internal friction. The kinetic energy of a system moving in them may, accordingly, be transferred to the medium itself, thereby necessitating the introduction of a term in the equation of motion that will represent this transfer of kinetic energy.

On the condition that the velocity of the sphere is small so that the square of the velocity may be neglected, Stokes first obtained the solution for a sphere in a viscous fluid in terms of the potential†

$$\psi=-\tfrac{1}{2}V\left\{1-\frac{3}{2}\frac{a}{r}+\frac{1}{2}\frac{a^3}{r^3}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

* Wied. *Ann.* Band xlii. pp. 356, 549; Band xliii. p. 43.

† Camb. Trans. ix. p. 8 (1850); Math. and Physical Papers, vol. iii. p. 56.

The expression for the resistance of a pendulum moving in a viscous fluid is, according to the same author *,

$$F = -\tfrac{2}{3}\pi\rho a^3 cn\sqrt{-1}\left(1+\frac{9}{ma}+\frac{9}{m^2a^2}\right)e\sqrt{-1}nt, \quad . \quad (10)$$

which, when the conditions for steady motion are applied, becomes

$$-F = 6\pi\mu'\rho a V, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for the resultant force on a sphere parallel to the direction of motion.

These results are obtained on the assumption that there is no slip at the surface, and that the inertia term

$$u\frac{\partial u}{\partial \chi}$$

may be neglected in comparison with the viscous term

$$V\nabla^2 u.$$

The general form of the results obtained by Stokes † from theory has been recently verified by Mr. H. S. Allen ‡. Mr. Allen allowed air-bubbles of various size to escape from a small opening under water. The size of the bubble was varied until the velocity with which the bubbles rose in the water or other fluid became constant. The force on the sphere due to its motion in the viscous fluid could then be measured in terms of gravity. Mr. Allen also allowed bicycle bearing-balls to fall through viscous fluids, varying the diameter until constant velocity was obtained. From results thus obtained Mr. Allen concludes that for very small velocities the motion agrees with that deduced theoretically by Stokes.

When, however, the velocity is greater than a certain definite velocity given by the formula

$$V = \tfrac{2}{9}\,ga^2\frac{\sigma\rho}{\mu}\frac{\beta a+3\mu}{\beta a+2\mu'}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the resistance is proportional to the radius to the three-halves power, and when the velocities are considerably greater than the critical velocity the resistance follows the law deduced by Sir Isaac Newton:

$$R = k\rho a^2 V^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

* Math. and Physical Papers, vol. iii. p. 33.

† *L. c.* p. 4.

‡ Phil. Mag. [5] vol. l. pp. 338, 519 (1900).

2. *The Method.*—The method of allowing air-bubbles to ascend or solid spheres to descend in a viscous fluid gives only the total resultant pressure on the sphere, and does not give the distribution of the pressure over the surface of the sphere. It occurred to the writer while experimenting with spheres in a Kundt-tube that the distribution of pressure around a sphere might be obtained by using a hollow sphere in which there was a small opening, the interior of the sphere being connected to a manometer.

3. *Apparatus.*—A glass sphere of uniform diameter was blown on a capillary tube. At a point in one of the equators of the sphere a small hole was drilled, and it was then mounted in a tube 160 cms. in length and 3·5 cms. in diameter, through which a constant flow of air was maintained. The arrangement in general is shown in Plate XVIII. fig. 1.

Great care was taken that all sharp edges which would tend to form surfaces of discontinuity around the opening *o* (fig. 2) were rounded *. The diameter of the capillary tube *c* leading to the manometer *m* (fig. 4) was small compared with the diameter of the sphere, being in general less than one-twentieth. The sphere was inserted into the tube through an opening in the side, which was so closed that the inner surface of the tube was smooth and continuous. A constant current of air was maintained in the tube by keeping the two ends of the tube at a constant difference of pressure, the end B (fig. 1) being open to the atmosphere while the end A was connected to a mercury manometer *m*, not shown in the figure. The air was furnished from a gas-reservoir maintained at constant pressure by means of weights.

4. *Methods of Determining the Pressure.*—The pressure in the interior of the spheres was determined by water-manometers (fig. 4) made from glass of uniform diameter and connected to the sphere through the capillary tube *c*, the difference between the levels of the two columns of water being read by a cathetometer reading to 0·1 mm. The manometers were so arranged that the difference between the pressure normal to the inner surface of the tube AB at the point at which the spheres were situated, and the pressure in the sphere could be determined. The difference of pressure between two spheres at any time could also be measured.

* Von Helmholtz, "Ueber Discontinuirliche Flussigkeitsbewegungen," *Berl. Monatsber.* April 1868; Phil. Mag. Nov. 1868. (See Lamb's 'Hydrodynamics,' pp. 100 to 102, reference.)

5. *Distribution of Pressure around a Single Sphere.*—A glass sphere five millimetres in diameter with an opening two-tenths millimetre in diameter was mounted in the tube; the opening was in the equator whose plane was parallel to the direction of the stream-lines in the tube, and could be rotated through 360° in this plane (fig. 2). The pressure over unit surface of the sphere at all points in this equator could then be observed; and since by symmetry this plane is identical with any other equatorial plane parallel to the same straight lines, the total distribution of pressure around the sphere may be obtained by rotating the pressure-distribution curve obtained in this plane through 180° around an axis parallel to the axis of the tube.

The full-line curve *mm*, Plate XVIII. fig. 9, shows the distribution of pressure in a plane parallel to the stream-lines, in terms of the pressure normal to the surface of the tube.

In all the diagrams, unless otherwise stated, the curves of the observed pressures are plotted to a scale in which the pressure of one millimetre of water is represented by each 5 mm. circle measured from the double circle marked *a.p.* in the diagram.

6. *Pressure around two Spheres whose Line of Centres is Parallel to the Stream-lines.*—Two similar spheres of 5 mm. diameter were placed in the tube with the line of their centres parallel to the direction of flow, the distance apart of their surfaces being 1·5 cm. They were first placed with their openings up stream, making $\theta=0$ (fig. 5). The openings were then rotated through an angle of 180°. Readings of the pressure normal to the surface of the spheres, as given by the water-manometer *m*, were taken for each 15°. The velocity of the air-current, as measured by the pressure at the ends of the tube, being the same for each reading.

In the following table columns 2 and 3 give the pressure normal to the surface of the sphere, in millimetres of water, for a normal pressure in the tube of three millimetres of water. Columns 4 and 5 give the same for a pressure in the tube of one and eight-tenths millimetres of water.

The pressure diagrams plotted from these readings are exhibited in fig. 10.

The general form for the pressure-distribution around Sphere A is similar to that for a single sphere. The distribution around B is slightly modified by the presence of A. The distributions for the two normal pressures (curves *mm* and *m′m′*) are similar.

Table I.

Angle θ.	Normal Pressure 3 mm.		Normal Pressure 1·8 mm.	
	Sphere A. P.	Sphere B. P.	Sphere A. P.	Sphere B. P.
1.	2.	3.	4.	5.
0°	5·5	4·5	3·5	2·0
15	5·3	4·2	3·3	2·0
30	4·7	3·7	3·1	1·7
45	4·3	3·5	2·5	0·9
60	3·4	2·3	1·5	0·5
75	2·9	2·3	1·6 ?	0·5
90	2·1	1·8	0·5	0·9 ?
105	2·4	2·4	0·5	0·6
120	2·5	2·5	0·7	0·9
135	2·5	2·5	0·7	1·1
150	2·5	2·5	0·6	1·0
165	2·5	2·5	0·8	1·2
180	2·5	2·5	0·8	0·5 ?

The two spheres were then moved until their distance apart was one-tenth the former distance, *i. e.* 1·5 mm., making the distance apart less than half the diameter of the spheres.

The following table obtains at the normal pressures 2·8 mm. and 1·4 mm. of water.

Table II.

Angle θ.	Normal Pressure 2·8 mm.		Normal Pressure 1·4 mm.	
	Sphere A.	Sphere B.	Sphere A.	Sphere B.
1.	2.	3.	4.	5.
0°	4·7	2·7	3·7	0·5
15	4·1	2·9	3·0	0·6
30	3·8	2·9	1·5 ?	0·6
45	2·9 ?	2·9	1·0	1·0
60	2·8	3·2	1·0	1·1
75	2·3	3·3	1·0	1·4
90	2·0	3·0	1·0	1·0
105	2·1	2·4	1·0	0·6
120	2·1	2·1	1·0	1·0
135	2·0	2·3	0·9	0·9
150	2·1	2·2	1·0	0·7
165	2·1	2·1	1·0	1·0
180	2·2	2·2	0·9	0·9

The pressure-curves plotted from the readings for normal pressure 2·8 mm. are given by the full-line curve *mm*, fig. 11. The pressure-distribution around sphere A is quite similar to the pressure-distribution around a single sphere. The pressure-distribution around sphere B differs from that of a single sphere owing to the proximity of sphere A.

7. *The Pressure Distribution around two Spheres whose Line of Centres is Perpendicular to the Stream-lines.*—Two glass spheres of five millimetres diameter were mounted so that their line of centres was perpendicular to the direction of flow, the distance apart of their surfaces being 2 mm., Plate XVIII. fig. 3.

As the pressure-distribution around the two spheres is not symmetrical with respect to the horizontal and vertical planes containing the line of centres, the distribution was obtained in these two planes; the distribution in the horizontal plane is given in Table III.

Since the distribution of pressure is not symmetrical with respect to any plane, the opening was rotated through 360°, readings being taken for each 10° for the first 90° and for each 30° thereafter.

TABLE III.

Angle θ.	Pressure in mm. of water.	Angle θ.	Pressure in mm. of water.
0°	1·50	120°	0·05
10	1·40	150	0·15
20	1·20	180	0·22
30	1·20	210	0·15
40	1·20	240	0·15
50	1·05	270	0·05
60	0·60	300	0·50
70	0·30	330	1·10
80	0·15	360	1·50
90	0·05		

The distribution of the pressure according to the above data is exhibited in Plate XVIII. fig. 12, curve *mm*. As the curve about B would be the image of that about A the readings were only taken over A.

The curves of the distribution of pressure around these spheres differ from the curve around a single sphere in protruding slightly at a position between 30° and 60° from the line of centres on the side nearest the companion sphere.

The curves are plotted to a scale of one centimetre to one millimetre of water pressure. The double circle *ap* being taken as atmospheric pressure.

The distribution of the pressure in a vertical plane is given in Table IV. The manometers were arranged to give the differential effect, the excess of the pressure on sphere A over the pressure on sphere B in the line of centres being recorded. The arrangement is shown in Plate XVIII. fig. 6.

TABLE IV.

Angle θ.	Difference in pressure in mm. of water. $P_A - P_B$.	Angle θ.	Difference in pressure in mm. of water. $P_A - P_B$.
0	0·30	190	0·25
10	0·40	200	0·20
20	0·40	210	0·15
30	0·50	220	0·05
40	0·34	230 to 310	0·00
50	0·30	320	0·05
60 to 120	0·00	330	0·15
130	0·30	340	0·20
140	0·34	350	0·25
150	0·50	360	0·30
160	0·40		
170	0·40		
180	0·30		

8. *The Distribution of Pressure for a Perfect Fluid.*—The velocity potential for a single sphere moving through a perfect fluid, at rest at infinity, with velocity u is

$$\phi = \tfrac{1}{2} u \frac{a^3}{r^2} \cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where θ is measured from the direction of motion of the sphere. The pressure at any point of the sphere is

$$\frac{P}{\rho} = F(t) + \frac{d\phi}{dt} - \tfrac{1}{2}q^2, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where $F(t)$ is a function of the time.

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial r}\frac{dr}{dt} + \frac{\partial\phi}{\partial\theta}\frac{d\theta}{dt} \quad . \quad . \quad . \quad (17)$$

and

$$\tfrac{1}{2}q^2 = \frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial\theta}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Where the velocity is constant

$$\frac{\partial\phi}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

From (15) $$\frac{\partial \phi}{\partial r} = -u\frac{a^3}{r^3}\cos\theta, \quad \ldots \quad (20)$$

and $$\frac{\partial \phi}{\partial \theta} = -\tfrac{1}{2}u\frac{a^3}{r^2}\sin\theta, \quad \ldots \quad (21)$$

also $$\frac{\partial r}{\partial t} = -u\cos\theta, \quad \ldots \quad (22)$$

and $$\frac{\partial \theta}{\partial t} = \frac{u\sin\theta}{r}. \quad \ldots \quad (23)$$

From (16) the pressure around a single sphere moving in a perfect fluid is

$$\frac{\mathrm{P}}{\rho} = \tfrac{9}{8}\cos^2\theta - \tfrac{5}{8}. \quad \ldots \quad (24)$$

The broken-line curve *nn* in Plate XVIII. fig. 9 exhibits the pressure of a perfect fluid around a single sphere when moving with constant velocity.

In order to obtain the pressure-distribution around two spheres in a perfect fluid we determine the velocity potential and solve equation (16). The velocity potential for two spheres moving in their line of centres may be obtained approximately by the theory of images*. Using only the terms in the expansion of the first image so far as $\left(\frac{a}{c}\right)^6$ we obtain for the velocity potential of two spheres moving in their line of centres, fig. 7,

$$\phi = \tfrac{1}{2}u\frac{a^3}{r^2}\cos\theta + \tfrac{1}{2}v\frac{b^3}{c^3}(c - r\cos\theta)\left\{1 + 3\frac{r}{c}\cos\theta + 3\frac{r^2}{c^2}\frac{5\cos^2\theta - 1}{2} + 5\frac{r^3}{c^3}\frac{7\cos^3\theta - 3\cos\theta}{2}\right\}, \quad (25)$$

where *a* and *b* are the radii of the spheres, *r* the distance from the centre of sphere A, *c* the distance apart of the two spheres, and *u* and *v* are the respective velocities. For constant velocity $u=v$ and at the surface of sphere A, $r=b=a$ and

$$\frac{\partial \phi}{\partial r} = -u\cos\theta - \tfrac{1}{2}u\frac{a^3}{c^3}\cos\theta\left\{1 + 3\frac{a}{c}\cos\theta + 3\frac{a^2}{c^2}\frac{5\cos^2\theta - 1}{2} + 5\frac{a^3}{c^3}\frac{7\cos^3\theta - 3\cos\theta}{2}\right\} + \tfrac{1}{2}u\frac{a^3}{c^3}\left(\frac{c}{a} - \cos\theta\right)\left\{3\frac{a}{c}\cos\theta + 6\frac{a^2}{c^2}\frac{5\cos^2\theta - 1}{2} + 15\frac{a^3}{c^3}\frac{7\cos^3\theta - 3\cos\theta}{2}\right\}, \quad (26)$$

* Stokes, *l. c.* p. 1; Hicks, *l. c.* p. 2.

and

$$\frac{\partial\phi}{\partial\theta}=-\tfrac{1}{2}ua\sin\theta+\tfrac{1}{2}u\frac{a^3}{c^3}a\sin\theta\left\{1+3\frac{a}{c}\cos\theta+3\frac{a^3}{c^3}\frac{5\cos^2\theta-1}{2}\right.$$
$$\left.+5\frac{a^3}{c^3}\frac{7\cos^3\theta-3\cos\theta}{2}\right\}-\tfrac{1}{2}u\frac{a^3}{c^3}(c-a\cos\theta)\left\{3\frac{a}{c}\sin\theta\right.$$
$$\left.+3\frac{a^2}{c^2}5\cos\theta\sin\theta+15\frac{a^3}{c^3}\frac{7\cos^2\theta-1}{2}\sin\theta\right\}. \quad . \quad (27)$$

The broken-line curve *nn* in Plate XVIII. fig. 11 exhibits the distribution of pressure as given by equation (16), to the approximation indicated, for two spheres moving with constant velocity in the line of their centres.

For two spheres whose direction of motion is perpendicular to the line joining their centres (fig. 8) to the same degree of approximation

$$\phi=\tfrac{1}{2}u\frac{a^3}{r^2}\cos\theta+\tfrac{1}{2}v\frac{b^3}{c^3}r\cos\theta\left\{1+3\frac{r}{c}\sin\theta+3\frac{r^2}{c^2}\frac{5\sin^2\theta-1}{2}\right.$$
$$\left.+5\frac{r^3}{c^3}\frac{7\sin^3\theta-3\sin\theta}{2}+\dots\right\} \quad . \quad . \quad . \quad (28)$$

$$\frac{\partial\phi}{\partial r}=-u\cos\theta+\tfrac{1}{2}u\frac{a^3}{c^3}\cos\theta\left\{1+3\frac{a}{c}\sin\theta+3\frac{a^2}{c^2}\frac{5\sin^2\theta-1}{2}\right.$$
$$\left.+5\frac{a^3}{c^3}\frac{7\sin^3\theta-3\sin\theta}{2}\right\}+\tfrac{1}{2}u\frac{a^3}{c^3}\cos\theta\left\{3\frac{a}{c}\sin\theta\right.$$
$$\left.+6\frac{a^2}{c^2}\frac{5\sin^2\theta-1}{2}+5\frac{a^3}{c^3}\frac{7\sin^3\theta-3\sin\theta}{2}\right\}, \quad . \quad (29)$$

and

$$\frac{\partial\phi}{\partial\theta}=-\tfrac{1}{2}ua\sin\theta-\tfrac{1}{2}u\frac{a^3}{c^3}a\sin\theta\left\{1+3\frac{a}{c}\sin\theta+3\frac{a^2}{c^2}\frac{5\sin^2\theta-1}{2}\right.$$
$$\left.+5\frac{a^3}{c^3}\frac{7\sin^3\theta-3\sin\theta}{2}\right\}+\tfrac{1}{2}u\frac{a^3}{c^3}a\cos\theta\left\{3\frac{a}{c}\cos\theta\right.$$
$$\left.+3\frac{a^2}{c^2}\frac{5\sin^2\theta-1}{2}+15\frac{a^3}{c^3}\frac{7\sin^3\theta-1\sin\theta}{2}\right\}. \quad (30)$$

The broken-line curve *nn* in fig. 12 exhibits the approximate distribution of pressure for two spheres in a perfect fluid moving perpendicular to their line of centres.

For each set of spheres observations were made when the distance apart of the spheres was somewhat less than three times their radius. The ratio of the radius to the distance apart of the spheres used in computing the curves for the distribution of pressure in a perfect fluid was 1/3. All terms in the expansion of the second image will contain this ratio to the sixth and higher powers, but all terms in the first image containing this ratio to a higher power than the sixth were neglected; that is, terms containing a factor less than $8{\cdot}6 \times 10^{-4}$ have been omitted in the computation of the broken-line curves in figs. 11 and 12.

9. *Comparison of the Distribution of Pressure for a Perfect Fluid with the Pressure obtained for a Viscous Fluid.*—For a single sphere moving with constant velocity in a perfect fluid at rest at infinity the curve of distribution of pressure is symmetrical with respect to each plane of the three rectangular axes whose origin is at the centre of the sphere. And hence the resulting force in any direction is zero. For the viscous fluid the curve is asymmetrical with respect to the plane perpendicular to the direction of motion, but symmetrical with respect to the line of motion; and the resultant force is such as would tend to bring the sphere to rest.

For two spheres moving in their line of centres in a perfect fluid the curves of distribution of pressure are asymmetrical with respect to the axial planes which are perpendicular to their direction of motion, the force on the inner hemisphere being the greater. The normal pressure and the resultant component pressures along the line of motion over the inner and outer hemispheres at different points are given in the following table for sphere B.

Table V.

Angle.	Inner hemisphere.	Angle.	Outer hemisphere.	Difference.	Resultant component.
0°	·5028	180°	·499	·0038	·0038
30	·2940	150	·220	·0540	·0468
60	−·2550	120	−·320	·0650	·0320
90	−·5880	90	−·5880	·0000	·0000

The resultant force on both spheres is tending to separate the spheres, *i. e*, gives repulsion. The results for the two spheres in a viscous fluid are exhibited in Table I., and it is evident that the two spheres would have a relative motion such that they would approach each other, *i. e.* attract.

For two spheres moving in a perfect fluid perpendicular to their line of centres the curves of distribution of pressure are asymmetrical with respect to a plane perpendicular to the line joining them, the pressure in the outer hemisphere being the greater.

The following table gives the normal pressures and the resultant component pressures over the outer and inner hemispheres of the two spheres at corresponding points. The curve being asymmetrical with respect to the line joining them, only the figures for the first quadrant are given.

TABLE VI.

Angle.	Outer hemisphere.	Angle.	Inner hemisphere.	Dif-ference.	Resultant component.
360°	·5000	0°	·5000	·0000	·0000
330	·2202	30	·1790	·0412	·0206
300	−·3480	60	−·4211	·0731	·0617
270	−·6340	90	−·7130	·0790	·0790

Table III. gives the results for a viscous fluid. The curve (fig. 12) is asymmetrical with respect to both axial planes, and it is clear from the form of the curve that the pressure in the inner hemisphere is greater than the pressure on the outer hemisphere. The pressure on the inner hemisphere is at 30°, 3·3 per cent. of the normal pressure at 0° in excess of the pressure at 360°, and at 60° it is 5·7 per cent. of the normal pressure greater than the corresponding pressure at 300°.

For a perfect fluid, therefore, two spheres moving with constant velocity perpendicular to the line joining their centres attract, and for a viscous fluid they repel.

I have shown in a former paper* that when two particles in a sound-wave are a certain critical distance apart they are attracted when their line of centres is parallel to the stream-lines and repelled when their line of centres is perpendicular to the stream-lines. The spheres used in these experiments were relatively large compared with particles or sphere that would form flutings in a sound-wave. The results, however, agree with the results obtained with the smaller sphere in a sound-wave. I hope soon to be able to determine the pressure around spheres small enough to form flutings in a sound-wave.

The experimental work included in this paper was conducted under the direction of Dr. Brace in the Physical Laboratory of the University of Nebraska, and my sincere thanks are due to him for valuable suggestions during the progress of the experiments, and also for his assistance in determining the curves of distribution for a perfect fluid.

Physical Laboratory, Case School of Applied Science,
Cleveland, Ohio, April 23, 1903.

* Phil. Mag. May 1902.

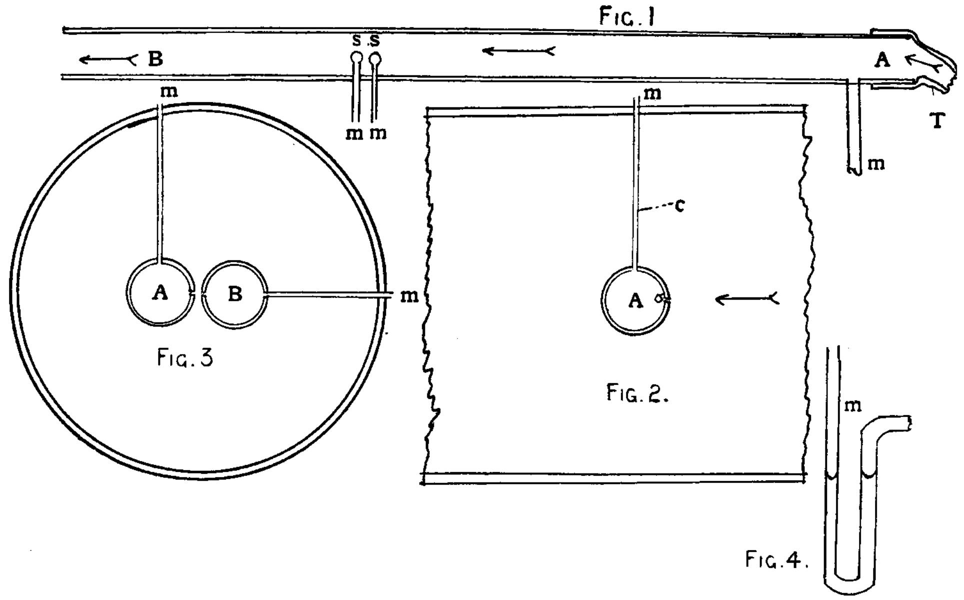

Fig. 1
Fig. 2.
Fig. 3
Fig. 4.

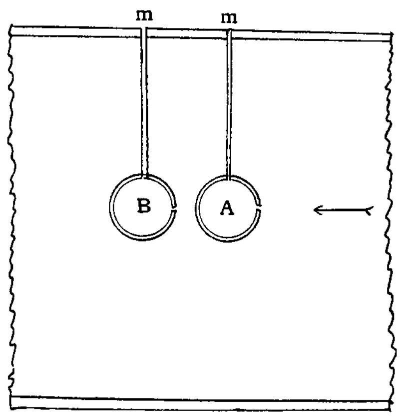

Fig. 5

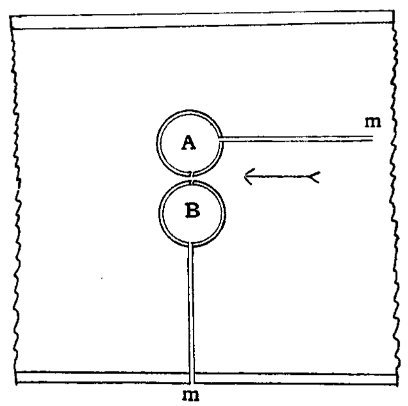

Fig. 6.

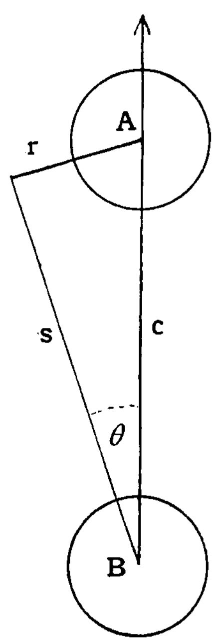

Fig. 7

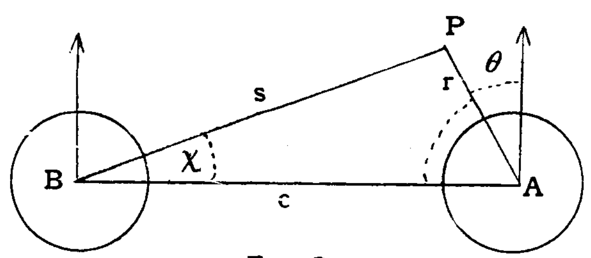

Fig. 8.

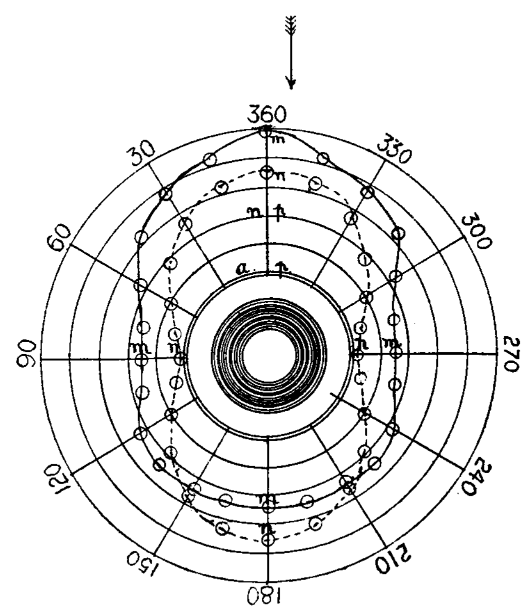

Fig. 9.

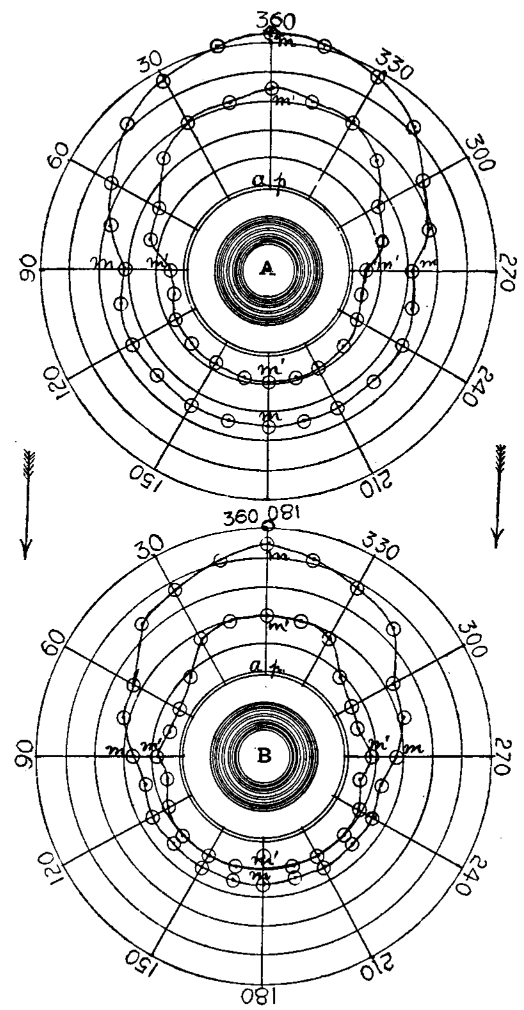

Fig. 10.

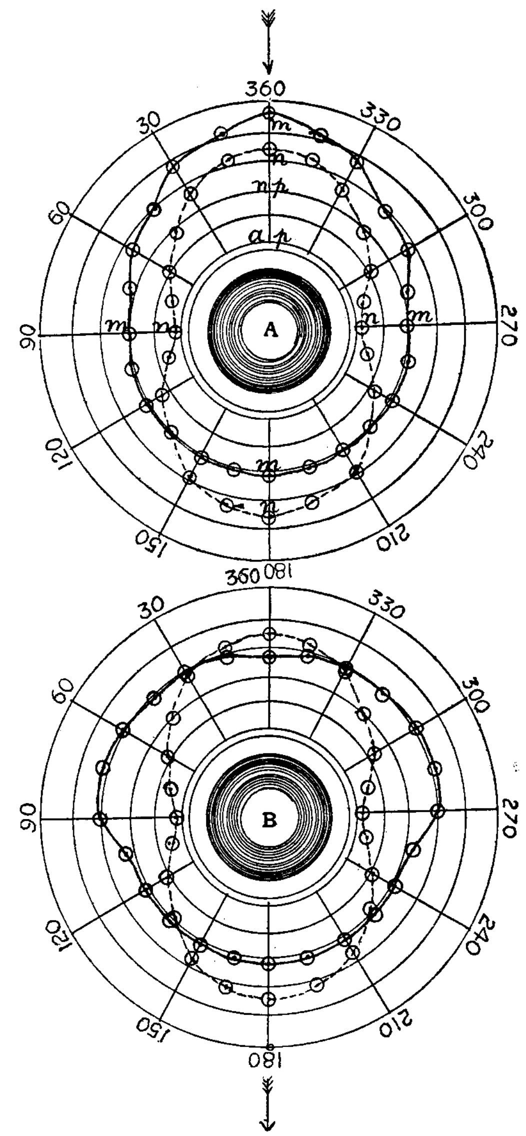

Fig. 11.

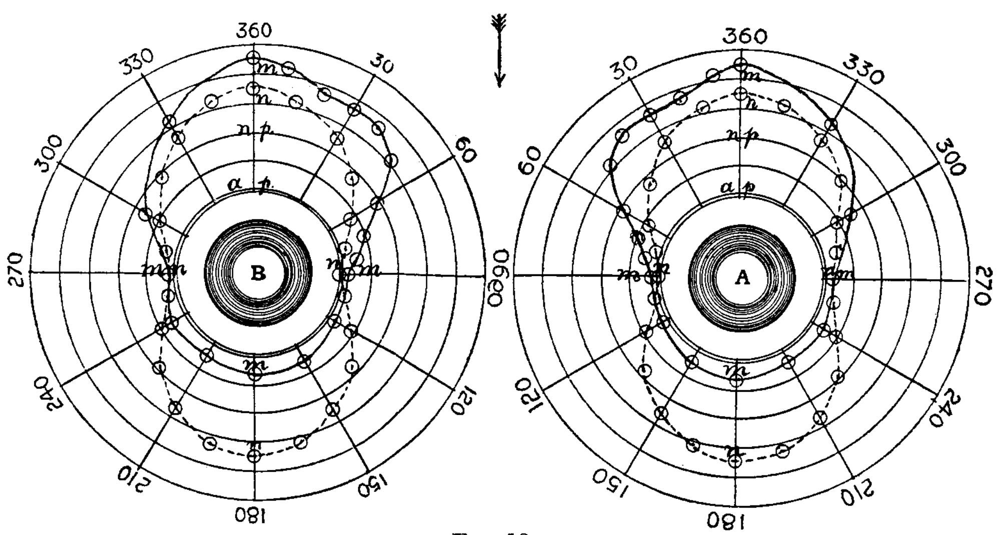

Fig. 12.