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XVI. Calculation of the magnetic field of a current running in a cylindrical coil

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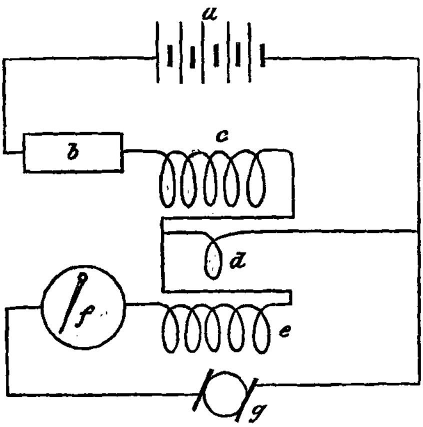
c and *e* are wound close together so that there is no appreciable time-lag in any current induced in *c* by *e*, the small error due to the mutual induction of these coils cancels out.

Fig. 2.



Of course a double dynamometer might be used, but then four mercury connexions are needed. It is not unlikely that a method with no mercury contacts can be made available, and greater accuracy could then be attained.

XVI. *Calculation of the Magnetic Field of a Current running in a Cylindrical Coil. By* Professor G. M. Minchin, *M.A.**

LET there be a series of very close circular currents running in the same sense and lying on a right cylinder of radius a whose axis is OO′ (fig. 1), and let it be required to find the magnetic potential of this system at any point, P, in space.

Replace each of these circular currents by its equivalent magnetic shell, which we shall take as a uniform circular plate coinciding with the aperture of the circle. Supposing the currents to circulate in the sense ACB, the upper surface of each plate (as seen in the figure) will be positive and the lower negative. Each circle being touched all round by the one below it, the negative surface of any plate will coincide with the positive of the one next below it; so that we shall be left with a terminal positive plate, ACB, and a terminal

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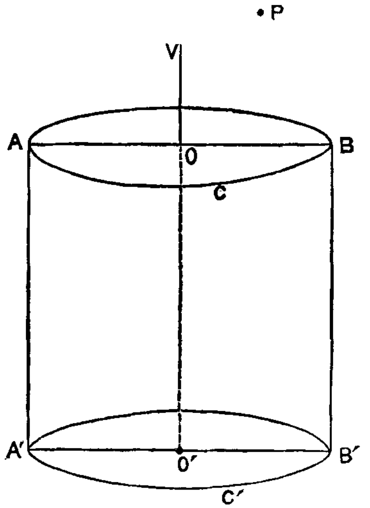
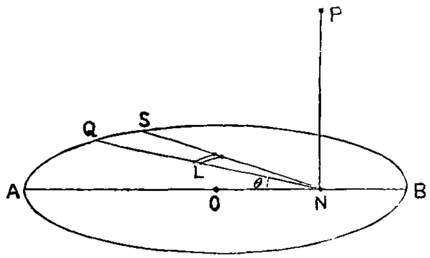
negative plate, A′C′B′, as the approximate equivalent of the given coil of wire. We shall now calculate the potential

Fig. 1.



produced at any point, P, by a uniform thin plate represented by AQB (fig. 2), the surface-density of attracting matter on this plate being m.

Fig. 2.



Of course it is well known that this potential can be exhibited by a series of Zonal Harmonics proceeding by powers, positive or negative, of the ratio $\frac{\mathrm{OP}}{a}$, where O is the centre of the plate and a its radius, according as P is near O or distant from it ; but when OP is not *very* much greater or *very* much less than a, the series is inconvenient, owing to the enormous number of terms that have to be taken to give a good approximation.

Hence we shall not use Spherical Harmonics.

Let PN be the perpendicular on the plate from P; let AB be the diameter in which it is cut by the plane through the axis of the plate and P; let Q, S be two very close points on the circumference, NQ making the angle θ with NA. Then we shall suppose the plate broken up into triangular strips such as QNS, and calculate the potential of each strip at P.

Let L be any point on NQ, let $\mathrm{NL}=\xi$; then the potential of the element $m\xi d\xi d\theta$ at P is $\frac{m\xi d\xi d\theta}{\mathrm{LP}}$, or $md\theta\frac{\xi d\xi}{\sqrt{z^2+\xi^2}}$, where $\mathrm{PN}=z$. If $\mathrm{NQ}=r$ we find by integration that the potential produced by the strip QNS is

$$md\theta(\sqrt{r^2+z^2}-z), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the potential, V, of the whole plate is

$$2m\int(\sqrt{r^2+z^2}-z)d\theta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now if, as in the figure, the point N falls within the plate, the limits of θ are 0 and π; if N falls on the edge of the plate, at B, the limits are 0 and $\frac{\pi}{2}$; and if it falls outside the plate, the limits are 0 and 0. Taking θ as the independent variable would, then, give us three different expressions for V, according to the position of N; and hence we must choose a more convenient variable than θ. Let ϕ be the angle QOA, and change the expression (2) into one in which ϕ is the independent variable. We shall then have

$$r^2=a^2+x^2+2ax\cos\phi\,; \quad \tan\theta=\frac{a\sin\phi}{x+a\cos\phi};$$

so that

$$\mathrm{V}=2ma\int_0^\pi\frac{\sqrt{a^2+x^2+z^2+2ax\cos\phi}-z}{a^2+x^2+2ax\cos\phi}(a+x\cos\phi)d\phi\,;$$

or if we put $\mathrm{D}=a^2+x^2+2ax\cos\phi$,

$$\frac{\mathrm{V}}{m}=\int_0^\pi\frac{\sqrt{z^2+\mathrm{D}}-z}{\mathrm{D}}(\mathrm{D}+a^2-x^2)d\phi$$

$$=\int_0^\pi\sqrt{z^2+\mathrm{D}}\,d\phi+(a^2-x^2)\int_0^\pi\frac{\sqrt{z^2+\mathrm{D}}}{\mathrm{D}}d\phi-z\int_0^\pi\left(1+\frac{a^2-x^2}{\mathrm{D}}\right)d\phi.$$

Now

$$(a^2-x^2)\int_0^\pi\frac{d\phi}{\mathrm{D}}=\pi\,;$$

so that

$$\frac{V}{m} = -2\pi z + \int_0^\pi \sqrt{z^2+D}\, d\phi + (a^2-x^2)\int_0^\pi \frac{\sqrt{z^2+D}}{D}\, d\phi$$

$$= -2\pi z + \int_0^\pi \sqrt{z^2+D}\, d\phi + (a^2-x^2)\int_0^\pi \frac{d\phi}{\sqrt{z^2+D}}$$

$$+ (a^2-x^2)z^2\int_0^\pi \frac{d\phi}{D\sqrt{z^2+D}}. \quad . \quad (3)$$

Again, if we put $\phi = 2\omega$, we have

$$D = (a+x)^2 - 4ax\sin^2\omega\ ;$$

and if the distances PA, PB are denoted by ρ, ρ', respectively, we have

$$4ax = \rho^2 - \rho'^2,$$

so that

$$z^2 + D = \rho^2 - (\rho^2 - \rho'^2)\sin^2\omega.$$

Let

$$k^2 = 1 - \frac{\rho'^2}{\rho^2}\ ; \quad k'^2 = \frac{\rho'^2}{\rho^2}\ ; \quad \Delta\omega = \sqrt{1-k^2\sin^2\omega}. \quad . \quad (4)$$

Then (3) becomes

$$\frac{V}{m} = -2\pi z + 2\rho E + \frac{2}{\rho}(a^2-x^2)K + 2\frac{z^2}{\rho}\frac{a-x}{a+x} \cdot \int_0^{\frac{\pi}{2}} \frac{1}{1-\frac{4ax}{(a+x)^2}\sin^2\omega} \cdot \frac{d\omega}{\Delta\omega}, \quad (5)$$

where E and K are the complete elliptic integrals of the second and first kinds with modulus k.

The integral in (5) is the complete elliptic integral of the third kind with modulus k and parameter $-\frac{4ax}{(a+x)^2}$. This parameter is numerically greater than the modulus ; and we shall find it convenient to convert the integral into one in which the parameter is less than the modulus by the well-known rule that a function with parameter n can be converted into one with parameter $\frac{k^2}{n}$. If the angles PBA and PAB are denoted by θ and θ', respectively, we see that n, the parameter in (5), is $-\frac{k^2}{\cos^2\theta'}$; so that the new parameter will be simply $-\cos^2\theta'$.

Now we have the general result that

$$\Pi(n, \epsilon) + \Pi\left(\frac{k^2}{n}, \epsilon\right) = \frac{1}{\sqrt{\alpha}}\tan^{-1}\left(\frac{\sqrt{\alpha}\tan\epsilon}{\Delta\epsilon}\right) + K(\epsilon),$$

where ϵ is the amplitude of each of the two functions of the third kind (denoted by Π), and $\alpha=(1+n)\left(1+\frac{k^2}{n}\right)$. Hence for complete functions $\left(\epsilon=\frac{\pi}{2}\right)$ we have

$$\Pi(n)+\Pi\left(\frac{k^2}{n}\right)=\frac{\pi}{2\sqrt{\alpha}}+\mathrm{K}.$$

But here

$$\alpha=\left(1-\frac{k^2}{\cos^2\theta'}\right)\sin^2\theta'=\left(\frac{a-x}{a+x}\right)^2\cdot\frac{z^2}{\rho^2};$$

so that

$$\Pi\left(-\frac{k^2}{\cos^2\theta'}\right)=\frac{\pi}{2}\frac{\rho}{z}\frac{a+x}{a-x}+\mathrm{K}-\Pi(-\cos^2\theta'), \quad . \quad . \quad (6)$$

and (5) becomes

$$\frac{\mathrm{V}}{m}=-\pi z+2\rho\mathrm{E}+2\rho\frac{a-x}{a+x}\cdot\mathrm{K}-2\frac{z^2}{\rho}\frac{a-x}{a+x}\Pi(-\cos^2\theta', k). \quad (7)$$

This expression holds without ambiguity for all positions of the point P, and it shows that for all points

on the axis, OV, of the plate, . $\frac{\mathrm{V}}{m}=2\pi(\rho-z)$, . . (8)

on the perpendicular through B, $\frac{\mathrm{V}}{m}=2\rho\mathrm{E}-\pi z$, . . (9)

in the plane, between O and B, . $\frac{\mathrm{V}}{m}=2\rho\mathrm{E}+2\rho'\mathrm{K}$, . (10)

in the plane, beyond B, ... $\frac{\mathrm{V}}{m}=2\rho\mathrm{E}-2\rho'\mathrm{K}$; . (11)

so that the points, occupying any of these positions, at which V has any assigned value can be easily found. Thus, to find the point on OV at which $\frac{\mathrm{V}}{m}$ has the value C, we have for this point

$$\rho=z+\frac{\mathrm{C}}{2\pi}.$$

Hence draw below AB, parallel to it and at the distance $\frac{\mathrm{C}}{2\pi}$, a right line, meeting VO produced in O′; then the perpendicular to AO′ at its middle point meets OV in the required point.

Now every complete elliptic integral of the third kind can be expressed in terms of complete and incomplete functions of the first and second kinds. Thus, for a complete function with the parameter $-m$ it is known that, if we put $m=-k'^2\sin^2\epsilon$,

$$\Pi(-m,k)=\mathrm{K}+\frac{\Delta'_\epsilon}{k'^2\sin\epsilon\cos\epsilon}\left\{\frac{\pi}{2}+\mathrm{KK}'_\epsilon-\mathrm{KE}'_\epsilon-\mathrm{EK}'_\epsilon\right\}, \quad (12)$$

where Δ'_ϵ stands for $\sqrt{1-k'^2\sin^2\epsilon}$, and K'_ϵ, E'_ϵ stand for incomplete functions of the first and second kinds with modulus k' and amplitude ϵ.

In the present case,

$$m=\cos^2\theta', \quad k'=\frac{\rho'}{\rho}=\frac{\sin\theta'}{\sin\theta};$$

$\therefore\ \epsilon=\theta$, and

$$\Pi(-\cos^2\theta',k)=\mathrm{K}+\frac{\cos\theta'}{k'^2\sin\theta\cos\theta}\left\{\frac{\pi}{2}+\mathrm{KK}'_\theta-\mathrm{KE}'_\theta-\mathrm{EK}'_\theta\right\}; \quad (13)$$

so that (7) becomes

$$\frac{\mathrm{V}}{2m}=z\{\mathrm{KE}'_\theta+\mathrm{EK}'_\theta-\mathrm{KK}'_\theta-\pi\}+\rho\mathrm{E}+\rho'\cos\theta\cos\theta'.\mathrm{K}. \quad (14)$$

It is evident that we may define the position of any point, P, in the plane of the figure by means of the two coordinates k and θ. Thus we have

$$\rho=\frac{2a}{\Delta'_\theta+k'\cos\theta},$$

$$\cos\theta'=\Delta'_\theta,$$

$$z=k'\rho\sin\theta, \quad \rho'=k'\rho.$$

Hence

$$\frac{\mathrm{V}}{4m}=\frac{a}{\Delta'_\theta+k'\cos\theta}\{\mathrm{E}+k'\sin\theta\,(\mathrm{KE}'_\theta+\mathrm{EK}'_\theta-\mathrm{KK}'_\theta-\pi)$$
$$+k'\cos\theta\,.\,\mathrm{K}\Delta'_\theta\}. \quad . \quad (15)$$

This, then, is the expression for the potential at any point in terms of the coordinates (k, θ) of the point. In particular, it gives the value (9) for any point on the perpendicular through B to the plate, since for such a point $\theta=\frac{\pi}{2}$, and then the coefficient of $k'\sin\theta$ within the brackets is equal to $-\frac{\pi}{2}$, by Legendre's well-known relation between the complete complementary functions, viz.,

$$\mathrm{KE}'+\mathrm{EK}'-\mathrm{KK}'=\frac{\pi}{2},$$

whatever the modulus k may be.

From (14) we can derive an expression for the conical angle subtended at any point, P, in space by a circle, *i. e.*, for the magnetic potential due to a current coinciding with the circle. It is well known that this conical angle is numerically equal to the component of the attraction, perpendicular to the plate, at P due to a uniform circular plate coinciding with the aperture of the circle—a result which is evident from the principle that the current can be replaced by a magnetic shell, or thin plate, the upper and lower surfaces of which are, of course, of opposite signs. But the resultant potential of these two indefinitely close plates is the difference between the value of V in (14) and the value which (14) assumes when $z+\Delta z$ is substituted for z; that is, the magnetic potential at P due to the current is $-\frac{dV}{dz}\,.\,\Delta z$, and the strength of the magnetic shell is $m\,.\,\Delta z$, which is i, the current in the circle; so that the magnetic potential is i multiplied by minus the differential coefficient of the right-hand side of (14) with respect to z.

Denote the function $\pi+KK'-KE'_\theta-EK'_\theta$ by the symbol Λ_θ, and for simplicity in the differentiation with respect to z (x being constant) write (14) in the form

$$\frac{V}{2m}=-z\Lambda_\theta+\rho E+(a^2-x^2)\frac{K}{\rho}. \quad . \quad . \quad . \qquad (16)$$

Now

$$\left.\begin{aligned}\frac{d}{dz}&=-\frac{kk'}{\rho}\sin\theta\,; & \frac{dk'}{dz}&=\frac{k^2}{\rho}\sin\theta\,;\\ \frac{d\rho}{dz}&=k\sin\theta\,; & \frac{d\theta}{dz}&=\frac{1}{k'\rho}\cos\theta\,;\end{aligned}\right\} \quad . \quad . \qquad (17)$$

and, regarding P as determined by the coordinates $(z,\ x)$ instead of (k,θ), we have

$$\frac{d}{dz}=\frac{dk}{dz}\cdot\frac{d}{dk}+\frac{d\theta}{dz}\cdot\frac{d}{d\theta};$$

but

$$\frac{d\Lambda}{dk}=\frac{K-E}{k}\,\frac{\sin\theta\cos\theta}{\Delta'_\theta},$$

$$\frac{d\Lambda}{d\theta}=\frac{Kk'^2\sin^2\theta-E}{\Delta'_\theta};$$

therefore

$$\frac{d\Lambda}{dz}=-\frac{1}{k'\rho}E\Delta'_\theta\cos\theta,$$

and we find $\frac{d}{dz}$ of the right-hand side of (16) equal to $-\Lambda_\theta + k'\text{K}\sin\theta$; so that if Ω is the conical angle subtended at P by the circle, or the magnetic potential per unit current in the circle, we have the very simple expression

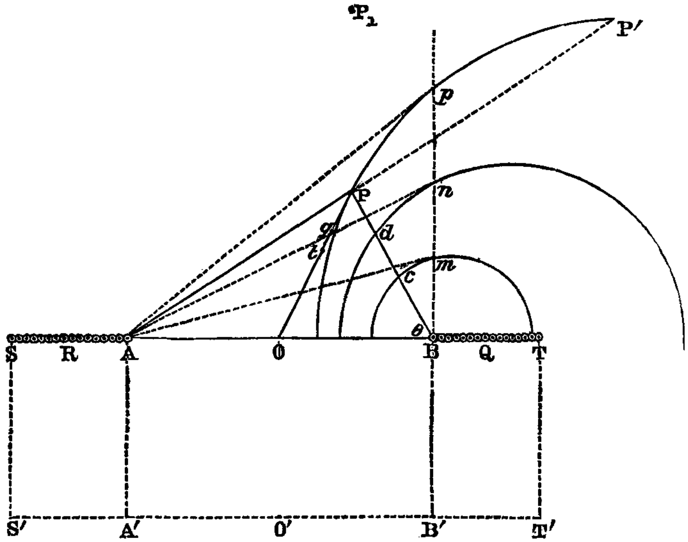
$$\Omega = 2\Lambda_\theta - 2k'\text{K}\sin\theta. \quad . \quad . \quad . \quad . \quad (18)$$

Again, supposing that the depth, OO′ (fig. 1), of a coil consisting of a single series of circular currents is small compared with the distance of the point P from any part of it, the two terminal plates, ACB, A′C′B′ may be considered as close together, and the potential of the coil at P is the value of V in (16) minus the value obtained by putting $z+h$ for z, where $h=$ OO′. Hence the potential in such a case is $-mh\Omega$, i. e., *at any point in space whose distance from every part of a coil is great compared with the depth of the coil, the potential is*

$$2mh(\Lambda_\theta - k'\text{K}\sin\theta). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The modulus k which appears in these equations, being $\left(1-\frac{\rho'^2}{\rho^2}\right)^{\frac{1}{2}}$, is constant at all points for which $\frac{\rho'}{\rho}$ is constant; *i. e.*, at all points on any circle which cuts that described on AB as diameter orthogonally. The circles which cut this

Fig. 3.



latter orthogonally, having their centres on AB, are most readily drawn by joining A to points, m, n, p, ... (fig. 3) on the perpendicular at B to AB, and drawing perpendiculars at $m, n, p, \ldots$ to Am, An, Ap, ... ; the points of intersection of AB produced with these perpendiculars are the centres of the circles. If C is the centre of the orthogonal circle through m, we know that the constant, $1-\frac{\rho'^2}{\rho^2}$, on this circle is $\frac{\text{AB}}{\text{AC}}$, *i. e.*, $\cos^2 m\text{AB}$. Hence if $m\text{AB}=\beta$, we have

$$k=\cos\beta; \qquad k'=\sin\beta. \quad . \quad . \quad . \quad . \quad (20)$$

The field due to the plate AB is most readily mapped out by describing a large number of very close circles of the orthogonal system for a regular gradation of the values mAB, nAB, pAB, ... of β, drawing a line BP in the assigned direction θ, and from Legendre's tables of Elliptic Integrals taking out the values of K, E, K'_θ, E'_θ.

The properties of the orthogonal circles lead to some simple results with regard to potentials. Thus, if any line, AP, is drawn from A cutting any circle of the series in P and P$'$, the lines joining P and P$'$ to B are equally inclined to AB, *i. e.*, $\angle \text{ABP}'=\pi-\theta$.

Now if in Λ_θ we put $\pi-\theta$ for θ, we have, in virtue of Legendre's relation between complete complementary integrals, $\Lambda_{\pi-\theta}=\pi-\Lambda_\theta$, *i. e.*,

$$\Lambda_{\pi-\theta}+\Lambda_\theta=\pi. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence, from (18), if Ω, Ω' are the conical angles subtended at P, P$'$ respectively by the circle (or plate) AB, we have the remarkable relation

$$\Omega+\Omega'=2\pi-4k'\text{K}\sin\theta. \quad . \quad . \quad . \quad . \quad (22)$$

Again, if V, V$'$ are the potentials at P, P$'$ due to the plate, we have from (15)

$$\frac{\text{V}}{\text{AP}}+\frac{\text{V}'}{\text{AP}'}=2m(2\text{E}-\pi k'\sin\theta), \quad . \quad . \quad . \quad (23)$$

a result which enables us to lay down the field at all points to the right of the perpendicular Bp when the field to the left of Bp is known.

Supposing now that instead of a single wire of diameter AB, we have a series of wires forming a coil contained between the diameter AB and the diameter ST, *i. e.*, the breadth of the coil is BT or AS ; then in calculating the potential at P we shall have to find the potentials due to a series of circular plates, each of surface-density m, and to add these potentials together. But observe that the potential at P due to any plate, AB, of radius a is of the form

$$a\,.\,\phi(k,\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

where $\phi(k,\theta)$ is the coefficient of a in (15), and $\phi(k,\theta)$ is a function of θ and θ', the angles PAB and PBA. Hence if we take a plate of radius OQ, and from B draw Bq parallel to QP and meeting OP in q, the potential of this plate at P is to the potential of the plate AB at q as OQ is to OA; for, if AR=BQ, the angles qBA and qAB are equal, respectively, to PQR and PRQ. Hence, if r=OQ and V_q is the potential at q due to the plate AB (of radius a), the resultant potential at P due to the series of plates of radii extending from OB to OT is

$$\frac{1}{a}\Sigma r\,.\,V_q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

the points q on OP ranging from t to P, where Bt is parallel to PT.

Of course any plate of the series may be taken instead of AB as the reference plate.

Thus, *the resultant potential, due to all the plates, is calculated from values of the potential of any one plate at a series of points ranged along the radius vector* OP.

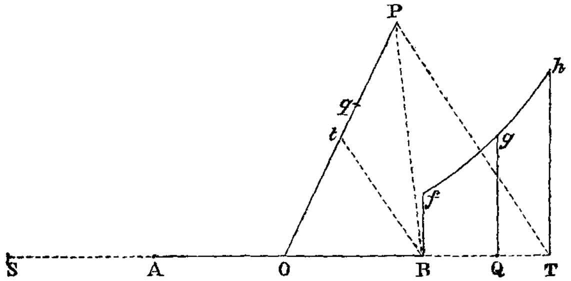
Pass now to the consideration of the practical problem in hand, viz., the potential at P due to a coil of depth OO′, *i. e.*, we have to consider the whole of the spaces BTT′B′ and ASS′A′ filled with wire traversed by a current of strength i. We have already seen that we have to subtract from the potential at P due to a series of uniform attracting plates, each of surface-density m, ranging from the radius OB to the radius OT, the potential at P due to the lower series, each of surface-density m, and ranging from radius OB′ to radius OT′. It merely remains to express m in terms of current-density. If C is the total quantity of current traversing (at right angles to the plane of the paper) a unit area (square centimetre) of the space BTT′B′, the quantity flowing in a filament of depth dy and breadth dr is $Cdydr$. Now this filament is replaced by the magnetic shell of radius r and thickness dy; and since we know that the strength of the shell is equal to the current in the filament, we have $m\,.\,dy=Cdydr$, $\therefore m=Cdr$; hence (25) becomes, from (15),

$$4C\int r\,.\,\phi(k,\theta)dr, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

which is the potential due to the upper series of plates, OB, . . . OT.

This quantity may be graphically represented and calculated as follows. Let a very close series of curves representing a series of constant values of the function $\phi(k, \theta)$ for the plate AB be drawn; draw OP, and at each point, T, Q, ... B, of the breadth BT of the coil draw an ordinate, Th, Qg, ... Bf (fig. 4), equal to the product of r and the

Fig. 4.



value of $\phi(k, \theta)$ at the corresponding point, $t, q, \ldots$ P, of the line OP: these ordinates will form by their extremities a curve, $hg \ldots f$, the area of which multiplied by four times the current-density in the space occupied by the coil is the potential at P due to the upper series of plates, OB, OQ, ... OT.

Now if we take the point P_1 such that PP_1 is equal and parallel to OO$'$, the depth of the coil, the potential at P due to the lower series of plates, OB$'$, ... OT$'$, is equal to that at P_1 due to the upper series. Hence, if A and A_1 are the areas of the curve $hg \ldots f$ and the corresponding curve for the point P_1, the total potential at P due to the complete coil is

$$4C(A-A_1). \quad \ldots \ldots \quad (27)$$

The curve hgf passes, of course, through the point O; and when the line OP coincides with the axis, OO$'$, of the coil, the curve is an hyperbola; for, in this case

$$\phi(k, \theta)=\frac{\pi}{2} \cdot \frac{1-\sin\theta}{\cos\theta},$$

and $r=\mathrm{OP}\cot\theta$.

We may, if we please, express r in terms of (k, θ), and draw the curve hgf by a different rule. Thus,

$$r=\mathrm{OP} \cdot \frac{\Delta'_\theta+k'\cos\theta}{\sqrt{(\Delta'_\theta-k'\cos\theta)^2+4k'^2\sin^2\theta}},$$

and we can make the ordinate of the curve equal to

$$\mathrm{OP} \cdot \frac{\mathrm{E}-k'\sin\theta \cdot \Lambda_\theta+k'\mathrm{K}\cos\theta \cdot \Delta'_\theta}{\sqrt{(\Delta'_\theta-k'\cos\theta)^2+4k'^2\sin^2\theta}}.$$