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On the optimality of functionals over triangulations of Delaunay sets

N. P. Dolbilin, H. Edelsbrunner, and O. R. Musin

In this short paper we consider the functional density on sets of uniformly bounded triangulations with fixed sets of vertices. We prove that if a functional attains its minimum on the Delaunay triangulation for every finite set in the plane, then for infinite sets the density of this functional attains its minimum also on the Delaunay triangulations.

A *Delaunay set* in $\mathbf{E}^d$ is a set of points X for which there are positive numbers r and R such that every open d-ball of radius r contains at most one point and every closed d-ball of radius R contains at least one point of X. In this paper we consider Delaunay sets in general position, that is, no $d+2$ points in X lie on a common $(d-1)$-sphere.

By a triangulation of X we mean a simplicial complex whose vertex set is X. For finite sets the simplices decompose the convex hull of the set, while for Delaunay sets X the simplices decompose $\mathbf{E}^d$. We say that a triangulation T is *uniformly bounded* if there exists a positive number $q = q(T)$ that is greater than or equal to the circumradii of all d-simplices in the triangulation: $R(S) \leqslant q$ for all d-simplices S of T. We denote the family of all uniformly bounded triangulations of X by $\Theta(X)$.

Delaunay sets were introduced by Boris Delaunay (1924), who called them *(r,R)-systems*. He proved that for any Delaunay set X there exists a unique Delaunay tesselation $DT(X)$ (see, for instance, [1]). If X is in general position, then $DT(X)$ is a triangulation of X in the sense defined above. Since the circumradius of any simplex is at most R, the Delaunay triangulation is uniformly bounded with $q = R$, that is, $DT(X) \in \Theta(X)$. We note that every Delaunay set also has triangulations that are not uniformly bounded, and it is not difficult to construct them.

We want to remind the reader of a related open problem about Delaunay sets: is it true that for every planar Delaunay set X and every positive number C there exists a triangle Δ that contains none of the points in X and has area greater than C? While we heard of this question from Michael Boshernitzan, it is sometimes referred to as *Danzer's problem*.

Let F be a functional defined on d-simplices S. (For instance, $F(S)$ may be the sum of squares of edge lengths multiplied by the volume of S.) We only consider functionals that are continuous with respect to the parameters describing the simplices, for example, the lengths of their edges. Let X be a finite set in $\mathbf{E}^d$ and T any triangulation of X. Then F can be defined on T as $F(T) = \sum_{S\in T} F(S)$.

It is clear that this definition cannot be used for infinite sets. We therefore define the (*lower*) *density* of F for a uniformly bounded triangulation T of a Delaunay set X as

$$\overline{F}(T) := \varliminf_{\alpha\to\infty} \frac{1}{\operatorname{vol}(B_\alpha)} \sum_{S\subset B_\alpha} F(S),$$

where B_α denotes the closed ball of radius α with centre at the origin of $\mathbf{E}^d$. For the rest of the paper, we limit ourselves to dimension $d = 2$.

Theorem. *Let F be a continuous functional that attains its minimum for every finite set $Y \subset \mathbf{E}^2$ for the Delaunay triangulation of Y. Then the density $\overline{F}$ on $\Theta(X)$, where $X \subset \mathbf{E}^2$ is a Delaunay set, attains its minimum for the Delaunay triangulation of X.*

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Proof. Let $T \in \Theta(X)$ be a triangulation with parameter q and consider the simplicial complex $\tau_\alpha(T)$ that consists of all triangles, edges, and vertices of T contained in B_α. We consider the convex hull C_α of the vertices of $\tau_\alpha(T)$.

The difference between C_α and the union of the triangles in $\tau_\alpha(T)$ consists of polygons, and since any polygon can be triangulated without adding vertices, τ_α can be extended to a triangulation ψ_α of the same set of vertices.

Write K_α for the number of triangles in $\tau_\alpha(T)$. Since the circumradius of each triangle is bounded from above, and the lengths of its edges are bounded from below, the area of each triangle is at least some constant. It follows that K_α is at most some constant times α^2. The circle bounding B_α intersects at most some constant times $\sqrt{K_\alpha}$ of the triangles in T, which implies that ψ_α has at most a constant times $\sqrt{K_\alpha}$ triangles in addition to those in τ_α. Using the continuity of the functional, it follows that

$$\lim_{\alpha\to\infty} \frac{F(\tau_\alpha)}{F(\psi_\alpha)} = 1. \tag{$*$}$$

By the assumption, $F(\psi_\alpha)$ is no less than the value of F on the Delaunay triangulation of the same set of vertices, which completes the proof.

We remark that there are non-convex polytopes in dimension $d > 2$ that cannot be triangulated without adding new vertices. They constitute the main difficulty in extending the theorem to general dimensions. The theorem and the results stated in the papers [1]– [5] yield the following result.

Corollary. *Let Δ be a triangle with barycentre b, circumcentre c, and edges of lengths a_1, a_2, a_3. Let us consider the following functionals*:

1) $F_1(\Delta) := \mathcal{R}^a(\Delta)$, *where $\mathcal{R}(\Delta)$ is the circumradius and $a > 0$*;
2) $F_2(\Delta) := \dfrac{a_1^2 + a_2^2 + a_3^2}{s(\Delta)}$, *where $s(\Delta)$ is the area of Δ*;
3) $F_3(\Delta) := -\rho(\Delta)$, *where $\rho(\Delta)$ is the inradius of Δ*;
4) $F_4(\Delta) := (a_1^2 + a_2^2 + a_3^2)s(\Delta)$;
5) $F_5(\Delta) := \mathcal{R}^a(\Delta)s(\Delta)$, *where $a \geqslant 1$*;
6) $F_6(\Delta) := \|b(\Delta) - c(\Delta)\|^2 s(\Delta)$.

Then the densities $\overline{F}_i$, $i = 1, \dots, 6$, attain their minima on the Delaunay triangulations of Delaunay sets in the plane.

For finite sets the optimality of the functionals F_1 and F_2 was shown in [3], the optimality of F_3 was shown in [2], the optimality of F_4 was shown [5], and the optimality of F_5 and F_6 was shown in [4].

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N. P. Dolbilin
Steklov Mathematical Institute
of the Russian Academy of Sciences

H. Edelsbrunner
Institute of Science and Technology Austria

O. R. Musin
University of Texas at Brownsville, USA

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and O. MUSIN