The Influence of Surface-Loading on the Flexure of Beams

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The curve from which the data are obtained is shown in fig. 1, Plate IV., the value of one division being equal to $\frac{1}{10}$ of a square therein represented.

In fig. 2, Plate IV., the results obtained above are multiplied by 100 and represented graphically. Fig. 3 represents the curves registered by the recording apparatus under varying conditions of light from June 13 to 16. It will be seen that such an instrument as this will record continuously the actinic intensity of the light under all conditions of weather throughout the year, and requires no attention further than winding the clock whereby the motion of the drum is maintained.

XXVI. *The Influence of Surface-Loading on the Flexure of Beams. By* Prof. C. A. Carus Wilson *.

[Plate V.]

The practical treatment of the problem of beam-flexure at the present time is based on the hypothesis enunciated by Bernoulli and Euler †, that the bending-moment is proportional to the curvature.

This assumes that the cross sections remain plane after flexure and neglects the surface-loading effect.

Saint-Venant has shown ‡ that the first assumption is untenable; but that, neglecting the surface-loading, Bernoulli's results are strictly true for one particular case of loading, that, namely, of a beam doubly supported and carrying a single isolated load, where, although the cross sections are distorted, the central displacement is zero.

I propose in this paper to describe some experiments made with a view to determining the actual state of strain in a beam doubly supported and centrally loaded, the surface-loading effect being taken into account.

The method of investigation adopted is based upon the following assumptions:—

* Read June 26, 1891.

† Todhunter and Pearson's 'History of Elasticity,' vol. i.

‡ Pearson's 'Elastical Researches of Saint-Venant.'

Fig. 1.

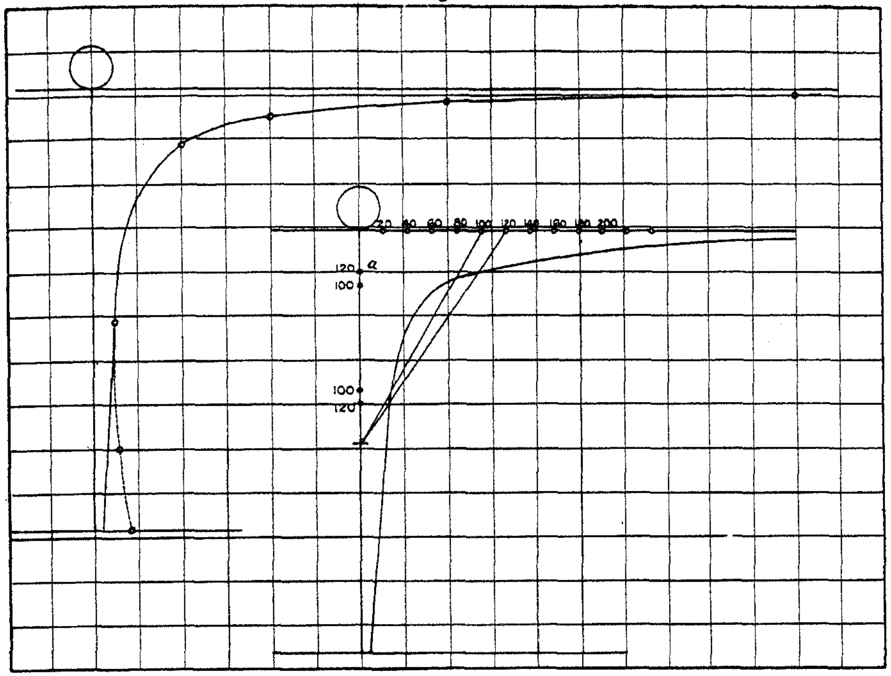


Fig. 2.

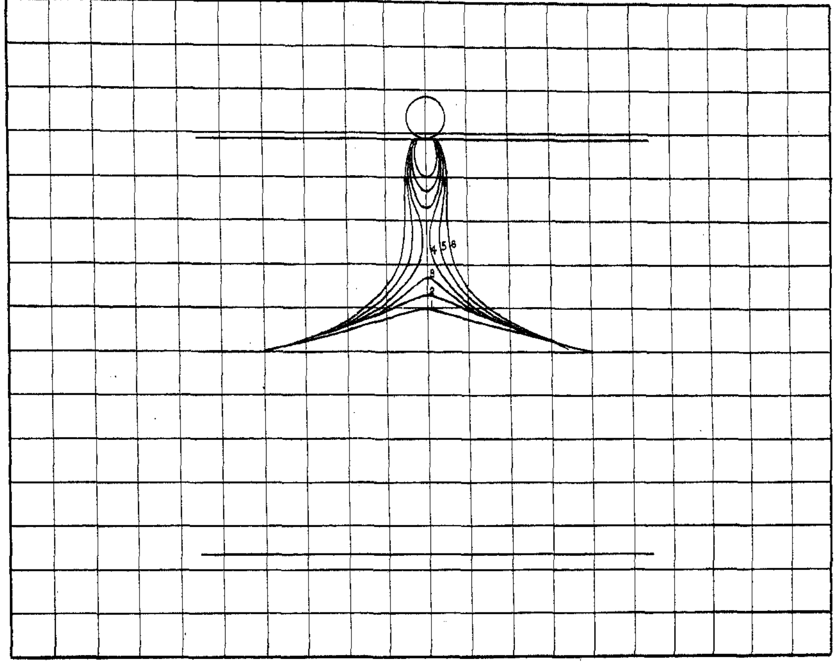


Fig. 3.

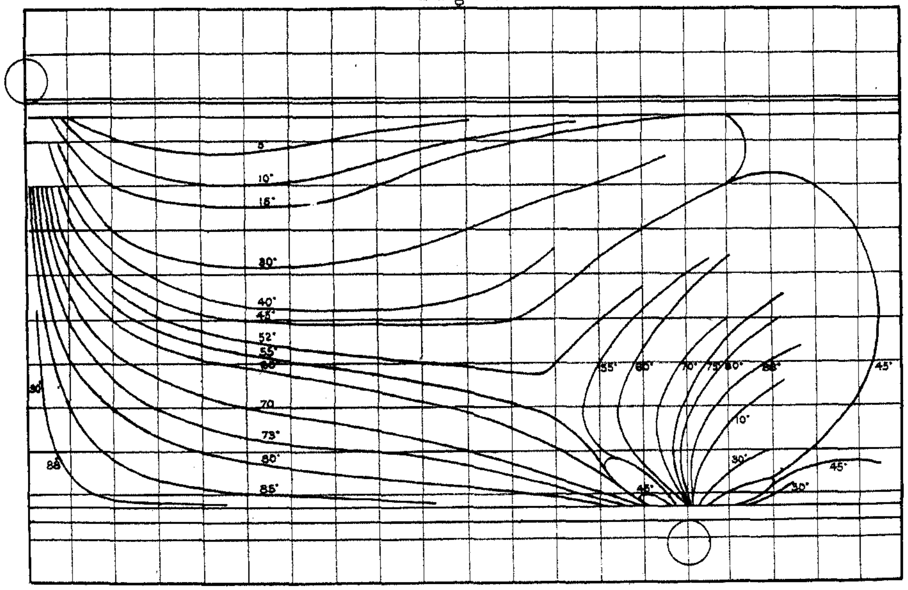
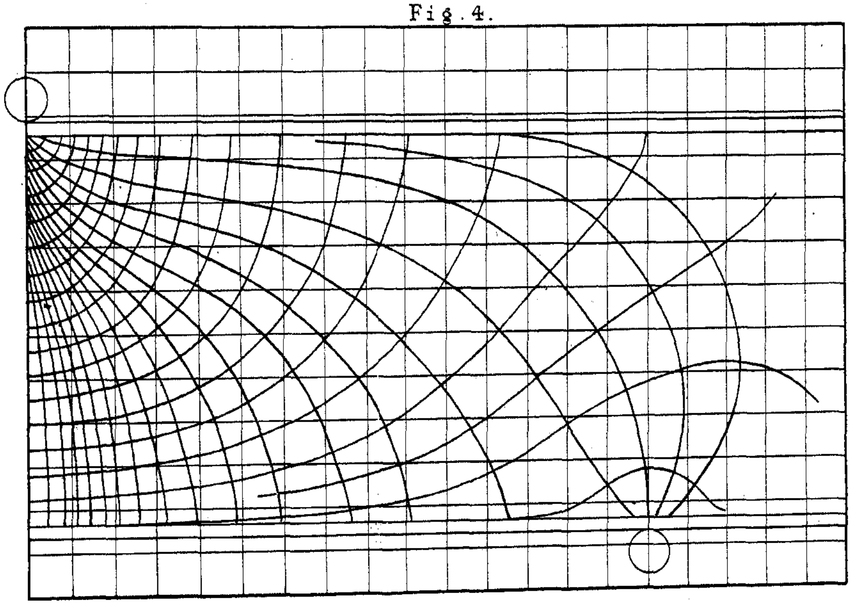


Fig. 4.



(1) The true state of strain at the centre of the beam may be found by superposing on the state of strain due to bending only, that due to surface-loading without bending.

(2) The state of strain due to surface-loading only may be found, with close approximation to truth, by resting the beam on a flat plane instead of on two supports.

(3) The strains due to bending only may be obtained from the Bernoulli-Saint-Venant results ; viz. :—

(α) The stretch for any cross section varies as the distance from the neutral axis.

(β) The central axis is unstretched.

(γ) For the same point in different cross sections the stretch varies as the bending-moment.

Saint-Venant has dealt with the shearing-strains at some little distance from the load in the case of a beam doubly supported and centrally loaded * ; and Professor Pearson has shown † that, in the case of beams continuously loaded, the results of the Bernoulli-Eulerian theory can only be considered as giving approximate formulæ when the span of the beam is not less than ten times its depth ‡.

The mathematical determination of the state of strain produced by the loading of a beam as it rests on a flat plane is one of considerable analytical difficulty.

MM. Lamé and Clapeyron have attempted the solution of a more general problem in their "Mémoire sur l'équilibre intérieur des corps solides homogènes." § The object of this paper is stated to be "to investigate the way in which the interior of a body is affected by the transmission through it of the action of forces." Here they treat the problem of a solid extending to infinity on one side of a plane, on which is a given distribution of tractive load, and also of a solid con-

* Pearson's 'Elastical Researches of Saint-Venant,' §§ 69–99.

† Pearson, "On the Flexure of Heavy Beams subject to continuous systems of Load," Quarterly Journal of Mathematics, No. 93 (1889).

‡ Rankine assumed that the surface-loading effect might be neglected. See his 'Applied Mechanics,' § 311.

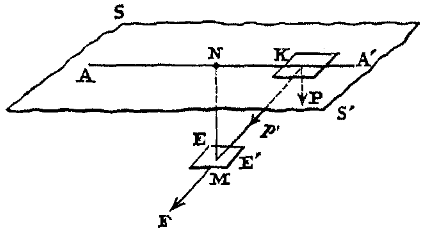
See also *Résumé des Leçons* &c. by Navier (Paris: Dunod, 1864), vol. i. p. 41:—"Observation sur le mode d'application et de distribution des forces qui font fléchir," where the same assumption is made.

§ Crelle's *Journal*, vol. vii. p. 145 *et seq.*

tained between two parallel infinite planes. They obtain as a result a set of definite integrals giving the displacements, introducing a function involving the distribution of tractive load, from which the stresses may be deduced, but concerning which they add: "Les formules précédentes, pour être obtenues en séries numériques et immédiatement applicables, exigent la connaissance des valeurs d'un genre particulier d'intégrales définies, dont il ne nous paraît que les géomètres se soient encore occupés."

The most successful attempt at a solution of this problem is to be found in a more recent work by Professor Boussinesq, published in 1885*. The following is a brief account of the results obtained.

Fig. 1.



S S′ being the surface of the solid (infinite below in length, width, and depth), M a point within, situate at a distance M N $=x$ below the surface, K any element of the surface, situate at the distance K M $=r$ from the point M, and subject to a given exterior pressure K P $=p$, having the component K P′ $=p'$ along K M, the pressure which a plane element E E′ taken through M parallel to the surface S S′ will support, per unit of area, in consequence of the pressure p, will be found directed along the direction of K M produced, and will be equal to

$$\mathrm{MF}=\frac{3p'x}{2\pi r^3}. \qquad (1)$$

* *Application des Potentiels à l'étude de l'Equilibre et du Mouvement des Solides élastiques* (Gauthier-Villars, Paris, 1885).

See also *Théorie de l'Elasticité des Corps solides*, Clebsch; translated and annotated by MM. de Saint-Venant and Flamant. (Paris: Dunod, 1883, p. 374, note to art. 46.)

If, as a particular case, the pressure KP $=p$ be normal, then $p'=p\cos \text{NMK}=p\,\frac{x}{r}$, and

$$\text{MF}=\frac{3px^2}{2\pi r^4}. \quad \cdots\cdots \quad (2)$$

If, further, it is required to find the vertical component of MF, we have $(\text{MF})\,\frac{x}{r}$, or

$$\frac{3px^3}{2\pi r^5}. \quad \cdots\cdots\cdots \quad (3)$$

The treatment of this particular problem is not pursued any further in this work; but Professor Boussinesq has kindly furnished me with a solution more nearly applicable to the case in point, and one which will be found to agree closely with the experimental results I had previously arrived at, and which are given later on.

Suppose there to be a uniform pressure p exerted over every element du of bearing-surface between two extremities A, A′ (see figure), having abscissæ $u=-\text{NA}=-a$, $u=\text{NA}'=+a$, and let $p=\text{P}du$, calling P the constant exterior pressure per unit of length $\text{AA}'=2a$.

The total pressure over unit of surface of an element E E′ will be, from equation (3),

$$\Sigma\frac{3px^3}{2\pi r^5}=\int\frac{3\text{P}x^3du}{2\pi r^5}=\frac{3\text{P}x^3}{2\pi}\int_{-a}^{+a}\frac{du}{r^5}$$

$$=\frac{3\text{P}x^3}{\pi}\int_0^{a}\frac{du}{(x^2+u^2)^{\frac{5}{2}}}\ [\text{NK}=u].$$

Putting $\frac{u}{x}=\alpha$, $du=x\,d\alpha$, we get as the normal pressure per unit of area on E E′,

$$\frac{3\text{P}}{\pi x}\int_0^{\frac{a}{x}}(1+\alpha^2)^{-\frac{5}{2}}d\alpha\ ;$$

or, very nearly, if x is much smaller than a,

$$\frac{3P}{\pi x}\int_0^\infty (1+\alpha^2)^{-\frac{5}{2}}\,d\alpha.$$

The value of the integral is $\frac{2\alpha^3+3\alpha}{3(1+\alpha^2)^{\frac{3}{2}}}$ or $\frac{2+\frac{3}{\alpha^2}}{3\left(1+\frac{1}{\alpha^2}\right)^{\frac{3}{2}}}$, which,

between the limits $\alpha=0$ and $\alpha=\infty$, becomes $\frac{2}{3}$. Thus the pressure per unit of area on an element E E′ becomes $\frac{2P}{\pi x_{\prime}}$, or

$$0{\cdot}64\frac{P}{x}. \qquad (4)$$

This expression has the form of that given below, though, inasmuch as the problem is not altogether the same as that treated experimentally*, a difference in the coefficients is only what might have been expected.

The value of the integral between the limits $\alpha=0$ and $\alpha=\infty$ is, as has been stated, $\frac{2}{3}$, or 0·667. For $\alpha=5$, *i. e.* for $u=5x$ as the upper limit, the integral = 0·666, and for $u=2x$ the integral =0·656; so that this solution is approximately correct for elements lying at a distance of ¼ the width of the beam from the point of contact.

Hence for a beam where the length A A′ is 5·5 millim., this solution would be applicable up to points lying at a distance of about 1·4 millim. from the top surface.

I have investigated the law up to within 0·5 millim. of the top surface, and find it to be

$$y=0{\cdot}726\,\frac{P}{x}.$$

The investigation of the state of strain in glass beams by means of polarized light was first suggested by Sir David Brewster †, and his experiments are usually quoted as proving

* The mathematical solution assumes the length of bearing A A′ on an infinite surface.

† Phil. Trans. 1818, p. 156.

the truth of the Bernoulli-Eulerian theory of flexure. It is, however, easy to show experimentally that these experiments must have been made under conditions where the surface-loading effect was inappreciable; though very accurate reasoning on this point is impossible, as the drawings accompanying Sir David Brewster's paper are not to scale, and the span of the beams and the precise method of application of the loads are not indicated.

M. Neumann developed a theory of the action of strained glass in the polariscope*, and found that the velocity of light in a medium is increased by compressing it. He bases his calculations on the measurement of the deflexions of glass beams supposed to obey the Bernoulli-Eulerian theory; the beams are doubly supported and centrally loaded, having the proportions 66 × 8·5 × 2, the latter being the depth. It is not in all cases stated what spans were employed, so it is impossible to say how far the results were influenced by surface-loading.

Professor Clerk-Maxwell † has examined the state of strain in pieces of unannealed glass of various shapes, the lines of equal intensity of strain being deduced from the isochromatic lines.

The lines of Principal Stress are found from those of Equal Inclination in the manner described later on in this paper.

It has already been pointed out ‡ that "Neither Neumann nor Maxwell seems to have remarked that the difference of the velocities of the ordinary and extraordinary rays depends solely on the maximum slide of planes perpendicular to the wave-front."

An important work on this subject is found in a paper by Dr. John Kerr §. He establishes the fact that "If a plate of glass, compressed or extended in one direction parallel to its faces, be traversed normally by two pencils of light, which are polarized in planes respectively parallel and perpendicular to the direction of strain, then both pencils are retarded by

* *Abhandlungen der k. Akademie der Wissenschaften zu Berlin*, 1841, vol. ii. pp. 50–61.

† Trans. Roy. Soc. Edinburgh, vol. xx. (1853) p. 117.

‡ Hist. of Elasticity, vol. i. p. 643.

§ Phil. Mag. October 1888.

the strain in the case of compression, and both are accelerated by the strain in the case of tension." Also that "strain-generated retardations, absolute as well as relative, are sensibly proportional to the strain," thus confirming Wertheim's results.

Dr. Kerr employs in his experiments a bent glass beam, doubly supported and centrally loaded, having the ratio of span to depth* of 8·4 to 1, and assumed to obey the Bernoulli-Eulerian theory.

I would draw attention to the disagreement between the results arrived at by M. Neumann and Dr. Kerr, the former stating that the velocity of light in a medium is increased by compressing it, while the latter states that the velocity is diminished.

Dr. Kerr examined a beam having a span equal to 8·4 depths, and at a point where the surface-loading effect would be least; whereas M. Neumann examined a beam—span to depth ratio not stated—immediately under the load.

I can only attempt to account for the discrepancy by pointing out that if the span is diminished to less than four depths, the elements of glass that M. Neumann assumed to be in a state of squeeze are actually, as will be shown later, in a state of stretch.

The instrument with which the following experiments were made consists of a steel straining-frame in which the beam to be examined is placed; the beam rests—for flexure—on two steel rollers, and is loaded by a micrometer-screw which bears on a third central roller. The base of the frame is divided, from the centre, in divisions of 2 millim. so that the supports can be set for any required span. A micrometer-screw is placed in the base of the frame opposite the load, so that deflexions can be measured to one ten thousandth of an inch. Two screws in the two sides of the frame enable lateral pressure to be applied. The whole frame can be moved in any direction in its own plane, so that all parts of the beam may be examined. The optical arrangements consist of two nicols, of which the upper is provided with a graduated disk on which the angle of rotation can be observed; a microscope

* According to the figure.

with micrometer-eyepiece can be fitted when it is desired to measure the fringes; circularly polarized light can be used when required.

The beams used were marked on one side with 2 millim. squares; they were covered with paraffin and marked in a dividing-engine and then etched; the lines thus formed enabled the position of dark bands to be determined with accuracy.

Proposition I.

If a beam of glass be laid on a flat surface and loaded across its upper surface, the shear at any point on the normal at the point of contact of the load is inversely proportional to the distance from the point of contact.

Experiment 1. A beam of annealed glass 61 millim. × 6·5 millim. × 20 millim. deep was placed in the steel straining-frame with its narrow side resting on a piece of thin paper.

A steel roller 2 millim. in diameter, 10 millim. long, was placed across the middle of the top surface and loaded by the screw.

The nicols were crossed and at 45° to the axis of the beam.

A quarter-wave mica plate was placed between the beam and the analyser, with the plane containing the optic axes at right angles to the length of the beam.

At that point *a* on the normal where the difference of phase between the ordinary and extraordinary pencils traversing the beam is equal and opposite to the difference of phase produced by the mica plate—the effect will be as if there were no strained glass between the two nicols, and there will therefore be a black spot as the nicols are crossed.

The position of this spot on the normal is plotted on a sheet of squared paper, and an ordinate parallel to the axis chosen to represent the shear.

A second quarter-wave plate is now superposed on the first, and the black spot consequently moves up the normal to where the shear is twice what it was at *a*; this point, *b*, is noted, the second mica plate removed, and the load reduced until the black spot with one mica plate is brought to *b*. In this way a series of points *a*, *b*, *c*, *d* on the normal are found at any one of which the shear is twice what it is at the point below.

Now it is proved later on that the strain at any point varies as the load on the beam; hence by taking the ordinate at b twice that at a, at c four times, and at d eight times, and so on, we get points on the curve of loading along the normal for the load that give a difference of phase at a equal to that of one-quarter wave-plate.

The results are plotted on Plate V. fig. 1: the observed points are indicated by circles, through one of which an hyperbola has been drawn taking the normal and the upper surface of the beam as asymptotes.

It will be seen that the six upper circles lie very nearly on the hyperbola.

It is clear that the upper surface of the beam is an asymptote only when the surface of contact between the beam and the roller is a line—making the stress there infinite; but in practice this cannot be so, the smallest pressure giving a bearing surface—as the roller indents the beam—making the stress there finite, *i. e.* the asymptote will be at some finite distance θ, say, above the point of contact, and θ will vary with the load. I have calculated below that with a load of 115·3 lb. on this same beam, the value of θ is 0·044 millim.

The apparently irregular position of the two lower points observed indicates the amount of error made in the assumption (2) above that the surface-loading effect may be found by substituting a flat plane instead of two supports.

This assumption would be correct only if the beam were of infinite depth and the surface-loading effect of the support infinitely small; here, however, the steel frame itself produces a surface effect, and this, added to that due to the load, makes the points observed lie off the hyperbola, which would be the true curve (as drawn) if the beam were of infinite depth.

The effect of the steel frame must be very small compared with that due to the load for points in the upper half of the beam. In drawing the hyperbola I have considered it as negligible at the centre of the beam; in other words, I consider that the correction of the position of the six upper points, required to allow for the surface effect of the frame, would not make them deviate seriously from the hyperbola.

It must be noted, however, that when the beam is resting on two supports the surface effect of the frame disappears, since the beam only touches the supports and surface effect can only be caused by actual contact; hence I conclude that the surface effect due to loading only is strictly represented by the hyperbola and is as if the beam were of infinite depth*.

In order to establish the hyperbolic law with greater certainty, experiments were made enabling as many as seven points on the curve to be obtained within 3·5 millim. of the point of contact, the highest point being about ·5 millim. from the top of the beam.

Within this range the effect due to the steel frame may with accuracy be neglected.

Experiment 2. A beam of annealed glass, 61 millim. × 6·5 millim. × 20 millim. deep, was placed in the steel straining-frame, on a piece of thick paper, and loaded as before with the steel roller 2 millim. in diameter.

Nicols crossed and at 45° to the axis of the beam.

The screw load was applied until six interference-fringes appeared under the roller; these were examined through a microscope with a micrometer-eyepiece divided to thousandths of an inch. Light from a sodium-flame was used, and the distance between the point of contact and the intersection of each fringe with the normal was measured in micrometer-divisions.

I. Distances in micrometer-divisions to successive fringes:

11·0   13·5   17·0   23·0   35·0   71·5,

but the shears are as 6 . 5 . 4 . 3 . 2 . 1, since there is a difference of phase of only $\frac{1}{2}$ a wave-length required to produce a fringe, hence taking the products of distances into shears we get

66·0   67·5   68·0   69·0   70·0   71·5.

But we have so far neglected the value of θ, the distance of the axis of shears from the point of contact.

By taking the two most reliable observations, where the distance from the point of contact is large and yet where the

* According to this reasoning there appears to be a shear of finite amount at the bottom of the beam—when doubly supported—due to loading only, but this does not seem to me to be inconsistent with the surface conditions.

fringes are well defined, we should have, if the law is hyperbolic,

$$3(23+\theta)=4(17+\theta),$$

or

$$\theta=1^{*}.$$

Correcting the original readings by adding θ to each, we get

12	14·5	18	24	36	72·5,

and the products become

72	72·5	72	72	72	72·5.

II. Same beam, &c., as before, roller and load readjusted. Distance to successive fringes :—

11·5	14·25	17·75	24·0	36·0	75·0

To find θ, take

$$3(24+\theta)=4(17\cdot75+\theta), \text{ or } \theta=1.$$

Correcting the distances, we have

12·5	15·25	18·75	25·0	37·0	76·0,

and the products of the distances into the shears become

75·0	76·25	75·0	75·0	74·0	76·0.

III. Same beam, &c., as before, roller and load readjusted. Distance to successive fringes :—

10·75	12·5	15·25	19·25	26·0	39·0	80·5.

To find θ take $3(26+\theta)=4(19\cdot25+\theta)$, whence $\theta=1$.

Correcting the distances, we have

11·75	13·5	16·25	20·25	27·0	40·0	81·5,

and the products become

82·25	81·0	81·25	81·0	81·0	80·0	81·5.

The law of variation of shear along the normal is thus hown to be hyperbolic.

* One micrometer-division = 0·044 millim.

Experiment 3. The steel straining-frame was removed from the instrument and—by a screw inserted in the place of the straining-screw—hung from a balance, which could be loaded with shot and had a leverage of 50 to 1 : a steel stirrup was hung over the frame with two hardened points resting on the two guiding-pins ; one lower end of the stirrup was secured to the body of the balance, the beam inserted and balanced, and shot put in the pan. This lifted the straining-frame and loaded the beam.

Beam [B] 56 millim. × 20 millim. × 6·5 millim. placed on the base of the steel frame on a piece of thin paper : loaded by a steel roller 2 millim. in diameter. Viewed through nicols crossed and at 45° to the horizontal axis of the beam.

The balance was loaded until the first blue fringe was brought down to a given position on the beam, and the weight of shot observed ; the same fringe was then brought down to a lower given position, and the weight of shot again observed, and so on for successive points.

Distance (α) of given points on normal from top of beam, in millim.	Load (β) on roller (weight of shot) × 50 in lb.			β/α.
	1.	2.	Mean.	
1·15	40	39	39·5	34·34
3·2	114	105	109·5	34·22
4·2	145	149	147	35·00
5·2	182	180	181	34·80
6·2	218	......	218	35·16

If the shear at 4·2 millim. with 147 lb. be taken as unity, the shear at 5·2 millim. with this same load will be $\frac{147}{181}$, since the same shear is produced at 5·2 millim. with 181 lb. as is produced at 4·2 millim. with 147 lb. Hence if the curve of loading is an hyperbola, we should have

$$4{\cdot}2 \times 1 = \frac{147}{181} \times 5{\cdot}2 \text{ or } \beta/\alpha \text{ a constant.}$$

From the third column given above the values of β/α will be seen to be nearly equal in each case ; the value of θ has here been neglected ; if we put $\theta = 0{\cdot}04$ millim., the values of β/α become

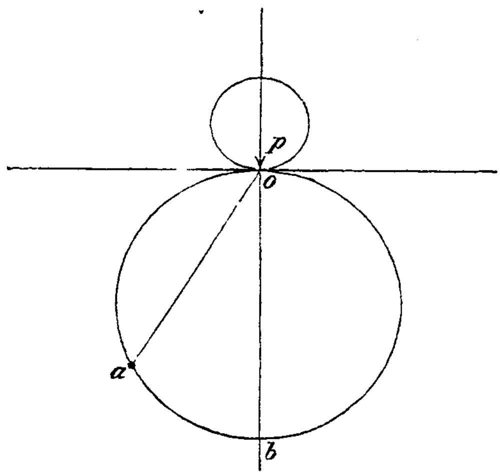
34·6 33·8 34·7 34·5 34·9.

Proposition II.

Things being arranged as in Proposition I., it is required to determine the locus of points of equal intensity of shear, and to show that at any point whatever the shear is inversely proportional to its distance from the point of contact.

Experiment 4. The beam was examined under circularly polarized light, as in Clerk-Maxwell's experiments, in order to obtain the variations in the amount of the strain uncomplicated by variations in the directions of the principal stress-axes; white light was used.

Fig. 2.



The loci of points of equal shear were found to be circles, as in the figure; circles of equal shear were obtained up to 8 millim. diameter with this beam.

Hence the shear at any point a equals the shear at b, if $o\,b\,a$ is a circle, and $o\,b$ the normal at o; *i. e.* shear at a

$$=k\frac{p}{ob}=k\frac{p\cos\theta}{o\,a},$$

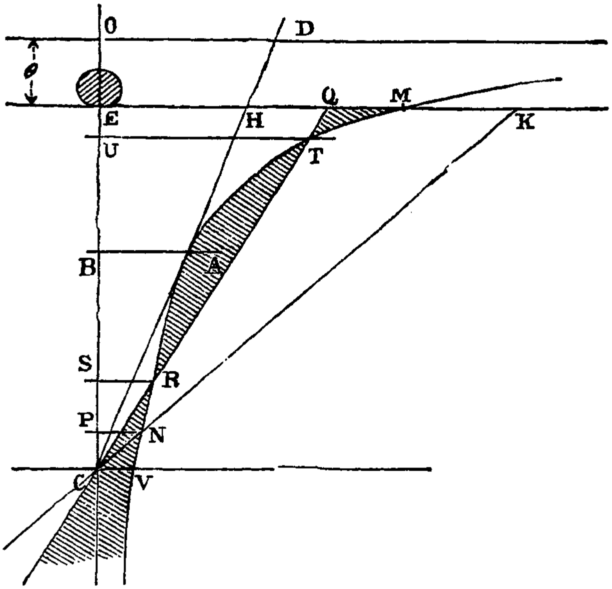
k being some constant, but $p\cos\theta$ is the resolved part of the pressure at o* along $o\,a$; hence the shear at any point is inversely proportional to its distance from the point of contact.

* See Professor Boussinesq's results quoted already.

Proposition III.

The state of strain at the centre of the beam when doubly supported may be found by superposing on the state of strain

Fig. 3.



due to bending only, that due to surface-loading without bending.

It has been proved that the state of strain along the normal at the point of contact due to the surface-loading may be represented by an hyperbola whose asymptotes are respectively the normal itself and a line parallel to the axis of the beam at a distance θ from the point of contact. Let O C, O D in fig. 3 represent these asymptotes, O E $=\theta$; let an hyperbola be drawn whose ordinates parallel to O D represent the shear at any point along E C for a given load: since the shear is proportional to the compressive stress, these ordinates may be considered as proportional to the compressive stress at any point along E C.

By our (α) assumption we may represent the stresses at any point along E C, due to bending, by a right line drawn through C, the centre of the depth.

Let C K be such a line, drawn on the same scale as the hyperbola, so that E K represents the shear (vertical stretch) at E due to bending*, while E M represents the shear (vertical squeeze) due to loading.

These two curves must intersect at some point N; at the corresponding point P on the normal the shear (vertical squeeze) due to the loading is equal to the shear (vertical stretch) due to the bending: an element of volume at P will therefore be subject to voluminal compression only, and the shear will be zero, there will therefore be no birefringent action, and when viewed with crossed nicols there should be a dark spot on a white field.

If the load is kept constant and the span diminished, E K will decrease until C K cuts the hyperbola at a second point; we should now get two points of darkness. As the span is still diminished these dark points should rise and fall respectively until they coincide, when C K is a tangent to the hyperbola; after this they should separate out at right angles.

Plate V. fig. 2 gives the results of an experiment (5) made with constant load and varying spans. The beam was 128 millim. × 19 millim. deep × 5·5 millim. thick, supported on two steel rollers 2 millim. in diameter and centrally loaded over a similar roller: the nicols were crossed and at 45° to the axis. The following table gives the spans:—

Curve.	Span in millim.	Ratio of span to depth.
1	120	6·31
2	100	5·26
3	88	4·63
4	80	4·21
5	78	4·10
6	72	3·79

This experiment shows that there are, generally, two points

* The compressive stress due to bending, at any point on C E, produces a shear (vertical stretch) and a voluminal compression, and both are proportional to the stress, similarly for the shear (vertical squeeze) and voluminal compression produced by the stress due to the loading; so for this purpose it is indifferent whether the ordinates of the two curves represent the compressive stresses or the shears produced.

of zero shear which close up as the span diminishes and then open at right angles.

The same phenomena may be observed by placing a beam on a flat surface and loading it, and then placing over this beam a second, which may be bent with a very long span, or by two couples at the end ; the effect is the same for different degrees of bending as for varying spans in the former experiment.

Thus for spans of four to five depths the normal under the load is divided into three parts by two points of zero shear, elements between these points being subject to shear (vertical stretch), while elements above and below them are subject to shear (vertical squeeze).

When, however, the span is less than four depths, every element in the cross section under the load is subject to shear (vertical squeeze) and the greatest strained element is immediately under the load.

These results may be further checked and confirmed by examining each part of the normal by placing over it a beam bent in the hand ; if the part under examination is in shear, say (vertical squeeze), darkness may be produced by superposing a part of the second beam oppositely strained; if the strains were similar, increased brightness would result.

I exhibit also the results of experiments made to determine the position of the black bands for lower ratios of span to depth.

The dimensions of the beam were 124 millim. × 20 millim. deep × 6·5 millim. thick, loaded on rollers like the others ; nicols crossed and at 45° to the axis.

Here the effect of the supports is very marked, so that when $p=2$ the black band only just touches the axis.

It must be remembered that at the point where the black band cuts the normal the shear is zero, but that everywhere else on the band all that is indicated is that the directions of resultant tension and compression are at 45° to the axis of the beam.

Experiment 6 was made to establish Proposition III. with greater certainty.

Beam 128 millim. × 19 millim. × 5·5 millim. was placed on the base of the straining-frame, on a piece of thin paper and loaded with shot until the first blue fringe came down to a point 1·7 millim. from the top. The load was 65 lb.

The same beam was then supported on two steel rollers 2 millim. in diameter and 120 millim. apart, and centrally loaded over a similar roller until the same blue fringe appeared at the bottom of the beam. The load was 55 lb.

An hyperbola has been drawn (see fig. 1, Plate V.) of convenient proportions, cutting the horizontal through the above-mentioned point at 28·5 divisions from the normal; the shear corresponding to the blue fringe is thus represented by 28·5 divisions, and there is that shear at the point with a load of 65 lb.

Now the stress due to bending, at the extreme bottom fibre of a beam 19 millim. deep, 120 millim. span, and 5·5 millim. thick, with a load of 55 lb., is 1·436 tons per square inch.

The vertical compressive stress at this point, due to the load of 55 lb., is, as is shown later on, 0·121 ton per square inch; but we are not justified in superposing the shears produced by these two stresses, being tensile and compressive at right angles, and the former as much as twelve times the latter, so I shall take the stress at the blue fringe as 1·436 tons per square inch.

Hence the compressive stress produced by 65 lb. over a span of 120 millim., at the top fibre, is

$$1{\cdot}436 \times \frac{65}{55},$$

and the corresponding value in scale-divisions is

$$1{\cdot}436 \times \frac{65}{55} \times \frac{28{\cdot}5}{1{\cdot}436} = 33{\cdot}7 \text{ divisions.}$$

This distance is set off along the top surface in the figure, and the point so found joined to the centre of the middle section: where it cuts the hyperbola we should get darkness on the normal with a span of 120 millim. We can also draw lines representing the bending-stresses for other spans for the same load of 65 lb.

The position of the black bands on the normal, as found by experiment for spans of 120 and 100 millim., are indicated on the normal, and will be found to agree very closely with those points found independently by the intersection of the two curves.

The curve of bending-stresses is a tangent to the curve of loading at a span of 73 millim., as measured from the figure, whereas it is apparently 82 millim. when actually observed ; it would appear more correct to determine this span by drawing the curve through two points which can be observed with accuracy, and then drawing the tangent and measuring the intercept, since the experimental determination of the span giving coincidence of the two dark bands is one liable to considerable error.

By drawing lines from the centre to the points along the top surface corresponding to longer spans we see that the deviation of the so-called "neutral axis" from the centre is considerable : thus even at a span = 10 depths = 190 millim. it should be 1 millim. above the centre.

Proposition IV.

The strain at every point along the normal due to loading varies directly as the load.

Experiment 7. The beam is placed on two supports as before, with a small central load, and the points of intersection of the black bands with the normal are noted. The load is now increased up to the safe limit when the points of intersection are observed to remain unaltered.

We know that the strain at any point on the normal due to bending is proportional to the load ; hence if the point of intersection of the curves of bending and loading remains the same when the load is increased, we know that the strain at any point due to the loading must vary as the load.

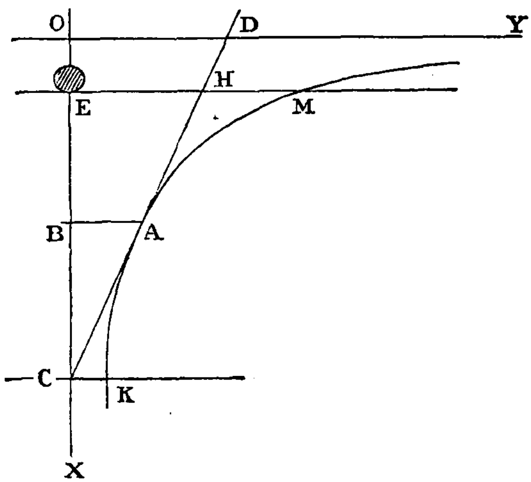
Proposition V.

To determine the constant in the equation to the curve of loading along the normal for any beam.

Let O X represent the vertical through the centre of a beam centrally loaded, E the point of contact of the load with the top of the beam E K ; O Y the axis of shear, $OE=\theta$; K A M the hyperbola of loading for any given load, C A H D the line of stresses due to bending along C E, for the same load, the span being chosen so that C A H is a tangent to the hyperbola at A; *i. e.* so that the dark bands coincide at B. Then O Y and O X are the asymptotes of the hyperbola.

It has been proved that the equation is of the form $y=k\frac{1}{x}$, where y is the compressive stress at a point on the normal E C at a distance X from O. If W is the load and b

Fig. 4.



the width of the beam = length of bearing of loaded roller, we have

$$y=k\,.\frac{W}{bx}\text{ —for the given beam.}$$

Then O D = B A = $2k\frac{W}{bx}$ (since B A represents the stress at B due to the load W). Also E H = $\frac{3}{2}\frac{Wl}{h^2b}$, where E H represents the stress at E due to a load W on a beam of depth h and width b and span l; and

$$\frac{EH}{OD}=\frac{CE}{CO};\quad\therefore\ CO=CE\,\frac{OD}{EH},$$

$$\text{or }\frac{h}{2}+\theta=\frac{h}{2}\times 2k\frac{W}{bx}\times\frac{2}{3}\frac{h^2b}{Wl}=\frac{2}{3}\cdot\frac{h^3k}{lx}$$

$$=\frac{2}{3}\frac{h^3k}{\frac{l}{2}\left(\frac{h}{2}+\theta\right)} \quad [\text{since } x=\tfrac{1}{2}\text{CO}]\ ;$$

$$\therefore\ \left(\frac{h}{2}+\theta\right)^2=\frac{4}{3}\frac{h^3k}{l}\ ;$$

$$\therefore\quad \frac{1}{4}+\frac{\theta}{h}+\frac{\theta^2}{h^2}=\frac{4}{3}\cdot\frac{h}{l}\cdot k\ ;\quad \text{put}\ \frac{l}{h}=\rho\ ;$$

$$\therefore\quad k=\frac{3}{4}\rho\left(\frac{1}{4}+\frac{\theta}{h}+\frac{\theta^2}{h^2}\right).$$

To find k the beam is placed on two supports and centrally loaded ; the two points where the black bands cross the normal are observed (the span being longer than four depths), and plotted, and an hyperbola drawn through them ; a tangent is then drawn to this curve from the centre of the section and its intercept on the upper edge measured, the span giving coincidence of the black bands can then be calculated.

Experiment 8. For a beam 128 millim. × 19 millim. deep × 5·5 millim. I find this span to be 73 millim. ; hence

$$\rho=\frac{73}{19}=3\cdot84.$$

Taking θ at 0·04 millim., $\frac{\theta}{h}=$ 0·002 millim., and neglecting $\frac{\theta^2}{h^2}$, we have

$$h=\tfrac{3}{4}\times3\cdot84\times0\cdot252=0\cdot726.$$

Proposition VI.

To verify the equation to the curve of loading.

Experiment 9. Beam 128 millim. × 19 millim. × 5·5 millim. The stress corresponding to the blue fringe with this beam was found, as already explained, by loading the beam over a span of 120 millim., until the blue fringe appeared at the bottom of the beam ; the load required was 55 lb. ; hence the corresponding stress is 1·436 tons per square inch *.

* From the equation $y=k\frac{\text{W}}{bx}$, there is a compressive stress of 0·121 ton per square inch here due to the load. I have not added the effect of this to that of the bending, as there is no proof that the superposition of small strains holds when the strains themselves are so unequal.

When laid on the base of the steel frame, the same fringe was observed at 1·7 millim. from the top with a load of 65 lb.

From the equation to the curve of loading, taking $k=0{\cdot}726$, $\theta=0{\cdot}04$ millim., we ought to have a stress at 1·7 millim. from the top equal to

$$y=0{\cdot}726\times\frac{25{\cdot}4^2}{2240}\times\frac{65}{1{\cdot}74}\times\frac{1}{5{\cdot}5}=1{\cdot}419 \text{ tons per square inch.}$$

The lines of Principal Stress afford a convenient means of studying the condition of strain in a bent beam.

In a memoir published in 1838* Lamé discussed the problem of the equilibrium of an elastic solid, and investigated the properties of what he termed "isostatic surfaces," or surfaces where only normal "actions" are applied.

In 1870 Saint-Venant† examined the differential equations to which the subject of "isostatic surfaces" gave rise, and in 1872 Professor Boussinesq‡ gave a geometric method for constructing isostatic lines passing through any given point. This memoir was shortly followed by a second §, treating of the integration of the equations involved.

Rankine has examined the form of the curves of Principal Stress, and given an expression from which the curves can be drawn ||. He neglects the surface-loading effect as "in most cases practically of small intensity when compared with the other elements of stress." On comparing his curves with those in Plate V. it will be noticed how closely the curves of tension agree, while the curves of compression are very dissimilar.

Sir George Airy has calculated and drawn the curves of principal stress for several cases of flexure, including that of a beam doubly supported and centrally loaded ¶. He assumes "that there is a neutral point in the centre of the depth; that on the upper side of this neutral point the forces are forces of tension, and on the lower side are forces of compression, and that these forces are proportional to the distances from the

* *Comptes Rendus,* vol. vii. p. 778: "Mémoire sur les surfaces isostatiques dans les corps solides en équilibre d'élasticité."

† Ibid. vol. lxx. ‡ Ibid. vol. lxxiv. p. 242. § Ibid. vol. lxxiv. p. 318.

|| 'Applied Mechanics,' §§ 310 and 311.

¶ Phil. Trans. 1863, part 1.

neutral point;" but he says "These suppositions seem to imply that the actual extensions or compressions correspond exactly to the curvature of the edge of the lamina." The surface-loading effect is not here taken into account; and it would have been interesting to compare the results as shown in fig. 6, for a beam in which the span equals ten depths, with the actual curve as found by experiment. This comparison, however, would lead to erroneous conclusions, since it has been shown* that the results arrived at are not consistent with the fundamental equations, and the form of the curves can be accepted only as a very general approximation.

Proposition VII.

To determine the lines of Principal Stress in a glass beam doubly supported and centrally loaded.

Experiment 10.—A glass beam, 128 millim. × 19 millim. deep × 5·5 millim. thick, was placed in the steel straining-frame on two steel rollers 2 millim. in diameter, and centrally loaded over a similar steel roller.

The span chosen was 60 millim., giving for ρ the value 3·15.

The nicols were crossed and set at an observed angle, and the black band plotted on squared paper corresponding to the squares on the glass beam. This band of course represents the locus of points where the axes of principal stress are parallel to the directions of the planes of the nicol.

The nicols were then turned through a small angle α, the new position of the black bands plotted, and so on for several different angles. These curves are shown in Plate V. The lines of principal stress are easily deduced from these and are shown in Plate V. fig. 4.

Since communicating the above, Sir George Stokes has gone very fully into this problem, and has kindly allowed me to quote the following extracts from letters I have received from him on the subject:—

"Let A be the point in the upper surface where the pres-

* See criticism on Sir George Airy's solution in Ibbetson's 'Mathematical Theory of Elasticity,' note on p. 358.

sure (P) is applied; B, C the points of support below, which I suppose to be equidistant from A; D the middle point of BC. Let y be measured downwards from A; denote BD or DC by a, and AB by b. You have the expressions for the stresses produced by P in an infinite solid $\left(x = \frac{2P}{\pi} \cdot \frac{1}{y}\right)$, and the question is, What system must we superpose on this to pass to the actual case? This, as I showed you, is the system of stresses produced by a system of forces applied to the surface. The forces consist—(1) of the two pressures $\frac{1}{2}$P at B and C; (2) of a continuous oblique tension below, represented in drawing by a fan of tensions directed at every point of the lower surface from the point A.

"Imagine now the beam cut into two by a plane along A D. Consider one half only, say that on the B side. Everything will remain the same as before, provided we supply to the surface A D forces representing the pressures or tensions which existed in the undivided beam. On account of the symmetry, the direction of these must be normal.

"At D the vertical pressure on a horizontal plane in the infinite solid is compounded with an equal vertical tension due to the fan. Hence, of the vertical pressure in A D which must be superposed on the vertical pressure in the infinite solid, we know thus much without obtaining a complete solusion of the problem, namely, that it must equal minus $2P/\pi b$ at D and 0 at A. If we suppose it to vary uniformly between, we are not likely to be far wrong.

"This leads to the following expression for the vertical pressure in A D:—

$$\frac{2P}{\pi}\left(\frac{1}{y} - \frac{y}{b^2}\right).$$

"Now for the horizontal. We know that the complete system of external forces must satisfy the conditions of equilibrium of a rigid body. The direction in each element of the fan passes through A, about which therefore the fan has no moment. Hence the moment of the horizontal forces along A D taken about A must equal $\frac{1}{2}$Pa. Again, the resultant of the semi-fan is a force passing through A, and its vertical component is $\frac{1}{2}$P. Its horizontal component is the integral of

$$\frac{2Pb^2}{\pi} \cdot \frac{x\,dn}{(b^2+x^2)^2},$$

taken from 0 to infinity, or $\frac{P}{\pi}$.

"Hence of the horizontal forces along A D we know these two things :—

(1) The sum must equal $\frac{P}{\pi}$,

(2) The moment round A must equal $\frac{1}{2}Pa$.

"In default of a knowledge of the law according to which the force varies with y, it is natural to take it, for a more or less close approximation, to be expressed by the linear function $A+By$, or say Y. To determine the arbitrary constants A, B, we have only to equate the integral of $Y.dy$ to $\frac{P}{\pi}$, and that of $Yy.dy$ to $\frac{1}{2}Pa$, the limits being 0 to b. We thus get for the expression for the tension at any point of A D,

$$\frac{P}{b}\left(\frac{4}{\pi}-\frac{3a}{b}\right)+\frac{6P}{b}\left(\frac{a}{b}-\frac{1}{\pi}\right)\frac{y}{b}.$$

"At neutral points the vertical pressure equals minus the horizontal tension, giving

$$\left(\frac{6\pi a}{b}-8\right)\frac{y^2}{b^2}+\left(4-\frac{3\pi a}{b}\right)\frac{y}{b}+2=0\cdot$$

or, putting for shortness $\frac{3\pi a}{b}-4=m$,

$$2m\left(\frac{y}{b}\right)^2-m\frac{y}{b}+2=0, \quad \therefore \frac{y}{b}=\frac{1}{4}\pm\sqrt{\frac{1}{16}-\frac{1}{m}}.$$

For the neutral points to be real and different, we must have

$$m>16, \quad \frac{2a}{b}>\frac{40}{3\pi}.$$

When the neutral points coalesce into one, we have m equal 16, y equal $\frac{b}{4}$; and for the ratio of the span to the depth,

$\frac{2a}{b}$ equal $\frac{40}{3\pi}$, equal 4·245, or, say, the span is $4\frac{1}{4}$ times the depth.

"As regards the horizontal tension at points along A D, you take a linear function of y as I do, and your condition of moments is the same as my (2), but in lieu of my (1) you do what is equivalent to taking the total tension *nil*. You further omit the correction to the vertical pressure when we pass from a solid of infinite depth to one terminated by a plane below. You further take the coefficient of $\frac{P}{y}$ as k, a constant to be determined by the observations, instead of $\frac{2}{\pi}$.

"Taking the place of the neutral point (at one fourth of the depth) and the ratio of span to depth as given by my formulæ, and then treating them as if they had been the results of experiment, and substituting in your formulæ for the determination of k, I got 0·7947 instead of 0·64. The largeness of your coefficient is I think fully accounted for by the employment of the formulæ which you used.

"In your method you take the stress belonging to the solid supposed infinitely deep, and superpose it on the stress corresponding to a pure bend.

"This comes to the same thing as retaining three terms only in the equation I gave in my letter for determining the y of the neutral points.

"The equation thus becomes

$$\frac{6\pi a}{b} \cdot \frac{y^2}{b^2} - \frac{3\pi a}{b} \cdot \frac{y}{b} + 2 = 0,$$

or

$$2m\frac{y^2}{b^2} - m\frac{y}{b} + 2 = 0,$$

where

$$m = \frac{3\pi a}{b} \text{ instead of } \frac{3\pi a}{b} - 4.$$

"When the two neutral points merge into one, we have in both cases alike y equal $\frac{1}{4}\,b$, and the only difference is that $3\pi\,\frac{a}{b}$ equals m instead of m plus 4.

"If you had supposed the coefficient for the infinite solid to be an unknown quantity k, and had applied your observations to determine it, using my formulæ instead of your own, you would have got something very close indeed to 0·64.

"It is noteworthy that in your problem, taken as one in two dimensions, the theoretical stresses in the planes of displacement are independent of the ratio between the two elastic constants; in other words, independent of the value of Poisson's ratio."

I have calculated the positions of the neutral points from Sir George Stokes's formula

$$\frac{y}{b}=\frac{1}{4}\pm\sqrt{\frac{1}{16}-\frac{1}{m}}$$

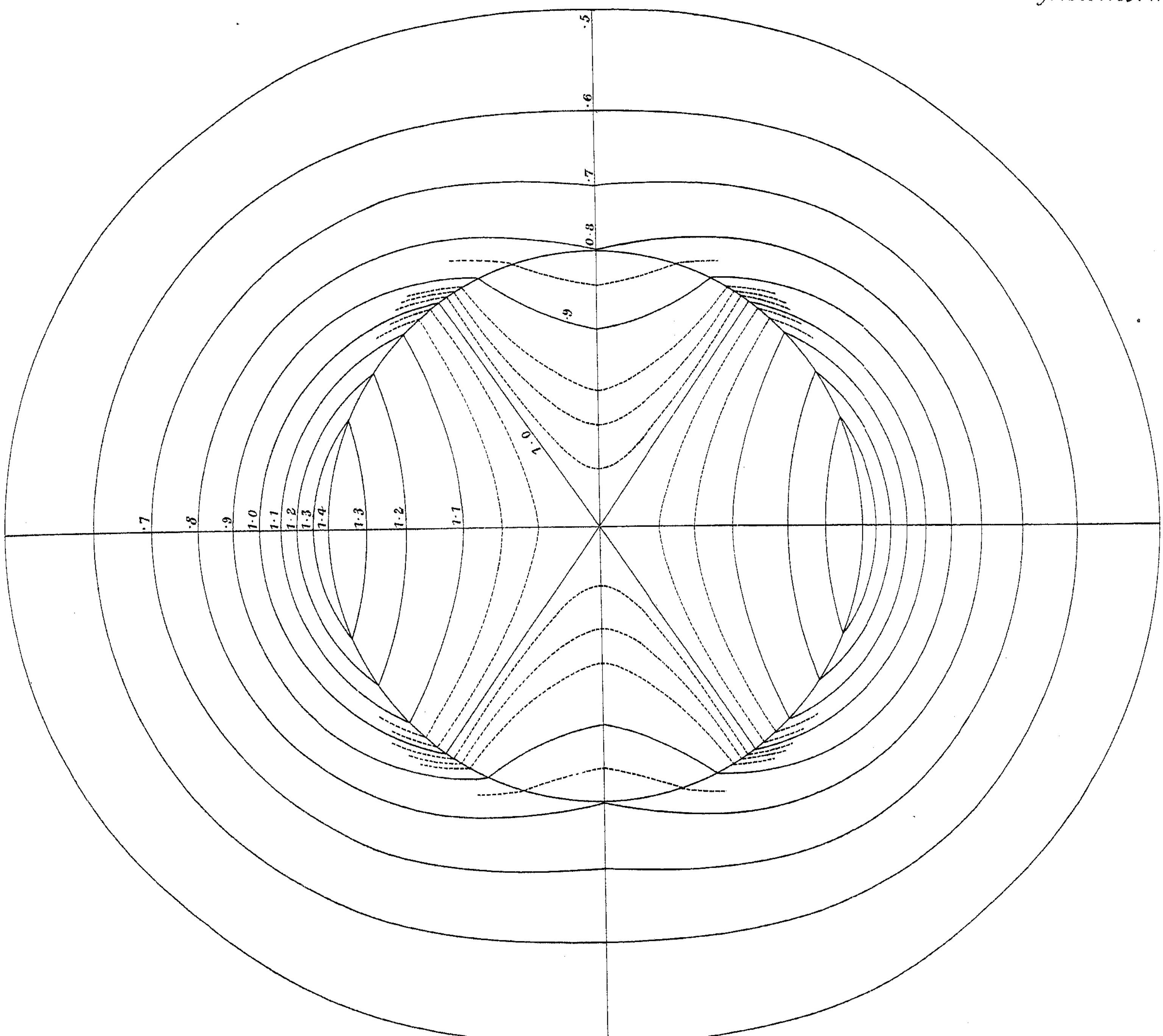
for spans of 88, 100, and 120 millim. in a beam 128 millim. long × 5·5 millim. wide × 19 millim. deep. These are given in the following Table in the 2nd and 3rd columns. The results of actual observations (see p. 192) are given in columns 4 and 5; while columns 6 and 7 give the same points as found by plotting the intersection of the curves of pure bending and loading (infinite solid assumed):—

Span.	Distance of Neutral Points from top edge, by					
	Sir George Stokes's formula.		Observation.		Intersection of curves.	
88.........	6·3	3·2	6·4	3·3	6·9	2·7
100.........	7·0	2·5	7·2	2·5	7·3	2·3
120.........	7·7	1·8	7·8	1·8	7·8	1·75

The error by the intersection method is greater in proportion as the span is smaller, as might have been expected.

If the observed positions of the neutral points are inserted in Sir George Stokes's formula, the value 0·64 is obtained for the constant k in the equation $x=\frac{2P}{\pi}\cdot\frac{1}{y}$.

M'Gill University, Montreal,
October 12, 1891.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