Criteria for the irreducibility of groups of linear homogeneous transformations.

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§ 1.

In what follows G will denote an arbitrarily given group of linear homogeneous transformations in n variables, and A the general transformation

(1) $$x_\varkappa' = a_{\varkappa 1} x_1 + a_{\varkappa 2} x_2 + \cdots + a_{\varkappa n} x_n \qquad (\varkappa = 1, 2, \cdots, n)$$

of G. The group can be either continuous or discontinuous. I shall demonstrate a theorem (theorem 1 below), embodying conditions necessary and sufficient for the irreducibility of a group of linear homogeneous transformations, and supplementary to H. Maschke's well known theorem, that such a group is reducible if some one or more of the non-diagonal coefficients of the group are zero throughout in all transformations of the group.*) Further, I apply the theorem in question to show, when G is generated by infinitesimal transformations, that certain conditions not involving a knowledge of the invariants of the group, or necessarily its finite equations, are sufficient for irreducibility. See theorem 3. Finally, I show by the aid of the latter theorem that no group in n variables $x_1, x_2, \cdots, x_n$, generated by r infinitesimal transformations

$$X_i = \xi_{i1}(x)\frac{\partial}{\partial x_1} + \xi_{i2}(x)\frac{\partial}{\partial x_2} + \cdots + \xi_{in}(x)\frac{\partial}{\partial x_n} \qquad (i = 1, 2, \cdots, r),$$

whose constants of multiplication are c_{ijk} $(i, j, k = 1, 2, \cdots, r)$, contains a subgroup invariant to the adjoined of G, if an integer i can be found, for each pair of distinct integers j and k from 1 to n, for which $c_{ijk} \neq 0$, and if, at the same time, the adjoined contains an infini-

*) Math. Ann., vol. 52 (1899), p. 363.

tesimal transformation $\sum_{i=1}^{r} g_i e_i \frac{\partial}{e_i}$ in which the coefficients $g_1, g_2, \cdots, g_r$ are all distinct. See theorem 4.

Conformally with the notion set forth by Cayley in his 'Memoir on Matrices'*), linear homogeneous transformations, or their matrices, will be regarded as capable of being subjected to the fundamental operations of algebra; and, in general, a linear homogeneous transformation will be identified with its matrix.

We may consider equations (1) as representing a group of deformations (linear homogeneous strains) of n-fold space; and G is termed *reducible* if there is a ν-flat $(0 < \nu < n)$ invariant to G, otherwise *irreducible.* When G is reducible, but not otherwise, the matrices of the collective transformations of G, for a proper choice of coordinates, take the form

(2) $$A' = \begin{vmatrix} A_{11}, & 0 \\ A_{21}, & A_{22} \end{vmatrix},$$

where A_{11} and A_{22} are square matrices of order ν and $n - \nu$ respectively, and A_{12} is a rectangular matrix with $n - \nu$ rows and n columns; and, thus, we have

(3) $$A = CA'C^{-1},$$

where C represents the transformation of coordinates in question.**)

The collective coefficients of the group G may be restricted to an arbitrarily given domain of rationality. In this case, G is said to be *reducible with respect to R* if there is a ν-flat $(0 < \nu < n)$ invariant to G, the coefficients of whose equations belong to R; otherwise, *irreducible with respect to R.* If G is irreducible with respect to R, but not otherwise, we have as above $A = CA'C^{-1}$, for a properly chosen transformation C of coordinates (which may be so taken that its coefficients all belong to R), where now all the coefficients of A' belong to R.***) If G is reducible with respect to R, it is reducible with respect to the domain of all scalars real and imaginary,. that is, is reducible according to the foregoing definition.

In what follows I shall denote by T_{ij} $(i, j = 1, 2, \cdots, n)$ the transformation

(4) $$x'_1 = 0, \cdots, x'_{i-1} = 0, x'_i = x_j, x'_{i+1} = 0, \cdots, x'_n = 0,$$

the coefficients of whose matrix are all zero, except that in the i^{th} row

*) Phil. Trans., 1858, p. 17; see also 'Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function', ibid., p. 39.

**) Cf. A. Loewy, Trans. Am. Math. Soc., vol. 4 (1903), p. 44.

***) Cf. A. Loewy, loc. cit., p. 59.

and j^{th} column, which is equal to unity. This transformation may be represented by the bilinear form $x_i y_j$. The n^2 transformations which we obtain by giving to i and j all integer values from 1 to n are linearly independent; and we have

(5) $$T_{ij}T_{jk} = T_{ik}, \quad T_{ij}T_{hk} = 0 \qquad (i, j, h, k = 1, 2, \cdots, n;\ h \neq j).$$

Every linear transformation whatever in n variables may be expressed linearly in terms of these n^2 transformations; and, in particular, for the general transformation of G, we have

(6) $$A = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}T_{ij}.$$

I shall denote by m the maximum number of linear independent transformations of G; and by $A_1, A_2, \cdots, A_m$, where A_i $(i = 1, 2, \cdots, m)$ is defined by

(7) $$x_\varkappa' = a_{\varkappa 1}^{(i)}x_1 + a_{\varkappa 2}^{(i)}x_2 + \cdots + a_{\varkappa n}^{(i)}x_n \qquad (\varkappa = 1, 2, \cdots, n),$$

any arbitrarily chosen system of m linearly independent transformations of G. From (6), it follows that *the maximum number m of linearly independent transformations of G cannot exceed n^2.* Every transformation of G is expressible linearly in terms of $A_1, A_2, \cdots, A_m$; since, otherwise, G would contain more than m linearly independent transformations. For the general transformation A of G we have

(8) $$A = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_m A_m,$$

that is,

(9) $$a_{\varkappa\lambda} = \alpha_1 a_{\varkappa\lambda}^{(1)} + \alpha_2 a_{\varkappa\lambda}^{(2)} + \cdots + \alpha_m a_{\varkappa\lambda}^{(m)} \qquad (\varkappa, \lambda = 1, 2, \cdots, n).$$

Moreover, we have

(10) $$A_iA_j = \gamma_{ij1}A_1 + \gamma_{ij2}A_2 + \cdots + \gamma_{ijm}A_m \qquad (i, j = 1, 2, \cdots, r),$$

since A_iA_j is a transformation of G for every pair of values of i and j from 1 to m. From (5) and (6) we obtain

(11) $$T_{ii}AT_{jj} = a_{ij}T_{ij} \qquad (i, j = 1, 2, \cdots, n);$$

therefore, in particular,

(12) $$T_{ii}A_hT_{jj} = a_{ij}^{(h)}T_{ij} \qquad (i, j = 1, 2, \cdots, n;\ h = 1, 2, \cdots, m).$$

The totality of linear homogeneous transformations

(13) $$\mathfrak{A} = \mathfrak{a}_1A_1 + \mathfrak{a}_2A_2 + \cdots + \mathfrak{a}_mA_m,$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_m$ are arbitrary scalars, constitutes a group; since, if

(14) $$\mathfrak{B} = \mathfrak{b}_1A_1 + \mathfrak{b}_2A_2 + \cdots + \mathfrak{b}_mA_m,$$

we have, by (10),

(15) $$\mathfrak{A}\mathfrak{B} = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \mathfrak{a}_i\mathfrak{b}_j\gamma_{ijk}A_k.$$

Since $A_1, A_2, \cdots, A_m$ are linearly independent, this group contains m essential parameters. Every transformation of G is a transformation of this group. Therefore, *if G contains r essential parameters $m \geqq r$.*

By a theorem of W. Burnside's, *the group G is irreducible if, and only if, $m = n^2$, that is, if, and only if, it contains n^2 linearly independent transformations.**) In what follows I shall establish, by the aid of Burnside's theorem, a theorem supplementary to Maschke's theorem referred to above. Namely, I shall show that *the group G is reducible if, and only if, either some one, or more, non-diagonal coefficients, of G are zero throughout in all transformations of G, or if the n transformations*

$$x'_1 = 0, \cdots, x'_{i-1} = 0, x'_i = x_i, x'_{i+1} = 0, \cdots, x'_n = 0,$$

for $i = 1, 2, \cdots, n$, cannot all be expressed linearly in terms of the transformations of G.

Let us first assume that G is irreducible. Then, by Burnside's theorem, $m = n^2$, that is, G contains n^2 linearly independent transformations $A_1, A_2, \cdots, A_{n^2}$. By (6), the transformations $A_1, A_2, \cdots, A_{n^2}$ are expressible linearly in terms of the n^2 linear transformations

$$T_{ij} \qquad (i, j = 1, 2, \cdots, n),$$

namely, the n^2 transformations

$$x'_1 = 0, \cdots, x'_{i-1} = 0, x'_i = x_j, x'_{i+1} = 0, \cdots, x'_n = 0 \quad (i, j = 1, 2, \cdots, n); \tag{16}$$

and, since the former are linearly independent, the latter, in particular the n transformations T_{ii}, for $i = 1, 2, \cdots, n$, can be expressed linearly in terms of $A_1, A_2, \cdots, A_{n^2}$. Thus, let

$$T_{ij} = c_{ij}^{(1)} A_1 + c_{ij}^{(2)} A_2 + \cdots + c_{ij}^{(n^2)} A_{n^2} \qquad (i, j = 1, 2, \cdots, n). \tag{17}$$

If possible, let some one coefficient a_{hk} of the general transformation A of G be zero throughout in all transformations of this group for some definite pair of values of h and k $(1 \leqq h \leqq n,\ 1 \leqq k \leqq n)$. Then in particular,

$$a_{hk}^{(1)} = a_{hk}^{(2)} = \cdots = a_{hk}^{(n^2)} = 0;$$

and, therefore, by (5), (12), and (17),

$$\begin{aligned} T_{hk} &= T_{hh} T_{hk} T_{kk} \\ &= \sum_{i=1}^{n^2} c_{hk}^{(i)} T_{hh} A_i T_{kk} \\ &= \sum_{i=1}^{n^2} c_{hk}^{(i)} a_{hk}^{(i)} T_{hk} = 0, \end{aligned}$$

*) Proc. Lond. Math. Soc., 2nd ser., vol. 3 (1905), p. 433.

which is impossible. Therefore, if G is irreducible, the n transformations T_{ii} $(i=1, 2, \cdots, n)$ are expressible linearly in terms of transformations of G, and no coefficient of G is zero throughout.

Let us now assume that each of the n transformations

$$T_{ii} \qquad (i=1, 2, \cdots, n)$$

is expressible linearly in terms of transformations of G; in which case, by (8), each of these n transformations is expressible in terms of any system $A_1, A_2, \cdots, A_m$ of the maximum number of linearly independent transformations of G; thus let

$$(18) \qquad T_{ii} = c_i^{(1)} A_1 + c_i^{(2)} A_2 + \cdots + c_i^{(m)} A_m \qquad (i=1,2,\cdots,m).$$

Let us also assume that no non-diagonal coefficient of G is zero throughout; in which case, for each pair of distinct integers i and j from 1 to n, an integer k can be found $(1 \leqq k \leqq m)$ such that $a_{ij}^{(k)} \neq 0$. For, otherwise, if for a assigned pair of values of i and j $(j \neq i)$,

$$a_{ij}^{(1)} = a_{ij}^{(2)} = \cdots = a_{ij}^{(m)} = 0,$$

then, by (9), $a_{ij} = 0$, that is, some one coefficient of G, outside the diagonal, is zero throughout in each transformation A of G, which is contrary to supposition. Therefore, for each pair of integers i and j from 1 to n, $j \neq i$, we have, by (12) and (18),

$$(19) \qquad \begin{aligned} T_{ij} &= \frac{1}{a_{ij}^{(k)}} T_{ii} A_k T_{jj} \\ &= \frac{1}{a_{ij}^{(k)}} \sum_{h=1}^{m} \sum_{l=1}^{m} c_i^{(h)} c_j^{(l)} A_h A_k A_l \\ &= \frac{1}{a_{ij}^{(k)}} \sum_{h=1}^{m} \sum_{l=1}^{m} \sum_{p=1}^{m} \sum_{q=1}^{m} c_i^{(h)} c_i^{(l)} \gamma_{hkp} \gamma_{plq} A_q, \end{aligned}$$

by (10). Thus, the n^2 linearly independent transformations T_{ij} $(i,j=1,2,\cdots,n)$ are expressible linearly in terms of the m transformations $A_1, A_2, \cdots, A_m$ Whence it follows that $m = n^2$; and, therefore, by Burnside's theorem, G is irreducible.

We have, therefore, the following theorem:

Theorem 1. *An arbitrarily given group G of linear homogeneous transformations*

$$x'_\varkappa = a_{\varkappa 1} x_1 + a_{\varkappa 2} x_2 + \cdots + a_{\varkappa n} x_n \qquad (\varkappa = 1, 2, \cdots, n)$$

in n variables is irreducible if, and only if, each of the n transformations

$$x'_1 = 0, \cdots, x'_{i-1} = 0, \; x'_i = x_i, \; x'_{i+1} = 0, \cdots, x'_n = 0,$$

for $i = 1, 2, \cdots, n$, is expressible linearly in terms of transformations of *the group, and if, at the same time, no one, or more, non-diagonal coefficients of G are zero throughout in all transformations of G.*

Let now G be any group whatever of linear homogeneous transformations whose coefficients lie in an arbitrary domain R. If G is irreducible with respect to the domain of all scalars real and imaginary, a fortiori, it is irreducible with respect to R. Therefore, from theorem 1 we obtain the following theorem:

Theorem 2. *Let G be any group of linear homogeneous transformations in n variables whose coefficients are all contained in the arbitrarily given domain R of rationality. Then, G is irreducible if no non-diagonal coefficient of G is zero throughout, and if each of the n transformations*

$$x'_1 = 0, \cdots, x'_{i-1} = 0, \; x'_i = x_i, \; x'_{i+1} = 0, \cdots, x'_n = 0$$

for $i = 1, 2, \cdots, n$ is expressible linearly in terms of transformations of G.

We may apply this theorem to show very readily that the subgroup of proper orthogonal substitutions in $n > 2$ variables whose coefficients lie wholly in an arbitrarily given domain R is irreducible. It suffices to prove this theorem for $R = 1$. Let G denote this subgroup. The coefficients of a proper orthogonal substitution in n variables are functions, rational in the domain $R = 1$, of $\frac{1}{2} n(n-1)$ parameters; and, therefore, each system of rational values of the parameters gives a transformation of G. Moreover, no coefficient of a proper orthogonal transformation is zero for all values; and, therefore, no coefficient is zero for all rational values of the parameters. Wherefore, for each pair of integers i and j from 1 to n, there is a rational system of values of the parameters, and, therefore, a transformation A of G, for which $a_{ij} \neq 0$. Whence it follows, that no coefficient of G is zero throughout; and therefore, we have only to show, for $n > 2$, that each of the n transformations T_{ii} $(i = 1, 2, \cdots, n)$ can be expressed linearly in terms of transformations of the group. To establish this, let S_0 denote the identical transformation, and let

$$S_{i-1} \qquad (i = 2, 3, \cdots, n)$$

denote the proper orthogonal substitution

$$x'_1 = -x_1, \; x'_2 = x_2, \cdots, x'_{i-1} = x_{i-1}, \; x'_i = -x_i, \; x'_{i+1} = x_{i+1}, \cdots, x'_n = x_n.$$

The n substitutions $S_0, S_1, \cdots, S_{n-1}$ are all transformations of G. If now

$$c_0 S_0 + c_1 S_1 + \cdots + c_{n-1} S_{n-1} = 0$$

then

$$c_0 - c_1 - c_2 - \cdots - c_{n-1} = 0,$$

and, for $k = 2, 3, \cdots, n-1$,

$$c_0 + c_1 + \cdots + c_{k-1} - c_k + c_{k+1} + \cdots + c_{n-1} = 0.$$

But the resultant of this system of equations is equal to $(-1)^n 2^{n-1} (n-2)$, being the determinant whose constituents in the principal diagonal and in the first row are all equal to -1, except that in the first row and column, which, and also the remaining constituents, is equal to $+1$. Therefore, if $n > 2$, the transformations $S_0, S_1, \cdots, S_{n-1}$ are linearly independent; and since these transformations are expressible linearly in terms of $T_{11}, T_{22}, \cdots, T_{nn}$, the latter, if $n > 2$, can be expressed linearly in terms of $S_0, S_1, \cdots, S_{n-1}$, and, thus, in terms of proper orthogonal substitutions, which was to be proved.

§ 2.

By the aid of theorem 1, I shall establish certain criteria for the reducibility of a group of linear homogeneous transformations generated by infinitesimal transformations. Let G be generated by the r $(r \leqq m)$ independent infinitesimal transformations $X_1, X_2, \cdots, X_r$, where

$$(20) \qquad X_i = \sum_{\varkappa=1}^{n} \sum_{\lambda=1}^{n} b_{\varkappa\lambda}^{(i)} x_\lambda \frac{\partial}{\partial x_\varkappa} \qquad (i = 1, 2, \cdots, r).$$

I shall denote by B_i the matrix of X_i $(i = 1, 2, \cdots, r)$: thus,

$$(21) \qquad B_i = \begin{vmatrix} b_{11}^{(i)}, & b_{12}^{(i)}, & \cdots, & b_{1n}^{(i)} \\ b_{21}^{(i)}, & b_{22}^{(i)}, & \cdots, & b_{2n}^{(i)} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1}^{(i)}, & b_{n2}^{(i)}, & \cdots, & b_{nn}^{(i)} \end{vmatrix} \qquad (i = 1, 2, \cdots, r).$$

Then for the general transformation A of G we have

$$(22) \qquad A = e^{\sum_{i=1}^{r} t_i B_i} = 1 + \sum_{i_1=1}^{r} t_{i_1} B_{i_1} + \tfrac{1}{2} \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} t_{i_1} t_{i_2} B_{i_1} B_{i_2} + \cdots,$$

where $t_1, t_2, \cdots, t_r$ are arbitrary scalars. The matrices $B_1, B_2, \cdots, B_r$ are linearly independent, being the matrices of independent infinitesimal transformations of G. In what follows, it is not necessary to distinguish between an infinitesimal transformation and its matrix.

If $\sum_{\varkappa=1}^{n} x_\varkappa \frac{\partial}{\partial x_\varkappa}$ is not an infinitesimal transformation of G, then m is at least as great as $r+1$. In the first place, if $\sum_{\varkappa=1}^{n} x_\varkappa \frac{\partial}{\partial x_\varkappa}$ is an infinitesimal transformation of the group, the matrix unity is expressible linearly in terms of $B_1, B_2, \cdots, B_r$; and conversely. For, if $\sum_{\varkappa=1}^{n} x_\varkappa \frac{\partial}{\partial x_\varkappa}$ is an infinitesimal transformation of G, we have

$$\sum_{\varkappa=1}^{n} x_\varkappa \frac{\partial}{\partial x_\varkappa} = \gamma_1 X_1 + \gamma_2 X_2 + \cdots + \gamma_r X_r,$$

which is equivalent to

$$\varepsilon_{\varkappa\lambda} = \gamma_1 b_{\varkappa\lambda}^{(1)} + \gamma_2 b_{\varkappa\lambda}^{(2)} + \cdots + \gamma_r b_{\varkappa\lambda}^{(r)} \qquad (\varkappa, \lambda = 1, 2, \cdots, n)$$

where $\varepsilon_{\varkappa\varkappa} = 1$, $\varepsilon_{\varkappa\lambda} = 0$ $(\lambda \neq \varkappa)$; and this system of equations is equivalent to

$$1 = \gamma_1 B_1 + \gamma_2 B_2 + \cdots + \gamma_r B_r.$$

Next, let

(23) $$A_i = e^{\varrho B_i} = 1 + \varrho B_i + \frac{1}{2} \varrho^2 B_i^2 + \cdots \qquad (i = 1, 2, \cdots, r),$$

where ϱ is an arbitrary scalar; and, if possible, let

(24) $$c_0 + c_1 A_1 + c_2 A_2 + \cdots + c_r A_r = 0$$

for all values of ϱ; that is, for all values of ϱ let, simultaneously,

(25) $$c_0 \varepsilon_{\varkappa\lambda} + c_1 a_{\varkappa\lambda}^{(1)} + \cdots + c_r a_{\varkappa\lambda}^{(i)} = 0 \qquad (\varkappa, \lambda = 1, 2, \cdots, n),$$

where $\varepsilon_{\varkappa\varkappa} = 1$, $\varepsilon_{\varkappa\lambda} = 0$ $(\lambda \neq \varkappa)$, the scalars $a_{\varkappa\lambda}^{(i)}$ $(i = 1, 2, \cdots, r)$ being the coefficients of A_i. Since these coefficients are transcendental integral functions of ϱ, if equations (25) have a solution other than

$$c_0 = c_1 = \cdots = c_r = 0,$$

we may take $c_0, c_1, \cdots, c_r$ to be integral functions of ϱ; and, thus, we may put

(26) $$c_i = c_i^{(0)} + c_i^{(1)} \varrho + \frac{1}{2} c_i^{(2)} \varrho^2 + \cdots \qquad (i = 1, 2, \cdots, r),$$

in which case equation (24) becomes

(27) $$\left[c_0^{(0)} + \sum_{i=1}^{r} c_i^{(0)}\right] + \varrho\left[\left(c_0^{(1)} + \sum_{i=1}^{r} c_i^{(1)}\right) + \sum_{i=1}^{r} c_i^{(0)} B_i\right]$$
$$+ \frac{1}{2}\varrho^2\left[\left(c_0^{(2)} + \sum_{i=1}^{r} c_i^{(2)}\right) + 2\sum_{i=1}^{r} c_i^{(1)} B_i + \sum_{i=1}^{r} c_i^{(0)} B_i^2\right] + \cdots = 0,$$

on substituting for A_1, A_2, etc., their expressions in terms of B_1, B_2, etc. Since this equation holds for all values of ϱ, we have, in particular,

(28) $$c_0^{(0)} + \sum_{i=1}^{r} c_i^{(0)} = 0,$$

(29) $$\left(c_0^{(1)} + \sum_{i=1}^{r} c_i^{(1)}\right) + \sum_{i=1}^{r} c_i^{(0)} B_i = 0,$$

(30) $$\left(c_0^{(2)} + \sum_{i=1}^{r} c_i^{(2)}\right) + 2\sum_{i=1}^{r} c_i^{(1)} B_i + \sum_{i=1}^{r} c_i^{(0)} B_i^2 = 0.$$

We have

$$c_0^{(1)} + \sum_{i=1}^{r} c_i^{(1)} = 0, \tag{31}$$

since, otherwise, by (29) the matrix unity is expressible linearly in terms of $B_1, B_2, \cdots, B_r$, which is contrary to supposition. Therefore, since $B_1, B_2, \cdots, B_r$ are linearly independent, $c_i^{(0)} = 0$ $(i = 1, 2, \cdots, r)$; whence, by (28), $c_0^{(0)} = 0$. Thus we have

$$c_0^{(0)} = c_1^{(0)} = \cdots = c_r^{(0)} = 0. \tag{32}$$

Further,

$$c_0^{(1)} = c_1^{(1)} = \cdots = c_r^{(1)} = 0. \tag{33}$$

For, from (30) and (32), we derive

$$\left(c_0^{(2)} + \sum_{i=1}^{r} c_i^{(2)}\right) + 2\sum_{i=1}^{r} c_i^{(1)} B_i = 0;$$

whence follows

$$c_0^{(2)} + \sum_{i=1}^{r} c_i^{(2)} = 0, \tag{34}$$

since, otherwise, the matrix unity is expressible linearly in terms of $B_1, B_2, \cdots, B_r$. Therefore, since $B_1, B_2, \cdots, B_r$ are linearly independent, $c_i^{(1)} = 0$ $(i = 1, 2, \cdots, r)$; whence, by (31), we have $c_0^{(1)} = 0$. Again, by the aid of the preceding equations and those obtained by putting equal to zero the terms in (27) involving ϱ^3, we have

$$c_0^{(2)} = c_1^{(2)} = \cdots = c_r^{(2)} = 0; \tag{35}$$

etc., etc. Wherefore,

$$c_0 = c_1 = \cdots = c_r = 0 \tag{36}$$

for all values of ϱ; and, thus, $A_1, A_2, \cdots, A_m$ and the identical transformation are linearly independent; that is, $m \geqq r + 1$. Consequently, if $r = n^2 - 1$, and G does not contain the infinitesimal transformation $\sum_{\varkappa=1}^{n} x_\varkappa \frac{\partial}{\partial x_\varkappa}$, the group is irreducible, by Burnside's theorem, since in this case $m = n^2$.

If $r = n^2$, since $m \geqq r$, we have $m = n^2$, and G is irreducible, which is otherwise evident, since in this case G is the general linear homogeneous group.

Let us assume that no non-diagonal coefficient of the matrices of the collective infinitesimal transformations of G is zero throughout for all the infinitesimal transformations of this group. That is, let us assume that no non-diagonal coefficient is zero in each of the matrices $B_1, B_2, \cdots, B_r$

of the respective infinitesimal transformations $x_1, x_2, \cdots, x_r$ of G; and, therefore, for each pair of distinct integers i and j from 1 to n, we can find an infinitesimal transformation B_k $(1 \leqq k \leqq r)$ for which $b_{ij}^{(k)} \neq 0$. Or, otherwise expressed, let us assume that

$$b_{12}^{(k_{12})} \neq 0,\ b_{13}^{(k_{13})} \neq 0,\ \cdots,\ b_{1r}^{(k_{1r})} \neq 0,$$
$$b_{21}^{(k_{21})} \neq 0,\ b_{23}^{(k_{23})} \neq 0,\ \cdots,\ b_{2r}^{(k_{2r})} \neq 0,$$
$$\cdots\cdots\cdots\cdots$$

where k_{pq} $(p, q = 1, 2, \cdots, r;\ q \neq p)$ are integers not less than 1 nor greater than r. Let i and j be any definite but arbitrary pair of distinct integers from 1 to n; and let

$$(37) \qquad A' = e^{\varrho B_k} = 1 + \varrho B_k + \frac{1}{2}\varrho^2 B_k^2 + \cdots,$$

where ϱ is an arbitrary scalar. For ϱ sufficiently small, the non-diagonal coefficients of A' will differ as little as we please from the corresponding non-diagonal coefficients of B_k. Therefore, since $b_{ij}^{(k)} \neq 0$, we have $a'_{ij} \neq 0$ for ϱ sufficiently small. Whence, since $A' = e^{\varrho B_k}$ is a transformation of G, it follows that no non-diagonal coefficient of G is zero throughout. Therefore, by theorem 1, if also each of the n transformations T_{ii} $(i = 1, 2, \cdots, n)$ can be expressed linearly in terms of transformations of G, this group is irreducible.

Let us next assume that $r \geqq n-1$ and that G contains $n-1$ infinitesimal transformations

$$(38) \qquad Y_i = \sum_{j=1}^{r} \tau_{ij} X_j = \mathfrak{b}_1^{(i)} x_1 \frac{\partial}{\partial x_1} + \mathfrak{b}_2^{(i)} x_2 \frac{\partial}{\partial x_2} + \cdots + \mathfrak{b}_n^{(i)} x_n \frac{\partial}{\partial x_n} \qquad (i = 1, 2, \cdots, n-1),$$

such that

$$\begin{vmatrix} 1, & e^{\mathfrak{b}_1^{(1)}}, & \cdots, & e^{\mathfrak{b}_1^{(n-1)}} \\ 1, & e^{\mathfrak{b}_2^{(1)}}, & \cdots, & e^{\mathfrak{b}_2^{(n-1)}} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & e^{\mathfrak{b}_n^{(1)}}, & \cdots, & e^{\mathfrak{b}_n^{(n-1)}} \end{vmatrix} \neq 0.$$

Let $\mathfrak{B}_i$ $(i = 1, 2, \cdots, n-1)$ denote the matrix of Y_i; and let

$$\mathfrak{A}_i = e^{\mathfrak{B}_i} = \begin{vmatrix} e^{\mathfrak{b}_1^{(i)}}, & 0, & \cdots, & 0 \\ 0, & e^{\mathfrak{b}_2^{(i)}}, & \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \cdots, & e^{\mathfrak{b}_n^{(i)}} \end{vmatrix} \qquad (i = 1, 2, \cdots, n-1).$$

The transformations $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_{n-1}$ are transformations of G, being generated by infinitesimal transformations of G; and, together with the identical transformation (which is also a transformation of G), are linear in $T_{11}, T_{22}, \cdots, T_{nn}$. Moreover, $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_{n-1}$, and the identical transformation, are linearly independent. For, if

$$c_0 + c_1 \mathfrak{A}_1 + \cdots + c_{n-1}\mathfrak{A}_{n-1} = 0,$$

we have, simultaneously,

$$c_0 + c_1 e^{\mathfrak{b}_\varkappa^{(1)}} + \cdots + c_{n-1} e^{\mathfrak{b}_\varkappa^{(n-1)}} = 0 \qquad (\varkappa = 1, 2, \cdots, n),$$

which is impossible, since the resultant of these equations is, by supposition, not equal to zero. Therefore, in the case supposed, the n transformations $T_{11}, T_{22}, \cdots, T_{nn}$ are expressible linearly in terms of transformations of G.

We have, therefore, the following theorem.

Theorem 3. *Let G be any group of linear homogeneous transformations in n variables generated by r independent infinitesimal transformations $X_1, X_2, \cdots, X_r$ where*

$$X_i = \sum_{\varkappa=1}^{n} \sum_{\lambda=1}^{n} b_{\varkappa\lambda}^{(i)} x_\lambda \frac{\partial}{\partial x_\varkappa} \qquad (i = 1, 2, \cdots, r).$$

Then G is irreducible if $r = n^2 - 1$ and $\sum_{\varkappa=1}^{n} x_\varkappa \frac{\partial}{\partial x_\varkappa}$ is not an infinitesimal transformation of the group. Further, G is irreducible if no non-diagonal coefficient is zero in each of the matrices of the respective infinitesimal transformations $X_1, X_2, \cdots, X_r$, that is, if an integer k $(1 \leqq k \leqq r)$ can be found, corresponding to each pair of distinct integers i and j from 1 to n, such that $b_{ij}^{(k)} \neq 0$, and if, at the same time, each of the n transformations

$$x_1' = 0, \cdots, x_{i-1}' = 0, \; x_i' = x_i, \; x_{i+1}' = 0, \cdots, x_n' = 0,$$

for $i = 1, 2, \cdots, n$, can be expressed linearly in terms of transformations of G. Therefore, in particular, G is irreducible if, corresponding to each pair of distinct integers i and j from 1 to n, an integer k can be found such that $b_{ij}^{(k)} \neq 0$, and if, at the same time, G contains $n-1$ infinitesimal transformations

$$Y_i = \sum_{\varkappa=1}^{n} \mathfrak{b}_k^{(i)} x_\varkappa \frac{\partial}{\partial x_\varkappa} \qquad (i = 1, 2, \cdots, n-1),$$

such that

$$\begin{vmatrix} 1, & e^{\mathfrak{b}_1^{(1)}}, & \cdots, & e^{\mathfrak{b}_1^{(n-1)}} \\ 1, & e^{\mathfrak{b}_2^{(1)}}, & \cdots, & e^{\mathfrak{b}_2^{(n-1)}} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & e^{\mathfrak{b}_n^{(1)}}, & \cdots, & e^{\mathfrak{b}_n^{(n-1)}} \end{vmatrix} \neq 0.$$

We can determine whether or not the group G, generated by infinitesimal transformations, is reducible, as soon as we know its invariants general and special; namely, we have only to ascertain whether among these invariants is a ν-flat $(0 < \nu < n)$. It is to be noted that the above theorem does not require a knowledge of the invariants of the group. To illustrate the application of this theorem, it may be applied to the proper orthogonal group in n variables. As is well known, this group has no finite invariant flat (except the origin) when $n > 2$ and, thus, for $n > 2$, is irreducible. The $r = \frac{1}{2} n(n-1)$ infinitesimal transformations of this group are

$$X_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} \qquad (i=1, 2, \cdots, n;\ j=i+1, i+2, \cdots, n).$$

Therefore, no coefficient in the collective infinitesimal transformations of the group is zero throughout. Moreover, as shown pag. 362, when $n > 2$, there are n linearly independent transformations $S_0, S_1, \cdots, S_{n-1}$ of the group of the form

$$x_1' = \rho_1 x_1,\ x_2' = \rho_2 x_2,\ \cdots,\ x_n' = \rho_n x_n;$$

and, therefore, the transformations T_{ii} $(i = 1, 2, \cdots, n)$ can be expressed linearly in terms of transformations of the group. Whence, by theorem 3, the group is irreducible if $n > 2$.

Finally, let $\mathfrak{G}$ be any group in n variables generated by r independent infinitesimal transformations

$$\mathfrak{X}_i = \xi_{i1}(x) \frac{\partial}{\partial x_1} + \xi_{i2}(x) \frac{\partial}{\partial x_2} + \cdots + \xi_{in}(x) \frac{\partial}{\partial x_n} \qquad (i = 1, 2, \cdots, r),$$

whose constants of composition are c_{ijk} $(i, j, k = 1, 2, \cdots, r)$. The infinitesimal transformations of the adjoined of $\mathfrak{G}$ are

$$E_i = \sum_{j=1}^{r} \sum_{k=1}^{r} c_{ijk} e_k \frac{\partial}{\partial e_j} \qquad (i = 1, 2, \cdots, r);$$

and, if this group contains no invariant ν-flat $(\nu < r)$, that is, if the adjoined is irreducible, there is no subgroup of $\mathfrak{G}$ invariant to the adjoined. It is to be noted that the adjoined will contain $r - 1$ infinitesimal transformations

$$g_1^{(i)} e_1 \frac{\partial}{\partial e_1} + g_2^{(i)} e_2 \frac{\partial}{\partial e_2} + \cdots + g_r^{(i)} e_r \frac{\partial}{\partial e_r} \quad (i = 1, 2, \cdots, r-1),$$

for which

$$\begin{vmatrix} 1, & e^{g_1^{(1)}}, & \cdots, & e^{g_1^{(r-1)}} \\ 1, & e^{g_2^{(1)}}, & \cdots, & e^{g_2^{(r-1)}} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & e^{g_r^{(1)}}, & \cdots, & e^{g_r^{(r-1)}} \end{vmatrix} \neq 0,$$

if it contains an infinitesimal transformation

$$g_1 e_1 \frac{\partial}{\partial e_1} + g_2 e_2 \frac{\partial}{\partial e_2} + \cdots + g_r e_r \frac{\partial}{\partial e_r}$$

in which no two of the g's are equal. We have now, as a consequence of theorem 3, the following theorem:

Theorem 4. *Let $\mathfrak{G}$ be any group in n variables generated by the r independent infinitesimal transformations $\mathfrak{X}_1, \mathfrak{X}_2, \cdots, \mathfrak{X}_r$, where*

$$\mathfrak{X}_i = \xi_{i1}(x) \frac{\partial}{\partial x_1} + \xi_{i2}(x) \frac{\partial}{\partial x_2} + \cdots + \xi_{in}(x) \frac{\partial}{\partial x_n} \qquad (i = 1, 2, \cdots, r),$$

whose constants of composition are c_{ijk} $(i, j, k = 1, 2, \cdots, r)$. Then $\mathfrak{G}$ contains no subgroup invariant to the adjoined group, if, for each pair of distinct integers j and k from 1 to r, an integer i $(1 \leqq i \leqq r)$ can be found such that $c_{ijk} \neq 0$, and if, at the same time, each of the r transformations

$$e'_1 = 0, \cdots, e'_{i-1} = 0, \; e'_i = e_i, \; e'_{i+1} = 0, \cdots, e'_r = 0,$$

for $i = 1, 2, \cdots, r$, can be expressed linearly in terms of transformations of the adjoined. The latter condition is satisfied if the adjoined contains $r - 1$ infinitesimal transformations

$$g_1^{(i)} e_1 \frac{\partial}{\partial e_1} + g_2^{(i)} e_2 \frac{\partial}{\partial e_2} + \cdots + g_r^{(i)} e_r \frac{\partial}{\partial e_r} \qquad (i = 1, 2, \cdots, r-1)$$

such that

$$\begin{vmatrix} 1, & e^{g_1^{(1)}}, & \cdots, & e^{g_1^{(r-1)}} \\ 1, & e^{g_2^{(1)}}, & \cdots, & e^{g_2^{(r-1)}} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & e^{g_r^{(1)}}, & \cdots, & e^{g_r^{(r-1)}} \end{vmatrix} \neq 0;$$

and, therefore, in particular, if the adjoined contains an infinitesimal transformation

$$g_1 e_1 \frac{\partial}{\partial e_1} + g_2 e_2 \frac{\partial}{\partial e_2} + \cdots + g_r e_r \frac{\partial}{\partial e_r}$$

for which the coefficients $g_1, g_2, \cdots, g_r$ are all distinct.