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XXXVII. Turbines

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XXXVII. *Turbines.* *By* J. LESTER WOODBRIDGE, *B.S.*, *M.E.**

I PROPOSE to discuss the action of turbines in general.

Consider first the water after it has entered the wheel and is passing along its vanes. Conceive the water to be divided into an infinite number of filaments by vanes similar to those of the wheel, but subjected to the condition that, at each point, their width, *ac* (fig. 1), measured on the arc whose centre is O, shall subtend at the centre a constant angle $d\theta$. Conceive each filament to be divided into small prisms whose bases are represented by the shaded areas $a'b'c'd'$, $d'c'c''d''$, and *abcd*, by vertical planes normal to the vanes, making the divisions *ae*, *ef*, intercepted on the radius by circles passing through the consecutive vertices on the same vane a', d', d'', &c., equal. The variable height of a prism represent by x, and let ρ be the variable distance from the centre.

Then $d\rho = ae,\ ef$, &c.

$\rho d\theta d\rho = abcd$, &c. = area of the base of an infinitesimal prism;

$x\rho d\theta d\rho$ = volume of infinitesimal prism;

$x\delta\rho d\theta d\rho = m$ = the mass of prism, δ being its density;

$\gamma = san$ = angle between the normal to the vane at any point ρ, and the radius O*a* prolonged through that point;

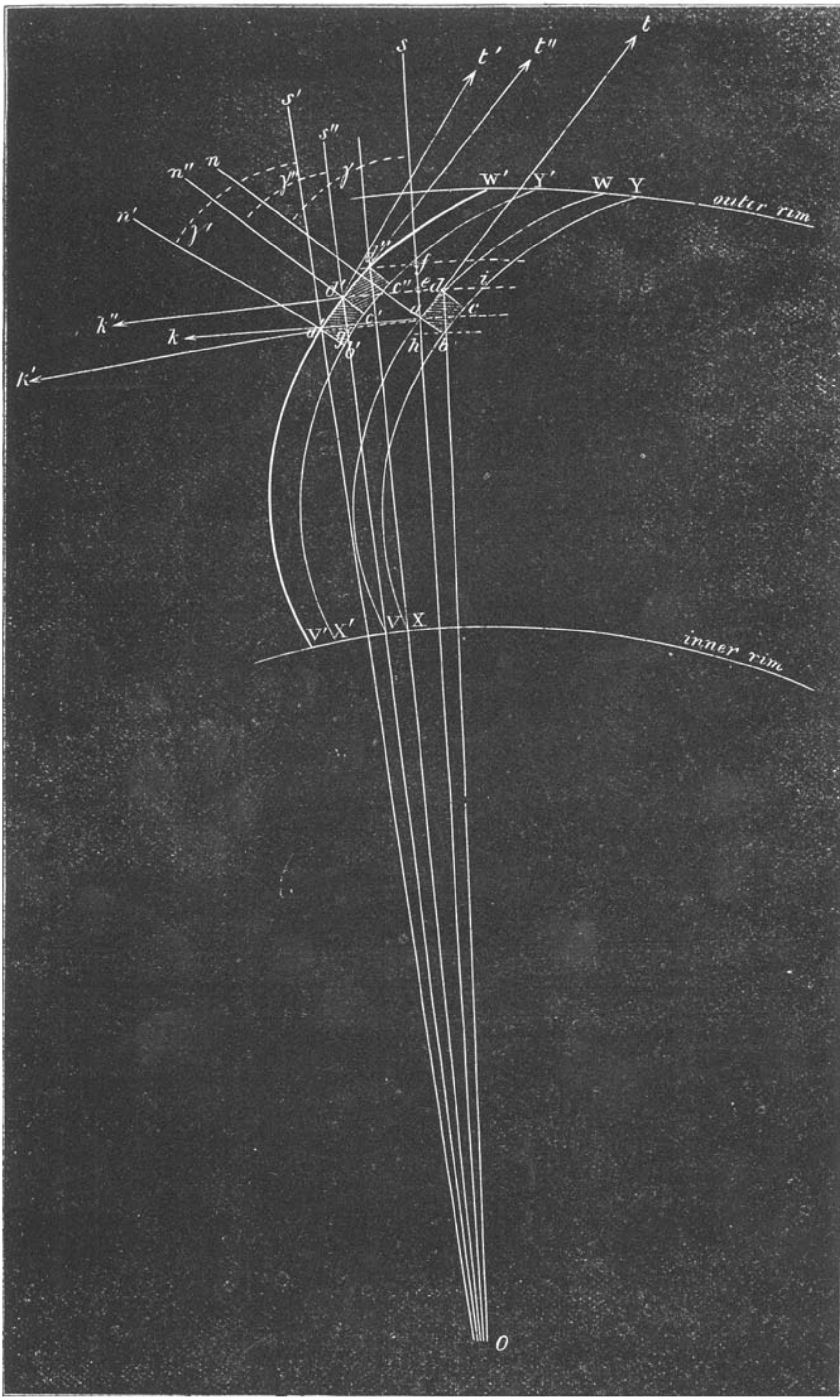
v = the velocity of a particle along the vane at ρ;

ω = the uniform angular velocity of the wheel; and

p = the pressure of the water at the point ρ.

* Communicated by De Volson Wood, Professor of Engineering in Stevens Institute of Technology, Hoboken, N.J.

Fig. 1.



For the element of time, dt, we will take the time occupied by one of the liquid prisms in passing along the vane through a distance equal to its own length, or outwardly a distance $d\rho$, thus making t a function of ρ, the latter being considered the independent variable. We have

$$\frac{dt}{d\rho}d\rho=\frac{d\rho}{\frac{d\rho}{dt}}=\frac{d\rho}{v\sin\gamma}.$$

In the figure let aa' be the distance passed through by the turbine in the time dt, then

$$aa'=\omega\rho dt=\omega\rho\frac{dt}{d\rho}d\rho.$$

The mass m will have two motions; one along the vane, the other with the wheel perpendicular to the radius. By changing its position successively in each of these directions, both its velocity with the wheel and its velocity along the vane may suffer changes both in *amount* and *direction*, and will give rise to the following eight possible reactions:—

I. By moving from a to a' in the arc of a circle.

1. $\omega\rho$ may be increased or diminished;
2. $\omega\rho$ may be changed in direction;
3. v may be increased or diminished;
4. v may be changed in direction.

II. By moving from a' to d' along the vane.

5. $\omega\rho$ may be increased or diminished;
6. $\omega\rho$ may be changed in direction;
7. v may be increased or diminished;
8. v may be changed in direction.

By the conditions imposed, 1 and 3 are zero. For the others we have:—

No. 2. By moving from a to a', the velocity $\omega\rho$ is changed in direction from ak to $a'k'$ in the time dt. The momentum is $m\omega\rho$, and the rate of angular change is

$$\frac{kak'}{dt}=\frac{\omega dt}{dt}=\omega,$$

and hence the reaction will be $m\omega^2\rho$ in a direction radially outwards. This is the *centrifugal force* as designated by most writers. Resolving into two components, we have

$$m\omega^2\rho\sin\gamma \text{ along the vane,}$$

$$m\omega^2\rho\cos\gamma \text{ normal to the vane.}$$

No. 4. In moving from a to a', the velocity along the vane, v, is changed in direction from at to $a't'$ at the rate ω as in No. 3. The momentum is mv, and the force will be $mv\omega$, which acts in the direction na, and being resolved gives

$$0 \text{ along the vane,}$$

$$-mv\omega \text{ normal to the vane.}$$

No. 5. In passing from a' to d' the *increase* of $\omega\rho$ will be $\omega d\rho$ in a time dt, and the reaction will be $m\omega\frac{d\rho}{dt}$, in a direction tangential to the motion of the wheel but backwards, and its components will be

$$m\omega\frac{d\rho}{dt}\cos\gamma \text{ along the vane,}$$

$$-m\omega\frac{d\rho}{dt}\sin\gamma \text{ normal to the vane.}$$

No. 6. In passing from a' to d', $\omega\rho$ will be changed in direction by the angle $a'\mathrm{O}d'=\frac{a'g}{\rho}=\frac{d\rho\cot\gamma}{\rho}$, and the rate of angular change will be

$$\frac{\cot\gamma}{\rho}\cdot\frac{d\rho}{dt},$$

and the momentum will be

$$m\omega\rho\,;$$

hence the reaction will be

$$m\omega\cot\gamma\frac{d\rho}{dt},$$

which acts radially inward, and its components are

$$-m\omega\cos\gamma\frac{d\rho}{dt} \text{ along the vane,}$$

$$-m\omega\cot\gamma\cos\gamma\frac{d\rho}{dt} \text{ normal to the vane.}$$

No. 7. By moving from a' to d', v will be increased by an amount $\frac{dv}{d\rho}d\rho$, in the time dt, and the reaction will be $m\frac{dv}{d\rho}\cdot\frac{d\rho}{dt}$, which acts backward along the vane, and its components are

$$-m\frac{dv}{d\rho}\cdot\frac{d\rho}{dt} \text{ along the vane,}$$

$$0 \text{ normal to the vane.}$$

No. 8. In passing from a' to d', v is changed in direction by two amounts :—1st, the angle γ changes an amount $-\frac{d\gamma}{d\rho}d\rho$, and 2nd, the radius changes in direction an amount

$\dfrac{d\rho \cot\gamma}{\rho}$, as in No. 6; hence the total change will be the sum of these, and the *rate* of change will be the sum divided by dt, which result multiplied by the momentum, mv, will give the reaction, the components of which will be

$$0 \text{ along the vane,}$$

$$mv\left[\frac{\cot\gamma}{\rho}\cdot\frac{d\rho}{dt}-\frac{d\gamma}{d\rho}\cdot\frac{d\rho}{dt}\right] \text{ normal to the vane.}$$

This completes the reactions. Next consider the *pressure* in the wheel. The *intensity* of the pressure on the two sides ab and cd differs by an amount $dp=\dfrac{dp}{d\rho}d\rho$. The area of the face is $dc\times x=x\rho d\theta\sin\gamma$, and the force due to the difference of pressures will be

$$x\rho d\theta \sin\gamma\frac{dp}{d\rho}d\rho.$$

If dp is positive, which will be the case when the pressure on dc exceeds that on ab, the force acts backwards, and the preceding expression will be *minus* along the vane.

In regard to the pressure normal to the vane, if a uniform pressure p existed from one end of the vane VW to the other, the resultant effect would be zero, since the pressure in one direction on VW would equal the opposite pressure on XY. If, however, the pressure at a exceeds that at d by an amount $-dp$, since Va is longer than Xb, the pressure on Va, due to this $-dp$, will exceed that on Xb by an amount

$$-dp\,.\,x\times ah=-dp\,.\,x\,.\,\rho d\theta\cos\gamma=-x\rho\cos\gamma\, d\theta\frac{dp}{d\rho}d\rho.$$

Collecting these several reactions, we have

	Normal to the vane.	Along the vane.
(2)	$+m\omega^2\rho\cos\gamma.$	$+m\omega^2\rho\sin\gamma.$
(4)	$-m\omega v.$	$0.$
(5)	$-m\omega\sin\gamma\dfrac{d\rho}{dt}.$	$+m\omega\cos\gamma\dfrac{d\rho}{dt}.$
(6)	$-m\omega\cot\gamma\cos\gamma\dfrac{d\rho}{dt}.$	$-m\omega\cos\gamma\dfrac{d\rho}{dt}.$
(7)	$0.$	$-m\dfrac{d\rho}{dt}\cdot\dfrac{dv}{d\rho}.$
(8)	$+mv\left[\dfrac{\cot\gamma}{\rho}\cdot\dfrac{d\rho}{dt}-\dfrac{d\gamma}{d\rho}\cdot\dfrac{d\rho}{dt}\right].$	$0.$
(9)	$-x\rho\cos\gamma\dfrac{dp}{d\rho}d\rho d\theta.$	$-x\rho\sin\gamma\dfrac{dp}{d\rho}d\rho d\theta.$

The sum of the quantities in the second column will be zero; hence

$$m\omega^2\rho\sin\gamma - m\frac{d\rho}{dt}\cdot\frac{dv}{d\rho} - x\rho\sin\gamma\frac{dp}{d\rho}d\rho d\theta = 0. \quad . \quad (1)$$

Substituting

$$\frac{d\rho}{dt} = v\sin\gamma, \text{ and } x\rho d\theta d\rho = \frac{m}{\delta},$$

and dividing by $m\sin\gamma$, we have

$$\omega^2\rho d\rho - \frac{1}{\delta}dp = vdv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating,

$$\left[\tfrac{1}{2}\omega^2\rho - \frac{p}{\delta}\right]_{\text{limit}}^{\text{limit}} = \left[\tfrac{1}{2}v^2\right]_{\text{limit}}^{\text{limit}}. \quad . \quad . \quad . \quad (3)$$

The sum of the quantities in the first column gives the pressure normal to the vane, which multiplied by $\rho\sin\gamma$ gives the moment. This done, and substituting as before, we have

$$d^2\text{M} = mv\left[\omega\rho\left(\frac{\rho}{v}\omega\cos\gamma - 2\right) - \rho v\sin\gamma\frac{d\gamma}{d\rho} + v\cos\gamma - \rho\frac{\cos\gamma}{v\delta}\frac{dp}{d\rho}\right]\sin\gamma.$$

Putting $mv\sin\gamma = \frac{\delta\text{Q}}{2\pi}d\theta d\rho$, where Q is the quantity of water flowing through the wheel per second, and integrating in reference to θ between 0 and 2π, we have

$$d\text{M} = \delta\text{Q}\left[\omega\rho\left(\frac{\rho}{v}\omega\cos\gamma - 2\right) - \rho v\sin\gamma\frac{d\gamma}{d\rho} + v\cos\gamma - \rho\frac{\cos\gamma}{v\delta}\frac{dp}{d\rho}\right]d\rho.$$

Multiplying (2) by $\frac{\rho}{v}\cos\gamma$, we have

$$\frac{\omega^2\rho^2}{v}\cos\gamma d\rho - \frac{\rho\cos\gamma}{v\delta}\frac{dp}{d\rho}d\rho = \rho\cos\gamma\frac{dv}{d\rho}d\rho,$$

which, substituted above, gives

$$d\text{M} = \delta\text{Q}\left[-2\omega\rho d\rho + \rho\cos\gamma\frac{dv}{d\rho}d\rho + v\cos\gamma d\rho - \rho v\sin\gamma\frac{d\gamma}{d\rho}d\rho\right], \quad (4)$$

and integrating,

$$\begin{aligned}\text{M} &= \delta\text{Q}[-\omega\rho^2 + \rho v\cos\gamma] \\ &= -\delta\text{Q}\rho[\omega\rho - v\cos\gamma]_{\text{lim.}}^{\text{lim.}} \quad . \quad . \quad . \quad . \quad (5)\end{aligned}$$

But $\omega\rho - v\cos\gamma$ is the circumferential velocity in space of the water at any point, and $\delta Q\rho[\omega\rho - v\cos\gamma]$ is the moment of the momentum; hence, integrating between limits for the inner and outer rims, *the moment exerted by the water on the wheel equals the difference in its moment of momentum on entering and leaving the wheel.* Thus we have deduced an expression which some writers have made the basis of their investigations.

Let the values of the variables at the entrance of the wheel be

$$\rho_1, \gamma_1, v_1, p_1,$$

and at exit be

$$\rho_2, \gamma_2, v_2, p_2.$$

Then equations (3) and (5) become

$$\tfrac{1}{2}\omega^2(\rho_1^2-\rho_2^2)-\frac{p_1-p_2}{\delta}=\tfrac{1}{2}(v_1^2-v_2^2). \quad . \quad . \quad . \quad . \quad (6)$$

$$M=\delta Q[\omega(\rho_1^2-\rho_2^2)-\rho_1 v_1\cos\gamma_1+\rho_2 v_2\cos\gamma_2]. \quad (7)$$

$$\therefore U=M\omega=\delta Q\omega[\omega(\rho_1^2-\rho_2^2)-\rho_1 v_1\cos\gamma_1+\rho_2 v_2\cos\gamma_2]. \quad (8)$$

Equation (8) gives the work per second in terms of the known quantities δ, ω, ρ_1, γ_1, ρ_2, γ_2, and the three quantities Q, v_1, v_2 as yet unknown. These three quantities are, however, connected by the condition that the quantity of water flowing through all the sections radially is constant. Calling a_1 the entire area of all the orifices at the entrance of the wheel $(=2\pi\rho_1 x_1)$, and a_2 those at exit $(=2\pi\rho_2 x_2)$, we have

$$Q=a_1 v_1\sin\gamma_1=a_2 v_2\sin\gamma_2, \quad . \quad . \quad . \quad . \quad (9)$$

which reduces this number of unknown quantities to one.

Equation (6) is the equation of the motion of the water in the wheel. Besides the velocities v_1 and v_2, it contains p_1 and p_2.

Let

p_a = the atmospheric pressure,

h = the mean depth of the wheel below the surface of the tail-race,

$p_t=\delta gh$ = the pressure due to flooding in the tail-race;

then

$$p_2=p_a+p_t.$$

The pressure p_1 where the water passes from the guide-plates into the wheel is unknown. Another condition is necessary, which may be found by considering the passage of the water from the guide-plates into the wheel. In fig. 2 let A C be the tangent to the guide-plate at its extremity, V the actual velocity of the water on leaving the guide-plate, $\omega\rho_1$ = A D the velocity of the initial rim of the wheel; then will A B be the velocity of the stream relative to the wheel. Now if A B does not coincide with the tangent to the vane at A, the stream cannot suddenly be made to change its direction into that of the vane, or float; and the water, by cushioning in the angles, will make its own angles, as roughly shown in fig. 3.

Fig. 2.

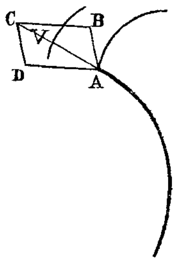
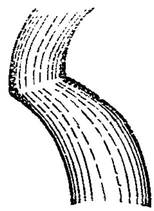


Fig. 3.



It is impossible, either practically or theoretically, to determine the new angles, and probably they are not constant; neither is it possible to determine the loss of energy due to eddying; we therefore make the hypothesis that the final direction of the guide-plates, the initial direction of the vanes, the angular velocity of the wheel, and the velocity of the flow, are so related that the water on leaving the guide-plates shall coincide in direction with the initial elements of the vanes. Any three of the four quantities above mentioned being fixed, the fourth becomes known by this relation. We will leave the angle of the guide-plates to be determined later.

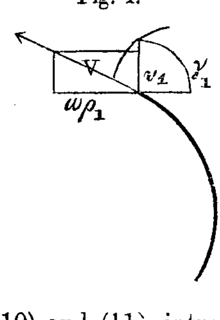
Let

V = the actual velocity of the water on leaving the guide-plates.
v_1 = the velocity relative to the vane, as before.

Then V must be the resultant of $\omega\rho_1$ and v_1, and we have

$$V^2 = v_1^2 + \omega^2\rho_1^2 - 2v_1\omega\rho_1 \cos\gamma_1. \quad (10)$$

Fig. 4.



From Bernouilli's theorem we have for the flow of water in the head-race the equation

$$\frac{p_a}{\delta g} + \mathfrak{h} = \frac{p_1}{\delta g} + \frac{V^2}{2g}, \quad . \quad (11)$$

$\mathfrak{h}$ being the mean height of the surface of the water in the reservoir above the wheel. We thus have two equations, (10) and (11), introducing one new unknown quantity.

Eliminating V in (10) and (11), we have

$$v_1^2+\omega^2\rho_1^2-2v_1\omega\rho_1\cos\gamma_1=2g\mathfrak{h}-\frac{2p_1}{\delta}+\frac{2p_a}{\delta}. \quad . \quad (12)$$

Substituting in equation (6), $p_2=p_a+g\delta h$, we have

$$\omega^2\rho_1^2-\omega^2\rho_2^2-\frac{2p_1}{\delta}+\frac{2p_a}{\delta}+2gh=v_1^2-v_2^2. \quad . \quad (13)$$

Adding (12) and (13), we have

$$2\omega^2\rho_1^2-\omega^2\rho_2^2=2g(\mathfrak{h}-h)+2v_1\omega\rho_1\cos\gamma_1-v_2^2. \quad (14)$$

From (9) and (14), H being substituted for $(\mathfrak{h}-h)$, we find

$$v_1=\frac{a_2^2\sin^2\gamma_2}{a_1^2\sin^2\gamma_1}\omega\rho_1\cos\gamma_1$$
$$+\sqrt{\frac{a_2^2\sin^2\gamma_2(\omega^2\rho_2^2-2\omega^2\rho_1^2+2g\mathrm{H})}{a_1^2\sin^2\gamma_1}+\frac{a_2^4\sin^4\gamma_2}{a_1^4\sin^4\gamma_1}\omega^2\rho_1^2\cos^2\gamma_1}. \quad (15)$$

$$v_2=\omega\rho_1\frac{a_2\sin\gamma_2}{a_1\sin\gamma_1}\cos\gamma_1$$
$$+\sqrt{\omega^2\rho_2^2-2\omega^2\rho_1^2+2g\mathrm{H}+\frac{a_2^2\sin^2\gamma_2}{a_1^2\sin^2\gamma_1}\omega^2\rho_1^2\cos^2\gamma_1}. \quad . \quad . \quad . \quad (15a)$$

$$\mathrm{Q}=\omega\rho_1\frac{a_2^2\sin^2\gamma_2}{a_1\sin\gamma_1}\cos\gamma_1$$
$$+a_2\sin\gamma_2\sqrt{\omega^2\rho_2^2-2\omega^2\rho_1^2+2g\mathrm{H}+\frac{a_2^2\sin^2\gamma_2}{a_1^2\sin^2\gamma_1}\omega^2\rho_1^2\cos^2\gamma_1} \quad . \quad (16)$$

The efficiency will be, from equation (8),

$$\mathrm{E}=\frac{\mathrm{U}}{g\delta\mathrm{QH}}=\frac{\omega}{g\mathrm{H}}[\omega^2\rho_1^2-\omega^2\rho_2^2-\rho_1v_1\cos\gamma_1+\rho_2v_2\cos\gamma_2], \quad . \quad . \quad . \quad (17)$$

which, substituting the values of v_1 and v_2, gives

$$\mathrm{E}=\frac{1}{g\mathrm{H}}\left\{\omega^2\rho_1^2-\omega^2\rho_2^2+\omega\left[\rho_2\cos\gamma_2-\frac{a_2\sin\gamma_2}{a_1\sin\gamma_1}\rho_1\cos\gamma_1\right]\right.$$
$$\left.\times\left[\frac{a_2\sin\gamma_2}{a_1\sin\gamma_1}\omega\rho_1\cos\gamma_1+\sqrt{\omega^2\rho_2^2-2\omega^2\rho_1^2+2g\mathrm{H}+\frac{a_2^2\sin^2\gamma_2}{a_1^2\sin^2\gamma_1}\omega^2\rho_1^2\cos^2\gamma_1}\right]\right\}. \quad (18)$$

To find the angular velocity that will give a maximum efficiency, make $\frac{d\mathrm{E}}{d\omega}=0$, in (18).

For brevity make

$$n=\rho_2\cos\gamma_2-\frac{a_2\sin\gamma_2}{a_1\sin\gamma_1}\rho_1\cos\gamma_1, \quad . \quad . \quad . \quad . \quad (19)$$

$$b=\frac{a_2\sin\gamma_2}{a_1\sin\gamma_1}\rho_1\cos\gamma_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$l^2=\rho_2^2-2\rho_1^2+\frac{a_2^2\sin^2\gamma_2}{a_1^2\sin^2\gamma_1}\rho_1^2\cos^2\gamma_1; \quad . \quad . \quad (21)$$

$$s^2=\mathrm{N}\frac{a_2\sin\gamma_2}{a_1\sin\gamma_1}\rho_1\cos\gamma_1+\rho_1^2-\rho_2^2, \quad . \quad . \quad . \quad (22)$$

then $\frac{d\mathrm{E}}{d\omega}=0$ will give

$$-\omega s^2\sqrt{\omega^2l^2+2g\mathrm{H}}=n^2\omega l^2+ng\mathrm{H}, \quad . \quad . \quad . \quad (23)$$

$$\omega^2=\frac{g\mathrm{H}}{l^2}\left[\frac{s^2-\sqrt{s^4-n^2l^2}}{\sqrt{s^2-n^2l^2}}\right]; \quad . \quad . \quad . \quad . \quad (24)$$

and this substituted in (18) will give the maximum efficiency for any turbine, and in (16) will give the quantity of water discharged. It would be simpler to find ω numerically for any particular case before making the substitution.

We will now use these equations to show some errors made by Rankine and Weisbach. Rankine, in his 'Steam-Engine and other Prime Movers,' discusses a special wheel of the Fourneyron type, in which he assumes

$$a_2=a_1 \text{ and } \gamma_1=90°.$$

These in equation (15) give

$$v_1=\sin\gamma_2\sqrt{\omega^2\rho_2^2-2\omega^2\rho_1^2+2g\mathrm{H}}. \quad . \quad . \quad . \quad (25)$$

This velocity is radial along the vane, and is called by Rankine the "velocity of flow." The tangential component of the actual velocity must, in this case, be $\omega\rho_1$, and this, by Rankine, is called the velocity of whirl (v); and his value of v on page 196, equation (9) of the 'Steam-Engine' is not only incorrect but meaningless, even for the turbine he is considering.

From equation (15a) we also have

$$v_2=\sqrt{\omega^2\rho_2^2-2\omega^2\rho_1^2+2g\mathrm{H}}. \quad . \quad . \quad . \quad (26)$$

The expression for the efficiency becomes

$$\mathrm{E}=\frac{\omega}{g\mathrm{H}}\left[\omega\rho_1^2-\omega\rho_2^2+\rho_2\cos\gamma_2\sqrt{\omega^2\rho_2^2-2\omega^2\rho_1^2+2g\mathrm{H}}\right]. \quad (27)$$

This expression differs entirely from Rankine's equation (10), page 196. But running this wheel at the same speed as Rankine does his, Art. 175, that is, making the final velocity of whirl zero, we shall have

$$v_2 \cos \gamma_2 = \omega \rho_2,$$

and equation (26) becomes

$$\omega \rho_2 = \cos \gamma_2 \sqrt{\omega^2 \rho_2^2 - 2\omega^2 \rho_1^2 + 2g\mathrm{H}}\,;$$

from which we find

$$\omega = \sqrt{\frac{2g\mathrm{H}}{2\rho_1^2 + \rho_2^2 \tan^2 \gamma_2}}\,; \quad . \quad . \quad . \quad (28)$$

$$\therefore\ \omega \rho_1 = \sqrt{\frac{2g\mathrm{H}}{2 + \frac{\rho_2^2}{\rho_1^2} \tan^2 \gamma_2}},$$

which is the same as Rankine's equation (3), Art. 175. Substituting these in (27), we have

$$\mathrm{E} = \frac{2\rho_1^2}{2\rho_2^2 + \rho_2^2 \tan^2 \gamma_2},$$

which is Rankine's equation (4), Art. 175.

We thus see that Rankine's equations not only do not fit any wheel except the one he is considering, but they apply to that only at one particular speed. These conclusions agree with those in an article on Turbines, by Professor Wood, in the Journal of the Franklin Institute for June 1884.

In regard to the speed for maximum efficiency, Rankine, in the 'Steam-Engine,' Art. 173, says, "In order that the water may work to the best advantage, it should leave the wheel without whirling motion, for which purpose the velocity of whirl relative to the wheel should be equal and contrary to that of the second circumference of the wheel." Plausible as this appears, it is true only for special cases even for his wheel. Also Weisbach makes the erroneous statement that the velocity of the second rim of the wheel should equal the relative velocity of discharge. Thus in 'The Mechanics of Engineering and of Machinery,' vol. ii. page 400 (Wiley and Sons) he says (substituting my notation for his)

$$w = \sqrt{\omega^2 \rho_2^2 + v_2^2 - 2v_2 \omega \rho_2 \cos \gamma_2} = \sqrt{(\omega \rho_2 - v_1)^2 + 4\omega \rho_2 v_2 \sin^2 \frac{\gamma_2}{2}},$$

in regard to which he states that for w a minimum $\omega\rho_2$ must equal v_1, which is not *generally* true, and is true only when $\gamma_2=0$, or when $v_2\omega\rho_2$ and $(\omega\rho_2-v_2)^2$ happen to be a minimum together. The value of ω in equation (24) will not in general satisfy Rankine's condition

$$\omega\rho_2=v_2\cos\gamma_2, \quad \ldots \ldots \quad (29)$$

nor Weisbach's

$$\omega\rho_2=v_2. \quad \ldots \ldots \ldots \quad (29a)$$

Substituting in equation (24) Rankine's condition $\gamma_1=90°$, and making $r=\frac{\rho_1}{\rho_2}$, we find

$$\omega^2\rho_2{}^2=\frac{g\mathrm{H}}{1-2r^2}\left[\frac{1-r^2-\sqrt{(1-r^2)^2-\cos^2\gamma_2(1-2r^2)}}{\sqrt{(1-r^2)^2-\cos^2\gamma_2(1-2r^2)}}\right]. \quad (30)$$

Substituting this in (15a) gives

$$v^2\cos\gamma_2=\omega\rho_2[1-r^2+\sqrt{(1-r^2)^2-\cos^2\gamma_2(1-2r^2)}]. \quad (31)$$

This satisfies equation (29) only when $r^2=\frac{1}{2}$, and (29a) only when $\gamma_2=0$ or $r=1$. The latter condition is that of a parallel flow wheel, or of an infinitely narrow wheel, in which case we have

$$\omega\rho_2=v_2=\sqrt{g\mathrm{H}}.$$

These in (17) give for the efficiency

$$\mathrm{E}=\cos\gamma_2,$$

which always exceeds the value given by Rankine, when $r=1$,

$$\mathrm{E}=\frac{2\cos^2\gamma_2}{1+\cos^2\gamma_2},$$

except when $\gamma_2=0$, when both become unity, but the work done will be zero.

The pressure in the wheel may be found by integrating equation (3) between initial and general limits, and eliminating p_1 and v by means of equations (12) and (9), giving

$$\frac{2p}{\delta}=\omega^2\rho^2-2\omega^2\rho_1{}^2+\frac{2p_a}{\delta}+2g\mathfrak{h}+2v_1\omega\rho_1\cos\gamma_1-v_1{}^2\frac{a_1{}^2\sin^2\gamma_1}{a^2\sin^2\gamma}, \quad (32)$$

which may be discussed for the various conditions to which the wheel is subjected.