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XLVI. On the figure of the earth

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words, they are now twice as far apart as before, the spacing of the secondary maxima remaining the same.

The analogy between the secondary maxima and the fringes produced by a rectangular aperture of the same size as the ruled surface, can be studied to advantage by means of coarse gratings made by ruling four or five lines on a piece of smoked *plate* glass, and making the lower third of the grating clear by wiping out the lines. Sun or arc light filtered through red glass should be used with a small spectrometer, the grating and aperture being covered in succession or used simultaneously.

XLVI. *On the Figure of the Earth.* *By* J. Prescott, *M.A., Lecturer in Mathematics at the Manchester School of Technology**.

I SHALL assume that the figure of the earth is an oblate spheroid of small ellipticity and I propose to find this ellipticity. I shall also assume that the figure is the same as it would be if the earth were wholly liquid. There can be little doubt that the figure obtained on these assumptions is not far from the actual shape. For, the earth's crust is probably not rigid enough to resist the forces pulling it into this shape.

It is necessary first to find the potential, at an external point on the axis, of a thin shell bounded by similar oblate spheroids whose generating curves are

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1,$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1+\sigma,$$

where $$b^2(1+\sigma)=(b+\delta b)^2=b^2+2b\delta b,$$

or $$\sigma=2\frac{\delta b}{b};$$

and b is the polar semi-axis.

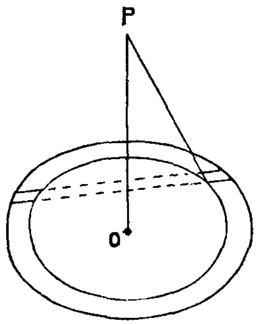
Let P be the external point, O the centre of the shells, $\text{OP}=r$, $(a-b)=eb$, K = the gravitation constant, ρ = density. The mass of a thin ring of the shell between the planes y and $y+\delta y$ is

$$\pi\rho(x_2^2-x_1^2)\delta y$$

$$=\pi\rho\sigma a^2\delta y.$$

* Communicated by the Author.

Fig. 1.



Taking potential to be $\Sigma\frac{\mathrm{K}m}{r}$, the potential of this ring at P

$$=\frac{\mathrm{K}\pi\rho\sigma a^2\delta y}{\sqrt{\{x_1{}^2+(r-y_1)^2\}}}$$

$$=\mathrm{K}\pi\rho\sigma a^2\delta y\left\{a^2\left(1-\frac{y^2}{b^2}\right)+r^2-2yr+y^2\right\}^{-\frac{1}{2}}$$

$$=\mathrm{K}\pi\rho\sigma a^2\delta y\left\{b^2(1+2e)\left(1-\frac{y^2}{b^2}\right)+r^2-2yr+y^2\right\}^{-\frac{1}{2}}\text{ nearly}$$

$$=\mathrm{K}\pi\rho\sigma a^2\delta y\{r^2-2yr+b^2+2e(b^2-y^2)\}^{-\frac{1}{2}}$$

$$=\mathrm{K}\pi\rho\sigma a^2\delta y(r^2-2yr+b^2)^{-\frac{1}{2}}\left\{1-e\frac{b^2-y^2}{r^2-2yr+b^2}\right\}$$

where only the first power of e has been retained. The potential of the whole shell

$$=\mathrm{K}\pi\rho\sigma a^2\int_{-b}^{+b}\{(r^2-2yr+b^2)^{-\frac{1}{2}}-e(b^2-y^2)(r^2-2yr+b^2)^{-\frac{3}{2}}\}dy$$

$$=\mathrm{K}\pi\rho\sigma a^2\left[-\frac{1}{r}(r^2-2yr+b^2)^{\frac{1}{2}}-e\frac{b^2}{r}(r^2-2yr+b^2)^{-\frac{1}{2}}\right]_{-b}^{+b}$$

$$+e\mathrm{K}\pi\rho\sigma a^2\int_{-b}^{+b}y^2(r^2-2yr+b^2)^{-\frac{3}{2}}dy.$$

Now $\int y^2(r^2-2yr+b^2)dy$

$$=\frac{y^2}{r}(r^2-2yr+b^2)^{-\frac{1}{2}}+\frac{2y}{r^2}(r^2-2yr+b^2)^{\frac{1}{2}}+\frac{2}{3r^3}(r^2-2yr+b^2)^{\frac{3}{2}}.$$

Hence the potential of the shell

$$=\mathrm{K}\pi\rho\sigma a^2\left\{\frac{2b}{r}-e\frac{b^2}{r}\left(\frac{1}{r-b}-\frac{1}{r+b}\right)+e\frac{b^2}{r}\left(\frac{1}{r-b}-\frac{1}{r+b}\right)\right.$$

$$\left.+\frac{2be}{r^2}(r-b+r+b)+\frac{2e}{3r^3}\{(r-b)^3-(r+b)^3\}\right\}$$

$$=2\mathrm{K}\pi\rho\sigma a^2\left\{\frac{b}{r}-\frac{2eb^3}{3r^3}\right\}$$

$$=2\mathrm{K}\pi\rho\sigma b^2[1+2e]\left\{\frac{b}{r}-\frac{2eb^3}{3r^3}\right\}$$

$$=2\mathrm{K}\pi\rho\sigma b^3\left\{\frac{1}{r}(1+2e)-\frac{2}{3}\frac{eb^2}{r^3}\right\}.$$

The potential, at an external point P on its axis, of a homogeneous spheroid whose polar radius is β and eccentricity e, is obtained by putting $2\frac{db}{b}$ for σ in the expression for the potential of a shell and integrating.

Thus the potential

$$=\mathrm{V}_1=2\mathrm{K}\pi\rho\int_0^\beta\frac{2db}{b}b^3\left\{\frac{1}{r}(1+2e)-\frac{2}{3}\frac{eb^3}{r^3}\right\}$$

$$=4\mathrm{K}\pi\rho\left\{\frac{1}{3}\frac{\beta^3}{r}(1+2e)-\frac{2}{3}\cdot\frac{1}{5}\frac{e\beta^5}{r^3}\right\}$$

$$=\frac{4\mathrm{K}\pi\rho\beta^3}{3}\left\{\frac{1}{r}(1+2e)-\frac{2}{5}\frac{e\beta^2}{r^3}\right\}.$$

Since the potential of the spheroid at an external point is a zonal harmonic, we find that the potential at any external point is

$$\mathrm{V}_2=\frac{4\mathrm{K}\pi\rho\beta^3}{3}\left\{\frac{1}{r}(1+2e)-\frac{2}{5}\frac{e\beta^2}{r^3}\left(\frac{3\cos^2\theta-1}{2}\right)\right\},$$

where r is the length of the radius vector from the centre of the spheroid to the point, and θ is the angle between this radius vector and the axis of symmetry. Now suppose the earth is so composed that layers of equal density are the surfaces of spheroids of varying ellipticity, the surface of the earth itself being one of these spheroids. We may consider the density of any layer and its ellipticity to be functions of the polar semi-axis of the layer. Let $\delta\mathrm{V}$ be the potential at r, θ of one of these layers. Then

$$\delta\mathrm{V}=\frac{4}{3}\mathrm{K}\pi\rho\frac{d}{d\beta}\left\{\frac{1}{r}(1+2e)\beta^3-\frac{1}{5r^3}e\beta^5(3\cos^2\theta-1)\right\}\delta\beta.$$

The potential of the earth, regarded as a heterogeneous spheroid such as I have supposed, its polar radius being R, is

$$V_3=\frac{4}{3}K\pi\int_0^R \rho\,\frac{d}{d\beta}\left\{\frac{1}{r}(1+2e)\beta^3-\frac{1}{5r^3}e\beta^5(3\cos^2\theta-1)\right\}d\beta$$

$$=\frac{4}{3}K\pi\left\{\frac{A}{r}-(3\cos^2\theta-1)\frac{B}{5r^3}\right\},$$

where

$$A=\int_0^R \rho\,\frac{d}{d\beta}(1+2e)\beta^3 d\beta$$

and

$$B=\int_0^R \rho\,\frac{d}{d\beta}(e\beta^5)d\beta.$$

By putting $r=R$ and $\theta=0$ in the expression for V_3 we find that the potential at the end of the polar radius

$$=\frac{4}{3}K\pi\left\{\frac{A}{R}-\frac{2B}{5R^3}\right\}.$$

And by putting $r=R(1+\epsilon)$ [ϵ being the value of e at the earth's surface] and $\theta=\frac{\pi}{2}$, the value of the potential at the earth's surface on the equator is found to be

$$\frac{4}{3}K\pi\left\{\frac{A}{R(1+\epsilon)}+\frac{B}{5R^3(1+\epsilon)^3}\right\}$$

$$=\frac{4}{3}K\pi\left\{\frac{A}{R}-\epsilon\frac{A}{R}+\frac{B}{5R^3}\right\}.$$

Since B already contains the first power of e we do not need to retain ϵB &c.

The potential of centrifugal force at the equator, due to the earth's rotation, is

$$\tfrac{1}{2}R^2(1+\epsilon)^2\omega^2 \text{ or } \tfrac{1}{2}R^2\omega^2,$$

(ω being the earth's angular velocity of rotation) provided we assume, as in the case of attractions, that the space variation of potential gives the force on unit mass, and not the negative of this force as is usual. Since the earth's surface is a level surface the whole potential at the equator is equal to the whole potential at the pole. Hence

$$\frac{4}{3}K\pi\left\{\epsilon\frac{A}{R}-\frac{3}{5}\frac{B}{R^3}\right\}=\frac{1}{2}R\omega^2.$$

Now

$$\frac{4}{3}\pi A=\int_0^R \rho\,\frac{d}{d\beta}\left(\frac{4}{3}\pi\beta^3\right)d\beta$$

$$=E, \text{ the earth's mass.}$$

Also

$$\frac{KE}{R^2}=g, \text{ the acceleration due to gravity at the earth's surface.}$$

Thus we get

$$\epsilon Rg-\frac{4}{3}K\pi\frac{3}{5}\frac{B}{R^2}=\frac{1}{2}R^2\omega^2,$$

or

$$\epsilon\frac{g}{R\omega^2}-\frac{4}{3}K\pi\frac{3}{5}\cdot\frac{B}{R^5\omega^2}=\frac{1}{2}\cdot \quad . \quad . \quad . \quad . \quad (\alpha)$$

If ρ is constant then e may be considered constant and $B=\rho\epsilon R^5$. Then our equation would give

$$\epsilon\frac{g}{R\omega^2}-\frac{3}{5}\epsilon\frac{g}{R\omega^2}=\frac{1}{2}.$$

But

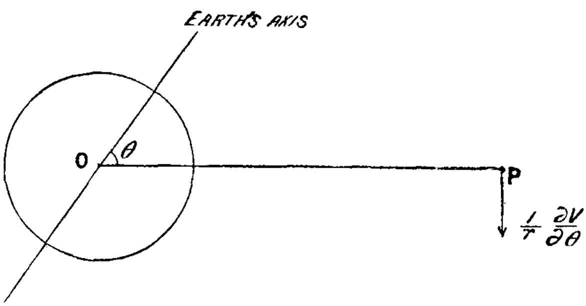
$$\frac{g}{R\omega^2}=289.$$

Hence

$$\frac{1}{\epsilon}=231.$$

It is certain, however, that e is not constant. We cannot find ϵ therefore without knowing the value of the coefficient B. Now this coefficient is given in terms of $\int_0^R \rho\beta^4 d\beta$, a quantity which does not contain e, by the theory of the precession of the earth's axis. By assuming a reasonable law for ρ we can then find B, and thence determine ϵ.

Fig. 2.



The potential of the earth at an external point P (fig. 2) is

$$V=\frac{4}{3}K\pi\left\{\frac{A}{r}-\frac{B}{5r^3}(3\cos^2\theta-1)\right\}.$$

The force exerted by the earth perpendicular to OP on unit mass at P is

$$\frac{\partial V}{r\partial\theta}.$$

The moment of this force about O is $\frac{\partial V}{\partial\theta}$. Now the reaction of the unit mass on the earth is exactly the opposite of the earth's action on the unit mass. Consequently the moment about O of the couple exerted by the unit mass on the earth is $\frac{\partial V}{\partial\theta}$ in the direction tending to increase the angle θ up to a right angle. If the sun is at P, the couple exerted by the sun, whose mass is S, is

$$S\frac{\partial V}{\partial\theta}=S\frac{8}{5}K\pi\frac{B}{r^3}\cos\theta\sin\theta.$$

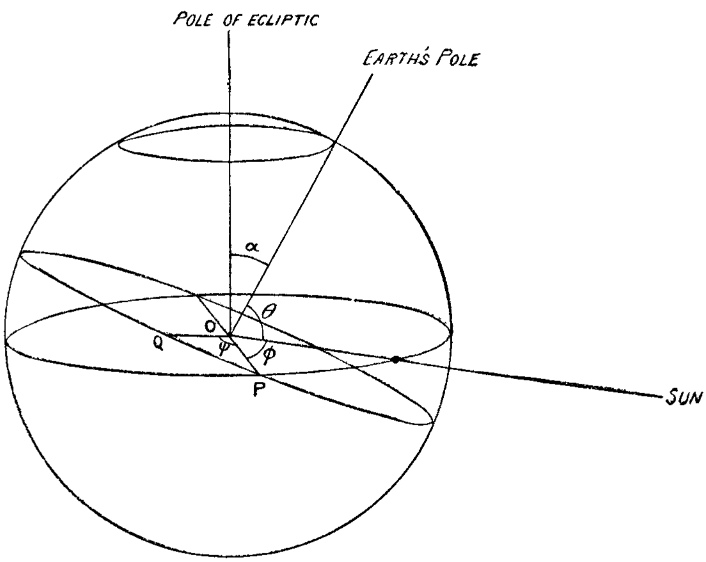
If n is the angular velocity of the earth in its orbit

$$\frac{KS}{r^2}=rn^2.$$

Thus the couple exerted by the sun on the earth becomes

$$\tfrac{8}{5}\pi Bn^2\sin\theta\cos\theta.$$

Fig. 3.



Referring to figure 3, by spherical trigonometry,

$$\cos\theta=\sin\alpha\sin\phi.$$

Therefore $\sin\theta = \sqrt{\{1 - \sin^2\alpha\sin^2\phi\}}$

$$= 1 - \tfrac{1}{2}\sin^2\alpha\sin^2\phi + \tfrac{3}{4}\sin^4\alpha\sin^4\phi$$

nearly, since $\sin\alpha = \sin 23\tfrac{1}{2}° = \cdot 4$ about, so that the terms ignored are small.

Hence the couple exerted by the sun

$$= \tfrac{8}{5}\pi B n^2 \sin\alpha\sin\phi\{1 - \tfrac{1}{2}\sin^2\alpha\sin^2\phi + \tfrac{3}{8}\sin^4\alpha\sin^4\phi\}.$$

The axis of this couple is OQ, perpendicular to the plane containing the earth's axis and the sun. The only effective component in precession is that parallel to OP, the intersection of the plane of the equator and the ecliptic. The other component causes nutation. Hence the effective couple for precession

$$= \text{moment of couple} \times \cos\psi.$$

By spherical trigonometry

$$\cos\psi = \frac{\sin\phi}{\sqrt{(1 + \tan^2\alpha\cos^2\phi)}}$$
$$= \sin\phi(1 - \tfrac{1}{2}\tan^2\alpha\cos^2\phi + \tfrac{3}{8}\tan^4\alpha\cos^4\phi).$$

Thus the effective couple is

$$L = \frac{8\pi}{5} B n^2 \sin\alpha\sin^2\phi\{1 - \tfrac{1}{2}\sin^2\alpha\sin^2\phi - \tfrac{1}{2}\tan^2\alpha\cos^2\phi$$
$$+ \tfrac{3}{8}\sin^4\alpha\sin^4\phi + \tfrac{1}{4}\tan^2\alpha\sin^2\alpha\sin^2\phi\cos^2\phi$$
$$+ \tfrac{3}{8}\tan^4\alpha\cos^4\phi\}.$$

Now $\dfrac{d\phi}{dt} = n$. Therefore $dt = \dfrac{d\phi}{n}$. Hence the moment of momentum generated by the sun's couple action in one year

$$= \int_0^{2\pi} L\frac{d\phi}{n}$$
$$= \frac{8}{5} B n\pi \sin\alpha \,.\, \pi \left\{ 1 - \frac{1}{2}\sin^2\alpha \,.\, \frac{3}{4}\,.\,\frac{1}{2}\,.\,2 - \frac{1}{2}\tan^2\alpha\,.\,\frac{1}{4}\,.\,\frac{1}{2}\,.\,2 \right.$$
$$+ \frac{3}{8}\sin^4\alpha\,\frac{5}{6}\,.\,\frac{3}{4}\,.\,\frac{1}{2}\,.\,2 + \frac{1}{4}\tan^2\alpha\sin^2\alpha\,\frac{1}{2}\,.\,\frac{3}{4}\,.\,\frac{1}{2}\,.\,2$$
$$\left. + \frac{3}{8}\tan^4\alpha\,.\,\frac{1}{2}\,.\,\frac{3}{4}\,.\,\frac{1}{2}\,.\,2 \right\},$$

which becomes, on putting $23\tfrac{1}{2}°$ for α,

$$\tfrac{8}{5} B n\pi^2 \sin\alpha\{\cdot 931\}.$$

This is the sun's effect. The moon's effect must be added to this. The mean place of the moon is in the ecliptic, and if the action of the moon be taken to be the same as that of a similar body always in the ecliptic at the moon's distance the error will be small.

Let S, E, M denote the sun's, earth's, and moon's masses; d_S, d_M the sun's and moon's distances from the earth: then

$$\frac{\text{Moon's effect}}{\text{Sun's effect}} = \frac{\frac{M}{d_M^3}}{\frac{S}{d_S^3}}$$

$$= \frac{M}{M+E} \frac{\frac{E+M}{d_M^3}}{\frac{S}{d_S^3}}$$

$$= \frac{1}{82} \frac{\frac{1}{(\text{Moon's Period})^2}}{\frac{1}{(\text{Earth's Period})^2}}$$

$$= \frac{1}{82}\left(\frac{365\frac{1}{4}}{27\cdot 3}\right)^2$$

$$= 2\cdot 18 \text{ about.}$$

Hence the effective moment of momentum created by the combined action of the sun and moon

$$= 3\cdot 18 \times \tfrac{8}{5}\,\pi^2 Bn \sin\alpha(\cdot 931).$$

If I is the moment of inertia of the earth and ω, as before, its angular velocity of rotation, this effect turns the earth's axis through

$$\frac{3\cdot 18 \times \frac{8}{5}\,\pi^2 nB \sin\alpha(\cdot 931)}{I\omega} \text{ radians.}$$

But the earth's axis describes a cone of semi-vertical angle α in 26,000 years. Hence the angle turned through by the earth's axis in one year is

$$\frac{2\pi \sin\alpha}{26000}.$$

Therefore $$\frac{\cdot 931 \times 3\cdot 18 \times \frac{8}{5}\,\pi^2 Bn \sin\alpha}{I\omega} = \frac{2\pi \sin\alpha}{26000}.$$

Whence

$$B = \frac{2 \times 5}{26000 \times \cdot 931 \times 3\cdot 18 \times 8\pi} \cdot \frac{\omega}{n} I$$

$$= \frac{10}{26000 \times \cdot 931 \times 3\cdot 18 \times 8\pi} \frac{365\frac{1}{4}}{1} \frac{2}{3} \int_0^R 4\pi\rho\beta^4 d\beta$$

$$= \frac{5}{316} \int_0^R \rho\beta^4 d\beta.$$

If ρ were constant, in which case we could consider e constant also, this would give

$$\epsilon = \frac{1}{316},$$

which does not agree with the result found by assuming ρ to be constant in the equation obtained by considering the equilibrium shape of a liquid earth.

We shall now have to assume an expression for ρ in terms of the radius which shall agree with all known facts, and give a density decreasing from the centre outwards. We know that, the density of water being taken as the unit, the mean density of the earth is about 5·5 and the surface density is 3 or 2·5.

Also it is quite certain that ρ decreases from the centre outwards. These facts tell us that

$$\int_0 \rho\beta^4 d\beta > 2\cdot 5 \times \tfrac{1}{5} R^5$$

and

$$< 5\cdot 5 \times \tfrac{1}{5} R^5.$$

The first of these conclusions is easily seen; for at every point in the earth

$$\rho\beta^4 > 2\cdot 5\beta^4;$$

and therefore $$\int_0^R \rho\beta^4 d\beta > 2\cdot 5 \times \tfrac{1}{5} R^5.$$

The second statement can be proved thus:—

We know that

$$\int_0^R \rho\beta^2 d\beta = 5\cdot 5 \times \tfrac{1}{3} R^3.$$

Therefore

$$\frac{3R^2}{5} \int_0^R \rho\beta^2 d\beta = \frac{5\cdot 5}{5} R^5.$$

Hence

$$\frac{5\cdot 5}{5}\mathrm{R}^5-\int_0^{\mathrm{R}}\rho\beta^4 d\beta=\frac{3\mathrm{R}^2}{5}\int_0^{\mathrm{R}}\rho\beta^2 d\beta-\int_0^{\mathrm{R}}\rho\beta^4 d\beta$$

$$=\frac{1}{5}\Big[\rho\mathrm{R}^2\beta^3-\rho\beta^5\Big]_0^{\mathrm{R}}-\int_0^{\mathrm{R}}\frac{1}{5}(\mathrm{R}^2\beta^3-\beta^5)\frac{d\rho}{d\beta}d\beta$$

$$=-\frac{1}{5}\int_0^{\mathrm{R}}\beta^3(\mathrm{R}^2-\beta^2)\frac{d\rho}{d\beta}d\beta.$$

Now $\frac{d\rho}{d\beta}$ is always negative and the remaining factors under the integral sign are positive. Thus

$$\frac{5\cdot 5}{5}\mathrm{R}^5=\int_0^{\mathrm{R}}\rho\beta^4 d\beta+\text{a positive quantity,}$$

which proves the second statement.

A law of density which gives roughly the type of variation to be expected is

$$\rho=\mathrm{C}-\mathrm{D}\left(\frac{\beta}{\mathrm{R}}\right)^n.$$

To make $\rho=3$ at the surface we get

$$3=\mathrm{C}-\mathrm{D}.$$

C is the density at the centre and can be given any arbitrary value. By making the mean density 5·5, n is determined thus:—

$$\int_0^{\mathrm{R}}\rho\beta^2 d\beta=5\cdot 5\int_0^{\mathrm{R}}\beta^2 d\beta.$$

This gives $$\mathrm{C}-\frac{3}{n+3}\mathrm{D}=5\cdot 5.$$

But also $$\mathrm{C}-\mathrm{D}=3.$$

Therefore $$n=\frac{15}{2\mathrm{C}-11}.$$

Let m denote the value of

$$\frac{\int_0^{\mathrm{R}}\rho\beta^4 d\beta}{\int_0^{\mathrm{R}}\beta^4 d\beta}.$$

Then

$$m = C - \frac{5}{n+5} D$$

$$= 4{\cdot}5 + \frac{3}{2C-8}$$

on substituting for n and D.

Now the densest materials we know in the earth's crust nave specific gravities only about 22 or 23. It is highly probable then that the density at the earth's centre is not more than 30. The fact that we find in the earth's crust matter so dense as platinum and osmium mingled with much lighter substances, rather indicates that the density at the centre is not nearly so great as 30. For, if substances of such different densities are to be found within a few feet of the earth's surface, it is very likely that lighter matter will be mingled with the denser in the interior. But at the same time there must be a gradual increase of density from the surface towards the centre.

When we have determined m the value of B can be calculated from the equation

$$B = \frac{5}{316} m \frac{1}{5} R^5$$

$$= \frac{m}{316} R^5.$$

Substituting for B in equation (α), obtained from the rotation of the earth, we find

$$\epsilon \frac{g}{R\omega^2} = \frac{4}{5} K\pi \frac{m}{316} \frac{R^5}{R^5\omega^2} + \frac{1}{2}$$

$$= \frac{3}{5} \frac{1}{316} \cdot \frac{m}{5{\cdot}5} \frac{KE}{R^2} \cdot \frac{1}{R\omega^2} + \frac{1}{2}$$

$$= \frac{3}{5} \frac{1}{316} \cdot \frac{m}{5{\cdot}5} \cdot \frac{g}{R\omega^2} + \frac{1}{2}.$$

Now $\frac{g}{R\omega^2} = 289.$

Hence $\epsilon = \frac{3m}{5 \times 316 \times 5{\cdot}5} + \frac{1}{2} \frac{1}{289}.$

The following table gives the calculated values of ϵ, using the value of m obtained by assuming

$$\rho = \mathrm{A} - \mathrm{B}\left(\frac{\beta}{\mathrm{R}}\right)^n$$

Density at the surface.	Density at the centre.	Mean density.	$\frac{1}{\epsilon}$.
3	30	5·5	303
3	25	5·5	303
3	20	5·5	302
3	15	5·5	300
3	13	5·5	300
3	12	5·5	299
3	10	5·5	297
2·5	30	5·5	310
2·5	25	5·5	309
2·5	20	5·5	307
2·5	15	5·5	306
2·5	10	5·5	299

It appears from the above table that the value assumed for the density at the centre does not greatly affect the result. I believe that any assumption concerning the density, which makes it decrease as the radius increases and which gives approximately the known mean and surface densities, cannot give a value of ϵ differing much from those in the table. I tried another completely different formula for the density to see what result it led to. The formula is

$$\rho = \frac{\mathrm{H}}{(\beta + \mathrm{K})^3}.$$

To make the surface-density 3 and the mean density 5·5, I found that H and K must be $16 \cdot 35 \mathrm{R}^3$ and $\cdot 76 \mathrm{R}$ respectively. These values make the density at the centre very large, namely 37·2. The value found for m is 4·68. This gives

$$\frac{1}{\epsilon} = 299.$$

The value I consider most probable from my calculations is

$$\frac{1}{\epsilon} = 303.$$