TROUBLE IN SOLID GEOMETRY.

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It is customary at the beginning of the subject of solid geometry to recall a definition of a plane. I doubt, however, if a plane can really be defined; but believe the conception of a plane is based directly on experience. A common so-called definition, that it is a surface such that a straight line through any two points of it lies entirely in the surface, is only a statement of a property which a plane—the mental picture of which must first be present in one's mind—is observed to possess. Other equally important properties, that there are points not in a plane and that a plane divides the points without it into two classes, are often passed over in silence, and surreptitiously assumed when needed. As a companion to the definition of a straight line that it is the shortest distance between two points, I would suggest, not seriously but only for comparison, that a plane is the smallest surface within a triangle. It can and should be shown that two planes having three non-collinear points in common are coincident.

A leading proposition is that "two *intersecting* planes *intersect* in a straight line," in which it is generally assumed that two planes with one common point have another common point; a fact which cannot be derived from the definition. Intersecting planes are left undefined.

The unfortunate custom of defining terms first and later proving the existence of figures having the assigned properties, is usually followed; for example, a perpendicular to a plane is first defined, with great redundancy, and the existence of a line and a plane with these properties is demonstrated afterward. In the demonstration it is assumed, however, that two intersecting lines have a common perpendicular—a fact not at all obvious. If anyone will furnish a demonstration independent of subsequent propositions the writer will be glad to receive it.

Parallel planes and a line parallel to a plane are also prematurely defined, and it would be well to *prove* that a line or a plane intersecting one of two parallel planes intersects the other. This property is, I believe, always assumed by authors.

In constructing a perpendicular to a given plane α from a given point P without it, we draw a line α in α, a perpendicular from P to x meeting it in a point A, a line y in x through A perpendicular to x, and finally a perpendicular from P to y meeting it in a point B. In showing that PB is the required perpendicular, authors speak of the *triangle* PAB, overlooking the fact that A and B may coincide; an emergency which deserves attention.

We all know what we mean by a closed surface. It is sometimes defined as a surface such that every plane section consists of one, or more, closed lines. Consider the surface of a coiled spring in the form of a helix and of unlimited extent in the direction of its axis. Every plane section would be one, or more, closed lines, the number being unlimited when the plane is parallel to the axis. Does the definition hold?

A prism is repeatedly defined as a solid, two of whose faces are parallel and congruent, and whose remaining faces are parallelograms. Some months after constructing a solid having these properties, a rhombohedron with twelve faces, parallel and congruent in pairs—that is, a twelve faced parallelopiped since its bases were parallelograms!—I saw a garnet crystal of the identical form I had devised; a solid refuting this definition had existed for ages. Doubtless all know how the hexagonal cell of a honey comb is closed by three congruent rhombuses. If we place the open ends of two cells together so that the faces closing one end are respectively parallel to those closing the other end, we have a solid satisfying the stated conditions. Is it a prism? Of course when we use the above definition all propositions in the subject of prisms are fallacious.

"Sections of a prism by parallel planes are congruent polygons."

Take, for example, a cube; a section of this prism by a plane intersecting internally three conterminous edges will be a triangle. The section by a parallel plane may be a triangle, a quadrilateral, a pentagon, or a hexagon; but evidently parallel sections are rarely congruent. In particular, four sections of a cube are regular hexagons.

We should obtain a good definition of a prismatic surface and then show that sections by parallel planes are congruent polygons.

In so doing, however, we should avoid the usual fallacy of showing these polygons to be mutually equilateral and equiangular and then following immediately with the statement that they must be congruent; for I have shown* that two polygons may be mutually equilateral and mutually equiangular without being congruent.

Some authors tell us that a right section of a prism (prismatic surface?) is the section made by a plane perpendicular to *all* the lateral edges. Are they careful lest we get it perpendicular to some of the lateral edges and not to the others?

"An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism."

The author usually supplies a figure in which the right section intersects all the lateral edges internally, and we see, literally, that the two prisms have one portion in common and the remaining portion of one is proven congruent to the remaining portion of the other. But suppose the given prism is so oblique that we cannot obtain a right section cutting all the lateral edges internally; the proof no longer holds.

Any demonstration which holds only under certain conditions must explicitly set forth those conditions or be open to the charge of fallacy.

Many other fallacies in solid geometry, including the use of such undefined terms as the area of a curved surface and the volume enclosed by a curved surface, are too obvious to be discussed. The treatment of limits and of incommensurable cases is not rigorous and probably never can be. But even the usual demonstration in the commensurable case that two rectangular parallelopipeds have the same ratio as the product of their three dimensions, is wholly unsound.

*Notes on Geometry. School Science and Mathematics, May, 1905.