On Simplicissima in Space of n Dimensions. (Third Paper.)

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SUMMARY OF THE PAPER.

Addenda to Arts. X., XIII., XXVII., XXIX., and XXXVI.

Articles.

XXXVII. Content of subsidiary simplicissima in terms of the co-ordinates of the vertices.

XXXVIII. The angle between two right lines. Condition of perpendicularity. Shortest distance between two right lines. Principal axes of a quadric.

XXXIX. Content of the simplicissimum vertices at any point and all but one of its projections on the faces of the simplicissimum of reference; of the pedal simplicissimum. Condition that this should vanish.

XL. Interpretation of various equations by the help of XXXIX., especially $\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\ldots=0$. The poles of infinity. All lines through these are such that the sum of the reciprocals of the intercepts is zero.

XLI. The identical relations between the products of complementary simplicissima of n vertices, situated at $2n$ points in space of $n-1$ dimensions. Triangles having their vertices at six or five points in a plane. Tetrahedron with vertices at eight, seven, or six points. Various classes of anharmonic systems. All up to the $n-1^{\text{th}}$ are applicable in the Geometry of space of n dimensions.

XLII. Analogue to harmonical progression. Complete system of 2^{n-1} points harmonically related to n points. Properties of such systems.

XLIII. Quadrics intersecting in two linear loci, one of which is infinity, are similar and similarly situated, and have two centres of similitude.

Articles.

XLIV. Any two quadrics have a common self-conjugate simplicissimum. Its vertices. When these become indeterminate.

XLV. The value of the discriminant of

$$U_{11}\lambda^2+U_{22}\mu^2+\ldots+2U_{12}\lambda\mu+\ldots.$$

XLVI. The invariants, contravariants, and covariants of two quadrics.

Addenda to Former Papers.

(Vol. XVIII., pp. 325–59; Vol. XIX., pp. 423–82.)

X. If $\Sigma\equiv S-(A_1\lambda+A_2\mu+A_3\nu+\ldots)(\lambda+\mu+\nu\ \ldots),$

then $$\Sigma-\left(\lambda'\frac{d\Sigma}{d\lambda}+\mu'\frac{d\Sigma}{d\mu}+\nu'\frac{d\Sigma}{d\nu}+\ldots\right)(\lambda+\mu+\nu+\ldots)=0\ \ldots\ldots(1)$$

represents the spheric with centre at $(\lambda', \mu', \nu' \ldots)$, and which cuts $\Sigma=0$ orthogonally. For at the centre of (1),

$$\frac{d\Sigma}{d\lambda}-\frac{d\Sigma'}{d\lambda'}=\frac{d\Sigma}{d\mu}-\frac{d\Sigma'}{d\mu'}=\frac{d\Sigma}{d\nu}-\frac{d\Sigma'}{d\nu'}=\ldots,$$

and the radical locus of (1) and $\Sigma=0$ is the polar of the centre of each with respect to the other.

XIII. The proof of this proposition, which is equally true if instead of spherics we say similar circumscribed quadrics (Art. XLIII.), is incomplete as it stands, as it is not proved that the common point on the radical loci lies upon the spherics. This is easily proved by multiplying the equations to the radical loci by $\lambda, \mu, \nu, \ldots$, respectively, and adding, when the result is $S=0$.

XXVII. The central axis is at right angles to the common radical locus of the spherics

$$A_1\lambda+A_2\mu+A_3\nu+\ldots=0,$$

for the equations to the axis may be written

$$\frac{\lambda-\dfrac{V}{n+1}}{\dfrac{1}{A_1}-\dfrac{1}{n+1}\Sigma\dfrac{1}{A_1}}=\frac{\mu-\dfrac{V}{n+1}}{\dfrac{1}{A_2}-\dfrac{1}{n+1}\Sigma\dfrac{1}{A_1}}=\frac{\nu-\dfrac{V}{n+1}}{\dfrac{1}{A_3}-\dfrac{1}{n+1}\Sigma\dfrac{1}{A_1}}=\&c.,$$

for which values of $a, b, c \ldots$ (Art. XVIII.),

$$\frac{d}{da} S(a, b, c \ldots) = n-1-\frac{1}{n+1}\Sigma\frac{1}{A_1}\Sigma A_1+\frac{2}{n+1}A_1\Sigma\frac{1}{A_1},$$

$$\frac{d}{db} S(a, b, c \ldots) = n-1-\frac{1}{n+1}\Sigma\frac{1}{A_1}\Sigma A_1+\frac{2}{n+1}A_2\Sigma\frac{1}{A_1},$$

&c. &c.

Therefore

$$\left\| \begin{matrix} \frac{d}{da} S(a, b, c \ldots), & \frac{d}{db} S(a, b, c \ldots), & \frac{d}{dc} S(a, b, c \ldots), & \ldots \\ A_1, & A_2, & A_3, & \ldots \\ 1, & 1, & 1, & \ldots \end{matrix} \right\| = 0,$$

therefore the axis is at right angles to the radical locus, and the centres of all the spherics lie upon it.

The shortest distance between two non-intersecting edges (p, q) and (r, s) of a rectangular simplicissimum is the square root of

$$\frac{A_pA_q}{A_p+A_q}+\frac{A_rA_s}{A_r+A_s}.$$

Let $(\lambda_1, \mu_1, 0, 0)$, $(0, 0, \pi_2, \rho_2, \ldots)$ be the points where the shortest distance between (1.2) and (3.4) meets these lines respectively; then, if d be this distance,

$$\begin{aligned} V^2d^2 &= -S_1-S_2+\left(\lambda_1\frac{dS_2}{d\lambda_2}+\mu_1\frac{dS_2}{d\mu_2}+\nu_1\frac{dS_2}{d\nu_2}-\ldots\right) \\ &= -\lambda_1\mu_1(A_1+A_2)-\pi_2\rho_2(A_3+A_4) \\ &\quad +\lambda_1\{\pi_2(A_1+A_3)+\rho_2(A_1+A_4)\} \\ &\quad +\mu_1\{\pi_2(A_2+A_3)+\rho_2(A_2+A_4)\} \\ &= -\lambda_1(V-\lambda_1)(A_1+A_2)-\pi_2(V-\pi_2)(A_3+A_4) \\ &\quad +\lambda_1\{\pi_2(A_1+A_3)+(V-\pi_2)(A_1+A_4)\} \\ &\quad +(V-\lambda_1)\{\pi_2(A_2+A_3)+(V-\pi_2)(A_2+A_4)\}; \end{aligned}$$

and, by differentiation,

$$0=-2VA_2+2\lambda_1(A_1+A_2), \quad 0=-2VA_4+2\pi_2(A_3+A_4),$$

and

$$d^2=\frac{A_1A_2}{A_1+A_2}+\frac{A_3A_4}{A_3+A_4}.$$

XXIX. If $\Sigma'=0$ be the bisecting spheric, $\Sigma''=0$ the spheric through the centres, and $\Sigma'''=0$ the orthogonal spheric, they are coaxal; for, from the equations as given,

$$\Sigma'+\Sigma'''\equiv 2S-2\left\{\frac{\lambda'+\lambda'''}{2}\frac{dS}{d\lambda}+\frac{\mu'+\mu'''}{2}\frac{dS}{d\mu}+\ldots\right\}\frac{\lambda+\mu+\ldots}{V}$$

$$+\frac{V^2r'^2+V^2r'''^2+S'+S'''}{V^2}(\lambda+\mu+\ldots)^2$$

$$\equiv 2\left\{S-\left(\lambda''\frac{dS}{d\lambda}+\mu''\frac{dS}{d\mu}+\ldots\right)\right\}\frac{\lambda+\mu+\ldots}{V}$$

$$+\frac{V^2r''^2+S''}{V^2}(\lambda+\mu+\ldots)^2\equiv 2\Sigma''.$$

XXXVI. If the equation to a linear locus be put into the form

$$A_1\frac{dS}{d\lambda}+A_2\frac{dS}{d\mu}+A_3\frac{dS}{d\nu}+\ldots=0\equiv L,$$

and $$\lambda':\mu':\nu'\ldots :: A_1:A_2:A_3,$$

[in other words, if $(\lambda'\mu'\nu'\ldots)$ be the pole of the linear locus with respect to the circumspheric], the equation may be written

$$\lambda\frac{dS'}{d\lambda'}+\mu\frac{dS'}{d\mu'}+\nu\frac{dS'}{d\nu'}+\ldots=0,$$

the determinant of which is (Art. XII.), $(-2)^{n+1}(n!)^2\,V^4d^2$, where d is the distance from $(\lambda',\mu'\ldots)$ to the circumcentre.

And hence the perpendicular from $(\lambda_1,\mu_1,\nu_1\ldots)$ upon $L=0$

$$=\sqrt{\left\{\frac{1}{(\Sigma A_1)^2}\cdot\frac{(-2)^{n-1}(n!)^2}{(-2)^{n+1}(n!)^2}\cdot\frac{L_1^2}{V^2d^2}\right\}}$$

$$=\frac{1}{2\Sigma A_1}\cdot\frac{L_1}{Vd}.$$

Hence the perpendiculars from $(\lambda_1,\mu_1\ldots)$ upon

$$\frac{dS}{d\lambda}=0,\quad \frac{dS}{d\mu}=0,\ \&c.,$$

the tangents to the circumspheric at the vertices, are

$$\frac{1}{2VR}\cdot\frac{dS_1}{d\lambda_1},\quad \frac{1}{2VR}\cdot\frac{dS_1}{d\mu_1},\ \&c.,$$

and the coordinates in the system mentioned in Art. XXXVI. are proportional to perpendiculars upon the faces of a simplicissimum and analogous to trilinear or quadruplanar coordinates.

[The following Articles are numbered in continuation of the former papers.]

XXXVII. Professor Sylvester's fundamental formula (Art. I., Vol. XVIII., p. 325) for the squared content of a simplicissimum of $n+1$ vertices, viz.,

$$\frac{-1}{(-2)^n (1 \,.\, 2 \ldots n)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \ldots \\ 1, & 0, & (1\,.\,2)^2, & (1\,.\,3)^2, & \ldots \\ 1, & (2\,.\,1)^2, & 0, & (2\,.\,3)^2, & \ldots \\ 1, & (3\,.\,1)^2, & (3\,.\,2)^2, & 0, & \ldots \\ & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

may be transformed so as to give the squared content of any simplicissimum in terms of the coordinates of the vertices.

For
$$V^2 (r\,.\,s)^2 = -S_r - S_s + \left(\lambda_r \frac{dS_s}{d\lambda_s} + \mu_r \frac{dS_s}{d\mu_s} + \ldots\right)$$
$$\equiv -S_r - S_s + S_{r.s}, \text{ say};$$

therefore the squared area of the triangle, the vertices of which are at

$$(\lambda_1, \mu_1, \nu_1, \ldots), (\lambda_2, \mu_2, \nu_2, \ldots), (\lambda_3, \mu_3, \nu_3, \ldots),$$

$$= \frac{-1}{16} \begin{vmatrix} 0, & 1, & 1, & 1, \\ 1, & 0, & \dfrac{-S_1 - S_2 + S_{1.2}}{V^2}, & \dfrac{-S_1 - S_3 + S_{1.3}}{V^2} \\ 1, & \dfrac{-S_2 - S_1 + S_{1.2}}{V^2}, & 0, & \dfrac{-S_2 - S_3 + S_{2.3}}{V^2} \\ 1, & \dfrac{-S_3 - S_1 + S_{1.3}}{V^2}, & \dfrac{-S_3 - S_2 + S_{2.3}}{V^2}, & 0, \end{vmatrix}$$

$$= \frac{-1}{16 V^4} \begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 2S_1, & S_{1.2}, & S_{1.3} \\ 1, & S_{1.2}, & 2S_2, & S_{2.3} \\ 1, & S_{1.3}, & S_{2,3}, & 2S_3 \end{vmatrix}.$$

In the same way it may be shown that the squared content of the tetrahedron, the vertices of which are at

$$(\lambda_1, \mu_1, \ldots), (\lambda_2, \mu_2, \ldots), (\lambda_3, \mu_3, \ldots), (\lambda_4, \mu_4, \ldots),$$

$$= \frac{-1}{(-2)^3(3!)^2}\frac{1}{V^6}\begin{vmatrix} 0, & 1, & 1, & 1, & 1, \\ 1, & 2S_1, & S_{1.2}, & S_{1.3}, & S_{1.4} \\ 1, & S_{1.2}, & 2S_2, & S_{2.3}, & S_{2.4} \\ 1, & S_{1.3}, & S_{2.3}, & 2S_3, & S_{3.4} \\ 1, & S_{1.4}, & S_{2.4}, & S_{3.4}, & 2S_4 \end{vmatrix},$$

and more generally that the squared content of the simplicissimum of $p+1$ vertices, situated at

$$(\lambda_1, \mu_1, \ldots), (\lambda_2, \mu_2, \ldots) \ldots (\lambda_{p+1}, \mu_{p+1}, \ldots),$$

$$= \frac{-1}{(-2)^p(p!)^2}\frac{1}{V^{2p}}\begin{vmatrix} 0, & 1, & 1, & 1, & \ldots & 1, \\ 1, & 2S_1, & S_{1.2}, & S_{1.3}, & \ldots & S_{1.p+1} \\ 1, & S_{1.2}, & 2S_2, & S_{2.3}, & \ldots & S_{2.p+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & S_{1.p+1}, & S_{2.p+1}, & S_{3.p+1} & \ldots & 2S_{p+1} \end{vmatrix}.$$

From this formula it appears that the locus of the $p+1^{\text{th}}$ vertex of a simplicissimum of $p+1$ vertices, the content of which is constant and the base (the other p vertices) fixed, is a locus of the second degree. The simplest examples of this class of loci are the circle (or spheric) and the right circular cylinder. Such loci pass through the intersection of the spherical locus at infinity, with the linear loci

$$\lambda\frac{dS_1}{d\lambda_1}+\mu\frac{dS_1}{d\mu_1}+\nu\frac{dS_1}{d\nu_1}+\ldots = 0,$$

$$\lambda\frac{dS_2}{d\lambda_2}+\mu\frac{dS_2}{d\mu_2}+\nu\frac{dS_2}{d\nu_2}+\ldots = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\lambda\frac{dS_p}{d\lambda_p}+\mu\frac{dS_p}{d\mu_p}+\nu\frac{dS_p}{d\nu_p}+\ldots = 0.$$

If $p = n$, in the formula of this article, the squared content of the simplicissimum vertices at

$$(\lambda_1, \mu_1 \ldots)\ (\lambda_2, \mu_2 \ldots) \ldots (\lambda_{n+1}, \mu_{n+1} \ldots)$$

$$= \frac{-1}{(-2)^n(n!)^2}\frac{1}{V^{2n}}\begin{vmatrix} 0, & 1, & 1, & \ldots\ldots & 1 \\ 1, & 2S_1, & S_{1.2}, & \ldots\ldots & S_{1.n+1} \\ 1, & S_{1.2}, & 2S_2 & \ldots\ldots & S_{2.n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & S_{1.n+1}, & S_{2.n+1}, & \ldots & 2S_{n+1} \end{vmatrix}$$

$$= \frac{-1}{(-2)^n (n!)^3} \frac{1}{V^{2n+1}} \begin{vmatrix} 1, & 0, & 0, & \dots\dots & 0 \\ 1, & \lambda_1, & \mu_1, & \dots\dots & \tau_1 \\ 1, & \lambda_2, & \mu_2, & \dots\dots & \tau_2 \\ \dots & \dots & \dots & \dots & \dots \\ 1, & \lambda_{n+1}, & \mu_{n+1}, & \dots & \tau_{n+1} \end{vmatrix} \times \begin{vmatrix} 0, & 1, & 1, & \dots\dots & 1 \\ 1, & \frac{dS_1}{d\lambda_1}, & \frac{dS_1}{d\mu_1}, & \dots & \frac{dS_1}{d\tau_1} \\ 1, & \frac{dS_2}{d\lambda_2}, & \frac{dS_2}{d\mu_2}, & \dots & \frac{dS_2}{d\tau_2} \\ \dots & \dots & \dots & \dots & \dots \\ 1, & \frac{dS_{n+1}}{d\lambda_{n+1}}, & \frac{dS_{n+1}}{d\mu_{n+1}}, & \dots & \frac{dS_{n+1}}{d\tau_{n+1}} \end{vmatrix}$$

and (Art. VI.)

$$\frac{1}{V^n} \begin{vmatrix} 1, & 0, & 0, & \dots & 0 \\ 1, & \lambda_1, & \mu_1, & \dots & \tau_1 \\ 1, & \lambda_2, & \mu_2, & \dots & \tau_2 \\ \dots & \dots & \dots & \dots & \dots \\ 1, & \lambda_{n+1}, & \mu_{n+1}, & \dots & \tau_{n+1} \end{vmatrix}$$

is equal to the content of the simplicissimum; therefore this is also

$$= \frac{-1}{(-2)^n (n!)^3 V^{n+1}} \begin{vmatrix} 0, & 1, & 1, & \dots\dots & 1 \\ 1, & \frac{dS_1}{d\lambda_1}, & \frac{dS_1}{d\mu_1}, & \dots & \frac{dS_1}{d\tau_1} \\ 1, & \frac{dS_2}{d\lambda_2}, & \frac{dS_2}{d\mu_2}, & \dots & \frac{dS_2}{d\tau_2} \\ \dots & \dots & \dots & \dots & \dots \\ 1, & \frac{dS_{n+1}}{d\lambda_{n+1}}, & \frac{dS_{n+1}}{d\mu_{n+1}}, & \dots & \frac{dS_{n+1}}{d\tau_{n+1}} \end{vmatrix},$$

which gives an expression in terms of the coordinates of Art. XXXVI.

XXXVIII. If $\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \dots\dots$ ....(1),

where $a+b+c+\dots = 0,$

and $\frac{\lambda-\lambda'}{a'} = \frac{\mu-\mu'}{b'} = \frac{\nu-\nu'}{c'} = \dots\dots$ ....(2),

where $a'+b'+c'+\dots = 0,$

be two intersecting straight lines, and A the angle between them,

$$\cos\theta = \frac{\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \dots\right) S(a, b, c\dots)}{2\sqrt{S(a, b, c\dots)\, S(a', b', c'\dots)}}.$$

The angle between the lines is the same as that between the parallels to them through $(V, 0, 0, \ldots)$ the vertex opposite $\lambda = 0$. Let those lines meet $\lambda = 0$ in $A_1(\lambda_1, \mu_1, \nu_1 \ldots)$ and $A_1'(\lambda_1', \mu_1', \nu_1' \ldots)$ respectively.

Now, at A_1,

$$\lambda_1 = 0, \quad \mu_1 = -\frac{b}{a}V, \quad \nu_1 = -\frac{c}{a}V, \text{ \&c.};$$

and at A_1',

$$\lambda_1' = 0, \quad \mu_1' = -\frac{b'}{a'}V, \quad \nu_1' = -\frac{c'}{a'}V, \text{ \&c.};$$

therefore

$$V^2 AA_1^2 = -S_1 + V\frac{dS_1}{d\lambda_1} \quad \ldots\ldots\ldots\ldots \quad \text{(X.)}$$

$$= -\frac{V^2}{a^2}S(a, b, c \ldots) + \frac{V^2}{a}\frac{d}{da}S(a, b, c \ldots)$$

$$- \frac{V^2}{a}\frac{d}{da}S(a, b, c \ldots)$$

$$= -\frac{V^2}{a^2}S(a, b, c \ldots),$$

and

$$AA_1^2 = -\frac{1}{a^2}S(a, b, c \ldots).$$

Similarly,

$$AA_1'^2 = -\frac{1}{a'^2}S(a', b', c' \ldots);$$

also,

$$V^2 A_1A_1'^2 = -S_1 - S_1' + \left(\lambda_1'\frac{d}{d\lambda_1} + \mu_1'\frac{d}{d\mu_1} + \ldots\right)S_1$$

$$= -\frac{V^2}{a^2}S(a, b, c \ldots) + \frac{V^2}{a}\frac{d}{da}S(a, b, c \ldots)$$

$$-\frac{V^2}{a'^2}S(a', b', c' \ldots) + \frac{V^2}{a'}\frac{d}{da'}S(a', b', c' \ldots)$$

$$+\frac{V^2}{aa'}\left\{(bc'+b'c)(2 \,.\, 3)^2 + (b'd+bd')(2 \,.\, 4)^2 + \ldots\right\}$$

$$= -\frac{V^2}{a^2}S(a, b, c \ldots) - \frac{V^2}{a'^2}S(a', b', c' \ldots)$$

$$+\frac{V^2}{aa'}\left\{a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \ldots\right\}S(a, b, c \ldots);$$

therefore

$$AA_1^2 + AA_1'^2 - A_1A_1'^2 = -\frac{1}{aa'}\left\{a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \ldots\right\}S(a, b, c \ldots);$$

therefore

$$\cos\theta = \frac{AA_1^2 + AA_1'^2 - A_1A_1'^2}{2\,.\,AA_1\,.\,AA_1'} = \frac{\left\{a'\frac{d}{da} + b'\frac{d}{db} + \dots\right\} S(a, b, c\dots)}{2\sqrt{S(a, b, c\dots)\,S(a', b', c'\dots)}}.$$

Hence, if AA_1, BB_1, CC_1, &c. be parallel lines through the vertices $A, B, C\dots$, meeting the opposite faces in A_1, B_1, C_1, &c. (Art. XVIII.),

$$\frac{1}{AA_1} + \frac{1}{BB_1} + \frac{1}{CC_1} + \dots = \frac{a+b+c\dots}{\sqrt{-S(a, b, c\dots)}} = 0,$$

and

$$\frac{(1\,.\,2)^2}{AA_1\,.\,BB_1} + \frac{(2\,.\,3)^2}{BB_1\,.\,CC_1} + \frac{(1\,.\,3)^2}{AA\,.\,CC_1} + \&c. = \frac{S(a, b, c, \dots)}{-S(a, b, c, \dots)} = -1.$$

Also, if the lines be at right angles,

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \dots\right) S(a, b, c\dots) = 0.$$

(Comp. Art. XXXV.) It appears too, that, if the line (2) lie upon the linear locus

$$A_1\lambda + A_2\mu + A_3\nu + \dots = 0,$$

to which

$$\frac{\lambda - \lambda''}{a} = \frac{\mu - \mu''}{b} = \frac{\nu - \nu''}{c} = \dots$$

is at right angles, the two lines are at right angles. For

$$A_1 a' + A_2 b' + A_3 c' + \dots = 0,$$

and

$$a' + b' + c' + \dots = 0;$$

therefore

$$(A_2 - A_1)\,b' + (A_3 - A_1)\,c' + \dots = 0,$$

and therefore (XVIII.)

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \dots\right) S(a, b, c\dots) = 0,$$

and any right line at right angles to a linear locus is at right angles to all right lines in that locus, and conversely.

Again, the shortest distance between two non-intersecting straight lines is at right angles to each of them. Let these be

$$\frac{\lambda - \lambda'}{a} = \frac{\mu - \mu'}{b} = \frac{\nu - \nu'}{c} = \dots \equiv p \text{ say},$$

$$\frac{\lambda - \lambda''}{a'} = \frac{\mu - \mu''}{b'} = \frac{\nu - \nu''}{c'} = \dots \equiv p' \text{ say},$$

and if $(\lambda_1, \mu_1 \dots)$, $(\lambda_2, \mu_2 \dots)$ be any points in these respectively, and d the distance between them,

$$V^2 d^2 = -S_1 - S_2 + \left(\lambda_2 \frac{dS_1}{d\lambda_1} + \mu_2 \frac{dS_1}{d\mu_1} + \dots\right)$$

$$= -S' - p\left(a\frac{dS'}{d\lambda'} + b\frac{dS'}{d\mu'} + c\frac{dS'}{d\nu'} + \dots\right)$$

$$-p^2 S(a, b, c \dots) - S'' - p'\left(a'\frac{dS''}{d\lambda''} + b'\frac{dS''}{d\mu''} + c'\frac{dS''}{d\nu''} + \dots\right)$$

$$-p'^2 S(a', b', c' \dots) + \left(\lambda''\frac{dS'}{d\lambda'} + \mu''\frac{dS'}{d\mu'} + \nu''\frac{dS'}{d\nu'} + \dots\right)$$

$$+p\left(a\frac{dS''}{d\lambda''} + b\frac{dS''}{d\mu''} + c\frac{dS''}{d\nu''} + \dots\right)$$

$$+p'\left(a'\frac{dS'}{d\lambda'} + b'\frac{dS'}{d\mu'} + c'\frac{dS'}{d\nu'} + \dots\right)$$

$$+pp'\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \dots\right) S(a, b, c \dots),$$

which is to be a minimum by the variation of p and p', which are independent. This leads to two simple equations in p and p' which determine these and d^2 and $(\lambda_1, \mu_1 \dots)$ $(\lambda_2, \mu_2 \dots)$, and which can be transformed into

$$\left\{(\lambda_2 - \lambda_1)\frac{d}{da} + (\mu_2 - \mu_1)\frac{d}{db} + (\nu_2 - \nu_1)\frac{d}{dc} + \dots\right\} S(a, b, c, \dots) = 0,$$

and

$$\left\{(\lambda_1 - \lambda_2)\frac{d}{da'} + (\mu_1 - \mu_2)\frac{d}{db'} + (\nu_1 - \nu_2)\frac{d}{dc'} + \dots\right\} S(a', b', c' \dots) = 0.$$

So that the shortest distance is at right angles to both lines.

By the help of this article, the theory of the principal axes of a quadric locus may be completed. It has been shown (XXX.) that they are generally n in number; (XXXVI.) that they are mutually at right angles; and now, since they are by definition at right angles to the diametral loci which bisect chords parallel to them, it appears by the above the diametral locus which is conjugate to any principal axis contains all the others.

XXXIX. If $P(\lambda', \mu', \nu' \dots)$ be any point, and A', B', $C' \dots$ its projections on the faces of the simplicissimum of reference; $(PB'C'D' \dots)$ the content of the simplicissimum with vertices at P, B', $C' \dots$

$$= \frac{n^{2(n-1)}}{\{(n-1)!\}^2} \cdot \frac{\mu'}{V_2^2} \cdot \frac{\nu'}{V_3^2} \dots V^{n-1},$$

and the pedal simplicissimum $A'B'C'...$

$$= \frac{n^{2(n-1)}}{\{(n-1)\,!\}^2} \cdot \frac{\lambda'}{V_1^2} \cdot \frac{\mu'}{V_2^2} \cdot \frac{\nu'}{V_3^2} \cdots V^{n-1} \left\{ \frac{V_1^2}{\lambda'} + \frac{V_2^2}{\mu'} + \frac{V_3^2}{\nu'} + \dots \right\}.$$

(Comp. Art. XXVI.)

Let
$$\frac{\lambda-\lambda'}{a_1} = \frac{\mu-\mu'}{b_1} = \frac{\nu-\nu'}{c_1} = \dots \quad \dots\dots\dots\dots (1)$$
$$(a_1+b_1+c_1\dots = 0),$$
$$\frac{\lambda-\lambda'}{a_2} = \frac{\mu-\mu'}{b_2} = \frac{\nu-\nu'}{c_2} = \dots \quad \dots\dots\dots\dots (2)$$
$$(a_2+b_2+c_2+\dots = 0),$$
$$\&c., \qquad \&c.$$

be the equations to the lines PA', PB', PC', &c.; then (1) is at right angles to the linear locus

$$a_1 \frac{dS}{d\lambda} + b_1 \frac{dS}{d\mu} + c_1 \frac{dS}{d\nu} + \dots = 0$$

through the circumcentre (Art. XXXVI.), or, writing S_1 for $S(a_1, b_1, c_1 \dots)$, to

$$\lambda \frac{dS_1}{da_1} + \mu \frac{dS_1}{db_1} + \nu \frac{dS_1}{dc_1} + \dots = 0,$$

and this is parallel to $\lambda = 0$; therefore

$$\frac{dS_1}{db_1} = \frac{dS_1}{dc_1} = \dots \equiv q_1, \text{ say.}$$

And the equations to determine a_1, b_1, c_1, &c. are

$$a_1+b_1+ \qquad c_1+\dots = 0,$$
$$(2\,.\,1)^2 a_1 + \qquad (2\,.\,3)^2 c_1 + \quad = q_1,$$
$$(3\,.\,1)^2 a_1 + (3\,.\,2)^2 \qquad b_1 + \dots = q_1,$$
$$\&c., \qquad \&c.,$$

and

$$a_1 : b_1 : c_1 : \dots :: \begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ 1, & (3.2)^2, & 0, & (3.4)^2 & \dots \\ 1, & (4.2)^2, & (4.3)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$: \begin{vmatrix} 1, & 0, & 1, & 1 & \dots \\ (2.1)^2, & 1, & (2.3)^2, & (2.4)^2 & \dots \\ (3.1)^2, & 1, & 0, & (3.4)^2 & \dots \\ (4.1)^2, & 1, & (4.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} : \begin{vmatrix} 1, & 1, & 0, & 1 & \dots \\ (2.1)^2, & 0, & 1, & (2.4)^2 & \dots \\ (3.1)^2, & (3.2)^2, & 1, & (3.4)^2 & \dots \\ (4.1)^2, & (4.2)^2, & 1, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$: \&c.;$$

and taking these as the values of a_1, b_1, c_1, &c.,

$$a_1 = -(-2)^{n-1}\{(n-1)!\}^2 V_1^2;$$

and

$$la_1 + mb_1 + nc_1 + \&c.$$

$$= -\begin{vmatrix} 0, & l, & m, & n & \dots \\ 0, & 1, & 1, & 1 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix};$$

therefore

$$\frac{dS_1}{db_1} = q_1 = \frac{dS_1}{dc_1} = \dots \equiv (2.1)^2 a_1 + (2.3)^2 c_1 + \dots$$

$$= -\begin{vmatrix} 0, & (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ 0, & 1, & 1, & 1, & 1 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0, & (3.4)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 1, & 1, & 1 & \dots \\ (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ (3.1)^2, & (3.2)^2, & 0, & (3.4)^2 & \dots \\ (4.1)^2, & (4.2)^2, & (4.3)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

and

$$\frac{dS_1}{da_1} = (1.2)^2 b_1 + (1.3)^2 c_1 + \dots$$

$$= -\begin{vmatrix} 0, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ 0, & 1, & 1, & 1 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 0, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} + \begin{vmatrix} 1, & 1, & 1, & 1 & \dots \\ (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ (3.1)^2, & (3.2)^2, & 0, & (3.4)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= -(-2)^n (n!)^2 V^2 + q_1,$$

and
$$2S_1 = a_1 \frac{dS_1}{da_1} + b_1 \frac{dS_1}{db_1} + c_1 \frac{dS_1}{dc_1} + \dots$$

$$= -(-2)^n (n!)^2 V^2 \times \left[-(-2)^{n-1} \{(n-1)!\}^2 V_1^2\right].$$

Thus, then, $a_1 = -(-2)^{n-1} \{(n-1)!\}^2 V_1^2,$

$$\frac{dS_1}{da_1} = -(-2)^n (n!)^2 V^2 + q_1,$$

$$S_1 = -(-2)^{2(n-1)} V^2 . V_1^2 (n!)^2 \{(n-1)!\}^2.$$

Similarly, $b_2 = -(-2)^{n-1} \{(n-1)!\}^2 V_2^2,$

$$\frac{dS_2}{db_2} = -(-2)^n (n!)^2 V^2 + q_2,$$

$$S_2 = -(-2)^{2(n-1)} (n!)^2 \{(n-1)!\}^2 V^2 V_2^2,$$

&c., &c.

Again,
$$\begin{vmatrix} b_2, & c_2, & \dots \\ b_3, & c_3, & \dots \\ \dots & \dots & \dots \\ b_{n+1}, & c_{n+1}, & \dots \end{vmatrix} \times \{-(-2)^n (n!)^2 V^2\}$$

$$= \begin{vmatrix} 1, & 0, & 0, & 0, & \dots \\ 0, & 1, & 0, & 0, & \dots \\ 0, & a_2, & b_2, & c_2, & \dots \\ 0, & a_3, & b_3, & c_3, & \dots \\ & \dots & \dots & \dots & \\ 0, & a_{n+1}, & b_{n+1}, & c_{n+1}, & \dots \end{vmatrix} \times \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2, & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix}$$

$$=\begin{vmatrix} 0, & 1, & 0, & 0, & \dots \\ 1, & 0, & \frac{dS_2}{da_2}, & \frac{dS_3}{da_3}, & \dots \\ 1, & (2.1)^2, & \frac{dS_2}{db_2}, & \frac{dS_3}{db_3}, & \dots \\ 1, & (3.1)^2, & \frac{dS_2}{dc_2}, & \frac{dS_3}{dc_3}, & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix} = -\begin{vmatrix} 1, & \frac{dS_2}{da_2}, & \frac{dS_3}{da_3}, & \dots \\ 1, & \frac{dS_2}{db_2}, & \frac{dS_3}{db_3}, & \dots \\ 1, & \frac{dS_2}{dc_2}, & \frac{dS_3}{dc_3}, & \dots \\ \dots & \dots & \dots & \end{vmatrix}$$

$$=-\begin{vmatrix} 1, & q_2, & q_3, & q_4, & \dots \\ 1, & \frac{dS_2}{db_2}, & q_3, & q_4, & \dots \\ 1, & q_2, & \frac{dS_3}{dc_3}, & q_4, & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix} = -\begin{vmatrix} 1, & q_2, & q_3, & \dots \\ 0, & \frac{dS_2}{db_2}-q_2, & 0, & \dots \\ 0, & 0, & \frac{dS_3}{dc_3}-q_3, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$=-\left(\frac{dS_2}{db_2}-q_2\right)\left(\frac{dS_3}{dc_3}-q_3\right)\dots\left(\frac{dS_{n+1}}{dl_{n+1}}-q_{n+1}\right)$$

$$=-2^{n^2}(n!)^{2n}V^{2n};$$

$$\therefore \begin{vmatrix} b_2, & c_2, & \dots \\ b_3, & c_3, & \dots \\ \dots & \dots & \dots \\ b_{n+1}, & _{(n+1)}, & \dots \end{vmatrix} = (-1)^n 2^{n(n-1)}(n!)^{2(n-1)}V^{2(n-1)}.$$

Now $$PA'=\frac{n\lambda'}{V_1},\ PB'=\frac{n\mu'}{V_2},\ \text{\&c.},$$

and the direction ratios of these lines are the

$$a_1 : b_1 : c_1 : \dots;\ a_2 : b_2 : c_2 \dots,$$

&c., &c., above.

And it was shown (Art. XX.) that $(PB'C'D'\dots)$

$$=\frac{1}{V^{n-1}}\frac{V^n}{\sqrt{(-1)^n S_2 S_3 \dots S_{n+1}}}\frac{n^n\mu'\nu'\dots}{V_2 . V_3}\begin{vmatrix} b_2, & c_2, & \dots \\ b_3, & c_3, & \dots \\ \dots & \dots & \dots \\ b_{n+1}, & c_{n+1}, & \dots \end{vmatrix}$$

$$=\frac{V}{(-1)^n 2^{n(n-1)}(n!)^n\{(n-1)!\}^n}\frac{1}{V_2.V_3\dots V^n}\times\frac{n^n\mu'\nu'}{V_2.V_3}\dots$$

$$\times\{(-1)^n 2^{n(n-1)}(n!)^{2(n-1)}V^{2(n-1)}\}$$

$$=\frac{n^{2(n-1)}}{\{(n-1)!\}^2}\frac{\mu'}{V_2^2}\frac{\nu'}{V_3^2}\dots V^{n-1},$$

with similar values for $(PA'C'D'\dots)$, $(PA'B'D'\dots)$, &c. And hence the pedal simplicissimum of P

$$=\frac{n^{2(n-1)}V^{n-1}}{\{(n-)1!\}^2}\frac{\lambda'}{V_1^2}\frac{\mu'}{V_2^2}\frac{\nu'}{V_3^2}\dots\left\{\frac{V_1^2}{\lambda'}+\frac{V_2^2}{\mu'}+\frac{V_3^2}{\nu'}+\text{\&c.}\right\},$$

which vanishes if $$\frac{V_1^2}{\lambda'}+\frac{V_2^2}{\mu'}+\frac{V_3^2}{\nu'}+\dots=0,$$

in which case the projections of P lie on a linear locus (the Simson locus of Art. XXVI.).

XL. The value of $(A'B'C'D'\dots)$ found in the last article leads to an interpretation of the equation

$$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\dots=0$$

to an n-ic locus circumscribed to the simplicissimum of reference, and having $n+1$ nodes at the vertices, the lines joining which lie wholly on the locus.

If points A'', B'', C'', &c. be taken on PA', PB', PC', &c., respectively, such that

$$PA''=f.PA',\quad PB''=g.PB',\quad PC''=h.PC',\ \text{\&c.},$$

then, if $$\frac{V_1^2}{f\lambda}+\frac{V_2^2}{g\mu}+\frac{V_3^2}{h\nu}+\dots=0$$

$\left(\textit{i.e.}, \text{ the isogonal conjugate of } \frac{\lambda}{f}+\frac{\mu}{g}+\frac{\nu}{h}+\dots=0\right)$

be equivalent to $$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\dots=0 \quad\dots\dots\dots\dots\dots\dots\quad (1);$$

i.e., if $$f:g:h:\dots::\frac{V_1^2}{A}:\frac{V_2^2}{B}:\frac{V_3^2}{C}:\dots;$$

then, for every position of P on (1), A'', B'', C'', &c. lie on a linear locus.

If the point lie upon the locus

$$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\ldots=\frac{M}{\lambda\mu\nu}\ldots\left(\lambda+\mu+\nu\ldots\right)^{2}\ldots\ldots\ldots\ldots(2)$$

(a locus concentric with the above, *i.e.*, having the same poles of infinity), the content of the simplicissimum whose vertices are at $A'', B'', C''+\ldots$ will be constant. (The above results are in analogy to those given for Conics in Questions 9816, 9817, 9818, and in the solution of Question 8177 in the *Reprint* from the *Educational Times*, Vol. XLIX.)

The linear polar of any point $(\lambda', \mu', \nu', \ldots)$ with respect to (1) is

$$\frac{\lambda}{\lambda'}\left(\frac{B}{\mu'}+\frac{C}{\nu'}+\ldots\right)+\frac{\mu}{\mu'}\left(\frac{A}{\lambda'}+\frac{C}{\nu'}+\ldots\right)+\&\text{c.}\ldots=0,$$

and hence the centres, the poles of infinity, are given by the equations

$$\frac{\lambda'}{A}\left(\frac{V}{n}-\lambda'\right)=\frac{\mu'}{B}\left(\frac{V}{n}-\mu'\right)=\frac{\nu'}{C}\left(\frac{V}{n}-\nu'\right)=\ldots.$$

If the point $(\lambda', \mu', \nu', \ldots)$ lie upon (1), this reduces to

$$\frac{A}{\lambda'^2}\lambda+\frac{B}{\mu'^2}\mu+\frac{C}{\nu'^2}\nu+\ldots=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(compare Salmon, *Geometry of Three Dimensions*, p. 416), and therefore (Art. XII.), if $p_1, p_2, p_3, \ldots$ be the perpendiculars from the vertices of the simplicissimum of reference upon (3),

$$\frac{A}{\lambda'^2}:\frac{B}{\mu'^2}:\frac{C}{\nu'^2}:\ldots::p_1:p_2:p_3:\ldots,$$

and if $$A=V_1^2,\ B=V_2^2,\ C=V_3^2,\ \&\text{c.},$$

so that the locus (1) is the isogonal conjugate of infinity, this implies that the perpendiculars from the vertices upon the tangent locus at any point are inversely proportional to the squares of the perpendiculars from the point of contact upon the corresponding faces of the simplicissimum (and hence this is true for the circumcircle of a triangle).

From (3), also, it follows that, if the linear locus

$$a\lambda+\beta\mu+\gamma\nu+\ldots=0$$

touches (1), $$(Aa)^{\frac{1}{2}}+(B\beta)^{\frac{1}{2}}+(C\gamma)^{\frac{1}{2}}+\ldots=0$$

(compare Salmon, *Geometry of Three Dimensions*, p. 416), and the reciprocal equation is of order 2^{n-1}.

If $(\lambda', \mu', \nu', \ldots)$ be any point, any line through it

$$\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \ldots \equiv p \text{ say}$$

$$(\text{where } a+b+c+\ldots = 0)$$

will meet (1) at the points where p has the values given by the equation

$$A(\mu'+bp)(\nu'+cp)(\pi'+dp)\ldots + B(\lambda'+ap)(\nu'+cp)(\pi'+dp)\ldots$$
$$+ C(\lambda'+ap)(\mu'+bp)(\pi'+dp)\ldots + \&c. = 0;$$

or
$$\lambda'\mu'\nu'\ldots\Big[\Big(\frac{A}{\lambda'}+\frac{B}{\mu'}+\frac{C}{\nu'}+\ldots\Big)$$

$$+p\left\{\frac{A}{\lambda'}\Big(\frac{b}{\mu'}+\frac{c}{\nu'}+\ldots\Big)+\frac{B}{\mu'}\Big(\frac{a}{\lambda'}+\frac{c}{\nu'}+\ldots\Big)+\frac{C}{\nu'}\Big(\frac{a}{\lambda'}+\frac{b}{\mu'}+\ldots\Big)+\&c.\right\}$$

$$+p^2\left\{\frac{A}{\lambda'}\Big(\frac{bc}{\mu'\nu'}+\ldots\Big)+\frac{B}{\mu'}\Big(\frac{ac}{\lambda'\nu'}+\ldots\Big)+\frac{C}{\nu'}\Big(\frac{ab}{\lambda'\mu'}+\ldots\Big)+\&c.\right\}$$

$$+\ \ldots\ \ldots\ \ldots\ \ldots$$

$$+p^n\frac{abc\ldots}{\lambda'\mu'\nu'\ldots}\Big(\frac{A}{a}+\frac{B}{b}+\frac{C}{c}+\ldots\Big)\Big]=0,$$

and the line will lie wholly on the locus if the $n+1$ coefficients all vanish. This involves that $(\lambda'\mu'\nu'\ldots)$ should lie on (1), and that the eliminant of the other n coefficients equated to zero and of

$$a+b+c+\ldots = 0$$

should vanish, which it must do since the edges lie on the locus. If $(\lambda'\mu'\nu'\ldots)$ be one of the poles of infinity, the coefficient of p in the above equation will vanish, and consequently the sum of the reciprocals of the intercepts made upon any line through one of them is zero. This is a generalisation of the property of the centre of a conic circumscribed to a triangle.

From the value of $(PB'C'D')$, found in the last article, it appears that
$$\mu\nu\ldots\tau = M(\lambda+\mu+\nu+\ldots+\tau)^n$$
represents a locus such that $(PB'C'D')$ is constant. The linear loci $\mu = 0$, $\nu = 0, \ldots$ are asymptotic to this, and their intersection, the vertex opposite to $\lambda = 0$, is the pole of infinity. Also, any line wholly on the surface must be parallel to a face (not $\lambda = 0$) of the simplicissimum.

Again,
$$\frac{A}{\lambda}+\frac{B}{\mu} = 0, \text{ or } A\mu+B\lambda = 0,$$

is a locus, such that $(PB'C'D'...) : (PA'C'D'...)$ is a constant ratio; and

$$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}=0$$

a locus of the second degree which contains the intersections of

$$\lambda=0,\ \mu=0;\quad \mu=0,\ \nu=0;\quad \nu=0,\ \lambda=0,$$

is the locus of a point P such that

$$(PB'C'D'...),\ (PA'C'D'...),\ (PA'B'D'...)$$

satisfy a linear relation.

Similarly, $$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\frac{D}{\pi}=0$$

is a locus of the third order, such that, if P be any point in it,

$$(PB'C'D'...),\ (PA'C'D'...),\ (PA'B'D'...),\ (PA'B'C'...)$$

are connected by a linear relation. This locus passes through the intersections of any three of the linear loci

$$\lambda=0,\ \mu=0,\ \nu=0,\ \pi=0,$$

and the process of interpretation may be carried on indefinitely.

XLI. It has been shown (Art. XX.) that, if a pencil of $2n$ coterminous lines, in space of n dimensions, be cut by any linear locus, the products of the contents of pairs of complementary $(n-1)$-ary simplicissima, having their vertices at the points of intersection, must satisfy certain identical relations, which lead to relations among the anharmonic ratios of the pencil. These may be obtained as follows :—

All the $(n-1)^{\text{th}}$ minors of

$$\begin{vmatrix} b_1, & b_2, & b_3, & \dots\dots & b_{2n} \\ c_1, & c_2, & c_3, & \dots\dots & c_{2n} \\ \dots & \dots & \dots & & \dots \\ l_1, & l_2, & l_3, & \dots\dots & l_{2n} \\ b_1, & b_2, & b_3, & \dots\dots & b_{2n} \\ c_1, & c_2, & c_3, & \dots\dots & c_{2n} \\ \dots & \dots & \dots & & \dots \\ l_1, & l_2, & l_3, & \dots\dots & l_{2n} \end{vmatrix}$$

must vanish. And therefore, writing $(p, q, r, \dots t)$ for

$$\begin{vmatrix} b_p, & b_q, & b_r, & \dots\dots & b_t \\ c_p, & c_q, & c_r, & \dots\dots & c_t \\ \dots & \dots & \dots & & \dots \\ l_p, & l_q, & l_r, & \dots\dots & l_t \end{vmatrix},$$

$$b_n\,(n+1,\ n+2,\ \dots\ 2n-1,\ 2n)+(-1)^n\,b_{n+1}\,(n+2,\ n+3,\ \dots\ 2n,\ n)$$
$$+\,b_{n+2}\,(n+3,\ n+4,\ \dots\ n,\ n+1)+(-1)^n\,b_{n+3}\,(n+4,\ n+5,\dots\ n+1,\ n+2)$$
$$+\,\&c. = 0,$$

$$c_n\,(n+1,\ n+2,\ \dots\ 2n-1,\ 2n)+(-1)^n\,c_{n+1}\,(n+2,\ n+3,\ \dots\ 2n,\ n)$$
$$+\,c_{n+2}\,(n+3,\ n+4,\ \dots\ n,\ n+1)+(-1)^n\,c_{n+3}\,(n+4,\ n+5,\dots\ n+1,\ n+2)$$
$$+\,\&c. = 0,$$

&c. &c. &c.

Now, if these equations be multiplied respectively by the first minors of

$$\begin{vmatrix} b_n, & c_n, & \dots\dots & l_n. \\ b_1, & c_1, & \dots\dots & l_1 \\ b_2, & c_2, & \dots\dots & l_2 \\ \dots & \dots & \dots & \dots \\ b_{n-1}, & c_{n-1}, & \dots\dots & l_{n-1} \end{vmatrix},$$

and the results added together,

$$(1,\ 2,\ \dots\ n-1,\ n)(n+1,\ n+2,\ \dots\ 2n-1,\ 2n)$$
$$+(-1)^n\,(1,\ 2,\ \dots\ n-1,\ n+1)(n+2,\ n+3,\ \dots\ 2n,\ n)$$
$$+(1,\ 2,\ \dots\ n-1,\ n+2)(n+3,\ n+4,\ \dots\ n,\ n+1)$$
$$+(-1)^n\,(1,2,\ \dots\ n-1,\ n+3)(n+4,\ n+5,\ \dots\ n+1,\ n+2)+\&c. = 0,$$

which is the general type of the identical relations required.

If $n = 2$, and 1, 2, 3, 4 denote four points in a straight line, the formula gives

$$(1,\ 2)(3,\ 4)+(1,\ 3)(4,\ 2)+(1,\ 4)(2,\ 3) = 0,$$

the well-known relation.

In general, the number of relations will be equal to the number of combinations of $2n$ things, $n-1$ together, *i.e.*,

$$\frac{2n\,!}{(n+1)!\ (n-1)!},$$

and therefore fifteen for tridimensional space. Thus, if 1, 2, 3, 4, 5, 6 denote any six points in a plane, and (1, 2, 3) the area of the triangle whose vertices are at 1, 2, and 3, taken in that order,

$$(1, 2, 3)(4, 5, 6)-(1, 2, 4)(5, 6, 3)+(1, 2, 5)(6, 3, 4) -(1, 2, 6)(3, 4, 5) = 0 \equiv \text{I.},$$

$$(1, 3, 4)(2, 5, 6)-(1, 3, 2)(5, 6, 4)+(1, 3, 5)(6, 4, 2) -(1, 3, 6)(4, 2, 5) = 0 \equiv \text{II.},$$

$$(1, 4, 5)(6, 2, 3)-(1, 4, 6)(2, 3, 5)+(1, 4, 2)(3, 5, 6) -(1, 4, 3)(5, 6, 2) = 0 \equiv \text{III.},$$

$$(1, 5, 6)(2, 3, 4)-(1, 5, 2)(3, 4, 6)+(1, 5, 3)(4, 6, 2) -(1, 5, 4)(6, 2, 3) = 0 \equiv \text{IV.},$$

$$(1, 6, 2)(3, 4, 5)-(1, 6, 3)(4, 5, 2)+(1, 6, 4)(5, 2, 3) -(1, 6, 5)(2, 3, 4) = 0 \equiv \text{V.},$$

$$(2, 3, 4)(6, 5, 1)-(2, 3, 6)(5, 1, 4)+(2, 3, 5)(1, 4, 6) -(2, 3, 1)(4, 6, 5) = 0 \equiv \text{VI.},$$

$$(2, 4, 5)(6, 3, 1)-(2, 4, 6)(3, 1, 5)+(2, 4, 3)(1, 5, 6) -(2, 4, 1)(5, 6, 3) = 0 \equiv \text{VII.},$$

$$(2, 5, 6)(4, 3, 1)-(2, 5, 4)(3, 1, 6)+(2, 5, 3)(1, 6, 4) -(2, 5, 1)(6, 4, 3) = 0 \equiv \text{VIII.},$$

$$(2, 6, 1)(5, 4, 3)-(2, 6, 5)(4, 3, 1)+(2, 6, 4)(3, 1, 5) -(2, 6, 3)(1, 5, 4) = 0 \equiv \text{IX.},$$

$$(3, 4, 5)(6, 1, 2)-(3, 4, 6)(1, 2, 5)+(3, 4, 1)(2, 5, 6) -(3, 4, 2)(5, 6, 1) = 0 \equiv \text{X.},$$

$$(3, 5, 6)(1, 2, 4)-(3, 5, 1)(2, 4, 6)+(3, 5, 2)(4, 6, 1) -(3, 5, 4)(6, 1, 2) = 0 \equiv \text{XI.},$$

$$(3, 6, 1)(2, 4, 5)-(3, 6, 2)(4, 5, 1)+(3, 6, 4)(5, 1, 2) -(3, 6, 5)(1, 2, 4) = 0 \equiv \text{XII.},$$

$$(4, 5, 6)(3, 2, 1)-(4, 5, 3)(2, 1, 6)+(4, 5, 2)(1, 6, 3) -(4, 5, 1)(6, 3, 2) = 0 \equiv \text{XIII.},$$

$$(4, 6, 1)(5, 3, 2)-(4, 6, 5)(3, 2, 1)+(4, 6, 3)(2, 1, 5) -(4, 6, 2)(1, 5, 3) = 0 \equiv \text{XIV.},$$

$$(5, 6, 1)(2, 3, 4)-(5, 6, 2)(3, 4, 1)+(5, 6, 3)(4, 1, 2) -(5, 6, 4)(1, 2, 3) = 0 \equiv \text{XV.};$$

and, denoting the sinisters of these equations by the Roman figures to which they are put equal,

$$\text{I.}+\text{X.}+\text{XV.}=0,\quad \text{I.}+\text{XI.}+\text{XIV.}=0,\quad \text{I.}+\text{XII.}+\text{XIII.}=0,$$
$$\text{II.}+\text{VII.}+\text{XV.}=0,\quad \text{II.}+\text{VIII.}+\text{XIV.}=0,\quad \text{II.}+\text{IX.}+\text{XIII.}=0,$$
$$\text{III.}+\text{VI.}+\text{XV.}=0,\quad \text{III.}+\text{VIII.}+\text{XII.}=0,\quad \text{III.}+\text{IX.}+\text{XI.}=0,$$
$$\text{IV.}+\text{VI.}+\text{XIV.}=0,\quad \text{IV.}+\text{VII.}+\text{XII.}=0,\quad \text{IV.}+\text{IX.}+\text{X.}=0,$$
$$\text{V.}+\text{VI.}+\text{XIII.}=0,\quad \text{V.}+\text{VII.}+\text{XI.}=0,\quad \text{V.}+\text{VIII.}+\text{X.}=0.$$

It is easy, by the help of these identities, to show that all the relations may be derived from five, but I have not succeeded in deriving them from four, as the theory (Art. XIX.) seems to require. This is probably due to my having overlooked some relation (probably not linear) between I., II., &c.

The above relations hold for any six points in a plane. If two points, say 5 and 6, coincide, those of the fifteen relations which do not become identically true, reduce to

$$(1, 2, 5)(3, 4, 5)-(1, 3, 5)(2, 4, 5)+(1, 4, 5)(2, 3, 5) = 0.$$

Hence, if 1, 2, 3, 4, 5 be any five points in a plane, the following relations hold among the triangles whose vertices are at these points:—

$$(1, 2, 5)(3, 4, 5)-(1, 3, 5)(2, 4, 5)+(1, 4, 5)(2, 3, 5) = 0,$$
$$(1, 2, 4)(3, 5, 4)-(1, 3, 4)(2, 5, 4)+(1, 5, 4)(2, 3, 4) = 0,$$
$$(1, 2, 3)(4, 5, 3)-(1, 4, 3)(2, 5, 3)+(1, 5, 3)(2, 4, 3) = 0,$$
$$(1, 3, 2)(4, 5, 2)-(1, 4, 2)(3, 5, 2)+(1, 5, 2)(3, 4, 2) = 0,$$
$$(2, 3, 1)(4, 5, 1)-(2, 4, 1)(3, 5, 1)+(2, 5, 1)(3, 4, 1) = 0.$$

If one of the anharmonic ratios be unity, others are so also; *e.g.*, if

$$(1, 2, 3)(4, 5, 6) = (1, 2, 4)(5, 6, 3),$$

from I., $(1, 2, 5)(6, 3, 4) = (1, 2, 6)(3, 4, 5),$

and from XV., $(1, 5, 6)(2, 3, 4) = (2, 5, 6)(1, 3, 5)$

and the lines 12, 34, and 56 are concurrent.

If another independent ratio be unity also, say

$$(1, 3, 4)(2, 5, 6) = (1, 3, 5)(2, 4, 6),$$

then $(1, 3, 4)(2, 5, 6) = (1, 3, 5)(2, 4, 6) = (1, 5, 6)(2, 3, 4)$,

by II., $(1, 3, 2)(4, 5, 6) = (1, 3, 6)(2, 4, 5) = (1, 4, 2)(3, 5, 6)$,

and by IX.,

$$(1, 2, 6)(3, 4, 5) = (1, 5, 4)(2, 3, 6) = (1, 2, 5)(6, 3, 4),$$

and the lines 15, 36, and 24 are concurrent, and also 13, 26, and 45 are concurrent.

Similarly, the typical relation when $n = 4$—in other words, among the products of complementary tetrahedra which have their vertices at any eight points in the same tridimensional space (*i.e.*, in the same linear locus in space of four dimensions)—is

$$(1, 2, 3, 4)(5, 6, 7, 8) + (1, 2, 3, 5)(6, 7, 8, 4)$$
$$+ (1, 2, 3, 6)(7, 8, 4, 5) + (1, 2, 3, 7)(8, 4, 5, 6)$$
$$+ (1, 2, 3, 8)(4, 5, 6, 7) = 0:$$

these are ${}^8C_3$ or 56 in number, and should be dependent (Art. XX.) upon 23 independent relations.

By supposing two of the points to coincide, the following typical relation is obtained among the tetrahedra the vertices of which are situated at any seven points :

$$(1, 2, 3, 4)(1, 5, 6, 7) + (1, 2, 3, 5)(6, 7, 4, 1)$$
$$+ (1, 2, 3, 6)(7, 4, 1, 5) + (1, 2, 3, 7)(4, 1, 5, 6) = 0;$$

and, by supposing two other points to coincide, this typical relation among the tetrahedra the vertices of which occupy any six points in space

$$(1, 2, 3, 4)(1, 2, 5, 6) - (1, 2, 3, 5)(1, 2, 6, 4)$$
$$+ (1, 2, 3, 6)(1, 2, 4, 5) = 0.$$

In the geometry of space of n dimensions all the anharmonic systems up to the n^{th} have their application. Pencils of $2n$ straight lines through a point cutting a linear locus, form a system of the $\overline{n-1}^{\text{th}}$ class; pencils of $2n-2$ planes through a line cut linear loci represented by two equations in a system of the $\overline{n-2}^{\text{th}}$ class; and so on, until pencils of four linear loci through a common intersection cut a line in points which form an ordinary anharmonic system, *i.e.*, in accordance with the nomenclature above in one of the first class.

XLII. A special case, which seems to be the proper generalization of a harmonic pencil arises, when the pencil meeting at $(V, 0, 0, \ldots)$ is composed of the $2n$ straight lines

$$\nu = \pi = \ldots = 0, \quad \mu = \pi = \ldots = 0, \quad \mu = \nu = \ldots = 0, \ \&c.;$$

the edges of the simplicissimum of reference which meet at the origin of the pencil, and the other n lines

$$\frac{\mu}{-a} = \frac{\nu}{b} = \frac{\pi}{c} = \ldots, \quad \frac{\mu}{a} = \frac{\nu}{-b} = \frac{\pi}{c} = \ldots,$$

$$\frac{\mu}{a} = \frac{\nu}{b} = \frac{\pi}{-c} = \ldots, \ \&c.,$$

(or more generally of the $2n$ lines

$$N = 0, \ P = 0, \ \ldots; \quad M = 0, \ P = 0, \ \ldots; \quad M = 0, \ N = 0, \ \ldots; \ \&c.,$$

and

$$\begin{aligned} -AM &= \quad BN = \quad CP = \ldots, \\ AM &= -BN = \quad CP = \ldots, \\ AM &= \quad BN = -CP = \ldots, \\ &\&c. \qquad \&c., \end{aligned}$$

all of which meet at the point

$$M = 0, \quad N = 0, \quad P = 0, \ \ldots).$$

In space of three dimensions the anharmonic ratios corresponding to such a system are given by

$$(1,2,3)(4,5,6) : (1,2,4)(3,5,6) : (1,2,5)(3,4,6) : (1,2,6)(3,4,5)$$
$$: (1,3,4)(2,5,6) : (1,3,5)(2,4,6) : (1,3,6)(2,4,5) : (1,4,5)(2,3,6)$$
$$: (1,4,6)(2,3,5) : (1,5,6)(2,3,4) :: 2 : 1 : -1 : 0 : -1$$
$$: 0 : -1 : 1 : -1 : 0;$$

and the other systems obtained by changing the signs of a, b, or c will, like the above, have their anharmonic ratios

$$\pm 2, \ \pm\tfrac{1}{2}, \ \pm 1, \ 0, \ \text{or} \ \infty.$$

The complete system, which includes all the rays of these, is composed of the first three of these rays, and four others, such that, if they cut any plane in the points 1, 2, 3 and 4, 5, 6, 7, respectively, then, if any three of the last four points be taken as the vertices of a triangle, the points 1, 2, 3 are the feet of concurrent lines from the vertices through the remaining point.

It is easily proved that, if the lines 56, 64, 45 touch a conic about 123 at these points respectively, the six points form a system of the kind, the seventh being the point of concurrence of 14, 25, 36. Therefore, if from any point three tangent planes be drawn to a quadric surface, the three points of contact, and the three points where the intersections of the tangent planes meet the polar plane of the original point, form a system of the kind stated.

Again, if two sets of points 4, 5, 6 and 4′, 5′, 6′ are each thus conjugate to the same three primary points 1, 2, 3, and if 7 and 7′ be the additional points connected with these systems, the eight points 4, 5, 6, 7 and 4′, 5′, 6′, 7′ all lie on the same conic, to which 123 is self-conjugate—viz., if the rays through 1, 2, 3 be the edges of the simplicissimum as above, and those through 4, 5, 6 the remaining three lines, these and that through 7, $\frac{\mu}{a} = \frac{\nu}{b} = \frac{\pi}{c}$ and 4′, 5′, 6′ and 7′ the same with a', b', c' written for a, b, c, on

$$\begin{vmatrix} \lambda^2, & \mu^2, & \nu^2 \\ a^2, & b^2, & c^2 \\ a'^2, & b'^2, & c'^2 \end{vmatrix} = 0.$$

In the same way, in space of n dimensions a pencil of $2n$ rays of the kind defined at the beginning of this article cuts any transversal linear locus in $2n$ points, such that first n points are the feet of concurrent lines from the vertices of the simplicissimum defined by the second n, and the anharmonic ratios are

$$\pm 2, \quad \pm \tfrac{1}{2}, \quad \pm 1, \quad 0, \quad \infty.$$

Any set of n points so conjugate with a given set of n points, involves, by changing the signs of a, &c., a system of 2^{n-1} points constituting 2^{n-1} sets of ratios.

All the $n \cdot 2^{n-1}$ points belonging to n such systems conjugate to the same n points, lie upon the same quadric locus in space of $(n-1)$ dimensions, to which the simplicissimum, of which the vertices are at the n points to which each system is conjugate, is self-reciprocal; and if $a, b, c \ldots$; $a', b' \ldots$, &c. be the constants determining these points, this locus is

$$\begin{vmatrix} \lambda^2, & \mu^2, & \nu^2 & \ldots \\ a^2, & b^2, & c^2 & \ldots \\ a'^2, & b'^2, & c'^2 & \ldots \\ a''^2, & b''^2, & c''^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0.$$

Each system of points upon such a quadric is analogous to a system of points in involution; the n vertices of the simplicissimum corresponding to the foci of the involution. Each point on the quadric determines the other 2^{n-1} points of its system.

It may also be shown that the four systems of this kind, determined on the faces of the tetrahedron of reference by the same point of concurrence, lie on the same quadric surface; for, if this point be $a:b:c:d$, the points on $\lambda = 0$ are

$$0:-b:c:d,\quad 0:b:-c:d,\quad 0:b:c-d;$$

those on $\mu = 0$,

$$-a:0:c:d,\quad a:0:-c:d,\quad a:0:c:-d$$

those on $\nu = 0$,

$$-a:b:0:d,\quad a:-b:0:d,\quad a:b:0:-d;$$

and on $\pi = 0$,

$$-a:b:c:0,\quad a:-b:c:0,\quad a:b:-c:0$$

these all lie upon

$$\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}+\frac{\pi^2}{d^2}+3\left\{\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\frac{\lambda\pi}{ad}+\frac{\mu\nu}{bc}+\frac{\mu\pi}{bd}+\frac{\nu\pi}{cd}\right\}=0,$$

or $$\left(\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c}+\frac{\pi}{d}\right)^2+\left(\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\frac{\lambda\pi}{ad}+\ldots\right)=0,$$

which has double contact with the circumscribed quadric

$$\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\frac{\lambda\pi}{ad}+\frac{\mu\nu}{bc}+\frac{\mu\pi}{bd}+\frac{\nu\pi}{cd}=0,$$

which passes through the points

$$(-a, b, c, d);\quad (a, -b, c, d);\quad (a, b, -c, d);\quad (a, b, c, -d).$$

In the same way, in space of four dimensions, the five systems determined on the faces of the simplicissimum of reference by the same point of concurrence,

$$(a:b:c:d:e),$$

all lie upon

$$\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\frac{\lambda\pi}{ad}+\frac{\lambda\rho}{ae}+\frac{\mu\nu}{bc}+\frac{\mu\pi}{bd}+\frac{\mu\rho}{be}+\frac{\nu\pi}{cd}+\frac{\nu\rho}{ce}+\frac{\pi\rho}{de}=0;$$

and, in space of five dimensions, the systems determined by the point

$$(a : b : c : d : e : f)$$

lie upon
$$\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}+\frac{\pi^2}{d^2}+\frac{\rho^2}{e^2}+\frac{\sigma^2}{f^2}$$

$$-\frac{5}{2}\left\{\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\frac{\lambda\pi}{ad}+\frac{\lambda\rho}{ae}+\frac{\lambda\sigma}{af}+\frac{\mu\nu}{bc}+\&c.\right\}=0\,;$$

and so on, the quadric locus being of the form

$$\left(\frac{\lambda}{a}+\frac{\mu}{b}+\dots\right)^2-\kappa\left(\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\dots\right)=0,$$

and having contact with

$$\left(\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}\dots\dots\right)=0$$

along its intersection with

$$\frac{\lambda}{a}+\frac{\mu}{b}+\dots=0,$$

a locus of which interpretations are given in Art. XXIII. *ad finem.*

XLIII. Two quadric loci, which intersect in the linear locus at infinity and another linear locus, are similar and similarly situated, having two centres of similitude in the line of centres.

$$U=A_{11}\lambda^2+A_{22}\mu^2+A_{33}\nu^2+\dots+2A_{12}\lambda\mu+\dots=0\dots\dots\dots(1)$$

and
$$U+(B_1\lambda+B_2\mu+B_3\nu+\dots)(\lambda+\mu+\nu+\dots)=0\ \dots\dots\dots(2)$$

are two such loci.

If $\dfrac{\lambda-\lambda'}{a}=\dfrac{\mu-\mu'}{b}=\dfrac{\nu-\nu'}{c}=\dots=\dfrac{Vp}{\sqrt{-S(a,b,c,\dots)}}\equiv mp,$

where $(a+b+c+\dots=0)$ (Art. XVIII.)

be any line through $(\lambda',\mu',\nu',\dots)$ cutting (1) and (2) at distances p and p', these are determined by the equations

$$U'+mp\left(a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+c\frac{dU'}{d\nu'}+\dots\right)$$

$$+m^2p^2(A_{11}a^2+A_{22}b^2+A_{33}c^2+\dots+2A_{12}bc+\dots)=0,$$

and $$U'+(B_1\lambda'+B_2\mu'+B_3\nu'+\ldots)(\lambda'+\mu'+\nu'+\ldots)$$

$$+mp'\left\{a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+c\frac{dU'}{d\nu'}\right.$$
$$\left.+\ldots+(B_1a+B_2b+B_3c+\ldots)(\lambda'+\mu'+\nu'+\ldots)\right\}$$
$$+m^2p'^2\{A_{11}a^2+A_{22}b^2+A_{33}c^2+\ldots+2A_{12}bc+\ldots\}.$$

In order that the values of p' should be equimultiples of those of p, it is necessary and sufficient that

$$1+\frac{(B_1\lambda'+B_2\mu'+B_3\nu'+\ldots)V}{U'}$$
$$=\left\{1+\frac{(B_1a+B_2b+B_3c+\ldots)V}{a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+c\frac{dU'}{d\nu'}+\ldots}\right\}^2 \ldots\ldots\ldots\ldots\ldots (3).$$

For similitude, these quantities must remain unchanged for all values of a, b, c, consistent with $a+b+c+\ldots=0$;

and therefore $$\frac{B_1a+B_2b+B_3c+\ldots}{a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+c\frac{dU'}{d\nu'}+\ldots}$$

must also remain unaltered. Putting this equal to q,

$$q=\frac{(B_2-B_1)\,b+(B_3-B_1)\,c+\ldots}{\left(\frac{dU'}{d\mu'}-\frac{dU}{d\lambda'}\right)b+\left(\frac{dU'}{d\nu'}-\frac{dU'}{d\lambda'}\right)c+\ldots}$$

and $$\frac{B_2-B_1}{\frac{dU'}{d\mu'}-\frac{dU'}{d\lambda'}}=\frac{B_3-B_1}{\frac{dU'}{d\nu'}-\frac{dU'}{d\lambda'}}=\ldots=q;$$

therefore
$$\frac{dU'}{d\lambda'}-\frac{B_1}{q}=\frac{dU'}{d\mu'}-\frac{B_2}{q}=\frac{dU'}{d\nu'}-\frac{B_3}{q}=\ldots\equiv h,\text{ say }\ldots\ldots (4);$$

therefore $$2U'-\frac{1}{q}(B_1\lambda'+B_2\mu'+B_3\nu'+\ldots)=hV.$$

But, by (3), $$\frac{B_1\lambda'+B_2\mu'+B_3\nu'\ldots}{U'}-2q-q^2V=0;$$

therefore $$qU'+h=0,$$

and the equations (4) may be written

$$q\frac{dU'}{d\lambda'}-B_1=q\frac{dU'}{d\mu'}-B_2=q\frac{dU'}{d\nu'}-B_3=\ldots$$
$$=-q^2U'=m\text{ suppose};$$

also, $$q(\lambda'+\mu'+\nu'+\ldots) = qV,$$

so that there are $n+3$ equations from which to eliminate

$$q\lambda',\ q\mu',\ q\nu' \ldots \text{ and } m,$$

and all but one are linear. Hence the resultant is a quadratic in q, each solution of which determines one position of $(\lambda', \mu', \nu', \ldots)$.

If $(\lambda'', \mu'', \nu'', \ldots)$ and $(\lambda''', \mu''', \nu''', \ldots)$

be the centres of (1) and (2),

therefore $$\frac{dU''}{d\lambda''} = \frac{dU''}{d\mu''} = \frac{dU''}{d\nu''} = \ldots,$$

and $$\frac{dU'''}{d\lambda'''} - B_1 V = \frac{dU'''}{d\mu'''} - B_2 V = \frac{dU'''}{d\nu'''} - B_3 V = \ldots;$$

and the line joining these points is therefore

$$\frac{\dfrac{dU}{d\mu} - \dfrac{dU}{d\lambda}}{B_2 - B_1} = \frac{\dfrac{dU}{d\nu} - \dfrac{dU}{d\lambda}}{B_3 - B_1} = \ldots,$$

and by (4) the centres of similitude lie upon this.

Other properties of similar and similarly situated quadric loci are given in Art. XXV. (note), Art. XXXI., and in the Addendum to Art. XIII., prefixed to this paper.

XLIV. Two quadric loci in space of n dimensions have a common self-conjugate simplicissimum. Let

$$U \equiv A_{11}\lambda^2 + A_{22}\mu^2 + A_{33}\nu^2 + \ldots + 2A_{12}\lambda\mu + \ldots = 0$$

$$W \equiv B_{11}\lambda^2 + B_{22}\mu^2 + B_{33}\nu^2 + \ldots + 2B_{12}\lambda\mu + \ldots = 0$$

be the quadrics, and let $(\lambda', \mu', \nu' \ldots)$ be a point which has the same linear polar with respect to each of them.

Therefore $$\lambda\frac{dU'}{d\lambda'} + \mu\frac{dU'}{d\mu'} + \nu\frac{dU'}{d\nu'} + \ldots = 0$$

and $$\lambda\frac{dW'}{d\lambda'} + \mu\frac{dW'}{d\mu'} + \nu\frac{dW'}{d\nu'} + \ldots = 0$$

are identical loci.

Therefore $$\frac{dU'}{d\lambda'} : \frac{dW'}{d\lambda'} :: \frac{dU'}{d\mu'} : \frac{dW'}{d\mu'} :: \frac{dU'}{d\nu'} : \frac{dW'}{d\nu'} :: \&c.,$$

$$:: r : 1, \text{ say};$$

and, eliminating λ', μ', ν', &c. from the resulting $n+1$ homogeneous linear equations,

$$\begin{vmatrix} A_{11}-B_{11}r, & A_{12}-B_{12}r, & A_{13}-B_{13}r, & \dots \\ A_{12}-B_{12}r, & A_{22}-B_{22}r, & A_{23}-B_{23}r, & \dots \\ A_{13}-B_{13}r, & A_{23}-B_{23}r, & A_{33}-B_{33}r, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0 \quad\dots\dots\dots\dots(1)$$

an equation of the $(n+1)^{\text{th}}$ degree in r.

(It is, indeed, the discriminant of $U-rW=0$, and the values of r are those for which this quadric has a double point.)

Each root r_1, r_2, &c. of (1) gives a separate solution for $\lambda', \mu', \nu', \dots$, so that, when the roots are all unequal, the $n+1$ corresponding points

$$(\lambda_1', \mu_1', \nu_1', \dots),\ (\lambda_2', \mu_2', \nu_2', \dots),\ (\lambda_{n+1}', \mu_{n+1}, \nu_{n+1}')$$

are the vertices of a simplicissimum, and each lies upon the linear polar of all the others with respect to both the quadrics.

For, if r_s and r_t be two values of r,

$$\frac{dU_s'}{d\lambda_s'} = r_s \frac{dW_s'}{d\lambda_s'},\quad \frac{dU_s'}{d\mu_s'} = r_s \frac{dW_s'}{d\mu_s'}, \text{ \&c.},$$

and
$$\frac{dU_t'}{d\lambda_t'} = r_t \frac{dW_t'}{d\lambda_t'},\quad \frac{dU_t'}{d\mu_t'} = r_t \frac{dW_t'}{d\mu_t'}, \text{ \&c.};$$

therefore
$$\lambda_t' \frac{dU_s'}{d\lambda_s'} + \mu_t' \frac{dU_s'}{d\mu_s'} + \nu_t' \frac{dU_s'}{d\nu_s'} + \dots$$
$$= r_s \left\{ \lambda_t' \frac{dW_s'}{d\lambda_s'} + \mu_t' \frac{dW_s'}{d\mu_s'} + \nu_t' \frac{dW_s'}{d\nu_s'} + \dots \right\},$$

or
$$\lambda_s' \frac{dU_t'}{d\lambda_t'} + \mu_s' \frac{dU_t'}{d\mu_t'} + \nu_s' \frac{dU_t'}{d\nu_t'} + \dots$$
$$= r_s \left\{ \lambda_s' \frac{dW_t'}{d\lambda_t'} + \mu_s' \frac{dW_t'}{d\mu_t'} + \nu_s' \frac{dW_t'}{d\nu_t'} + \dots \right\};$$

but
$$\lambda_s' \frac{dU_t'}{d\lambda_t'} + \mu_s' \frac{dU_t'}{d\mu_t'} + \nu_s' \frac{dU_t'}{d\nu_t'} + \dots$$
$$= r_t \left\{ \lambda_s' \frac{dW_t'}{d\lambda_t'} + \mu_s' \frac{dW_t'}{d\mu_t'} + \nu_s' \frac{dW_t'}{d\nu_t'} + \dots \right\},$$

and therefore, r_s and r_t being different,

$$\lambda_s' \frac{dU_t'}{d\lambda_t'} + \mu_s' \frac{dU_t'}{d\mu_t'} + \nu_t' \frac{dU_t'}{d\nu_t'} + \dots = 0,$$

$$\lambda_s' \frac{dW_t'}{d\lambda_t'} + \mu_s' \frac{dW_t'}{d\mu_t'} + \nu_s' \frac{dW_t'}{d\nu_t'} + \dots = 0,$$

and $(\lambda'_s, \mu'_s, \nu'_s \ldots)$ lies upon the polar of $(\lambda'_t, \mu'_t, \nu'_t \ldots)$ with respect to each of the quadrics $U = 0$ and $W = 0$.

Hence, when the roots of (1) are unequal, these two quadrics have a definite self-conjugate simplicissimum, the vertices of which are the double points of those quadrics of the system $U - rW = 0$ which have double points.

When two or more of the roots of (1) are equal, the corresponding vertices are no longer determinate. If two be equal, one of the quadrics $U - rW = 0$ is expressible as a linear homogeneous function of $n-1$ squares, and has double points along a straight line; if three of the roots be equal, one of the quadrics is expressible as a linear homogeneous function of $n-2$ squares, and has double points over a plane; and so on.

For if, when referred to a common self-conjugate simplicissimum,

$$U \equiv a\lambda^2 + b\mu^2 + c\nu^2 + d\pi^2 + e\rho^2 + \ldots = 0,$$

$$W \equiv a'\lambda^2 + b'\mu^2 + c'\nu^2 + d'\pi^2 + e'\rho^2 + \ldots = 0,$$

it is necessary and sufficient, in order that

$$a\lambda'\lambda + b\mu'\mu + c\nu'\nu + d\pi'\pi + e\rho'\rho + \ldots = 0$$

and

$$a'\lambda'\lambda + b'\mu'\mu + c'\nu'\nu + d'\pi'\pi + e'\rho'\rho + \ldots = 0$$

may be identical, that all but one of the quantities $\lambda', \mu', \nu', \ldots$ should vanish, unless some of the coefficients be such that some of the ratios $a : b : \ldots :: a' : b' : c' \ldots$ hold—*i.e.*, unless the equation (1) has equal roots. When this is the case, it is only necessary that those coordinates, of which the coefficients do not satisfy the proportions above, should vanish. Thus, if $a : b :: a' : b'$, all the coordinates but λ' and μ' must vanish; if $a : b : c :: a' : b' : c'$, all but λ', μ', ν'; and so on.

If, in the first case, $(\lambda_1, \mu_1, 0, 0)$, $(\lambda_2, \mu_2, 0, 0)$ be new positions of the two vertices on (1.2),

$$a\lambda_1\lambda_2 + b\mu_1\mu_2 = 0,$$

and if $(\lambda_1, \mu_1, \nu_1, 0)$, $(\lambda_2, \mu_2, \nu_2, 0)$, $(\lambda_3, \mu_3, \nu_3, 0)$ be new positions of the three vertices in the plane (1, 2, 3),

$$a\lambda_1\lambda_2 + b\mu_1\mu_2 + c\nu_1\nu_2 = 0,$$

$$a\lambda_2\lambda_3 + b\mu_2\mu_3 + c\nu_2\nu_3 = 0,$$

$$a\lambda_3\lambda_1 + b\mu_3\mu_1 + c\nu_3\nu_1 = 0;$$

and similar relations hold in other cases.

The intersections of quadric loci in space of n dimensions may be classified according as the roots (1) are equal or unequal. Thus, in ordinary space,

(i.) all the roots may be unequal,

(ii.) two roots may be equal and $W = AU + LM$,

(iii.) three roots may be equal and $W = AU + L^2$,

(iv.) two pairs of roots may be equal and

$$U = ALM + BL'M',$$
$$W = A'LM + B'L'M';$$

and so for space of any dimensions.

XLV. If the transformation of Art. V. be applied to $U = 0$, the result, the equation in simplicissimum content coordinates to the intersection of $U = 0$ with the linear locus determined by the $p+1$ points

$$(x_1, y_1, z_1, \ldots), \quad (x_2, y_2, z_2, \ldots), \ \ldots \ (x_{p+1}, y_{p+1}, z_{p+1}),$$

will be

$$U_{11}\lambda^2 + U_{22}\mu^2 + U_{33}\nu^2 + \ldots + 2U_{12}\lambda\mu + \ldots = 0 \ \ldots\ldots\ldots\ldots\ (\text{v.}),$$

where U_{11} stands for the value of U when $x_1, y_1, z_1, \ldots$ are written for $x, y, z, \ldots$ and $2U_{12}$ for the result of operating on this with

$$\left(x_2 \frac{d}{dx_1} + y_2 \frac{d}{dy_1} + z_2 \frac{d}{dz_1} + \ldots\right).$$

If $p+1 = n$, so that the locus determined by the points is a linear locus in space of n dimensions, and then, if this be the tangent locus at $(x_1, y_1, z_1, \ldots)$, a point on $U = 0$,

$$U_{11} = 0, \quad U_{12} = 0, \quad U_{13} = 0 \ \ldots,$$

and the discriminant of (1) vanishes; therefore the linear tangent locus at any point on a quadric locus, in space of n dimensions, intersects the quadric in a quadric in space of $n-1$ dimensions (*i.e.*, on the linear locus), having a double point at the point of contact (comp. Art. XXXIII.); and if the locus through the n points be a tangent, the discriminant of (1) vanishes; and conversely, for if the discriminant of (1) vanishes, (1) has a node, and if this be taken as the first vertex of the simplicissimum of n vertices

$$U_{11} = U_{12} = U_{13} = \ldots = 0,$$

and the linear locus touches at this point.

Since the equation to the linear locus is

$$\begin{vmatrix} x, & y, & z, & \dots \\ x_1, & y_1, & z_1, & \dots \\ x_2, & y_2, & z_2, & \dots \\ \dots & \dots & \dots \\ x_n, & y_n, & z_n, & \dots \end{vmatrix} \equiv \alpha x + \beta y + \gamma z + \dots = 0 \quad \dots\dots\dots\dots (2),$$

it follows that the vanishing of the discriminant of (1) is the condition that (2) should touch $U = 0$, and therefore α, β, γ, &c., the first minors of the determinant in (2), must satisfy the reciprocal equation $\sigma = 0$; and hence the discriminant of (1) can only differ by a simple multiplier from the value of σ when the above values are written for α, β, γ, &c. (*Educational Times*, Questions 8940, 8970 and 9828).

By an entirely similar proof it appears that, if

$$\Sigma \equiv \sigma_{11}\alpha^2 + \sigma_{22}\beta^2 + \sigma_{33}\gamma^2 + \dots + 2\sigma_{23}\alpha\beta + \dots ,$$

the discriminant of Σ can only differ by a multiplier from $\Delta^{n-1} U$, with the first minors of

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \dots \\ \alpha_1, & \beta_1, & \gamma_1, & \dots \\ \alpha_2, & \beta_2, & \gamma_2, & \dots \\ \dots & \dots & \dots \\ \alpha_n, & \beta_n, & \gamma_n, & \dots \end{vmatrix}$$

written for x, y, z, &c.

If $\qquad p = 1, \quad U_{11}\lambda^2 + 2U_{12}\lambda\mu + U_{22}\mu^2 = 0$

determines the points in which the line joining the two given points cuts $U = 0$; and, if $U_{11}U_{22} - U_{12}^2 = 0$, this line is a tangent, and therefore

$$4U_{11}U - \left\{ \left(x\frac{d}{dx_1} + y\frac{d}{dy_1} + z\frac{d}{dz_1} + \dots \right) U_{11} \right\}^2 = 0$$

is the equation to the group of linear tangents from the point $(x_1, y_1, z_1, \dots)$.

If $p = 2$, the plane through these given points cuts $U = 0$ in

$$U_{11}\lambda^2 + U_{22}\mu^2 + U_{33}\nu^2 + 2U_{12}\lambda\mu + \dots = 0,$$

and this will touch if

$$U_{11}U_{22}U_{33} + 2U_{12}U_{13}U_{23} - U_{11}U_{23}^2 - U_{22}U_{13}^2 - U_{33}U_{12}^2 = 0\,;$$

and

$$4U_{11}U_{22}U+2U_{12}\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}+\dots\right)U.\left(x_2\frac{d}{dx}+y_2\frac{d}{dy}+\dots\right)U$$
$$-4U_{12}^2U-U_{11}\left\{\left(x_2\frac{d}{dx}+y_2\frac{d}{dy}+\dots\right)U\right\}^2$$
$$-U_{22}\left\{\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}+\dots\right)U\right\}^2=0,$$

is the equation to the group of tangent planes which can be drawn to $U=0$, through the line joining $(x_1, y_1, z_1, \dots)$ and $(x_2, y_2, z_2, \dots)$; and so on.

XLVI. The $(n+1)$-ary quadric, the sinister of the equation to a quadric locus in space of n dimensions, has only one invariant, the discriminant; but two such quadrics have in all $(n+2)$ invariants—the two discriminants and n other functions involving the coefficients of both—these last are the coefficients of the powers 1, 2, ... n of r in (1), Art. XLIV. They are combinants and are derived from the discriminant of U by operating upon that function with

$$\frac{1}{p!}\left(B_{11}\frac{d}{dA_{11}}+B_{22}\frac{d}{dA_{22}}+\dots+B_{12}\frac{d}{dA_{12}}+\dots\right),$$

giving to p all values from 1 to n.

As stated above, the roots of (1) give the quadrics through the intersection of $U=0$ and $W=0$, which have double points.

If $a_{11}, a_{12}, \dots a_{1n}$, &c. be the first minors of the discriminant of U, the coefficient of r is

$$B_{11}a_{11}+B_{22}a_{22}+\dots+2B_{12}a_{12}+\dots,$$

and this vanishes if a simplicissimum self-conjugate with respect to $W=0$ can be inscribed in $U=0$; for then

$$B_{12}=B_{13}=B_{23}=\dots 0 \text{ and } a_{11}=a_{22}=\dots=0 \text{ (Art. XXX.)}.$$

If $V=0$ and $X=0$ be the equations to the sections of $U=0$ and $W=0$ respectively, by the linear locus

$$a\lambda+\beta\mu+\gamma\nu+\dots+\theta\tau=0,$$

—*i.e.*, through the n points

$$\left(\frac{V\theta}{\theta-a}, 0, 0 \dots \frac{-Va}{\theta-a}\right), \left(0, \frac{V\theta}{\theta-\beta}, 0, \dots \frac{-V\beta}{\theta-\beta}\right), \text{ \&c. (Art. V.)},$$

—the discriminant of $V-rX=0$ will be an equation in r of order n, the first and last coefficients when reduced being the discriminants of V and X; and so the sinisters of the reciprocal equations to $U=0$ and $W=0$, the meaning of the others may be obtained from the geometry of $(n-1)$-dimensions. Thus the vanishing of the coefficient of r implies, from the general result above, that all linear loci, the coefficients of which make it vanish, cut $U=0$, $W=0$ in sections such that a simplicissimum of n vertices self-conjugate with respect to the section of $W=0$ is inscribed in that of $U=0$.

In this way I interpreted the contravariants of two quadric surfaces in a paper read here in December 1883; and so, if U and W be quinary quadrics, and

$$\sigma-r\tau+r^2\pi-r^3\tau'+r^4\sigma'=0$$

(where σ and σ' are the reciprocals of U and W), the expanded form of the discriminant of $V-rX=0$, σ, τ, π, τ', σ' are the Δ, θ, ϕ, θ', Δ' of the quadric surfaces $V=0$ and $X=0$ (Salmon, *Geometry of Three Dimensions*, p. 145).

If $\tau=0$, as above, it is possible to inscribe in $V=0$ a tetrahedron self-conjugate with respect to $X=0$.

If $\tau'=0$, there is a tetrahedron self-conjugate with respect to $V=0$, the faces of which touch $X=0$.

If $\pi=0$, the edges of a tetrahedron, self-conjugate with respect to either $V=0$ or $X=0$, touch the other.

The condition that a tetrahedron may be inscribable in $X=0$, which has two pairs of opposite edges on $V=0$, is

$$4\sigma\tau\pi=\tau^3+8\sigma^2\tau',$$

and that it may be possible to find a tetrahedron having two pairs of opposite edges on one of the quadrics and its four faces touching the other

$$4\sigma'\tau'\pi=\tau'^3+8\sigma'^2\tau.$$

It thus appears that two $(n+1)$-ary quadrics have $(n+2)$ independent invariants, and $(n+1)$ independent contravariants.

If $u=0$ and $w=0$ be the tangential equations to $U=0$ and $W=0$ respectively, $u-rw=0$ will be the tangential equation to the system of quadrics which touch the same system of common tangent loci as $U=0$ and $W=0$.

The equation to $u-rw=0$ in simplicissimum coordinates will be of the n^{th} order in r, and of the form

$$U - rT + \dots \pm r^{n-1}T' \mp r^n W = 0,$$

and the coefficients T, &c. of all the powers of r except 0 and n, will be the covariants of the two quadrics, and therefore these are $n-1$ in number.

The contravariants above might have been obtained in a similar way by forming the tangential equation to $U - rW = 0$, when they would be the coefficients of all the powers of r from 1 to $n-1$, and these covariants might have been obtained by employing a transformation similar to that of Art. V. to the contragredient variables.

It has not seemed advisable to attempt to carry the general investigation of the concomitants further, as the results would be more easily obtained and more clearly intelligible by working them out successively for four, five, and n-dimensional space.

I regret to have to add the following corrigenda to the previous papers:—

Vol. XIX., p. 444, l. 17, *for* 2^{n-1} *read* 2^{n+1}.

p. 455, l. 20, *for* $\dfrac{n-2p-1}{(p+1)\,\Sigma\dfrac{1}{A_1}}$ *read* $\dfrac{n-2p-1}{(p+1)\,\Sigma\dfrac{1}{A_1}}V$.

p. 460, l. 11, *for* $r_1^2 V$, *read* $r_1^2 V^2$.

p. 468, l. 3, *for* $\dfrac{h}{m^2}$, *read* $\dfrac{h}{m^2}V^2$.

l. 5, *for* hp^2, *read* hp^2V^2.

p. 474, l. 14, *for* $U'U''$ *read* $4U'U''$.

l. 17, *for* UU', *read* $4UU'$.

p. 478, l. 27, *for* Q_1, *read* Q_1^2.

[*P.S.*—It is only during the printing of this paper that I have noticed Mr. Brill's question (8431) and its solution in the *Reprint* from the *Educational Times*, Vol. XLVIII., p. 111. In this the equations of p. 335 are enunciated for six points on a circle.]